# On the topology of pointwise convergence on the boundaries of $L_1$-preduals

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**Abstract.** In this paper we prove a theorem more general than the following. "If $(X, \|\cdot\|)$ is an $L_1$-predual, $B$ is any boundary of $X$ and $\{x_n : n \in \mathbb{N}\}$ is any subset of $X$ then the closure of $\{x_n : n \in \mathbb{N}\}$ with respect to the topology of pointwise convergence on $B$ is separable with respect to the topology generated by the norm, whenever $\mathrm{Ext}(B_{X^*})$ is Lindelöf." Several applications of this result are also presented.

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# 1 Introduction

We shall say that a Banach space $(X, \|\cdot\|)$ is an $L_1$-*predual* if $X^*$ is isometric to $L_1(\mu)$ for some suitable measure $\mu$. Some examples of $L_1$-preduals include $(C(K), \|\cdot\|_\infty)$, and more generally, the space of continuous affine functions on a Choquet simplex (see [9] for the definition) endowed with the supremum norm (see, [4, Proposition 3.23]). We shall also consider the notion of a boundary. Specifically, for a non-trivial Banach space $X$ over $\mathbb{R}$ we say that a subset $B$ of $B_{X^*}$, the closed unit ball of $X^*$, is a *boundary*, if for each $x \in X$ there exists a $b^* \in B$ such that $b^*(x) = \|x\|$. The prototypical example of a boundary is $\mathrm{Ext}(B_{X^*})$ - the set of all extreme points of $B_{X^*}$, but there are many other interesting examples given in [8].

In a recent paper [8] the authors investigate the topology on a Banach space $X$ that is generated by $\mathrm{Ext}(B_{X^*})$ and, more generally, the topology on $X$ generated by an arbitrary boundary of $X$. This paper continues this study.

To be more precise we must first introduce some notation. For a nonempty subset $Y$ of the dual of a Banach space $X$ we shall denote by $\sigma(X, Y)$ the weakest linear topology on $X$ that makes all the functionals from $Y$ continuous. In [8] the authors show (see, [8, Theorem 2.2]) using [2, Lemma 1] that for any compact Hausdorff space $K$, any countable subset $\{x_n : n \in \mathbb{N}\}$ of $C(K)$ and any boundary $B$ of $(C(K), \|\cdot\|_\infty)$, the closure of $\{x_n : n \in \mathbb{N}\}$ with respect to the $\sigma(C(K), B)$ topology is separable with respect to the topology generated by the norm. In this paper we extend this result by showing that if $(X, \|\cdot\|)$ is an $L_1$-predual, $B$ is any boundary of $X$ and $\{x_n : n \in \mathbb{N}\}$ is any subset of $X$ then the closure of $\{x_n : n \in \mathbb{N}\}$ in the $\sigma(X, B)$ topology is separable with respect to the topology generated by the norm whenever $\mathrm{Ext}(B_{X^*})$ is weak$^*$ Lindelöf.

We conclude this paper with some applications that indicate the utility of our results.

## 2 Preliminary Results

Let $X$ be a topological space and let $\mathcal{F}$ be a family of nonempty, closed and separable subsets of $X$. Then $\mathcal{F}$ is *rich* if the following two conditions are fulfilled:

(i) for every separable subspace $Y$ of $X$, there exists a $Z \in \mathcal{F}$ such that $Y \subseteq Z$;

(ii) for every increasing sequence $(Z_n : n \in \mathbb{N})$ in $\mathcal{F}$, $\overline{\bigcup_{n\in\mathbb{N}} Z_n} \in \mathcal{F}$.

For any topological space $X$, the collection of all rich families of subsets forms a partially ordered set, under the binary relation of set inclusion. This partially ordered set has a greatest element, namely,

$$\mathcal{G}_X := \{S \subseteq X : S \text{ is a nonempty, closed and separable subset of } X\}.$$

On the other hand, if $X$ is a separable space, then the partially ordered set has a least element, namely, $\mathcal{G}_\varnothing := \{X\}$.

