Weak compactness of sublevel sets

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Abstract. In this paper we provide a short proof of the fact that if $X$ is a Banach space and $f : X \to \mathbb{R} \cup \{\infty\}$ is a proper function such that $f - x^*$ attains its minimum for every $x^* \in X^*$, then all the sublevels of $f$ are relatively weakly compact. This result has many applications.


Keywords: weakly compact sets, James’ theorem, sublevel sets

Recently several authors (see, [1, 5, 7] to name just a few) have considered the result (or a special case of it) mentioned in the abstract. In this note we provide a short proof of this result. To date the only full proof of this result (known to the author) is [7, Theorem 2.4]; which is long and involved.

**Theorem 1** Let $X$ be a Banach space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper function on $X$. If $f - x^*$ attains minimum for every $x^* \in X^*$ then for each $a \in \mathbb{R}$, $S(a) := \{(y, s) \in X \times \mathbb{R} : f(y) \leq s \leq a\}$ is relatively weakly compact.

**Proof:** In this proof we will identify the dual of $X \times \mathbb{R}$ with $X^* \times \mathbb{R}$. We will also consider $X \times \mathbb{R}$ endowed with the norm $\|(x, r)\|_1 := \|x\| + |r|$ and note that with this norm, $(X \times \mathbb{R}, \|\cdot\|_1)$ is a Banach space. We shall apply James’ theorem, [3, Theorem 5], in $X \times \mathbb{R}$. Let $H := \{(x, r) \in X \times \mathbb{R} : r = 0\}$ and define $T : (X \times \mathbb{R}) \setminus H \to (X \times \mathbb{R}) \setminus H$ by, $T(x, r) := r^{-1}(x, -1)$. Then $T$ is a bijection. In fact, $T$ is a homeomorphism when $(X \times \mathbb{R}) \setminus H$ is considered with the relative weak topology. Note that since $f$ is bounded below we may assume, after possibly translating, that $1 = \inf_{x \in X} f(x)$.

Our proof relies upon the Fenchel conjugate, $p : X^* \to \mathbb{R}$ of $f$, which is defined by,

$$p(x^*) := \sup_{x \in X} [x^*(x) - f(x)] = -\inf_{x \in X} [f(x) - x^*(x)] = -\min_{x \in X} [f(x) - x^*(x)] = \max_{x \in X} [x^*(x) - f(x)].$$

It is routine to check that $p$ is convex on $X^*$. We claim that $\co(T(\text{epi}(f)) \cup \{(0, 0)\})$ is weakly compact. To show this, it is sufficient, because of James’ theorem, to show that every non-zero continuous linear functional attains its maximum value over $T(\text{epi}(f)) \cup \{(0, 0)\}$. To this end, let $(x^*, r) \in (X^* \times \mathbb{R}) \setminus \{(0, 0)\}$. We consider two cases.

**Case (I)** Suppose that for every $0 < \lambda$, $p(\lambda x^*) \leq \lambda r$. Then $x^*(x) - \lambda^{-1} f(x) \leq r$ for all $x \in X$ and all $0 < \lambda$. Let $(y, s) \in \text{epi}(f)$ and let $0 < \lambda$. Then,

$$(x^*, r)(T(y, s)) = s^{-1}(x^*(y) - r) \leq s^{-1}(x^*(y) - [x^*(y) - \lambda^{-1} f(y)]) = s^{-1} f(y) \lambda^{-1} \leq \lambda^{-1}$$

since $f(y) \leq s$. As $0 < \lambda$ was arbitrary, $(x^*, r)(T(y, s)) \leq 0 = (x^*, r)(0, 0)$. Thus, $(x^*, r)$ attains it maximum value over $T(\text{epi}(f)) \cup \{(0, 0)\}$ at $(0, 0)$.

