Weak compactness of sublevel sets

Warren B. Moors

Abstract. In this paper we provide a short proof of the fact that if X is a Banach space and $f : X \to \mathbb{R} \cup \{\infty\}$ is a proper function such that $f - x^*$ attains its minimum for every $x^* \in X^*$, then all the sublevels of f are relatively weakly compact. This result has many applications.

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Recently several authors (see, [1, 5, 7] to name just a few) have considered the result (or a special case of it) mentioned in the abstract. In this note we provide a short proof of this result. To date the only full proof of this result (known to the author) is [7, Theorem 2.4]; which is long and involved.

Theorem 1 *Let X be a Banach space and let $f : X \to \mathbb{R}\cup\{\infty\}$ be a proper function on X. If $f - x^*$ attains minimum for every $x^* \in X^*$ then for each $a \in \mathbb{R}$, $S(a) := \{(y, s) \in X \times \mathbb{R} : f(y) \leq s \leq a\}$ is relatively weakly compact.*

Proof: In this proof we will identify the dual of $X \times \mathbb{R}$ with $X^* \times \mathbb{R}$. We will also consider $X \times \mathbb{R}$ endowed with the norm $\|(x, r)\|_1 := \|x\| + |r|$ and note that with this norm, $(X \times \mathbb{R}, \|\cdot\|_1)$ is a Banach space. We shall apply James' theorem, [3, Theorem 5], in $X \times \mathbb{R}$. Let $H := \{(x, r) \in X \times \mathbb{R} : r = 0\}$ and define $T : (X \times \mathbb{R}) \setminus H \to (X \times \mathbb{R}) \setminus H$ by, $T(x, r) := r^{-1}(x, -1)$. Then T is a bijection. In fact, T is a homeomorphism when $(X \times \mathbb{R}) \setminus H$ is considered with the relative weak topology. Note that since f is bounded below we may assume, after possibly translating, that $1 = \inf_{x\in X} f(x)$. Our proof relies upon the *Fenchel conjugate*, $p : X^* \to \mathbb{R}$ *of* f, which is defined by,

$$p(x^*) := \sup_{x\in X}[x^*(x) - f(x)] = -\inf_{x\in X}[f(x) - x^*(x)] = -\min_{x\in X}[f(x) - x^*(x)] = \max_{x\in X}[x^*(x) - f(x)].$$

It is routine to check that p is convex on X^*. We claim that $\overline{\mathrm{co}}[T(\mathrm{epi}(f)) \cup \{(0, 0)\}]$ is weakly compact. To show this, it is sufficient, because of James' theorem, to show that every non-zero continuous linear functional attains its maximum value over $T(\mathrm{epi}(f)) \cup \{(0, 0)\}$. To this end, let $(x^*, r) \in (X^* \times \mathbb{R}) \setminus \{(0, 0)\}$. We consider two cases.

Case (I) Suppose that for every $0 < \lambda$, $p(\lambda x^*) \leq \lambda r$. Then $x^*(x) - \lambda^{-1} f(x) \leq r$ for all $x \in X$ and all $0 < \lambda$. Let $(y, s) \in \mathrm{epi}(f)$ and let $0 < \lambda$. Then,

$$(x^*, r)(T(y, s)) = s^{-1}(x^*(y) - r) \leq s^{-1}(x^*(y) - [x^*(y) - \lambda^{-1} f(y)]) = s^{-1} f(y)\lambda^{-1} \leq \lambda^{-1}$$

since $f(y) \leq s$. As $0 < \lambda$ was arbitrary, $(x^*, r)(T(y, s)) \leq 0 = (x^*, r)(0, 0)$. Thus, (x^*, r) attains it maximum value over $T(\mathrm{epi}(f)) \cup \{(0, 0)\}$ at $(0, 0)$.