The raison d'être for rich families is revealed next.

**Proposition 1** *Suppose that $X$ is a topological space. If $\{\mathcal{F}_n : n \in \mathbb{N}\}$ are rich families of $X$ then so is $\bigcap_{n\in\mathbb{N}} \mathcal{F}_n$.*

For a proof of this Proposition see [3, Proposition 1.1].

Throughout this paper we will be primarily working with Banach spaces so a natural class of rich families, given a Banach space $X$, is the family of all closed separable linear subspaces of $X$, which we denote by $\mathcal{S}_X$. There are however many other interesting examples of rich families that can be found in [3] and [7].

For our first result we will provide another non-trivial example of a rich family, but to achieve this we first need a preliminary result that characterises when a given Banach space is an $L_1$-predual.

**Lemma 1** [6, §21, Theorem 7] *For a Banach space $X$ the following are equivalent:*

(i) *$X$ is an $L_1$-predual;*

(ii) *for each weak* continuous convex function $f$ on $B_{X^*}$*

$$f^*(0) = \frac{1}{2}\max\{f(x^*) + f(-x^*) : x^* \in B_{X^*}\}$$

*where $f^* = \inf\{h : h \geq f$ and $h$ is weak* continuous and affine on $B_{X^*}\}$.*

Before proceeding further we shall introduce the following notation. If $X$ is a normed linear space then each $x \in X$ defines a weak$^*$ continuous affine function $\widehat{x}$ on $B_{X^*}$ via the canonical embbeding, that is, $\widehat{x}(x^*) := x^*(x)$ for all $x^* \in B_{X^*}$.

**Theorem 1** *Let $X$ be an $L_1$-predual. Then the set of all closed separable linear subspaces of $X$ that are themselves $L_1$-preduals forms a rich family.*

**Proof:** Let $\mathscr{L} := \{Z \in \mathcal{S}_X : \ Z$ is an $L_1$-predual$\}$. We shall verify that $\mathscr{L}$ is a rich family. So first let us consider an arbitrary separable closed linear subspace $Y$ of $X$. Then by [12, Lemma 3.1] there exists a closed separable subspace $Z \in \mathscr{L}$ such that $Y \subseteq Z$. Next, let us consider an increasing sequence $(Z_n : n \in \mathbb{N})$ in $\mathscr{L}$ and let $Z := \overline{\bigcup_{n\in\mathbb{N}} Z_n}$. To show that $Z \in \mathscr{L}$ we shall appeal to Lemma 1. Let $f$ be a weak* continuous convex function on $B_{Z^*}$. Since

$$\frac{1}{2}\max\{f(x^*) + f(-x^*) : x^* \in B_{Z^*}\} \leq f^*(0),$$

it is enough to verify that for each $\varepsilon > 0$, $f^*(0) \leq \frac{1}{2}\max\{f(x^*) + f(-x^*) : x^* \in B_{Z^*}\} + \varepsilon$. To this end, suppose that $\varepsilon > 0$. Since $f$ is weak* continuous and convex and $B_{Z^*}$ is weak* compact, by [1, Corollary I.1.3] there exist $z_i \in Z$ and $c_i \in \mathbb{R}$, $i = 1, \dots n$, such that the weak* convex continuous $g : B_{Z^*} \to \mathbb{R}$ defined by

$$g := \max\{\widehat{z_1} + c_1, \widehat{z_2} + c_2, \dots, \widehat{z_n} + c_n\}$$

satisfies

$$f(z^*) - \varepsilon < g(z^*) < f(z^*), \quad z^* \in B_{Z^*}.$$

Since $\bigcup_{n\in\mathbb{N}} Z_n$ is dense in $Z$ we may further assume that all the elements $z_i$ are contained in some fixed $Z_j$, $j \in \mathbb{N}$.

