Eberlein Theorem and Norm Continuity of Pointwise Continuous Mappings into Function Spaces

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Abstract

For a pseudocompact (strongly pseudocompact) space T we show that every strongly bounded (bounded) subset A of the space $C(T)$ of all continuous functions on T has compact closure with respect to the pointwise convergence topology. This generalization of the Eberlein-Grothendieck theorem allows us to prove that, for any strongly pseudocompact spaces T, there exist many points of norm continuity for any pointwise continuous, $C(T)$-valued mapping h, defined on a Baire space X, which is homeomorphic to a dense Borel subset of a pseudocompact space. In particular, this is so, if X is pseudocompact. In the case when T is pseudocompact the same "norm-continuity phenomenon" has place for every strongly pseudocompact space X or, for every Baire space X which is homeomorphic to a Borel subset of some countably compact space.

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1. Introduction

Let T be a completely regular topological space and $C(T)$ be the set of all continuous real-valued functions defined on T. By $C_p(T)$ we denote the set $C(T)$ endowed with the pointwise convergence topology on T. For a compact space T, Eberlein [14] has shown that the closure of every countably compact subset of $C_p(T)$ is compact. Grothendieck [16] proved that this result remains valid for countably compact spaces T. Another generalization was obtained by Asanov and Veličko [5]:

Theorem 1.1. *If A is bounded in $C_p(T)$ and T is countably compact, then $\overline{A}$ is a compact subset of $C_p(T)$.*

Recall that a subset A of a completely regular space X is said to be **bounded in** X if every continuous real-valued function defined on X is bounded on A. If a completely regular space X is bounded in itself, then it is called **pseudocompact**. Obviously, every countably compact space T is pseudocompact. There are many pseudocompact spaces however which are not countably compact.

The first goal of this article is to extend the validity of Theorem 1.1 to a larger class of spaces T and sets A. There are many indications that this is possible. As noted by Arhangel'skii ([2], Theorem III.4.9.) the result of Asanov and Veličko remains valid (with the same proof) for the class of spaces T that contain a dense subset D with the property that every infinite subset of D has a cluster point in T. Many other results also point in this direction (see Pryce [29], Pták [30], Preiss and Simon [28], Haydon [17], Arhangel'skii [3], Troallic [36] and many others). The limits for the possible extensions of Theorem 1.1 are set however by an example of Shachmatov [32] who constructed a pseudocompact space T such that the closed unit ball $B = \{x \in C(T) : \|x\| = \max_{t\in T} |x(t)| \leq 1\}$ is pseudocompact but not a compact subset of $C_p(T)$. Having Shachmatov's example in mind, it is reasonable to look for some subclasses of the class of all pseudocompact spaces for which Theorem 1.1 is still valid. As will become clear later in this paper, the following notion provides some opportunities in this direction.

Definition 1.2. A subset A of a topological space X is called **strongly bounded in** X, if it contains a dense subset D with the property that for every infinite subset C of D there exists some separable subspace S of X such that the set $S \cap C$ is infinite and bounded in S. A space X which is strongly bounded in itself is called **strongly pseudocompact**.

Countably compact spaces or, more generally, the spaces T that contain a dense subset D with the property that every every infinite subset of D has a cluster point in T obviously belong to the class of strongly pseudocompact spaces. Separable pseudocompact spaces are also obvious examples of strongly pseudocompact spaces. Furthermore, Watson [37] has exhibited an example of a separable pseudocompact T without any dense subsets D with the property that every every infinite subset of D has a cluster point in T. The free union of uncountably many strongly pseudocompact spaces T_γ, $\gamma \in \Gamma$, complemented with an"infinity point" ∞, is an example of a non-separable strongly pseudocompact space T, if the sets of the form $T \setminus (\cup_{\gamma\in F} T_\gamma)$, F a finite subset of Γ, are taken as a base of neighborhoods at the point ∞.

The role played by strong boundedness and strong pseudocompactness in the generalizations of the Eberlein theorem is clearly seen from the next assertion which is a corollary of the results in Section 4 below.

Theorem 1.3. *Let T be a strongly pseudocompact (pseudocompact) completely regular topological space and A be a set which is bounded (strongly bounded) in $C_p(T)$. Then $\overline{A}$ is a compact subset of $C_p(T)$.*

This means that the above mentioned pseudocompact space T of Shachmatov is not strongly pseudocompact and the respective closed unit ball B is bounded but not strongly bounded in $C_p(T)$.

For a compact space T and a continuous mapping $h : X \to C_p(T)$, defined on a space X satisfying some mild completeness condition, Namioka proved in [22] that there exists a dense G_δ-subset of X, at the points of which, the mapping h is continuous with respect to the much stronger topology generated by the sup-norm $\|x\| = \max_{t\in T} |x(t)|$ in $C(T)$ (sometimes called **uniform convergence topology**). The work of Namioka sparked intensive research in this direction and many similar results were obtained for different classes of compacts T and topological spaces X. The second goal of this paper is to show that some of these results remain valid, if the compactness of T is weakened to pseudocompactness or to strong pseudocompactness. To formulate these results for a possibly larger class of spaces X, we consider two topological games $G(X)$ and $G^*(X)$ where two players (α and β) play as in some variants of the famous Banach-Mazur game but where the winning rules (in both games) are changed in such a way that pseudocompact spaces are α-favorable for the game $G(X)$ and strongly pseudocompact spaces are

α-favorable for the game $G^*(X)$. The next statement is a corollary from the results in Section 5.

Theorem 1.4. *Let T be a pseudocompact (strongly pseudocompact) space and let X be a topological space that is β-unfavorable for the game $G^*(X)$ (for the game $G(X)$). Then for every continuous mapping $h : X \to C_p(T)$ there exists a dense G_δ subset $C \subset X$, at the points of which, h is continuous with respect to the sup-norm topology in $C(T)$.*

We provide also sufficient conditions for a space X to be β-unfavorable for the game $G(X)$ or for the game $G^*(X)$. Given a topological space Y, consider the smallest family of subsets of Y which is closed under the Souslin operation and contains all the closed and all the open subsets of Y. We call the sets from this family **Souslin generated subsets of** Y.

Proposition 1.5. *If X is a Baire space which is homeomorphic to a dense and Souslin generated subset of a pseudocompact space Y, then X is β-unfavorable for the game $G(X)$.*

If X is a Baire space which is homeomorphic to a Souslin generated subset of a countably compact space Y, then X is β-unfavorable for the game $G^*(X)$.

