A SURVEY ON TOPOLOGICAL GAMES AND THEIR APPLICATIONS IN ANALYSIS

JILING CAO AND WARREN B. MOORS*

Abstract. In this survey article we shall summarise some of the recent progress that has occurred in the study of topological games as well as their applications to abstract analysis. The topics given here do not necessarily represent the most important problems from the area of topological games, but rather, they represent a selection of problems that are of interest to the authors.

2000 *Mathematics Subject Classification*:Primary 46B20; Secondary 91A44

Keywords and Phrases: Banach-Mazur game, Baire spaces, fragmentability

1. Introduction

Although a combinatorial game, in a mathematical form, was described probably for the first time at the beginning of the 17^{th} century, the notion of a positional game with perfect information was introduced in the famous monograph of von Neumann and Morgenstern [37]. In that monograph, the authors considered finite positional games and proved that each such game can be reduced to a matrix game, and moreover, if the (finite) positional game is one with perfect information, then the corresponding matrix game has a saddle point. However, infinite positional games with perfect information were discovered a little earlier. In 1935, Stanistaw Mazur proposed a game related to the Baire Category Theorem, which is described in Problem No. 43 of the Scottish Book; its solution, given by Stefan Banach, is dated August 4, 1935. This game, now known as the *Banach-Mazur game*, is the first infinite positional game with the perfect information. Unfortunately, because of World War II, the problems in the Scottish Book were not widely known until the mid fifties. So although, historically, the Banach-Mazur game was the first infinite positional game with perfect knowledge we shall delay its description until Section 2. Instead, we shall begin with the description of the simpler Choquet game given in [10].

Let (X,τ) be a topological space. The *Choquet game* $Ch(X)$, played on (X,τ) is played between two players α and β who, alternately, select nonempty open subsets of X. Player β goes first (always!) and chooses a nonempty open subset B_1 of X. Player α must respond by selecting a nonempty open subset $A_1 \subseteq B_1$. Following this player β must select another

*The authors are supported by the Marsden Fund research grant, UOA0422, administered by the Royal Society of New Zealand.

nonempty open subset $B_2 \subseteq A_1 \subseteq B_1$ and in turn the player α must again respond by selecting a nonempty open subset $A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1$. In general, β selects any nonempty open subset B_n of the last move A_{n-1} of α and the latter player answers by choosing a nonempty open subset A_n of the set B_n, just chosen by β. Acting in this way, the players α and β "produce" a sequence of nonempty open sets

$$B_1 \supseteq A_1 \supseteq B_2 \supseteq A_2 \supseteq \cdots B_n \supseteq A_n \supseteq \cdots$$

which is called a *play* and will be denoted by $((A_n, B_n))_{n\in\mathbb{N}}$. We shall declare that the player α *wins* a play $((A_n, B_n))_{n\in\mathbb{N}}$ of the Choquet game $Ch(X)$ if $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$. Otherwise, the player β is said to have *won* this play. A finite sequence $((A_k, B_k))_{k=1}^n$ of pairs of nonempty open sets consisting of the first n moves of the Choquet game is called a *partial play*. It is clear that every partial play can be extended to a play.

By a *strategy* for the player α we mean a rule that species each move of the player α in every possible situation. More precisely, a strategy $t = (t_n : n \in \mathbb{N})$ for α is a sequence of τ-valued mappings such that

$$\varnothing \neq t_n(B_1, B_2, \ldots, B_n) \subseteq B_n \text{ for all } n \in \mathbb{N}.$$

The domains of the t_n's are families of finite sequences of nonempty open sets defined inductively as follows:

$$\mathrm{Dom}(t_1) := \{(B_1) : B_1 \text{ is a nonempty open subset of } X\};$$
$$\mathrm{Dom}(t_{n+1}) := \{(B_1, ..., B_n, B_{n+1}) : (B_1, ..., B_n) \in \mathrm{Dom}(t_n), \text{ and } B_{n+1} \subseteq t_n(B_1, ..., B_n)\}.$$

Each element of $\bigcup_{n\in\mathbb{N}} \mathrm{Dom}(t_n)$ is called a *partial t-play* and an infinite sequence $(B_n)_{n\in\mathbb{N}}$ of nonempty open subsets of X is called a *t-play* provided $(B_1 \ldots, B_n) \in \mathrm{Dom}(t_n)$ for all $n \in \mathbb{N}$. Of course, one can consider the space of all t-plays $P(t)$ endowed with the Baire metric d, that is, if $p = (B_n : n \in \mathbb{N})$ and $p' = (B'_n : n \in \mathbb{N})$ are two t-plays, then $d(p, p') = 0$ if $p = p'$ and otherwise $d(p, p') = 1/n$, where $n := \min\{k \in \mathbb{N} : B_k \neq B'_k\}$. It is clear that $(P(t), d)$ is a complete metric space. A strategy t is called a *winning strategy* for the player α if α wins each t-play in $Ch(X)$. Strategies and winning strategies for the player β in $Ch(X)$ can be defined similarly. A space X is called *weakly α-favourable* if α has a winning strategy in $Ch(X)$.