Next, let $r : B_{Z^*} \to B_{Z_j^*}$ be the restriction mapping (i.e., $r(z^*) = z^*|_{Z_j}$ for all $z^* \in B_{Z^*}$) and let $h : B_{Z_j^*} \to \mathbb{R}$ be defined by, $h := \max\{\widehat{z_1} + c_1, \widehat{z_2} + c_2, \dots, \widehat{z_n} + c_n\}$. Then $h$ is weak* continuous and convex on $B_{Z_j^*}$ and $g = h \circ r$. Moreover, by the definition of $g$ (and the fact that $r$ is weak*-to-weak* continuous and linear) we have that $g^*(z^*) \leq h^*(r(z^*))$ for all $z^* \in B_{Z^*}$. Now, by the assumption that $Z_j$ is an $L_1$-predual (and Lemma 1) there exists a $y^* \in B_{Z_j^*}$ such that

$$h^*(0) = \frac{1}{2}[h(y^*) + h(-y^*)].$$

Choose $z^* \in r^{-1}(y^*)$; which is nonempty by the Hahn-Banach extension theorem. Then,

$$g^*(0) \leq h^*(0) = \frac{1}{2}[h(y^*) + h(-y^*)] = \frac{1}{2}[g(z^*) + g(-z^*)] \leq g^*(0).$$

Therefore,

$$\begin{aligned} f^*(0) - \varepsilon = (f - \varepsilon)^*(0) &\leq g^*(0) = \frac{1}{2}[g(z^*) + g(-z^*)] \\ &\leq \frac{1}{2}[f(z^*) + f(-z^*)] \leq \frac{1}{2}\max\{f(x^*) + f(-x^*) : x^* \in B_{Z^*}\}. \end{aligned}$$

That is, $f^*(0) \leq \frac{1}{2}\max\{f(x^*) + f(-x^*) : x^* \in B_{Z^*}\} + \varepsilon$; which completes the proof. ☺

Before we can introduce another class of rich families we require the following lemma which is a Banach space version of [10, Theorem 2.10].

**Lemma 2** *Let $Y$ be a closed separable linear subspace of a Banach space $X$ and suppose that $L \subseteq \mathrm{Ext}(B_{X^*})$ is weak$^*$ Lindelöf. Then there exists a closed separable linear subspace $Z$ of $X$, containing $Y$, such that for any $l^* \in L$ and any $x^*$, $y^* \in B_{Z^*}$ if $l^*|_Z = \frac{1}{2}(x^* + y^*)$ then $x^*|_Y = y^*|_Y$.*

**Proof:** Let $\mathscr{B}$ be a countable base for the topology on $(B_{Y^*}, \text{weak}^*)$ consisting of closed convex sets. Recall that such a base exists because $(B_{Y^*}, \text{weak}^*)$ is compact, by the Banach-Alaoglu Theorem, and $(B_{Y^*}, \text{weak}^*)$ is metrizable, since $Y$ is separable. Let:

(i) $\mathscr{F} := \{r^{-1}(B) : B \in \mathscr{B}\}$, where $r : B_{X^*} \to B_{Y^*}$ is the restriction mapping;

(ii) $\mathscr{R} := \{\frac{1}{2}(F_1 + F_2) : F_1, F_2 \in \mathscr{F}$ and $F_1 \cap F_2 = \varnothing\}$.

By construction $\bigcup \mathscr{R} \subseteq B_{X^*} \setminus \mathrm{Ext}(B_{X^*})$ and so $L \cap \bigcup \mathscr{R} = \varnothing$. Furthermore, for each $l^* \in L$ and $F \in \mathscr{R}$ there exists a $y \in X$ such that

$$\sup\{\widehat{y}(f^*) : f^* \in F\} < \widehat{y}(l^*).$$

Therefore, since $L$ is weak$^*$ Lindelöf for each $F \in \mathscr{R}$ there exists a countable subset $C_F$ in $X$ such that, for each $l^* \in L$ there exists a $y \in C_F$ such that $\sup\{\widehat{y}(f^*) : f^* \in F\} < \widehat{y}(l^*)$. If we set $C := \bigcup\{C_F : F \in \mathscr{R}\}$ and $Z := \overline{\mathrm{span}}(C \cup X)$ then $X \subseteq Z$ and $Z$ is a closed separable linear subspace of $X$.