**Case (II)** Suppose that for some $0 < \lambda$, $\lambda r < p(\lambda x^*)$. Then, since the mapping, $\lambda' \mapsto p(\lambda' x^*)$, is real-valued and convex, it is continuous. Furthermore, it follows, from the intermediate value theorem applied to the function $g : [0, \lambda] \to \mathbb{R}$, defined by,

$$g(\lambda') := p(\lambda' x^*) - \lambda' r \quad \text{for all} \quad \lambda' \in [0, \lambda],$$

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that there exists a \( 0 < \mu < \lambda \) such that \( g(\mu) = 0 \), i.e., \( p(\mu x^*) = \mu r \), since \( g(0) = -1 < 0 < g(\lambda) \).

Thus, \( \mu(x^*, r) = (\mu x^*, p(\mu x^*)) \).

Choose \( z \in X \) such that \( p(\mu x^*) = \mu x^*(z) - f(z) \). We claim that \((x^*, r)\) attains its maximum value over \( T(\text{epi}(f)) \cup \{(0,0)\} \) at \( T(z, f(z)) = f(z)^{-1}(z, -1) \). Now, 

\[
(x^*, r)(T(z, f(z))) = f(z)^{-1}(x^*(z) - r) - f(z)^{-1}(x^*(z) - [\mu^{-1}p(\mu x^*)]) = f(z)^{-1}(x^*(z) - [x^*(z) - \mu^{-1}f(z)]) = \mu^{-1} > 0.
\]

On the other hand, if \((y, s) \in \text{epi}(f)\) then 

\[
(x^*, r)(T(y, s)) = s^{-1}(x^*(y) - r) - s^{-1}(x^*(y) - [\mu^{-1}p(\mu x^*)]) \\
\leq s^{-1}(x^*(y) - \mu^{-1}f(y))] = s^{-1}f(y)\mu^{-1} \leq \mu^{-1} = (x^*, r)(T(z, f(z)))
\]

since \( f(y) \leq s \). Note also that \((x^*, r)(0,0) = 0 < \mu^{-1} = (x^*, r)(T(z, f(z))) \). Therefore, by James’ theorem [3, Theorem 5], \( \text{co}[T(\text{epi}(f)) \cup \{(0,0)\}] \) is weakly compact.

Let \( 1 \leq a \), then \( T(S(a)) \subseteq \text{co}[T(\text{epi}(f)) \cup \{(0,0)\}] \cap \{(x, r) \in X \times \mathbb{R} : r \leq -a^{-1}\} \); which is weakly compact. Therefore, \( S(a) \subseteq T^{-1}(\text{co}[T(\text{epi}(f)) \cup \{(0,0)\}] \cap \{(x, r) \in X \times \mathbb{R} : r \leq -a^{-1}\}) \); which completes the proof. \( \square \)

For each \( a \in \mathbb{R} \), \( L(a) := \{ x \in X : f(x) \leq a \} \). It follows from Theorem 1 that if \( X \) is a Banach space, \( f : X \to \mathbb{R} \cup \{\infty\} \) is a proper function on \( X \) and \( f - x^* \) attains minimum for every \( x^* \in X^* \) then, for each \( a \in \mathbb{R} \), \( L(a) \) is relatively weakly compact.

Theorem 1 has many applications, see [1, Section 10.6] and [2,4–7] to name only some of them. It also recaptures James’ theorem on weak compactness. Indeed, if \( C \) is a closed and convex subset of a Banach space \( X \) that has the property that every continuous linear function on \( X \) attains its maximum value over \( C \), then the function \( f : X \to \mathbb{R} \cup \{\infty\} \) defined by,

\[
f(x) := \begin{cases} 
1 & \text{if } x \in C \\
\infty & \text{if } x \notin C
\end{cases}
\]

satisfies the hypotheses in Theorem 1. Hence, \( C = L(1) \) is weakly compact.

In the proof of Theorem 1 we exploited the simple, but under utilised fact, that if \( T : B \to B \) is a weak-to-weak (not necessarily linear) homeomorphism on a subset \( B \) of a Banach space \( X \), then a subset \( C \) of \( B \) is weakly compact in \( X \) if, and only if, \( T(C) \) is a weakly compact subset of \( B \), (endowed with the relative weak topology).

References