Given two strategies t and σ for the player α in $Ch(X)$, we say that σ *refines* t, denoted by $t \preceq \sigma$, if each σ-play is a t-play or, alternatively, if for each $n \in \mathbb{N}$, $\mathrm{Dom}(\sigma_n) \subseteq \mathrm{Dom}(t_n)$ and $\sigma_n(B_1, \ldots B_n) \subseteq t_n(B_1, \ldots B_n)$ for each $(B_1, \ldots B_n) \in \mathrm{Dom}(\sigma_n)$. Note that if t is a winning strategy for the player α and $t \preceq \sigma$ then σ is also a winning strategy for the player α. We shall call a family $\mathscr{P}$ of nonempty open subsets of a space X a *pseudo-base* (sometimes, *π-base*) for X if for every nonempty open set $U \subseteq X$, there is some $P \in \mathscr{P}$ with $P \subseteq U$. It is easy to see that if $\mathscr{P}$ is a pseudo-base for X, then for any strategy t for the player α in $Ch(X)$, there exists a strategy $\sigma := (\sigma_n : n \in \mathbb{N})$ for α such that $t \preceq \sigma$ and $\sigma_n(B_1, \ldots, B_n) \in \mathscr{P}$ for all

$(B_1, \ldots, B_n) \in \mathrm{Dom}(\sigma_n)$. Based on these observations we shall often restrict the moves of players in the Choquet game to a pre-chosen pseudo-base (or base) of a space.

The motivation of Choquet [10] for introducing the Choquet game was to characterise metric spaces with certain completeness properties and to study the set of extreme points of a compact convex set in a locally convex linear space. In the literature, there are many generalisations and extensions of the Choquet game. For instance, some of the results in [10] were extended to non-metrizable spaces in [7]. The readers should refer to the excellent survey articles [46] or [42] for more information. In the next few sections, we shall explore some modifications of the Choquet game and also some of their applications to abstract analysis and topology.

2. Applications of games to Baire spaces

In this section, we shall first present a characterisation of Baire spaces in terms of the Choquet game and then give some of its applications. Also, we shall present the original setting of the Banach-Mazur game, as well as, some of its applications. Recall that a space X is a *Baire space* if the intersection of any sequence of dense open subsets of X is dense. Further, if every closed subspace of X is Baire, then X is called a *hereditarily Baire* space. Of course, Baire spaces can be defined in terms of sets of the second category, refer to [14]. Among the known examples of Baire spaces are complete metric spaces, (locally) compact spaces and Čech complete spaces.

The following theorem, first discovered by Oxtoby [39], and later proved in [25] and [43] independently, gives a characterisation of Baire spaces in terms of the Choquet game.

Theorem 2.1 ([25, 39, 43])**.** *A space X is a Baire space if, and only if, the player β does* **not** *have a winning strategy in $Ch(X)$.*

It is an immediate consequence of Theorem 2.1 that weakly α-favourable spaces are Baire spaces. We begin our discussion of applications of Theorem 2.1 with the problem of whether the product of two (or a family of) Baire spaces is still Baire. This question can be tracked to Sikorski [44]. First of all, by Theorem 2.1 and remarks in Section 1, the product of a family of Baire spaces is Baire if, and only if, all countable subproducts are Baire. It is known that the product of a (hereditarily) Baire space with any complete metric space is (hereditarily) Baire. On the other hand, Oxtoby [40] showed that **CH** implies that there is a metric Baire space whose square is not Baire. Furthermore, Aarts and Lutzer [1] constructed a metric hereditarily Baire space whose square is not hereditarily Baire. Finally, in 1978, Fleissner and Kunen [13] presented a metric Baire space whose square is not Baire without using any additional hypothesis. Due to a mistake in [11] it was claimed that Example 1 of [13] gives two metric hereditarily Baire spaces whose product is not Baire. Recently however, Chaber and Pol [9] have corrected this error by showing that the product of any family of metric hereditarily Baire spaces is Baire, and further asked whether the product of a metric Baire space and a metric hereditarily Baire space must be Baire. By applying Theorem 2.1, Moors [33] provided an affirmative answer to this question.

Theorem 2.2 ([33]). *The product of a Baire space and a metric hereditarily Baire space is Baire.*

Now, we turn our attention to McCoy's problem on the Vietoris hyperspace of a Baire space. Given a space X, let 2^X denote the *hyperspace* of X consisting of all nonempty closed subsets of X endowed with the Vietoris topology. Recall that a canonical base for this topology is given by all subsets of $2^X \setminus \{\varnothing\}$ having the form

$$\langle \mathscr{U} \rangle := \{F \in 2^X : F \subseteq \bigcup \mathscr{U},\ F \cap V \neq \varnothing \text{ for any } V \in \mathscr{U}\},$$

where $\mathscr{U}$ runs over the finite families of nonempty open subsets of X [28]. The problem of when or whether the hyperspace 2^X of a Baire space must be Baire was first considered by McCoy [26]. He proved that if X is a Baire space with a countable pseudo-base, then 2^X is Baire, but left the general case as an open question. Recently, Cao et al have considered this question and proved the following two results with the help of Theorem 2.1.

Theorem 2.3 ([3]). *Let X be a Hausdorff space. If 2^X is Baire, then X^n is Baire for all $n \in \mathbb{N}$.*

Theorem 2.4 ([8]). *Let X be a Hausdorff space. If X^ω is Baire, then 2^X is Baire.*

The above two theorems establish a nice link between the Baireness of the hyperspace and that of the product spaces. As we mentioned previously, there is a metric Baire space whose square is not Baire. As a corollary of Theorem 2.3, there exists a metric Baire space X such that 2^X is not Baire, that is, the general answer to McCoy's problem is negative. On the other hand, Theorem 2.4 together with Theorem 1.1 of [9] implies that the hyperspace 2^X of a metric hereditarily Baire space is Baire, which answers affirmatively an oral question of Moors. Further, two examples were provided in [8] to show that neither of the converses of Theorem 2.3 and Theorem 2.4 hold.