The paper is organized as follows. In section 2 we present the basic properties of bounded sets needed in the sequel. It should be noted that the notion of boundedness used here is different from the widely used notion of boundedness

Sections 3 and 4 contain some generalizations of the Eberlein Theorem while Section 5 deals with the norm continuity of mappings into $C(T)$.

2. Sets bounded in a regular topological space

The definition of the notion of "boundedness".

A subset A of a topological space X is called **bounded in** X, if for every locally finite family γ of open subsets in X the subfamily $\{U \in \gamma : U \cap A \neq \emptyset\}$ is finite. A set A is bounded in X if, and only if, for any sequence $\{W_i\}_{i\geq 1}$ of open sets satisfying the requirements $W_i \cap A \neq \emptyset$ and $W_{i+1} \subseteq W_i$ for every $i \geq 1$, the intersection of the closures $\bigcap_{i\geq 1} \overline{W_i}$ is not empty. In a completely regular space X this definition is equivalent to the requirement that every continuous real-valued function defined on X is bounded on A. If the space X is bounded in itself, then X is called **feebly compact**. For completely regular spaces X the latter notion is equivalent to saying that X is **pseudocompact** (Engelking [15], Theorem 3.10.22).

Boundedness is preserved by continuous mappings and by taking the closure of the set. We present here some other properties of bounded sets which play a role in our considerations. We say that a mapping $f : X \longrightarrow Y$ has **compact fibers** if the set $f^{-1}(y)$ is compact for every $y \in f(X)$.

Proposition 2.1. *Let A be a set that is bounded in a space X and $f : X \longrightarrow Y$ be a continuous mapping, with compact fibers, from the space X into a space Y, each point of which is a G_δ-point (i.e. each point is the intersection of countably many open sets). Then*

1. $f(\overline{A}) = \overline{f(A)}$.

2. The restriction of f on $\overline{A}$ is a closed mapping (i.e. f sends closed subsets of $\overline{A}$ into closed subsets of $f(\overline{A}) = \overline{f(A)}$). In particular, if f is a one-to one mapping, then its restriction on $\overline{A}$ is homeomorphism.

3. The space $f(\overline{A}) = \overline{f(A)}$ with the topology inherited from Y is first countable (i.e. each point has a countable neighborhood base).

4. If, in addition, Y is a normal topological space, then $\overline{A}$ is a countably compact subset of X.

Proof. 1. Let $y_0 \in \overline{f(A)}$ and $\{U_i\}_{i\geq 1}$ be a sequence of open subsets of Y such that $\cap_{i\geq 1} U_i = \{y_0\}$. Without loss of generality we may assume that $\overline{U_{i+1}} \subset U_i$ for every $i \geq 1$. Suppose $f^{-1}(y_0) \cap \overline{A} = \emptyset$. Compactness of $f^{-1}(y_0)$ implies that there is some open subset V of X such that $f^{-1}(y_0) \subset V$ and $\overline{V} \cap \overline{A} = \emptyset$. The family $\{W_n = f^{-1}(U_n) \setminus \overline{V}\}_{n\geq 1}$ consists of open sets and $W_n \cap A \neq \emptyset$ for each $n \geq 1$. As A is bounded in X, there exists some $x_0 \in \cap_{i\geq 1} \overline{W_i}$. Clearly, $x_0 \notin V$ and therefore $f(x_0) \neq y_0$. On the other hand, continuity of f implies that $f(x_0) \in \cap_{i\geq 1} \overline{f(W_i)} \subset \cap_{i\geq 1} \overline{U_i} = \{y_0\}$, a contradiction. Therefore $f^{-1}(y_0) \cap \overline{A} \neq \emptyset$ and $f(\overline{A}) = \overline{f(A)}$.

2. Let B be a closed subset of $\overline{A}$. Then B is a bounded subset of X and, by what was already proved, we have $f(B) = f(\overline{B}) = \overline{f(B)}$.

3. Let $y_0 \in \overline{f(A)}$ and $\{U_i\}_{i\geq 1}$ be a family of open sets in X such that $\cap_{i\geq 1} U_i = \{y_0\}$ and $\overline{U_{i+1}} \subset U_i$ for every $i \geq 1$. We will show that the sets $U_i \cap f(\overline{A})$, $i \geq 1$, form a base of neighborhoods at y_0. Let U be an open set containing y_0. Since Y is regular, it suffices to show that there exists some positive integer n such that $U_n \cap \overline{f(A)} \subset \overline{U}$. Suppose this is not the case. Then each of the sets in the family $\{U_i \setminus \overline{U}\}_{i\geq 1}$ is open, non-empty and intersects the set $\overline{f(A)}$ which is bounded in Y. Then there exists a

point $y^* \in \cap_{i\geq 1}\overline{U_i \setminus \overline{U}} \subset \cap_{i\geq 1}\overline{U_i} = \{y_0\}$. It is clear however that $y^* \notin U$, a contradiction.

4. In a normal space any closed bounded set is countably compact. The preimage of a countably compact space under a mapping which is continuous, closed and with compact fibers, is countably compact ([Eng], Theorem 3.10.10). $\square$

Corollary 2.2. *Let A be a bounded subset of a space X and $f : X \longrightarrow Y$ be a one-to-one continuous mapping from X into a metrizable space Y. Then $\overline{A}$ is a compact metrizable subset of X.*

Another immediate corollary of Proposition 2.1 is the following statement.

Proposition 2.3. *Let $\varphi : X \to Y$ be a continuous one-to-one mapping from a feebly compact space X onto a space Y, each point of which is a G_δ-point. Then φ is a homeomorphism. In particular, if Y is metrizable, then X and Y are homeomorphic metrizable compact spaces.*

The next assertion, for the case when X is a pseudocompact space, could be derived from the results of [10].

Proposition 2.4. *Let $\varphi : X \to Y$ be a continuous mapping from a feebly compact space X onto a space Y, each point of which is a G_δ-point. Let $h : X \to R$ be a continuous real-valued function which is constant on each fiber $\varphi^{-1}(y) = \{x \in X : \varphi(x) = y\}$, $y \in Y$. Then the naturally defined function $h'(y) = h(\varphi^{-1}(y))$ is continuous on Y.*

Proof. Consider the mapping ψ which puts into correspondence to each $x \in X$ the point $(\varphi(x), h(x)) \in Y \times R$. Since h is continuous the mapping $\psi : X \longrightarrow \psi(X)$ is continuous with respect to the product topology of $Y \times R$ and $\psi(X)$ is feebly compact. Since h is constant on the fibers, the projection π_Y of $\psi(X)$ into Y is a one-to-one continuous (and onto) mapping which, due to Proposition 2.1, must be a homeomorphism. Since the projection π_R of $\psi(X)$ to R is also continuous we get that $\pi_R \circ \pi_Y^{-1} = h'$ is continuous. $\square$

3. Sets bounded in $C_p(T)$. The separable case.

We start with a simple and well-known observation.