Case(II) Suppose that for some $0 < \lambda$, $\lambda r < p(\lambda x^*)$. Then, since the mapping, $\lambda' \mapsto p(\lambda' x^*)$, is real-valued and convex, it is continuous. Furthermore, it follows, from the intermediate value theorem applied to the function $g : [0, \lambda] \to \mathbb{R}$, defined by,

$$g(\lambda') := p(\lambda' x^*) - \lambda' r \quad \text{for all } \lambda' \in [0, \lambda],$$

that there exists a $0 < \mu < \lambda$ such that $g(\mu) = 0$, i.e., $p(\mu x^*) = \mu r$, since $g(0) = -1 < 0 < g(\lambda)$. Thus, $\mu(x^*, r) = (\mu x^*, p(\mu x^*))$. Choose $z \in X$ such that $p(\mu x^*) = \mu x^*(z) - f(z)$. We claim that (x^*, r) attains its maximum value over $T(\text{epi}(f)) \cup \{(0,0)\}$ at $T(z, f(z)) = f(z)^{-1}(z, -1)$. Now,

$$\begin{aligned}(x^*, r)(T(z, f(z)) &= f(z)^{-1}(x^*(z) - r) = f(z)^{-1}(x^*(z) - [\mu^{-1}p(\mu x^*)]) \\ &= f(z)^{-1}(x^*(z) - [x^*(z) - \mu^{-1}f(z)]) = \mu^{-1} > 0.\end{aligned}$$

On the other hand, if $(y, s) \in \text{epi}(f)$ then

$$\begin{aligned}(x^*, r)(T(y, s)) &= s^{-1}(x^*(y) - r) = s^{-1}(x^*(y) - [\mu^{-1}p(\mu x^*)]) \\ &\leq s^{-1}(x^*(y) - [x^*(y) - \mu^{-1}f(y)]) = s^{-1}f(y)\mu^{-1} \leq \mu^{-1} = (x^*, r)(T(z, f(z)))\end{aligned}$$

since $f(y) \leq s$. Note also that $(x^*, r)(0, 0) = 0 < \mu^{-1} = (x^*, r)(T(z, f(z))$. Therefore, by James' theorem [3, Theorem 5], $\overline{\text{co}}[T(\text{epi}(f)) \cup \{(0,0)\}]$ is weakly compact.

Let $1 \leq a$, then $T(S(a)) \subseteq \overline{\text{co}}[T(\text{epi}(f)) \cup \{(0,0)\}] \cap \{(x, r) \in X \times \mathbb{R} : r \leq -a^{-1}\}$; which is weakly compact. Therefore, $S(a) \subseteq T^{-1}(\overline{\text{co}}[T(\text{epi}(f)) \cup \{(0,0)\}] \cap \{(x, r) \in X \times \mathbb{R} : r \leq -a^{-1}\})$; which completes the proof. □

For each $a \in \mathbb{R}$, let $L(a) := \{x \in X : f(x) \leq a\}$. It follows from Theorem 1 that if X is a Banach space, $f : X \to \mathbb{R} \cup \{\infty\}$ is a proper function on X and $f - x^*$ attains minimum for every $x^* \in X^*$ then, for each $a \in \mathbb{R}$, $L(a)$ is relatively weakly compact.

Theorem 1 has many applications, see [1, Section 10.6] and [2,4–7] to name only some of them. It also recaptures James' theorem on weak compactness. Indeed, if C is a closed and convex subset of a Banach space X that has the property that every continuous linear function on X attains it maximum value over C, then the function $f : X \to \mathbb{R} \cup \{\infty\}$ defined by,

$$f(x) := \begin{cases} 1 & \text{if } x \in C \\ \infty & \text{if } x \notin C \end{cases}$$

satisfies the hypotheses in Theorem 1. Hence, $C = L(1)$ is weakly compact.

In the proof of Theorem 1 we exploited the simple, but under utilised fact, that if $T : B \to B$ is a weak-to-weak (not necessarily linear) homeomorphism on a subset B of a Banach space X, then a subset C of B is weakly compact in X if, and only if, $T(C)$ is a weakly compact subset of B, (endowed with the relative weak topology).

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