Before we present some more problems and applications we need to introduce a variation of the Choquet game, called the *$G_S(D)$-game*. Let X be a topological space and let $D \subseteq X$ be a dense subset of X. The rules for playing the $G_S(D)$-game are the same as for the $Ch(X)$-game. The only distinction between them is in the definition of a win. We shall say that α *wins* a play $((A_n, B_n))_{n\in\mathbb{N}}$ of the $G_S(D)$-game if $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$, and each sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A_n \cap D$ has a cluster point in X. Otherwise the player β is said to have won this play. The space X is called a *strongly Baire* space if it is regular and there is a dense subset $D \subseteq X$ such that the player β does not possess a winning strategy in the $G_S(D)$-game played on X. The motivation to introducing the $G_S(D)$-game and the class of strongly Baire spaces was to study the problem when a semitopological group is a (paratopological) topological group. Recall that a *semitopological group* (*paratopological group*) is a group endowed with a topology for which multiplication is separately (jointly) continuous and a *topological group* is a paratopological group whose inversion is also continuous. In [2], Bouziad improved results of both Montgomery [31] and Ellis [12], and also answered a question of Pfister in [41] by showing that each Čech-complete semitopological group is a topological group. Since the Sorgenfrey line is a Baire paratopological group which is not a topological group, there is no hope to improve Bouziad's result by replacing "Čech-complete" with "Baire". However, by applying the notion of strongly Baire space, Kenderov et al [18] improved Bouziad by proving the following result.

Theorem 2.5 ([18])**.** *Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is a strongly Baire space, then $(G, \cdot)$ is a topological group.*

In a recent paper, Cao et al [4] used the strong Baireness and Baireness of function spaces to characterise metrizability of a manifold. By a *manifold M* it is meant a connected, Hausdorff, locally Euclidean space. The function space we shall consider is $C_k(M)$, the space of all continuous real-valued functions defined on M endowed with the compact-open topology.

Theorem 2.6 ([4])**.** *The following are equivalent for a manifold M:*

(i) *M is metrizable;*
(ii) *$C_k(M)$ is a strongly Baire space;*
(iii) *$C_k(M)$ is a Baire space.*

In the last part of this section, we shall present the original setting of the Banach-Mazur game as well as some of its applications. As mentioned in Section 1, the original version of the Banach-Mazur appeared in the Scottish Book under problem No. 43, where two players alternately select nonempty intervals of the real line. A more general setting for the Banach-Mazur game was given by Oxtoby in [39]. Let X be a topological space and let $A \subseteq X$. In the *Banach-Mazur game $BM(A)$*, two players α and β alternately select nonempty open sets $B_1 \supseteq A_1 \supseteq B_2 \supseteq A_2, \supseteq \cdots$ just as in the Choquet game $Ch(X)$. We shall declare that the player α *wins* a play $((A_n, B_n))_{n\in\mathbb{N}}$ if $\bigcap_{n\in\mathbb{N}} A_n \subseteq A$. Otherwise, the player β is said to have *won* the play. Strategies for both β and α in $BM(A)$ are defined in a similar fashion to those in $Ch(X)$. In contrast to the Choquet game, the Banach-Mazur can be used to test whether a given subset A is "big" in X, as shown in the next theorem.

Theorem 2.7 ([39, 17])**.** *Let X be a topological space and let $A \subseteq X$. Then player α has a winning strategy in $BM(A)$ if, and only if, A is residual in X.*

The Banach-Mazur game and Theorem 2.7 have been applied to obtain a topological closed graph theorem in [32]. Recall that a mapping $f : X \to Y$ from a space X into a space Y is said to be *nearly continuous* if $f^{-1}(U) \subseteq \mathrm{int}\overline{f^{-1}(U)}$ for each open set $U \subseteq Y$. A sequence $(\mathscr{V}_n : n \in \mathbb{N})$ of covers of a space X is said to be *complete* if each filter $\mathscr{F}$ on X that is $\mathscr{V}_n$-small for each $n \in \mathbb{N}$ has $\bigcap_{F\in\mathscr{F}} \overline{F} \neq \varnothing$. Moreover, a cover $\mathscr{V}$ of X is called *exhaustive* provided every nonempty set A of X has a nonempty relatively open subset of the form $A \cap V$ with $V \in \mathscr{V}$. Finally, a regular space X is called *partition complete* [30] if it has a complete sequence of exhaustive covers.

Theorem 2.8 ([32])**.** *Every nearly continuous mapping $f : X \to Y$ with a closed graph from a Baire space X into a partition complete space Y is continuous.*

3. Several modifications of the Choquet game

In this section we shall describe several variations of the Choquet game. These modifications will give new characterisations of some known topological properties such as fragmentability, the Namioka property and membership in the class of weakly Stegall spaces.