Corollary 3.1. *Let T be a separable topological space and A be a bounded set in $C_p(T)$. Then the closure of A in $C_p(T)$ is a metrizable compact.*

Proof. Let D be a countable dense subset of T. Consider in $C(T)$ the topology τ of pointwise convergence at the points of D. It is metrizable and the identity mapping $C_p(T) \longrightarrow C_\tau(T)$ is one-to-one and continuous. By Corollary 2.2 it follows that the closure of A is a metrizable compact in the topology of $C_p(T)$. □

Many of the results that follow are based upon the next statement.

Theorem 3.2. *Let T be a pseudocompact space and A be a bounded set in a separable subspace S of the space $C_p(T)$. Then*

a) $\overline{A}$ is a compact metrizable subset of $C_p(T)$;

b) If the sequence $\{f_i\}_{i\geq 1}$ where $f_i \in \overline{A}$, $i \geq 1$, is uniformly bounded in $C(T)$ and f_0 is a cluster point of this sequence in $C_p(T)$, then for every $\epsilon > 0$ there exist an integer $k > 0$ and nonnegative numbers λ_i, $1 \leq i \leq k$, such that $\sum_{i=1}^k \lambda_i = 1$ and $|f_0(t) - \sum_{i=1}^k \lambda_i f_i(t)| \leq \epsilon$ for every $t \in T$ (i.e. convex combinations of elements of $\{f_i\}_{i\geq 1}$ approximate f_o arbitrarily well with respect to the sup-norm in $C(T)$).

Proof. a) Let H be a countable dense subset of S. For every $g \in H$ the set $g(T) = \{g(t) : t \in T\}$ is a compact subset of the real line. Consider the continuous mapping $\varphi : T \longrightarrow T'$, where $\varphi(t) = \{g(t)\}_{g\in H}$ is a point in the product $\prod_{g\in H} g(T)$ and $T' = \varphi(T)$.

The continuous mapping φ generates a dual mapping $\psi : C_p(T') \longrightarrow C_p(T)$, where $\psi(f') = f' \circ \varphi$ for any $f' \in C_p(T')$. Note that every $g \in S$, as a cluster point of H, is constant on the fiber $\varphi^{-1}(t')$ for every $t' \in T'$. It follows from Proposition 2.4 that there exists a unique continuous function $g' \in C_p(T')$ such that $g(t) = g'(\varphi(t))$. Therefore $S \subset \psi(C_p(T'))$.

Put $S' = \psi^{-1}(S)$ and $A' = \psi^{-1}(A)$. Since ψ is a one-to-one and homeomorphic mapping, the set A' is bounded in S' with respect to the topology inherited from $C_p(T')$. By construction, T' is a metrizable pseudocompact (and so a metrizable compact). Denote by Z some dense countable subset of T'. The topology of pointwise convergence on Z is metrizable and weaker than the topology of $C_p(T')$. It follows from Corollary 2.2 that $\overline{A'}$ is a metrizable compact. This implies that $\overline{A} = \psi(\overline{A'})$ is a metrizable compact as well.

b) Because of a) there exists a subsequence converging to f_0. Without loss of generality we may think that $\{f_i(t)\}_{i\geq 1}$ converges to $f_0(t)$ for every $t \in T$. Uniform boundedness of the sequence means that there is some constant $a > 0$ such that $|f_i(t)| \leq a$ whenever $t \in T$ and $i \geq 1$. Then the sequence $\{f'_i = \psi^{-1}(f_i)\}_{i\geq 1}$ is uniformly bounded and converges to $f'_0 = \psi^{-1}(f_0)$ in $C_p(T')$. By Lebesgue dominated convergence theorem the sequence $\{f'_i\}_{i\geq 1}$ converges to f'_0 in the weak topology of the normed space $(C(T'), \|.\|)$ where $\|f'\| = \max\{|f'(t')| : t' \in T'\}$. In particular, f'_0 belongs to the weak closure of the convex hull C of the sequence $\{f'_i\}_{i\geq 1}$. By a known result (separation theorem for Banach spaces) it follows that f'_0 belongs also to the norm closure of C and this implies that for every $\epsilon > 0$ there exist a positive integer k and non-negative numbers λ_i, $1 \leq i \leq k$, such that $\sum_{i=1}^k \lambda_i = 1$ and $|f'_0(t') - \sum_{i=1}^k \lambda_i f'_i(t')| \leq \epsilon$ for every $t' \in T'$. This means that $|f_0(t) - \sum_{i=1}^k \lambda_i f_i(t)| \leq \epsilon$ for every $t \in T$. □

Let βT be the Čech-Stone compactification of a pseudocompact space T. Then $C(T)$ and $C(\beta T)$ can be considered as one and the same space. This space has three natural topologies: the topology of pointwise convergence in T, the topology of pointwise convergence in βT and the weak topology of the Banach space $(C(\beta T), \|\cdot\|)$ where $\|f\| = \max\{|f(t)| : t \in \beta T\}$. Since the natural extension $\tilde{\varphi} : \beta T \longrightarrow T'$ of the map $\varphi : T \longrightarrow T'$ from the proof of Theorem 3.2 is an onto mapping, the dual map $\psi : C_p(T') \longrightarrow C_p(\beta T)$ is a homeomorphism. This allows us to formulate the following assertion (a more general form of it will be given in the next section).

Corollary 3.3. *Let T be a pseudocompact space and A be a bounded set in some separable subset of the space $C_p(T)$. Then the closure of A in $C_p(\beta T)$ is a metrizable compact.*

If the set A is uniformly bounded, then A is metrizable and compact in the weak topology of $(C(\beta T), \|\cdot\|)$ as well.

4. Eberlein Theorem for Pseudocompact and for Strongly Pseudocompact Spaces T.

We show in this section that Theorem 1.1 remains valid for strongly pseudocompact spaces T and we prove a similar statement for pseudocompact spaces T.

