CHEBYSHEV SETS

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Abstract

A **Chebyshev set** is a subset of a normed linear space that admits unique best approximations. In the first part of this article we present some basic results concerning Chebyshev sets. In particular, we investigate properties of the metric projection map, sufficient conditions for a subset of a normed linear space to be a Chebyshev set, and sufficient conditions for a Chebyshev set to be convex. In the second half of the article we present a construction of a non-convex Chebyshev subset of an inner product space.

1. Introduction

Given a normed linear space $(X, \|\cdot\|)$, we shall say that a subset of X is a 'Chebyshev set' if every point outside of it admits a unique nearest point. In this article we will review some of the literature, and a little, a very little, of the history of the question of whether every Chebyshev set is convex (for a brief summary of the history of this problem, see [26, p.307]). The volume of research invested in this topic is vast and we will not even pretend to present, even a small amount, of what is known. Instead, we will be content to present some of the most basic results concerning Chebyshev sets (see Section 2) and then present what the authors believe are among some of the most interesting partial solutions to the following question, which from hence forth, will be referred to as the 'Chebyshev set problem':

"Is every Chebyshev set in a Hilbert space convex?".

In the latter part of this article we will present, in full gory detail, an example, due to V.S. Balaganskiĭ and L.P. Vlasov [7], of a nonconvex Chebyshev set in an infinite dimensional inner product space. While this result appears tantalisingly close to a solution, in the negative, of the Chebyshev set problem, it is not at all clear how to transfer this construction of Balaganskiĭ and Vlasov (which is based upon an earlier construction of Johnson [47] and later corrected by Jiang [46]) into the realm of Hilbert spaces.

So at the time of writing this article, the problem of whether every Chebyshev set in a Hilbert space is convex is still wildly open. We hope that the gentle introduction presented here might serve to inspire some of the next generation of mathematicians to tackle, and perhaps even solve, this long standing open problem.

This article is of course not the first survey article ever written on the Chebyshev set problem. Prior to this there have been many such articles, among the most prominent of these are the survey articles of [5, 7, 11, 25, 62, 82, 90], each of which had something different to offer.

In the present article, we believe that our point of difference from the aforementioned articles is that our approach is more naive and elementary and hence, hopefully, more accessible to students and researchers entering this area of research.

REMARK. Throughout this article all normed linear spaces/Hilbert spaces will be assumed to be non-trivial, i.e., contain more than just the zero vector, and be over the field of real numbers.

The structure of the reminder of this paper is as follows: Section 2 contains the preliminary results, which comprise: Basic results (subsection 2.1); Strict convexity (subsection 2.2) and Continuity of the metric projection (subsection 2.3). Section 3, titled "The convexity of Chebyshev sets" consists of four subsections: Proof by fixed point theorem (subsection 3.1); Proof using inversion in the unit sphere (subsection 3.2); A proof using convex analysis (subsection 3.3) and Vlasov's theorem (subsection 3.4). The final section; Section 4, is entitled "A non-convex Chebyshev set in an inner product space". This section has two subsections: Background and motivation (subsection 4.1) and, The construction (subsection 4.2). Finally, the paper ends with five appendices: A, B, C, D and E which contain the proofs of several technical results that are called upon within the main text of the paper.

2. Preliminary results

2.1. Basic facts We begin this section by formally defining how we measure the distance of a point to a subset of a normed linear space. From this we define the metric projection mapping, which in turn is used to formulate the concepts of proximinal sets and Chebyshev sets. A few simple examples of such sets are then given to familiarise the reader with these objects.

DEFINITION 2.1. Let $(X, \|\cdot\|)$ be a normed linear space and K be a nonempty subset of X. For any point $x \in X$ we define $d(x, K) := \inf_{y \in K} \|x - y\|$ and call this the **distance from x to K**. We will also refer to the map $x \mapsto d(x, K)$ as the **distance function for K**.

The following simple result regarding the distance function will be used repeatedly throughout the remainder of the article.

PROPOSITION 2.2. *Let K be a nonempty subset of a normed linear space $(X, \|\cdot\|)$. Then the distance function for K is nonexpansive (and hence continuous).*

PROOF. Let $x, y \in X$ and $k \in K$. By the triangle inequality and the definition of the distance function, we get that $d(x, K) \leq \|x - k\| \leq \|x - y\| + \|y - k\|$. By rearranging this equation, we obtain $d(x, K) - \|x - y\| \leq \|y - k\|$. Since $k \in K$ was arbitrary and the left hand side of the previous expression is independent of k, we see that $d(x, K) - \|x - y\| \leq d(y, K)$. By symmetry, $d(y, K) - \|x - y\| \leq d(x, K)$. Thus, $-\|x - y\| \leq d(x, K) - d(y, K) \leq \|x - y\|$, and so $|d(x, K) - d(y, K)| \leq \|x - y\|$. Therefore, we have the nonexpansive property. □

The fundamental concept behind the definition of a proximinal set, or a Chebyshev set, is that of 'nearest points'. The following definition makes this idea precise.