The first of these modifications is the "fragmenting game". Let τ and τ' be topologies on a set X. On X we shall consider the $G(X, \tau, \tau')$-game played between two players α and β. The player β goes first (always!) and chooses a nonempty subset B_1 of X. Player α must then respond by choosing a nonempty τ-relatively open subset A_1 of B_1. Following this player β must select another nonempty subset $B_2 \subseteq A_1 \subseteq B_1$ and in turn α must again respond by selecting a nonempty τ-relatively open subset A_2 of B_2. In general, β selects any nonempty subset B_n of the last move A_{n-1} of α and the latter player answers by choosing a nonempty τ-relatively open subset A_n of the set B_n, just chosen by β. Acting in this way, the players α and β "produce" a sequence of nonempty sets

$$B_1 \supseteq A_1 \supseteq B_2 \supseteq A_2 \supseteq \cdots B_n \supseteq A_n \supseteq \cdots$$

which is called a *play* and will be denoted by $((A_n, B_n))_{n\in\mathbb{N}}$. The winning rule is connected with the topology τ'. The player α is said to have *won* a play $((A_n, B_n))_{n\in\mathbb{N}}$ if the set $\bigcap_{n\in\mathbb{N}} A_n$ is either empty or contains exactly one point x and for every τ'-open neighbourhood U of x there exists an $n \in \mathbb{N}$ such that $A_n \subseteq U$. Otherwise the player β is said to have won. All other concepts related to this game, such as strategies, winning strategies, t-plays and partial t-plays etc. are defined in a similar fashion to those in the Choquet game. In the special case when τ' is the trivial topology (consisting of the empty set and the whole space X) we shall simply denote the $G(X, \tau, \tau')$-game by: $G(X, \tau)$.

Let X be a topological space and let ρ be some metric defined on it (not necessarily generating the topology on X). For any $\varepsilon > 0$ we will say that *X is fragmented by ρ down to ε* if for every nonempty subset A of X there exists a nonempty relatively open subset B of A such that $\rho - \text{diam}(B) < \varepsilon$. Following Jayne and Rogers [16], we say that a topological space X is *fragmentable* if there exists a metric ρ defined on X such that for every $\varepsilon > 0$, X is fragmented by ρ down to ε. In such a case it is said that the metric ρ *fragments* X. The next theorem, discovered by Kenderov and Moors in [21] justifies the name: "fragmenting game".

Theorem 3.1 ([21, 22])**.** *A topological space (X, τ) is fragmentable if, and only if, the player α has a winning strategy in the $G(X, \tau)$-game.*

A set-valued mapping $\varphi : X \to \mathscr{P}(Y)$ is said to be *minimal* if for every pair of open subsets $U \subseteq X$ and $V \subseteq Y$ with $\varphi(U) \cap V \neq \varnothing$ there exists a nonempty open subset $W \subseteq U$ such that $\varphi(W) \subseteq V$ and a topological space X is said to be a *weakly Stegall* space [34] if for every complete metric M and every nonempty-valued minimal mapping $\varphi : M \to \mathscr{P}(X)$, φ is single-valued at some point of M. The class of weakly Stegall spaces can be characterised in terms of the $G(X, \tau)$-game.

Theorem 3.2 ([19, 34])**.** *A topological space (X, τ) is weakly Stegall if, and only if, the player β does* **not** *have a winning strategy in the $G(X, \tau)$-game.*

The previous theorem enables us to establish the relationship between weakly Stegall and fragmentable spaces. Specifically, the distinction between being fragmentable and being weakly Stegall is precisely the distinction between the player α having a winning strategy in the $G(X, \tau)$-game and the player β not having a winning strategy in the $G(X, \tau)$-game.

Theorem 3.2 can also be used to obtain many new facts concerning weakly Stegall spaces, see [19] and [34].

The following theorem improves Theorem 3.1.

Theorem 3.3 ([20])**.** *A topological space (X, τ) is fragmentable by a metric ρ whose topology is at least as fine as a topology τ' if, and only if, there exists a winning strategy for the player α in the $G(X, \tau, \tau')$-game.*

Recall that a (single-valued) mapping $f : X \to Y$ between two spaces X and Y is said to be *quasicontinuous* if for every pair of open subsets $U \subseteq X$ and $V \subseteq Y$ with $f(U) \cap V \neq \varnothing$ there exists a nonempty open subset $W \subseteq U$ such that $f(W) \subseteq V$.

Theorem 3.4 ([20])**.** *Let τ, τ' be T_1 topologies on a set X. Suppose that for every τ'-open set U and every point $x \in U$ there exists a τ'-neighbourhood V of x such that $\overline{V}^{\tau} \subset U$. Then the following conditions are equivalent:*

(i) *β does* **not** *possess a winning strategy in the $G(X, \tau, \tau')$-game;*
(ii) *every quasicontinuous mapping $f : Z \to (X, \tau)$ from a complete metric space Z into (X, τ) has at least one point of τ'- continuity;*
(iii) *every quasicontinuous mapping $f : Z \to (X, \tau)$ from an α-favourable space Z into (X, τ) is τ'-continuous at the points of a subset which is of second category in every nonempty open subset of Z.*

Similarly to before we see that the distinction between (X,τ) being fragmentable by a metric whose topology is at least as fine as τ' and (X,τ) having the property that: "every quasicontinuous mapping $f : Z \to (X,\tau)$ from a complete metric space Z has at least one point of τ'-continuity", is the same as the distinction between α having a winning strategy in the $G(X,\tau,\tau')$-game and β not having a winning strategy in the $G(X,\tau,\tau')$-game.