Theorem 4.1. *Let T be a completely regular strongly pseudocompact space and let A be a non-empty set which is bounded in $C_p(T)$. Then*

(i) $\overline{A}$ is a non-empty compact subset of $C_p(T)$;

(ii) Every sequence $\{f_i\}_{i\geq 1}$ of functions $f_i \in A$, $i \geq 1$, has a subsequence converging to some f_0 in $C_p(T)$. If, in addition, the sequence $\{f_i\}_{i\geq 1}$ is uniformly bounded in $C(T)$, then for every $\epsilon > 0$ there exist an integer $k > 0$ and nonnegative numbers λ_i, $1 \leq i \leq k$, such that $\sum_{i=1}^k \lambda_i = 1$ and $|f_0(t) - \sum_{i=1}^k \lambda_i f_i(t)| \leq \epsilon$ for every $t \in T$.

Proof. Let D be a dense subset of T such that for every infinite subset C of D there exists some separable subspace S of X such that the set $S \cap C$ is infinite and bounded in S. The proof exploits the basic idea which goes back to the original proof of Eberlein theorem. Consider $C_p(T)$ as a subset of the product $Y = \prod_{t\in T} R_t$ where $R_t = R$ is the real line with its usual topology. Since A is bounded in $C_p(T)$, it follows from Tykhonoff theorem that the closure of A in Y (which is denoted by $\overline{A}^Y$) is a compact subset of Y. The proof will be completed if we show that each function $g_0 \in \overline{A}^Y$ is continuous (i.e. $\overline{A}^Y \subset C_p(T)$).

Let $g_0 \in \overline{A}^Y$ and $t_0 \in T$. It suffices to prove that, for every $\epsilon > 0$ there exists an open $V \ni t_0$ such that $|g_0(t) - g_0(t_0)| \leq \epsilon$ for every $t \in V \cap D$. Suppose this is not so for some $t_0 \in T$ and $\epsilon > 0$. Since $g_0 \in \overline{A}^Y$, there is some $g_1 \in A$ such that $|g_1(t_0) - g_0(t_0)| < 1$. Consider the open set $V_1 = \{t \in T : |g_1(t) - g_1(t_0)| < 1\}$ and find some $t_1 \in V_1 \cap D$ for which $|g_0(t_1) - g_0(t_0)| > \epsilon$. We define inductively sequences of: functions $\{g_i\}_{i\geq 0}$, of points $\{t_i\}_{i\geq 0}$ and of open sets $\{V_i\}_{i\geq 1}$ in such a way that, for each integer $k \geq 1$,

1. $g_k \in A$;
2. $|g_k(t_i) - g_0(t_i)| < 1/k$ whenever $0 \leq i < k$;
3. $t_k \in V_k \cap D$ where $V_k = \{t \in T : |g_i(t) - g_i(t_0)| < 1/k$ whenever $0 < i \leq k\}$;
4. $|g_0(t_k) - g_0(t_0)| > \epsilon$.

The objects g_0, g_1, t_0, t_1 and V_1 were defined above. Suppose that the functions $\{g_i\}_{i=0}^k$, the points $\{t_i\}_{i=0}^k$ and the sets $\{V_i\}_{i=1}^k$ with the listed properties have already been defined for all $k < n$. Since $g_0 \in \overline{A}^Y$, there is some $g_n \in A$ such that $|g_n(t_i) - g_0(t_i)| < 1/n$ whenever $0 \leq i < n$. Consider the open set $V_n = \{t \in T : |g_i(t) - g_i(t_0)| < 1/n$ whenever $0 < i \leq n\}$ and find some $t_n \in V_n \cap D$ for which $|g_0(t_n) - g_0(t_0)| > \epsilon$. This finishes the induction step.

Without loss of generality we may assume that $t_i \neq t_k$ for $i \neq k$. Property 2. implies that $\lim_{k\to\infty} g_k(t_i) = g_0(t_i)$ for every $i \geq 0$. Since $g_i \in A$ for every $i \geq 1$, the set $A' = \{g_i\}_{i\geq 1}$ is bounded in $C_p(T)$. Since $\{t_i\}_{i\geq 1}$ is an infinite subset of D, there exists a closed separable space $S \subset T$ such that the set $S \cap \{t_i\}_{i\geq 1} = \{t_{i_l}\}_{l\geq 1}$ is infinite and bounded in S. We may assume that $i_1 < i_2 < \dots$ and $i_l \geq l$ for every $l \geq 1$.

Consider the restriction operator $r_S : C_p(T) \longrightarrow C_p(S)$ which puts into correspondence to each function $f \in C_p(T)$ its restriction $f_{|S}$ on S. Clearly, $f_{|S} \in C_p(S)$. The mapping r_s is continuous and, therefore, the set $r_S(A')$ is bounded in $C_p(S)$. Since S is separable, there is some metrizable topology in $C(S)$ which is weaker than the topology of pointwise convergence in S. Proposition 2.1 implies that $\overline{r_S(A')}$ is a compact metrizable subset of $C_p(S)$. In particular, there is a function $g^* \in C(S)$ which is a cluster point in $C_p(S)$ of the restrictions of the functions g_i, $i \geq 1$, on S. In particular, $g^*(t_{i_l}) = \lim_{k\to\infty} g_k(t_{i_l}) = g_0(t_{i_l})$ for every $l \geq 1$. Then, by 4., we have $|g^*(t_{i_l}) - g_0(t_0)| = |g_0(t_{i_l}) - g_0(t_0)| > \epsilon$ for every $l \geq 1$.

Since g^* is continuous in S, for every $l \geq 1$ there exists an open (in S) set $U_{i_l} \subset V_{i_l}$ such that $t_{i_l} \in U_{i_l}$ and $|g^*(t) - g_0(t_0)| > \epsilon$ for every $t \in U_{i_l}$. The set $\{t_{i_l}\}_{l\geq 1}$ is bounded in S. Hence the set $\bigcap_{k\geq 1} \overline{\bigcup_{l\geq k} U_{i_l}}^S$ is not empty and contains some point $t^* \in S$. Clearly, $|g^*(t^*) - g_0(t_0)| \geq \epsilon$. On the other hand,$\{V_k\}_{k\geq 1}$ is a nested sequence of sets and, therefore, $\bigcup_{l\geq k} U_{i_l} \subset \bigcup_{l\geq k} V_{i_l} = V_{i_k} \subset V_k$. This means that $t^* \in \overline{V_k}$ and, by 3., $|g_i(t^*) - g_i(t_0)| \leq 1/k$ whenever $1 \leq i \leq k$. Therefore $g_i(t^*) = g_i(t_0)$ for every $i \geq 1$. Since the sequence $\{g_i(t_0)\}_{i\geq 0}$ converges to $g_0(t_0)$ and the sequence $\{g_i(t^*)\}_{i\geq 1}$ has $g^*(t^*)$ as cluster point we obtain that $g^*(t^*) = g_0(t_0)$. This contradicts the inequality $|g^*(t^*) - g_0(t_0)| \geq \epsilon$ obtained earlier. Therefore $\overline{A}^Y$ is a subset of $C_p(T)$.