It now only remains to verify that if $l^* \in L$, $x^*$, $y^* \in B_{Z^*}$ and $l^*|_Z = \frac{1}{2}(x^* + y^*)$ then $x^*|_Y = y^*|_Y$. So, in order to obtain a contradiction, suppose that for some $l^* \in L$ and $x^*$, $y^* \in B_{Z^*}$, $l^*|_Z = \frac{1}{2}(x^* + y^*)$ but $x^*|_Y \neq y^*|_Y$. Then there exists $B_1, B_2 \in \mathscr{B}$ such that $x^*|_Y \in B_1$ and $y^*|_Y \in B_2$ and $B_1 \cap B_2 = \varnothing$. Set $F_1 := r^{-1}(B_1)$ and $F_2 := r^{-1}(B_2)$. Then $F_1, F_2 \in \mathscr{F}$ and $F_1 \cap F_2 = \varnothing$. Now, by the Hahn-Banach extension Theorem there exist $x_1^* \in B_{X^*}$ and $y_1^* \in B_{X^*}$ such that $x_1^*|_Z = x^*$ and $y_1^*|_Z = y^*$. Moreover,

$$x_1^*|_Y = (x_1^*|_Z)|_Y = x^*|_Y \in B_1 \quad \text{and} \quad y_1^*|_Y = (y_1^*|_Z)|_Y = y^*|_Y \in B_2.$$

That is, $x_1^* \in F_1$ and $y_1^* \in F_2$. Therefore, $\frac{1}{2}(x_1^* + y_1^*) \in \frac{1}{2}(F_1 + F_2) =: F$. Since $F \in \mathscr{R}$, by the construction there exists a $y \in C_F \subseteq C \subseteq Z$ such that $\sup\{\widehat{y}(f^*) : f^* \in F\} < \widehat{y}(l^*)$. In particular,

$$\frac{1}{2}(x^* + y^*)(y) = \widehat{y}(\frac{1}{2}(x_1^* + y_1^*)) < l^*(y) = (l^*|_Z)(y).$$

However, this contradicts the fact that $\frac{1}{2}(x^* + y^*) = l^*|_Z$. ☺

**Theorem 2** *Let $X$ be a Banach space and let $L \subseteq \mathrm{Ext}(B_{X^*})$ be a weak$^*$ Lindelöf subset. Then the set of all $Z$ in $\mathcal{S}_X$ such that $\{l^*|_Z : l^* \in L\} \subseteq \mathrm{Ext}(B_{Z^*})$ forms a rich family.*

**Proof:** Let $\mathscr{L}$ denote the family of all closed separable linear subspaces $Z$ of $X$ such that $\{l^*|_Z : l^* \in L\} \subseteq \mathrm{Ext}(B_{Z^*})$. We shall verify that $\mathscr{L}$ is a rich family of closed separable linear subspaces of $X$. So first let us consider an arbitrary closed separable linear subspace $Y$ of $X$, with the aim of showing that there exists a subspace $Z \in \mathscr{L}$ such that $Y \subseteq Z$. We begin by inductively applying Lemma 2 to obtain an increasing sequence $(Z_n : n \in \mathbb{N})$ of closed

separable linear subspaces of $X$ such that: $Y \subseteq Z_1$ and for any $l^* \in L$ and any $x^*, y^* \in B_{Z^*_{n+1}}$ if $l^*|_{Z_{n+1}} = \frac{1}{2}(x^* + y^*)$ then $x^*|_{Z_n} = y^*|_{Z_n}$.
We now claim that if $Z := \overline{\bigcup_{n\in\mathbb{N}} Z_n}$ then $l^*|_Z \in \mathrm{Ext}(B_{Z^*})$ for each $l^* \in L$. To this end, suppose that $l^* \in L$ and $l^*|_Z = \frac{1}{2}(x^* + y^*)$ for some $x^*, y^* \in B_{Z^*}$. Then for each $n \in \mathbb{N}$