A space X (or its topology τ) is said to be *sigma-fragmented by a metric* ρ if, for every $\varepsilon > 0$, there exists a countable family $(X_i^\varepsilon)_{i\geq 1}$ of subsets of X such that:

(i) $X = \bigcup_{i\geq 1} X_i^\varepsilon$;
(ii) every X_i^ε, $i = 1, 2, 3, \ldots$, is fragmented by ρ down to ε.

Theorem 3.5 ([22]). *For a subset X of a Banach space E the following properties are equivalent:*

(i) *X admits a metric ρ which fragments the weak topology and whose topology is at least as fine as the norm topology (i.e., the player α has a winning strategy in the game $G(X, weak, norm)$);*
(ii) *X admits a metric ρ which fragments the weak topology and whose topology is at least as fine as the weak topology (i.e., the player α has a winning strategy in the game $G(X, weak, weak)$);*
(iii) *X is sigma-fragmented by the norm.*

In order to present some of the applications of this theorem we need another definition. We say that a subset Y of a topological space (X,τ) has *countable separation in X* if there is a countable family $\{O_n : n \in \mathbb{N}\}$ of open subsets of X such that for every pair $\{x,y\}$ with $y \in Y$ and $x \in X \setminus Y$, $\{x,y\} \cap O_n$ is a singleton for at least one $n \in \mathbb{N}$. If we denote by X_Σ the family of all subsets of X with countable separation in X then X_Σ is a σ-algebra that contains all then open subsets of X. Moreover, X_Σ is closed under the Souslin operation. For a completely regular topological space (X,τ) we shall say that X has *countable separation* if in some compactification bX, X has countable separation in bX. It is shown in [22] that if X has countable separation in one compactification then X has countable separation in every compactification and so we see that every Čech-analytic space has countable separation.

Theorem 3.6 ([22]). *Let B_E denote the closed unit ball of a Banach space E. If $(E, weak)$ has countable separation then E is sigma-fragmented by the norm.*

Theorem 3.7 ([22]). *If a regular Hausdorff space (X,τ) is sigma-fragmented by a metric ρ whose topology is at least as fine as τ then (X,τ) is fragmented by some metric d whose topology is at least as fine as τ.*

Theorem 3.8 ([22]). *Let (X,τ) be a topological space and let ρ be a metric which sigma-fragments X by means of sets with countable separation in X*

(i.e., the sets $(X_i^\varepsilon)_{i\geq 1}$ involved in the definition is sigma-fragmentability have countable separation in X). Then X is fragmentable.

Let $\mathscr{T}$ denote the class of all Banach spaces E for which every continuous mapping $f : Z \to (E, \text{weak})$ defined on a weakly α-favourable space Z is norm continuous at the points of a dense subset of Z.

Theorem 3.9 ([20])**.** *A Banach space E is in $\mathscr{T}$ if, and only if, the player β does* **not** *have a winning strategy in the $G(X, weak, weak)$-game.*

Yet again we see that games can be used to distinguish between topological properties. In this case we see that the distinction between a Banach space E being sigma-fragmented by the norm and being a member of $\mathscr{T}$ is equivalent to the distinction between α having a winning strategy in the $G(X, \text{weak}, \text{weak})$-game and β not having a winning strategy in the $G(X, \text{weak}, \text{weak})$-game.

Theorem 3.9 also has many other applications. For example it can be used to show that:

- $\mathscr{T}$ contains all the weakly Lindelöf Banach spaces;
- $E = l^\infty$ and $E = l^\infty/c_0$ do not belong to $\mathscr{T}$. In both cases there exists a weakly continuous mapping $h : Z \to E$ defined on a completely regular weakly α-favourable space Z which is nowhere norm continuous.
- $\mathscr{T}$ is stable under weak homeomorphisms (i.e., if $\mathscr{T}$ contains some Banach space E, then it contains any other Banach space that is weakly homeomorphic to E);
- A Banach space E is a member of $\mathscr{T}$ if, and only if, every quasi-continuous mapping $f : Z \to (E, \text{weak})$ defined on a complete metric space Z is densely norm continuous;
- A Banach space E is a member of $\mathscr{T}$ if, and only if, every quasi-continuous mapping $f : Z \to (E, \text{weak})$ defined on a complete metric space Z is weakly continuous at some point of Z.

4. Other games in abstract analysis

In this section we shall consider two more topological games which are perhaps more esoteric than the games considered previously.

Let X be a space, $\mathscr{F}$ a proper filter (or filterbase) in X. We shall consider the following *$G(\mathscr{F})$-game* played in X between players α and β: Player β goes first (always!) and chooses a point $x_1 \in X$. Player α responds by choosing a member $F_1 \in \mathscr{F}$. Following this, player β must select another (possibly the same) point $x_2 \in F_1$ and in turn player α must again respond to this by choosing a member $F_2 \in \mathscr{F}$. Repeating this procedure infinitely, the players α and β produce a sequence $p := ((x_n, F_n) : n \in \mathbb{N})$ with $x_{n+1} \in F_n$ for all $n \in \mathbb{N}$, called a *play* of the $G(\mathscr{F})$-game. We shall say that α *wins* a play of the $G(\mathscr{F})$-game if the sequence $(x_n : n \in \mathbb{N})$ has a cluster point in X. Otherwise, the player β is said to have *won* this play.