Proof of *(ii)*. What was already proved shows that every sequence of functions $f_i \in A$, $i \geq 1$, has a cluster point f_0 in $C_p(T)$. Therefore the set $A' = \{f_i\}_{i\geq 1}$ is bounded in its closure $\overline{A'}$ and the latter is a separable subset of $C_p(T)$. The claim now follows from Theorem 3.2.

□

Corollary 4.2. *Let T be a completely regular strongly pseudocompact space. If a set A is bounded in $C_p(T)$, then it is bounded in $C_p(\beta T)$. Moreover, the closure $\overline{A}$ in $C_p(\beta T)$ is compact and every closed separable subset of $\overline{A}$ is a metrizable compact.*

If, in addition, the set A is uniformly bounded, then $\overline{A}$ is compact with respect to the weak topology of $(C(\beta T), \|\cdot\|)$.

Proof. Theorem 4.1 yields that every sequence of functions $f_i \in A$, $i \geq 1$, has a cluster point f_0 in $C_p(T)$. Therefore the set $A' = \{f_i\}_{i\geq 1}$ is bounded in its closure $\overline{A'}$ which is a separable subset of $C_p(T)$. Corollary 3.3 implies that $\overline{A'}$ is a metrizable compact in $C_p(\beta T)$. □

To have a result similar to Theorem 4.1 for arbitrary pseudocompact spaces T, it is necessary to increase the requirements imposed upon the set A.

Theorem 4.3. *Let T be a pseudocompact completely regular space and A be a non-empty set which is strongly bounded in $C_p(T)$. Then*

(i) the set $\overline{A}$ is a non-empty compact subset of $C_p(T)$;

(ii) Every sequence $\{f_i\}_{i\geq 1}$ of functions $f_i \in A$, $i \geq 1$, has a subsequence converging to some f_0 in $C_p(T)$. If, in addition, the sequence $\{f_i\}_{i\geq 1}$ is uniformly bounded in T, then for every $\epsilon > 0$ there exist an integer $k > 0$ and nonnegative numbers λ_i, $1 \leq i \leq k$, such that $\sum_{i=1}^{k} \lambda_i = 1$ and $|f_0(t) - \sum_{i=1}^{k} \lambda_i f_i(t)| \leq \epsilon$ for every $t \in T$.

Proof. Without loss of generality we may assume that the set A itself (rather then its dense subset) has the property from the definition of strong boundedness: for every infinite sequence $\{g_i\}_{i\geq 1} \subset A$ there exists an infinite subsequence $\{g_{i_l}\}_{l\geq 1}$ which is bounded in some separable subspace S' of $C_p(T)$. Then we proceed as in the proof of Theorem 4.1 and construct the sequences $\{g_i\}_{i\geq 0}$, $\{t_i\}_{i\geq 0}$ $\{V_i\}_{i\geq 1}$ having the four properties listed in that proof. We show next that $\{g_i\}_{i\geq 1}$ has a convergent subsequence. This is obvious, if $\{g_i\}_{i\geq 1}$ is a finite set. If it is infinite, then the strong boundedness implies the existence of a separable subset S' of $C_p(T)$ such that the set $S' \cap \{g_i\}_{i\geq 1} = \{g_{i_l}\}_{l\geq 1}$ is infinite and bounded in S'. By Theorem 3.2 the sequence $\{g_{i_l}\}_{l\geq 1}$ has a subsequence converging in $C_p(T)$ to some g^*. Further the proof runs precisely as the proof of Theorem 4.1 (with the set S replaced by T).

□

Corollary 4.4. *Let T be a completely regular pseudocompact space and let A be a strongly bounded set in $C_p(T)$. Then A is strongly bounded in $C_p(\beta T)$ as well. Moreover, the closure $\overline{A}$ in $C_p(\beta T)$ is compact and every closed separable subset of $\overline{A}$ is a metrizable compact.*

If, in addition, the set A is uniformly bounded, then $\overline{A}$ is compact with respect to the weak topology of $(C(\beta T), \|\cdot\|)$.

As mentioned in the introduction, Shachmatov [32] exhibited a pseudocompact space T such that the closed unit ball B of $C_p(T)$ is pseudocompact but not compact. It follows from Theorem 4.1 that Shachmatov's space T is not strongly pseudocompact. Similarly, Theorem 4.3 implies that the pseudocompact unit ball B is not strongly bounded in $C_p(T)$.

5. Norm Continuity of Pointwise Continuous Mappings

In order to formulate the results in this section we need two games G and G^*, played in a given topological space X. Both games have identical playing rules but differ in the definition of a win.

Let X be a topological space. Two players, α and β, play a game similar to the famous Banach-Mazur game. The player β begins the game by selecting some non-empty open subset U_1 of X. In response, player α picks an open non-empty subset V_1 of U_1 and a point $x_1 \in V_1$. Then β selects some non-empty open $U_2 \subset V_1$ and, in turn, α selects some open nonempty set $V_2 \subset U_2$ and a point $x_2 \in V_2$. Proceeding in this way infinitely many times, the players "produce" a nested sequence $U_1 \supset V_1 \supset U_2 \supset V_2 \supset \cdots \supset U_n \supset V_n \supset \cdots$ of open non-empty subsets of X and a sequence of points $\{x_i \in V_i\}_{i\geq 1}$. The sequence $\{U_i, V_i, x_i\}_{i\geq 1}$ is called *a play*.

Definition 5.1. Player α is said to have won a play $\{U_i, V_i, x_i\}_{i\geq 1}$ of the game $G(X)$, if the sequence of sets $\{V_n\}_{n\geq 1}$ has non-empty intersection and the sequence of points $\{x_i\}_{i\geq 1}$ has a subsequence which is bounded in X. Otherwise, the player β is declared to be the winner of this play of the game $G(X)$.

Definition 5.2. Player α is said to have won a play $\{U_i, V_i, x_i\}_{i\geq 1}$ of the game $G^*(X)$, if the sequence of sets $\{V_n\}_{n\geq 1}$ has non-empty intersection and there exists a subsequence of $\{x_i\}_{i\geq 1}$ which is bounded in some separable subspace $S \subset X$. Otherwise, the player β is declared to be the winner of this play of the game $G^*(X)$.

A very similar game to this was considered in [11].