$$l^*|_{Z_{n+1}} = (l^*|_Z)|_{Z_{n+1}} = \frac{1}{2}(x^* + y^*)|_{Z_{n+1}} = \frac{1}{2}(x^*|_{Z_{n+1}} + y^*|_{Z_{n+1}})$$

and $x^*|_{Z_{n+1}}, y^*|_{Z_{n+1}} \in B_{Z^*_{n+1}}$ Therefore, by construction $x^*|_{Z_n} = y^*|_{Z_n}$. Now since $\bigcup_{n\in\mathbb{N}} Z_n$ is dense in $Z$ and both $x^*$ and $y^*$ are continuous we may deduce that $x^* = y^*$; which in turn implies that $l^*|_Z \in \mathrm{Ext}(B_{Z^*})$. This shows that $Y \subseteq Z$ and $Z \in \mathscr{L}$.

To complete this proof we must verify that for each increasing sequence of closed separable subspaces $(Z_n : n \in \mathbb{N})$ in $\mathscr{L}$, $\overline{\bigcup_{n\in\mathbb{N}} Z_n} \in \mathscr{L}$. This however, follows easily from the definition of the family $\mathscr{L}$. ☺

Let $X$ be a normed linear space. Then we say that an element $x^* \in B_{X^*}$ is *weak* exposed* if there exists an element $x \in X$ such that $y^*(x) < x^*(x)$ for all $y^* \in B_{X^*} \setminus \{x^*\}$. It is not difficult to show that if $\mathrm{Exp}(B_{X^*})$ denotes the set of all weak* exposed points of $B_{X^*}$ then $\mathrm{Exp}(B_{X^*}) \subseteq \mathrm{Ext}(B_{X^*})$. However, if $X$ is a separable $L_1$-predual then the relationship between $\mathrm{Exp}(B_{X^*})$ and $\mathrm{Ext}(B_{X^*})$ is much closer.

**Lemma 3** [12, Lemma 3.3(b)] *If $X$ is a separable $L_1$-predual, then* $\mathrm{Exp}(B_{X^*}) = \mathrm{Ext}(B_{X^*})$.

Let us also pause for a moment to recall that if $B$ is any boundary of a Banach space $X$ then

$$\mathrm{Exp}(B_{X^*}) \subseteq B \cap \mathrm{Ext}(B_{X^*}) \subseteq \mathrm{Ext}(B_{X^*}) \subseteq \overline{B}^{\text{weak}^*}.$$

The fact that $\mathrm{Ext}(B_{X^*}) \subseteq \overline{B}^{\text{weak}^*}$ follows from Milman's theorem, [9, page 8] and the fact that $B_{X^*} = \overline{\mathrm{co}}^{\text{weak}^*}(B)$; which in turn follows from a separation argument. Let us also take this opportunity to observe that if $B_X$ denotes the closed unit ball in $X$ then $B_X$ is closed in the $\sigma(X, B)$ topology for any boundary $B$ of $X$. Finally, let us end this section with one more simple observation that will turn out to be useful in our later endeavours.