We shall call a pair $(\mathscr{F}, \sigma)$ a *σ-filter* (*σ-filterbase*) if $\mathscr{F}$ is a proper filter (filterbase) in X and σ is a winning strategy for player α in the $G(\mathscr{F})$-game. Finally, we say that a space X has *property* $(**)$ if $\bigcap\{\overline{F} : F \in \mathscr{F}\} \neq \varnothing$ for each σ-filterbase $(\mathscr{F}, \sigma)$ in X. The class of spaces having property $(**)$ includes all metric spaces [6], all Dieudoné-complete spaces, all function spaces $C_p(X)$ for compact Hausdorff spaces X, and all Banach spaces in their weak topologies [6]. Recall that a space X is a q-space if for every point $x \in X$, there is a sequence $(U_n : n \in \mathbb{N})$ of neighbourhoods of x such that if $x_n \in U_n$ for all $n \in \mathbb{N}$, the sequence $(x_n : n \in \mathbb{N})$ has a cluster point in X (which is not necessarily x itself). All first countable spaces and all Čech complete spaces are q-spaces.

The $G(\mathscr{F})$-game can be used to deduce some selection theorems.

Theorem 4.1 ([5]). *Let $f : X \to Y$ be a closed mapping from a regular T_1-space X with property $(**)$ onto a regular q-space Y. If $f^{-1}(y)$ is closed for every $y \in Y$, then there exists a quasicontinuous mapping $\varphi : Y \to X$ such that $(f \circ \varphi)(y) = y$ for all $y \in Y$.*

The last game we shall consider is the "Cantor-game" which was used in [35] to show that there exist Gâteaux differentiability spaces that are not weak Asplund.

Recall that a Banach space X is a called a *weak Asplund space* (*Gâteaux differentiability space*) if each continuous convex function defined on it is Gâteaux differentiable at the points of a dense G_δ subset (dense subset) of its domain.

We will say that a σ-ideal $\mathscr{A}$ of subsets on a topological space (X, τ) is *topologically stable* if $h(A) \in \mathscr{A}$ for each homeomorphism $h : (X, \tau) \to (X, \tau)$ and $A \in \mathscr{A}$. In the reminder of this paper, $\mathscr{A}$ will always denote a topologically stable σ-ideal on $(\{0,1\}^{\mathbb{N}}, \tau_p)$, where τ_p denotes the topology of pointwise convergence on $\mathbb{N}$. With this understanding, we can introduce the following notation.

Given a topological space (X, τ) that is homeomorphic to $(\{0,1\}^{\mathbb{N}}, \tau_p)$ and a topologically stable σ-ideal $\mathscr{A}$ on $(\{0,1\}^{\mathbb{N}}, \tau_p)$, we shall denote by $\mathscr{A}_{(X,\tau)}$, the induced σ-ideal on X defined by, $\mathscr{A}_{(X,\tau)} := \{h^{-1}(A) : A \in \mathscr{A}\}$ for some homeomorphism $h : (X, \tau) \to (\{0,1\}^{\mathbb{N}}, \tau_p)$. (Note: Since $\mathscr{A}$ is topologically stable, the definition of $\mathscr{A}_{(X,\tau)}$ is independent of the particular choice of homeomorphism $h : (X, \tau) \to (\{0,1\}^{\mathbb{N}}, \tau_p)$). When there is no ambiguity, we shall simply denote $\mathscr{A}_{(X,\tau)}$ by $\mathscr{A}_X$. In terms of this notation we can introduce a stronger notion of topological stability. A σ-ideal $\mathscr{A}$ on $(\{0,1\}^{\mathbb{N}}, \tau_p)$ is said to be *strongly topologically stable* if (i) $\mathscr{A}$ is topologically stable and (ii) for each clopen subset Y of $\{0,1\}^{\mathbb{N}}$ that is homeomorphic to $(\{0,1\}^{\mathbb{N}}, \tau_p)$, we have that $\mathscr{A}_Y \subseteq \mathscr{A}$.

Let (M, d) be a complete metric space without isolated points, R be a subset of M and $\mathscr{A}$ be a strongly topologically stable proper σ-ideal on $(\{0,1\}^{\mathbb{N}}, \tau_p)$. On M we consider the *Cantor-game* $\mathcal{C}^{\mathscr{A}}(R)$*-game* played between two players α and β. Player β goes first (always!) and chooses a family $B_0 := \{B_0^t : t = \varnothing\}$ consisting of a nonempty open set $B_0^{\varnothing}$ with d-diam$(B_0^{\varnothing}) < 1/2^0$. Player α must respond to this by choosing a family $A_0 := \{A_0^t : t = \varnothing\}$ consisting of a nonempty open set $A_0^{\varnothing}$ of $B_0^{\varnothing}$. Following this player β must select another family $B_1 := \{B_1^t : t \in \{0,1\}^1\}$ of nonempty open subsets such that; (i) $\varnothing = \overline{B_1^0} \cap \overline{B_1^1} \subseteq \overline{B_1^0} \cup \overline{B_1^1} \subseteq A_0^{\varnothing}$ and (ii) d-diam$(B_1^t) < 1/2^1$ for all $t \in \{0,1\}^1$. In turn, player α must again respond by selecting a family $A_1 := \{A_1^t : t \in \{0,1\}^1\}$ of nonempty open subsets such that $A_1^t \subseteq B_1^t$ for all $t \in \{0,1\}^1$.