Under *a strategy* s for player β (in any of the considered games) we mean "a rule" that specifies each move of player β in every possible situation. If the player β applies a certain strategy s, then the resulting plays $\{U_i, V_i, x_i\}_{i\geq 1}$ are called *s-plays.* A strategy s for the player β is called *winning*, if the player β wins all s-plays. A space X is called *β-unfavorable*, if the player β does not have a winning strategy. For more information on topological games see [9].

Theorem 5.3. *Let T be a pseudocompact space and let X be a topological space that is β-unfavorable for the game $G^*(X)$. Then for every continuous mapping $h : X \to C_p(T)$ there exists a dense G_δ subset $C \subset X$, at the points of which, h is continuous with respect to the sup-norm topology in $C(T)$.*

Proof. Without loss of generality we may assume that $h(X)$ is a subset of the closed unit ball B of $C(T)$. It is also well known that the set C of points at which h is norm-continuous is a G_δ subset of X. Therefore it suffices to show that C is dense in X. To do this we take an arbitrary non-empty open set $U_1 \subset X$ and construct a special strategy s for the player β with U_1 as her/his first choice. Since X is β-unfavorable, there exists an s-play $\{(U_i, V_i, x_i)\}_{i\geq 1}$ which is won by the player α. In particular, the set $\cap_{i\geq 1} V_i$ is nonempty. The construction of the strategy s however is such that h is norm-continuous at the points of the set $\cap_{i\geq 1} V_i \subset U_1$ whenever the s-play is won by player α.

The strategy s will be constructed inductively. As mentioned above, the set U_1 is the first move of β. Suppose the strategy s has been defined "up to the stage n", where $n \geq 1$. Let $\{U_i, V_i, x_i\}_{i=1}^n$ be a finite s-play. To define the set U_{n+1}, the next move of β, we put $g_i = h(x_i)$, $i = 1, \ldots, n$, and

$$d_n := \inf\{t > 0 : \ co\{g_1, \ldots, g_n\} \ + \ tB \supset h(V_n)\},$$

where $co\{g_1, \ldots, g_n\}$ is the convex hull of the set $\{g_1, \ldots, g_n\}$.

If $d_n = 0$, then $h(V_n)$ is a subset of the finite-dimensional compact $co\{g_1, \ldots, g_n\}$ in which pointwise convergence topology and norm topology in $C(X)$ coincide. In such a case h is norm-continuous at the points of $V_n \subset U_1$. Therefore, without loss of generality, we may assume that $d_n > 0$.

Consider the nonempty set

$$h(V_n) \setminus \{co\{g_1, \ldots, g_n\} + \frac{n}{n+1} d_n B\}.$$

It is relatively open in $h(V_n)$. Take some nonempty open subset $U \subset V_n$ such that

$$\overline{h(U)} \bigcap \{co\{g_1, \ldots, g_n\} + \frac{n}{n+1} d_n B\} = \emptyset.$$

Since $co\{g_1, \ldots, g_n\}$ is compact, there is a finite set M such that

$$co\{g_1, \ldots, g_n\} \subset M + \frac{1}{n+1}B.$$

Since

$$h(U) \subset h(V_n) \subset co\{g_1, \ldots, g_n\} + d_n B,$$

we have $h(U) \subset M + (d_n + \frac{1}{n+1})B$. Without loss of generality we can assume that M is a minimal (with respect to its cardinality) finite set with this property.

Then, for any $m_0 \in M$, the set

$$h(U) \setminus \left\{ \{M \setminus \{m_0\}\} + (d_n + \frac{1}{n+1})B \right\} \neq \emptyset.$$

is a nonempty relatively open subset of $h(U)$. Then there exists some nonempty open set $U_{n+1} \subset U$ such that $h(U_{n+1}) \subset m_0 + (d_n + \frac{1}{n+1})B$. We take this set U_{n+1} to be the next move of player β. This completes the construction of the strategy s.

Note that, for every s-play $\{U_i, V_i, x_i\}_{i \geq 1}$ and $n \geq 1$ we have:

a) $\| \cdot \|$ -diam$(h(U_{n+1})) \leq 2(d_n + \frac{1}{n+1})$;

b) $\overline{h(U_{n+1})} \bigcap \{co\{g_1, \ldots, g_n\} + \frac{n}{n+1} d_n B\} = \emptyset$ and

c) $g_{n+1} \notin co\{g_1, \ldots, g_n\}$.

Since X is β-unfavorable for the game $G^*(X)$, there exists some s-play $\{U_i, V_i, x_i\}_{i \geq 1}$ which is won by α. In particular, there exists some separable subset S of X and a subsequence of $\{x_i\}_{i \geq 1}$ which is bounded in S. Since h is continuous, the set $h(S)$ is separable in $C_p(X)$ and $\{g_i = h(x_i)\}_{i \geq 1}$ contains a subsequence which is bounded in $h(S)$. By Theorem 3.2, the sequence $\{g_i\}_{i \geq 1}$ contains a subsequence converging in $C_p(T)$ to some function g_∞ which, necessarily, belongs to $\bigcap_{i \geq 1} \overline{h(U_i)}$.

The sequence $\{d_i\}_{i \geq 1}$ of non-negative numbers is non-increasing. Put $d_\infty := \lim_{n \to \infty} d_n$. It suffices to show that $d_\infty = 0$. Suppose that $d_\infty > 0$ and take some positive number $\varepsilon < \frac{1}{2} d_\infty$. Then we have, for every $i \geq 1$,

$$\{g_\infty + \varepsilon B\} \cap co\{g_1, \ldots, g_i\} \subset \{\overline{h(U_i)} + \frac{i}{i+1} d_i B\} \cap co\{g_1, \ldots, g_i\} = \emptyset.$$

Therefore $\{g_\infty + \varepsilon B\} \cap \{\bigcup_{i \geq 1} co\{g_1, \ldots, g_i\}\} = \emptyset$.

This contradicts part b) of Theorem 3.2. Hence $d_\infty = 0$. This completes the proof of Theorem 5.3. □

Theorem 5.4. *Let T be a strongly pseudocompact space and let X be a topological space that is β-unfavorable for the game $G(X)$. Then for every continuous mapping $h : X \to C_p(T)$ there exists a dense G_δ subset $C \subset X$, at the points of which, h is continuous with respect to the sup-norm topology in $C(T)$.*

Proof. The proof is almost identical with the proof of Theorem 5.3. We construct a strategy s for player β with U_1 as a first move and such that every s-play $\{U_i, V_i, x_i\}_{i\geq 1}$ satisfies the properties a), b) and c) whenever $n \geq 1$.