**Proposition 2** *Suppose that $Y$ is a linear subspace of a Banach space $(X, \|\cdot\|)$ and $B$ is any boundary for $X$. Then for each $e^* \in \mathrm{Exp}(B_{Y^*})$ there exists $b^* \in B$ such that $e^* = b^*|_Y$.*

**Proof:** Suppose that $e^* \in \mathrm{Exp}(B_{Y^*})$ then there exists an $x \in Y$ such that $y^*(x) < e^*(x)$ for each $y^* \in B_{Y^*} \setminus \{e^*\}$. By the fact that $B$ is a boundary of $(X, \|\cdot\|)$ there exists a $b^* \in B$ such that $b^*(x) = \|x\| \neq 0$. Then for any $y^* \in B_{Y^*}$ we have

$$y^*(x) \leq |y^*(x)| \leq \|y^*\|\|x\| \leq \|x\| = b^*(x) = (b^*|_Y)(x).$$

In particular, $e^*(x) \leq b^*|_Y(x)$. Since $b^*|_Y \in B_{Y^*}$ and $y^*(x) < e^*(x)$ for all $y^* \in B_{Y^*} \setminus \{e^*\}$, it must be the case that $e^* = b^*|_Y$. ☺

This ends our preliminary section.

## 3 The Main Results

**Theorem 3** *Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual and suppose that $\{x_n : n \in \mathbb{N}\} \subseteq X$, then $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \subseteq \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}$.*

**Proof:** In order to obtain a contradiction let us suppose that

$$\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \not\subseteq \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}.$$

Choose $x \in \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \setminus \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}$. Then there exists a finite set $\{e_1^*, e_2^*, \ldots, e_m^*\} \subseteq \mathrm{Ext}(B_{X^*})$ and $\varepsilon > 0$ so that

$$\{y \in X : |e_k^*(x) - e_k^*(y)| < \varepsilon \text{ for all } 1 \leq k \leq m\} \cap \{x_n : n \in \mathbb{N}\} = \varnothing.$$

Let $Y := \overline{\mathrm{span}}(\{x_n : n \in \mathbb{N}\} \cup \{x\})$, let $\mathcal{F}_1$ be any rich family of $L_1$-preduals; whose existence is guaranteed by Theorem 1, and let $\mathcal{F}_2$ be any rich family such that for every $Z \in \mathcal{F}_2$ and every $1 \leq k \leq m$, $e_k^*|_Z \in \mathrm{Ext}(B_{Z^*})$; whose existence is guaranteed by Theorem 2. Next, let us choose $Z \in \mathcal{F}_1 \cap \mathcal{F}_2$ so that $Y \subseteq Z$. Recall that this is possible because, by Proposition 1, $\mathcal{F}_1 \cap \mathcal{F}_2$ is a rich family. Since $Z$ is a separable $L_1$-predual we have by Lemma 3 that $e_k^*|_Z \in \mathrm{Exp}(B_{Z^*})$ for each $1 \leq k \leq m$. Now, by Proposition 2 for each $1 \leq k \leq m$ there exists a $b_k^* \in B$ such that $e_k^*|_Z = b_k^*|_Z$. Therefore,

$$|b_k^*(x) - b_k^*(x_j)| = |(b_k^*|_Z)(x) - (b_k^*|_Z)(x_j)| = |(e_k^*|_Z)(x) - (e_k^*|_Z)(x_j)| = |e_k^*(x) - e_k^*)(x_j)|.$$

for all $j \in \mathbb{N}$ and all $1 \leq k \leq m$. Thus,

$$\{y \in X : |b_k^*(x) - b_k^*(y)| < \varepsilon \text{ for all } 1 \leq k \leq m\} \cap \{x_n : n \in \mathbb{N}\} = \varnothing.$$

This contradicts the fact that $x \in \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$; which completes the proof. ☺

**Corollary 1** [12, Theorem 1.1(a)] *Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual. Then every relatively countably $\sigma(X,B)$-compact subset is relatively countably $\sigma(X, \mathrm{Ext}(B_{X^*}))$-compact. In particular, every norm bounded, relatively countably $\sigma(X,B)$-compact subset is relatively weakly compact.*

**Proof:** Suppose that a nonempty set $C \subseteq X$ is relatively countably $\sigma(X,B)$-compact. Let $\{c_n : n \in \mathbb{N}\}$ be any sequence in $C$ then by Theorem 3

$$\varnothing \neq \bigcap_{n\in\mathbb{N}} \overline{\{c_k : k \geq n\}}^{\sigma(X,B)} \subseteq \bigcap_{n\in\mathbb{N}} \overline{\{c_k : k \geq n\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}$$