Continuing this procedure indefinitely the players α and β produce a sequence $\{(A_n, B_n) : n \in \omega\}$ of ordered pairs of indexed families of nonempty open subsets with $A_n := \{A_n^t : t \in \{0,1\}^n\}$ and $B_n := \{B_n^t : t \in \{0,1\}^n\}$ that satisfy the following conditions; (i) $\varnothing = \overline{B_{n+1}^{t0}} \cap \overline{B_{n+1}^{t1}} \subseteq \overline{B_{n+1}^{t0}} \cup \overline{B_{n+1}^{t1}} \subseteq A_n^t \subseteq B_n^t$ for all $t \in \{0,1\}^n$ and (ii) d-diam$(B_n^t) < 1/2^n$ for all $t \in \{0,1\}^n$. Such a sequence will be called a *play* of the $\mathcal{C}^{\mathscr{A}}(R)$-game. We shall declare that α *wins* a play $\{(A_n, B_n) : n \in \omega\}$ of the $\mathcal{C}^{\mathscr{A}}(R)$-game if the set $K \backslash R \in \mathscr{A}_K$, where $K := \bigcap_{n=0}^{\infty} K_n$ and $K_n := \bigcup\{B_n^t : t \in \{0,1\}^n\}$. Otherwise the player β is said to have won this play. By a *strategy* σ for the player α, we mean a 'rule' that specifies each move of the player α in every possible situation. More precisely a strategy $\sigma := (\sigma_n : n \in \omega)$ for α is a sequence of functions such that (i) $\sigma_n(B_0, B_1, \dots B_n) := \{\sigma_n^t(B_0, B_1, \dots B_n) : t \in \{0,1\}^n\}$; (ii) $\varnothing \neq \sigma_n^t(B_0, B_1, \dots B_n) \subseteq B_n^t$ for all $t \in \{0,1\}^n$ and (iii) $\sigma_n^t(B_0, B_1, \dots B_n)$ is open for all $t \in \{0,1\}^n$. The domain of each function σ_n is precisely the set of all finite sequences $(B_0, B_1, \dots B_n)$ of indexed families $B_j := \{B_j^t : t \in \{0,1\}^j\}$ of nonempty open subsets that satisfy the following conditions; (i) $\varnothing = \overline{B_{j+1}^{t0}} \cap \overline{B_{j+1}^{t1}} \subseteq \overline{B_{j+1}^{t0}} \cup \overline{B_{j+1}^{t1}} \subseteq \sigma_j^t(B_0, B_1, \dots B_j)$ for all $t \in \{0,1\}^j$ and $0 \leq j < n$ and (ii) d-diam$(B_j^t) < 1/2^j$ for all $t \in \{0,1\}^j$ and $0 \leq j \leq n$. Such a finite sequence $(B_0, B_1, \dots B_n)$ (infinite sequence $(B_n : n \in \omega)$) is called a *partial* σ*-play* (σ*-play*). A strategy $\sigma := (\sigma_n : n \in \omega)$ for the player α is called a *winning strategy* if each σ-play is won by α.

This Cantor-game is used to prove the following theorem.

Theorem 4.2 ([35])**.** *There exists a Banach space $(X, \|\cdot\|)$ such that $(X^*, weak^*)$ is weakly Stegall but $(X, \|\cdot\|)$ is not weak Asplund. In particular, $(X, \|\cdot\|)$ is a Gâteaux differentiability space that is not weak Asplund.*

There are many other games and applications that are not mentioned here. For example, games have been successfully used in Optimisation and in the theory of selections. For an excellent account of this area the reader is referred to the article [42] by J. Revalski. Topological games (which are variations on the Choquet game) have also been used extensively in the study of separate and joint continuity, see [27] for further information in this direction.

Finally, let us also mention here that a game very similar to the $G(X, \tau)$ was considered by E. Michael in [29, 30] to characterise the class of partition complete spaces. The only difference between these games is the definition of a win. In [29] E.Michael says that the player α *wins* if the sequence $(A_n)_{n\in\mathbb{N}}$ is complete. Then he obtains the result that a regular space X is partition complete if, and only if, the player α has a winning strategy.