Since X is β-unfavorable for the game $G(X)$, there exists some s-play $\{U_i, V_i, x_i\}_{i\geq 1}$ which is won by α in the game $G(X)$. I.e. $\cap_{i\geq 1} V_i \neq \emptyset$ and the sequence $\{x_i\}_{i\geq 1}$ has a subsequence bounded in X. Since h is continuous, the sequence $\{g_i = h(x_i)\}_{i\geq 1}$ has a subsequence bounded in $C_p(X)$. By Theorem 4.1, $\{x_i\}_{i\geq 1}$ has a subsequence converging in $C_p(T)$ to some function g_∞ which, necessarily, belongs to $\bigcap_{i\geq 1} \overline{h(U_i)}$. Using part (iii) of Theorem 4.1 Instead of part b) of Theorem 3.2) we establish as above that $d_\infty = 0$.

□

At the end of this section we give some sufficient conditions for a space X to be β-unfavorable in the games $G(X)$ and $G^*(X)$.

For a point x and a family δ of subsets of some space put $St(x, \delta) = \cup\{W \in \delta : x \in W\}$. This set is **the star of x with respect to** δ. We need the following separation property (see [4] Definition 4.4).

Definition 5.5. Let X be a subset of a topological space Z. X is said to have **star separation in** Z, if there exists a sequence $\{\delta_i\}_{i\geq 1}$ of families of open subsets of Z, such that the points of X are separated from the points of $Z \setminus X$ in the following sense: for every pair of points $x \in X$ and $z \in Z \setminus X$, there exists some $n \geq 1$ such that at least one of the stars $St(x, \delta_n)$, $St(z, \delta_n)$ is not empty and does not contain the other point. In such a case the sequence of families $\{\delta_i\}_{i\geq 1}$ is called a **star separation for X in Z**.

If the families $\{\delta_i\}_{i\geq 1}$ are a star separation for X in Z, then they are a star separation for $Z \setminus X$ in Z as well. An arbitrary open subset U of any space Z has star separation in Z (all δ_i consist of the set U only). Hence, every closed subset of Z also has star separation in Z. It is easy to realize that the collection of all sets with star separation in a certain space Z is closed under taking countable unions and countable intersections. In particular, Borel subsets of Z have star separation in this space. Moreover, the Souslin scheme, applied to sets with star separation in Z, also produces a set with star separation in Z. In particular, Souslin generated subsets of Z have star separation in it.

It is easy to see that a space X has star separation in some Z if, and only if, X has star separation in $cl_Z X$.

Some partial cases of this notion have been studied and used earlier. For instance, spaces admitting star separation in a compact space Z by families δ_n which are open covers for X are called **p-spaces** [1] (or also **feathered spaces**). The class of p-spaces is very large. It contains all metric spaces as well as all locally compact spaces.

Spaces X admitting star separation in a compact space Z by families δ_i each consisting of just one open subset of Z have been used in the study of fragmentability and σ-fragmentability of Banach spaces under the name **spaces with countable separation** (see [19] page 213).

According to Theorem 4.9 in [4] a completely regular space X has star separation in some compactification bX if, and only if, it has star separation in its Stone-Ĉech compactification βX.

The notion"star separation" allows to formulate the following two statements which imply Proposition 1.5 from the Introduction section of the paper.

Theorem 5.6. *Let X be a Baire space that is dense in some feebly compact space Z, and which has star separation in it. Let T be a strongly pseudocompact space. Then for every continuous mapping $h : X \longrightarrow C_p(T)$ there exists a dense G_δ subset $C \subset X$, at the points of which, h is continuous with respect to the norm topology in $C(T)$.*

Proof. Corollary 4.6 c) from [4] implies that X is β-unfavorable for the game $G(X)$. It remain to apply Theorem 5.4. □

Theorem 5.7. *Let X be a Baire space that is dense in some countably compact space Z, and which has star separation in it. Let T be a pseudocompact space. Then for every continuous mapping $h : X \longrightarrow C_p(T)$ there exists a dense G_δ subset $C \subset X$, at the points of which, h is continuous with respect to the norm topology in $C(T)$.*

Proof. Corollary 4.6 b) from [4] implies that X is β-unfavorable for the game $G^*(X)$. It remain to apply Theorem 5.4. □

The topic *norm-continuity of pointwise continuous mappings* stems from another, perhaps more popular, topic known under the name *joint continuity of separately continuous functions.* Recall that a real-valued function $f(x,t)$ defined on the product $X \times T$ of two topological spaces X and T is **separately continuous** if for every pair $(x^*, t^*) \in X \times T$ the function $f(x^*, t)$ (of the variable $t \in T$) and the function $f(x, t^*)$ (of the variable $x \in X$) are continuous. Well-known examples show that a separately continuous function $f(x,t)$ need not be continuous. However, under some requirements imposed on the spaces X and T, it is possible to prove that the set

$$\{x \in X : f(x,t) \text{ is continuous at the point } (x,t) \text{ for every } t \in T\}$$

contains a dense G_δ-subset of X. This phenomenon received a lot of attention after the famous paper of Baire [7] was published. Information about historical developments in this topic can be found in the survey papers of Piotrowski [25, 26]. Interesting results in this area are contained in the papers of Namioka [22], Christensen [12], Saint-Raymond [31], Talagrand [33, 34], Debs [13], Troallic [35, 36], Namioka and Pol [23] and many others, including [20, 21]. The current state of the art together with many new results can be found in the paper of Bareche and Bouziad [6].

Each separately continuous function $f(x,t)$ determines a continuous mapping $h : X \to C_p(T)$ which puts into correspondence to every $x_0 \in X$ the function $h(x_0) = f(x_0, t) \in C(T)$. The norm continuity of h at $x_0 \in X$ implies the continuity of $f(x,t)$ at all points (x_0, t), $t \in T$. For compact spaces T the inverse implication also holds: continuity of $f(x,t)$ at all points (x_0,t), $t \in T$, implies norm continuity of the mapping h at x_0. For non-compact spaces, even for sequentially compact spaces T, this is not necessarily so. This is seen from the next example. In this sense the results obtained in this paper concerning norm continuity of h at many points $x \in X$ do not follow directly from the results concerning joint continuity of $f(x,t)$ at (x,t) for every $t \in T$.