Hence $C$ is relatively countably $\sigma(X, \mathrm{Ext}(B_{X^*}))$-compact. In the case when $C$ is also norm bounded the result follows from [5]. ☺

Recall that a *network* for a topological space $X$ is a family $\mathscr{N}$ of subsets of $X$ such that for any point $x \in X$ and any open neighbourhood $U$ of $x$ there is an $N \in \mathscr{N}$ such that $x \in N \subseteq U$, and a topological space $X$ is said to be $\aleph_0$-*monolithic* if the closure of every countable set has a countable network.

The next Corollary generalises [8, Theorem 2.2].

**Corollary 2** *Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual and suppose that $\{x_n : n \in \mathbb{N}\} \subseteq X$. Then $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$ is norm separable whenever $X$ is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology. In particular, $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$ is norm separable whenever $\mathrm{Ext}(B_{X^*})$ is weak* Lindelöf.*

**Proof:** This follows directly from Theorem 3 and [8, Theorem 2.6]; which states that if $M \subseteq X$ has a countable network with respect to the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology then $M$ is separable in $(X, \|\cdot\|)$. The last claim follows from [8, Theorem 2.14] where it is shown that if $\mathrm{Ext}(B_{X^*})$ is weak* Lindelöf then $X$ is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology. ☺

In [8] many conditions are given under which $\sigma(X, \mathrm{Ext}(B_{X^*}))$ is $\aleph_0$-monolithic.

To demonstrate how this last theorem may be applied we shall present some sample applications.

# 4 Applications

Our first application is to metrizability of compact convex sets. If $K$ is a compact convex set in a real locally convex space, let $A(K)$ stand for the space of all affine continuous functions on $K$.

**Proposition 3** *Let $K$ be a Choquet simplex in a separated locally convex space (over $\mathbb{R}$) such that every regular Borel probability measure carried on $\mathrm{Ext}(K)$ is atomic. Then $K$ is metrizable if, and only if, the space $(B_{A(K)}, \sigma(A(K), B))$ is separable, for some boundary $B$ of $(A(K), \|\cdot\|_\infty)$.*

**Proof:** This follows directly from Theorem 3 and [8, Theorem 2.19]. ☺

We remark that there exists a non-metrizable Choquet simplex $K$ and a boundary $B$ of $(A(K), \|\cdot\|_\infty)$ such that $(B_{A(K)}, \sigma(A(K), B))$ is separable. (It is shown in [12, Section 4] that the construction of [8, Example 2.10] yields the required example.)

Our final few results concern automatic continuity. In particular, the next result improves [11, Theorem 6].

**Proposition 4** *Let $B$ be any boundary for a Banach space $X$ that is an $L_1$-predual and suppose that $A$ is a a separable Baire space. If $X$ is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology then for each continuous mapping $f : A \to (X, \sigma(X, B))$ there exists a dense subset $D$ of $A$ such that $f$ is continuous with respect to the norm topology on $X$ at each point of $D$.*

**Proof:** Fix $\varepsilon > 0$ and consider the open set:

$$O_\varepsilon := \bigcup \{U \subseteq A : U \text{ is open and } \|\cdot\| - \mathrm{diam}[f(U)] \leq 2\varepsilon\}.$$