References

[1] J. M. Aarts and D. J. Lutzer, *The product of totally nonmeagre spaces*, Proc. Amer. Math. Soc. **38** (1973), 198–200.
[2] A. Bouziad, *Every Čech-analytic Baire semi-topological group is a topological group*, Proc. Amer. Math. Soc. **124** (1996), 953–959.
[3] J. Cao, S. Garcia-Ferreira and V. Gutev, *Baire spaces and Vietoris hyperspaces*, Proc. Amer. Math. Soc., to appear.
[4] J. Cao, D. Gauld, S. Greenwood and A. Mohamad, *Games and metrizability of manifolds*, preprint.
[5] J. Cao and W. B. Moors, *Quasi continuous selections of upper continuous set-valued mappings*, Real Anal. Exchange, to appear.
[6] J. Cao, W. B. Moors and I. L. Reilly, *Topological properties defined by games and their applications*, Topology Appl. **123** (2002), 47–55.
[7] J. Cao and Z. Piotrowski, *Two variations of the Choquet game*, Kyungpook Math. J. **44** (2004), 495–504.
[8] J. Cao and A. H. Tomita, *Baire space, Tychonoff powers and the Vietoris topology*, preprint.
[9] J. Chaber and R. Pol, *On hereditarily Baire spaces, σ-fragmentability of mappings and Namioka property*, Topology Appl. **151** (2005), 132–143.
[10] G. Choquet, *Lectures on Analysis I*, W. A. Benjamin Inc, New York, 1969.
[11] G. Debs, *Espaces héréditairement de Baire*, Fund Math. **129** (1988), 199–206.
[12] R. Ellis, *A note on the continuity of the inverse*, Proc. Amer. Math. Soc. **8** (1957), 372–373.
[13] W. G. Fleissner and K. Kunen, *Barely Baire spaces*, Fund Math. **101** (1978), 229–240.
[14] R. Haworth and R. McCoy, *Baire spaces*, Dissertationes Math. **141**, 1977.
[15] R. Haydon, *Baire trees, bad norms and the Namioka property*, Mathematika **42** (1995), 30–42.
[16] J. E. Jayne and C. A. Rogers, *Borel selectors for upper semicontinuous set-valued maps*, Acta Math. **155** (1985), 41–79.
[17] A. S. Kechris, *Classical descriptive set theory*, Springer-Verlag, New York, 1995.
[18] P. Kenderov, I. Kortezov and W. B. Moors, *Topological games and topological groups*, Topology Appl. **109** (2001)), 157–165.
[19] P. Kenderov, I. Kortezov and W. B. Moors, *Continuity points of quasi-continuous mappings*, Topology Appl. **109** (2001), 321–346.
[20] P. Kenderov, I. Kortezov and W. B. Moors, *Norm continuity of weakly continuous mappings into Banach spaces*, Topology Appl., to appear.
[21] P. S. Kenderov and W. B. Moors, *Game characterizations of fragmentability of topological spaces*, Mathematics and education in mathematics 8-18, Proceedings of the 25^{th} Spring Conference of the Union of Bulgarian Mathematicians, April 1996, Kazanlak, Bulgaria.

[22] P. S. Kenderov and W. B. Moors, *Fragmentability and sigma-fragmentability of Banach spaces*, J. London Math. Soc. (2) **60** (1999), 203–223.

[23] P. S. Kenderov and W. B. Moors, *Separate continuity, joint continuity and the Lindelöf property*, Proc. Amer Math. Soc. **134** (2006), 1503–1512.

[24] P. S. Kenderov and J. Orihuela, *A generic factorization theorem*, Mathematika **42** (1995), 56–66.

[25] M R. Krom, *Cartesian products of metric Baire spaces*, Proc. Amer. Math. Soc. **42** (1974), 588–594.

[26] R. McCoy, *Baire spaces and hyperspaces*, Pacific J. Math. **58** (1975), 133–142.

[27] S. Mercourakis and S. Negrepontis, *Banach spaces and Topology II.* Recent Progress in General Topology (Prague, 1991), 493–536, Amsterdam, 1992.

[28] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.

[29] E. Michael, *A note on completely metrizable spaces*, Proc. Amer. Math. Soc. **96** (1986), 513–522.

[30] E. Michael, *Almost complete spaces, hypercomplete spaces and related mapping theorems*, Topology Appl. **41** (1991), 113–130.

[31] D. Montgomery, *Continuity in topological groups*, Bull. Amer. Math. Soc. **42** (1936), 879–882.

[32] W. B. Moors, *Closed graph theorems and Baire spaces*, New Zealand J. Math. **31** (2002), 55–62.

[33] W. B. Moors, *The product of a Baire space with a hereditarily Baire metric space is Baire*, Proc. Amer. Math. Soc., to appear.

[34] W. B. Moors and S. Somasundaram, *A weakly Stegall space that is not a Stegall space*, Proc. Amer. Math. Soc. **131** (2003), 647–654.

[35] W. B. Moors and S. Somasundaram, *A Gâteaux differentiability space that is not weak Asplund*, Proc. Amer. Math. Soc., to appear.

[36] I. Namioka, *Separate continuity and joint continuity*, Pacific J. Math. **51** (1974), 515–531.

[37] J. von Neumann and O. Morgenstern, *Theorey of games and economic behavoirs*, Princeton University Press, Princeton, 1944.

[38] D. Noll, *Baire spaces and graph theorems*, Proc. Amer. Math. Soc. **96** (1986), 141–151.

[39] J. C. Oxtoby, *The Banach-Mazur game and Banach category theorem, Contribution to the theory of games* Vol III, Annals of Mathematics Studies **39**, Princeton University Press, 1957.

[40] J. C. Oxtoby, *Cartesian products of Baire spaces*, Fund. Math. **49** (1961), 157–166.

[41] H. Pfister, *Continuity of the inverse*, Proc. Amer. Math. Soc. **95** (1985), 312–314.

[42] J. P. Revalski, *The Banach-Mazur game: history and recent developments*, preprint.

[43] J. Saint-Raymond, *Jeux topologiques et espaces de Namioka*, Proc. Amer. Math. Soc. **87** (1983), 499–504.

[44] R. Sikorski, *On the Cartesian product of metric spaces*, Fund. Math. **34** (1947), 288–292.

[45] M. Talagrand, *Deux généralisations dún théoréme de I. Namioks*, Pacific J. Math. **81** (1979), 239–251.

[46] R. Telgársky, *Topological games: on the 50th anniversary of the Banach-Mazur game*, Rocky Mountain J. Math. **17** (1987), 227–276.

Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland, New Zealand

E-mail address: `cao@math.auckland.ac.nz, moors@math.auckland.ac.nz`