Example 5.8. Let ω_1 be the first non-countable ordinal. Let $X = [0, \omega_1]$ and $T = [0, \omega_1)$. Endow T with the usual order generated topology which turns it into a sequentially compact space. Consider in X a topology for which all points $x < \omega_1$ are isolated and the family $\{(x, \omega_1]\}_{x<\omega_1}$ is a base of neighborhoods at ω_1. Consider the function f which is equal to 0 at the points $(x, t), t \leq x$, and 1 otherwise. This function is continuous at the points of $X \times T$. The corresponding mapping h however is not norm continuous at the point $x_0 = \omega_1$.

References

[1] A. V. Arhangel'skii, *A class of spaces which contains all metric and all locally compact spaces*, Matem. Sb. 67 (1965), 55-88 (English translation: Amer. Math. Soc. Transl. 92 (1970), 1 - 39).

[2] A. V. Arhangel'skii, *Topological Function Spaces*, Kluwer Academic Publ., 1992 (Translated from Russian: Moskva: Izd-vo Moskov. Universiteta, 1989).

[3] A. V. Arhangel'skii, *On a theorem of Grothendieck in C_p-theory.* Topology and its Appl. 80 (1997), 21-41.

[4] A. V. Arhangel'skii, M. M. Choban, P. S. Kenderov, *Topological games and continuity of group operations*, Topology and its Appl. 157 (2010), 2542-2552.

[5] M. O. Asanov, N. V. Veličko, *Compact sets in $C_p(X)$*, Comment. Math. Univ. Carolinae, Vol. 22 No. 2, (1981), 255–266 (in Russian).

[6] A. Bareche, A. Bouziad, *Some results on separate and joint continuity.* Topology and its Appl. 157 (2010), 327-335.

[7] R. Baire, *Sur les Functions des variables reelles,* Ann. Mat. Pura Appl., 3 (1899), 1-122.

[8] A. Bouziad, *Continuity of separately continuous group actions in the p-spaces*, Topology and its Appl. 71 (1996), 119-124.

[9] J. Cao and W. B. Moors, *A survey on topological games and their applications in analysis*, RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A. Mat., 100 (2006), 39-49.

[10] M. Choban, *Baire sets in complete topological spaces*, Ukrainskii Matem. Jurnal 22 (1970), 330-342 (English translation: Ukrainian Math. Journal 22 (1979), 286-295).

[11] M. M. Choban, P. S. Kenderov and W. B. Moors, *Pseudo-compact semi-topological groups*, Mathematics and Education in Mathematics, Proceedings of the Forty First Spring Conference of the Union of Bulgarian Mathematicians, Borovetz, April 9-12$^{\text{th}}$, (2012), 53 - 60.

[12] J. P. R. Christensen, *Theorems of I. Namioka and R. E. Johnson type for upper semi-continuous and compact-valued set-valued mappings,* Proc. Amer. Math. Soc., 86 (1982), 649-655.

[13] Gabriel Debs, *Fonctions separement continuous et the premiere classe sur un espace produit,* Math. Scand., 59 (1986), 122-130.

[14] W. F. Eberlein, *Weak Compactness in Banach Spaces,* Proc. Natl. Acad. Sci. U S A., 33(3) (1947), 51-53.

[15] R. Engelking, *General Topology*, PWN. Warszawa, 1977.

[16] A. Grothendieck, *Criteres de Compacite dans les Espaces Fonctionnels Generaux* American Journal of Mathematics, 74 (1) (1952) 168-186.

[17] R. Haydon, *Compactness in $C_s(T)$ and applications*, Publ. Dep. Math., 9 (1) (1972), 105-113.

[18] P. S. Kenderov, I. Kortezov and W. B. Moors, *Norm continuity of weakly continuous mappings into Banach spaces,* Topology and its Appl., 153, (2006), 2745-2759.

[19] P. S. Kenderov and W. B. Moors, *Fragmentability and sigma-fragmentability of Banach spaces*, J. London Math. Soc. 60 (1999), 203-223.

[20] P. S. Kenderov and W. B. Moors, *Separate continuity, joint continuity and the Lindelöf property*, Proc. Amer. Math. Soc., **134** (2006), 1503-1512.

[21] W. B. Moors, *Separate continuity, joint continuity, the Lindelöf property and p-spaces*, Topology and its Appl. **154** (2007), 428-433.

[22] I. Namioka, *Separate continuity and joint continuity,* Pacific. J. Math. 51 (1974), 515-531.

[23] I. Namioka, R. Pol, *Mappings of Baire spaces into function spaces and Kadeč renorming,* Israel J. Math. 78 (1992), 1-20.

[24] J. C. Oxtoby, *The Banach-Mazur game and Banach category theorem. Contribution to the theory of games,* vol. III. Annals of Mathematics Studies 39, Princeton University Press, 1957.

[25] Z. Piotrowski, *Separate and joint continuity,* Real Anal. Exch. 11 (1985/86), (1986), 293-322.

[26] Z. Piotrowski, *Separate and joint continuity II,* Real Anal. Exch., 15 No.1, (1990), 248-258.

[27] Z. Piotrowski, E. Wingler, *A note on continuity points of functions,* Real Anal. Exch. 16 No.2, (1991), 408-414.

[28] D. Preiss, P. Simon, *A weakly pseudocompact space of Banach space is weakly compact*, Comment. Math. Univ. Carolinae 15:4 (1974), 603-609.

[29] J. D. Pryce, *A device of R.J. Whitley s applied to pointwise compactness in spaces of continuous functions*, Proc. London Math. Soc. 23 (3) (1971), 532-546.

[30] V. Pták, *On a theorem of W.F. Eberlein*, Studia Math., 14 (1954), 276-284.

[31] J. Saint-Raymond, *Jeux topologiques et espaces de Namioka,* Pacific J. Math. 81 (1983), 499-504.

[32] D. B. Shakhmatov, *A pseudocompact Tychonoff space all countable subsets of which are closed and C*-embedded*, Topology and its Appl. 22 (2) (1986), 139-144.

[33] Michel Talagrand, *Espaces de Baire et espaces de Namioka,* Math. Ann., 270 (1985), 159-164.

[34] Talagrand, Michel, *Deux généralisations d'un théoreme de I. Namioka,* Pacific J. Math., 81 (1979), 239-251 .

[35] J. P. Troallic, *Quasi-continuité, continuite separée, et topologie éxtremale,* Proc. Amer. Math. Soc., 110 No.3, (1990), 819-827.

[36] J. P. Troallic, *Boundedness in $C_p(X,Y)$ and equicontinuity*, Topology and its Appl. 108 (2000), 79-89.

[37] Stephen Watson, *A connected pseudocompact space*, Topology and its Appl. 57 (1994), 151-162.