We shall show that $O_\varepsilon$ is dense in $A$. To this end, let $W$ be a nonempty open subset of $A$ and let $\{a_n : n \in \mathbb{N}\}$ be a countable dense subset of $W$). Then by continuity

$$f(W) \subseteq \overline{\{f(a_n) : n \in \mathbb{N}\}}^{\sigma(X,B)};$$

which is norm separable by Corollary 2. Therefore there exists a countable set $\{x_n : n \in \mathbb{N}\}$ in $X$ such that $f(W) \subseteq \bigcup_{n\in\mathbb{N}}(x_n + \varepsilon B_X)$. For each $n \in \mathbb{N}$, let $C_n := f^{-1}(x_n + \varepsilon B_X)$. Since each $x_n + \varepsilon B_X$ is closed in the $\sigma(X, B)$ topology each set $C_n$ is closed in $A$ and moreover, $W \subseteq \bigcup_{n\in\mathbb{N}} C_n$. Since $W$ is of the second Baire category in $A$ there exist a nonempty open set $U \subseteq W$ and a $k \in \mathbb{N}$ such that $U \subseteq C_k$. Then $U \subseteq O_\varepsilon \cap W$ and $O_\varepsilon$ is indeed dense in $A$. Hence $f$ is $\|\cdot\|$-continuous at each point of $\bigcap_{n\in\mathbb{N}} O_{1/n}$. ☺

**Theorem 4** *Suppose that $A$ is a topological space with countable tightness that possesses a rich family $\mathcal{F}$ of Baire subspaces and suppose that $X$ is an $L_1$-predual. Then for any boundary $B$ of $X$ and any continuous function $f : A \to (X, \sigma(X, B))$ there exists a dense subset $D$ of $A$ such that $f$ is continuous with respect to the norm topology on $X$ at each point of $D$ provided $X$ is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology.*

**Proof:** In order to obtain a contradiction let us suppose that $f$ does not have a dense set of points of continuity with respect to the norm topology on $X$. Since $A$ is a Baire space (by [7, Theorem 3.3]), this implies that for some $\varepsilon > 0$ the open set:

$$O_\varepsilon := \bigcup\{U \subseteq A : \ U \text{ is open and } \|\cdot\|\text{-diam}[f(U)] \leq 2\varepsilon\}$$

is not dense in $A$. That is, there exists a nonempty open subset $W$ of $A$ such that $W \cap O_\varepsilon = \varnothing$. For each $x \in A$, let $F_x := \{y \in A : \|f(y) - f(x)\| > \varepsilon\}$. Then $x \in \overline{F_x}$ for each $x \in W$. Moreover, since $A$ has countable tightness, for each $x \in W$, there exists a countable subset $C_x$ of $F_x$ such that $x \in \overline{C_x}$.

Next, we inductively define an increasing sequence of separable subspaces $(F_n : n \in \mathbb{N})$ of $A$ and countable sets $(D_n : n \in \mathbb{N})$ in $A$ such that:

(i) $W \cap F_1 \neq \varnothing$;

(ii) $\bigcup\{C_x : x \in D_n \cap W\} \cup F_n \subseteq F_{n+1} \in \mathcal{F}$ for all $n \in \mathbb{N}$, where $D_n$ is any countable dense subset of $F_n$.

Note that since the family $\mathcal{F}$ is rich this construction is possible.

Let $F := \overline{\bigcup_{n\in\mathbb{N}} F_n}$ and $D := \bigcup_{n\in\mathbb{N}} D_n$. Then $\overline{D} = F \in \mathcal{F}$ and $\|\cdot\|$-diam$[f(U)] \geq \varepsilon$ for every nonempty open subset $U$ of $F \cap W$. Therefore, $f|_F$ has no points of continuity in $F \cap W$ with respect to the $\|\cdot\|$-topology. This however, contradicts Proposition 4. ☺

Our final result improves [7, Theorem 4.7].

**Corollary 3** *Suppose that $A$ is a topological space with countable tightness that possesses a rich family of Baire subspaces and suppose that $K$ is a compact Hausdorff space. Then for any boundary of $(C(K), \|\cdot\|_\infty)$ and any continuous function $f : A \to (C(K), \sigma(C(K), B))$ there exists a dense subset $D$ of $A$ such that $f$ is continuous with respect to the $\|\cdot\|_\infty$-topology at each point of $D$.*

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