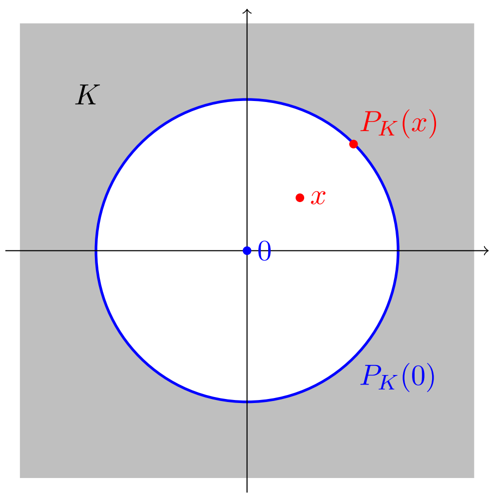
DEFINITION 2.3. Let $(X, \|\cdot\|)$ be a normed linear space and K be a subset of X. We define a set valued mapping $P_K : X \to \mathcal{P}(K)$ by $P_K(x) := \{y \in K : \|x - y\| = d(x, K)\}$, if K is nonempty and by $P_K(x) = \emptyset$ if K is the empty set. We refer to the elements of $P_K(x)$ as the **best approximations of x in K** (or, the **nearest points to x in K**). We say that K is a **proximinal set** if $P_K(x)$ is nonempty for each $x \in X$ and that K is a **Chebyshev set** if $P_K(x)$ is a singleton for each $x \in X$. For a Chebyshev set we define the map $p_K : X \to K$ as the map that assigns to each $x \in X$ the unique element of $P_K(x)$. We will refer to both P_K and p_K as the **metric projection mapping (for K)**.

Note: it should be immediately clear from the definition that every nonempty compact set in a normed linear space is proximinal. Moreover, $P_K(x) = \{x\}$ for any $x \in K$ and

$$P_K(X \setminus K) \subseteq \mathrm{Bd}(K) = \mathrm{Bd}(X \setminus K).$$

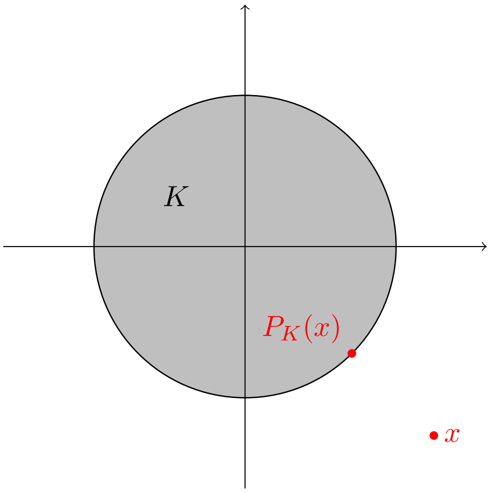
To familiarise ourselves with these notions we now present a few examples.

EXAMPLE 1. Consider $K := \mathbb{R}^2 \setminus B(0; 1) \subseteq \mathbb{R}^2$ equipped with the Euclidean norm.



It is easy to check that for any $x \in B(0; 1) \setminus \{0\}$, $P_K(x) = \left\{\frac{x}{\|x\|}\right\}$, whilst $P_K(0) = S_{\mathbb{R}^2}$. Hence, K is proximinal, but not a Chebyshev set.

EXAMPLE 2. Consider $K := B[0;1] \subseteq \mathbb{R}^2$ equipped with the Euclidean norm.



It is straightforward to check that for any $x \in \mathbb{R}^2 \setminus K$, $P_K(x) = \left\{\frac{x}{\|x\|}\right\}$. Therefore, K is a Chebyshev set.

EXAMPLE 3. Let $n \in \mathbb{N}$. Consider $K := B(0;1) \subseteq \mathbb{R}^n$ equipped with the Euclidean norm. Choose $x \in S_{\mathbb{R}^n}$. Clearly, $d(x, K) = 0$, but since $x \notin K$, $P_K(x) = \emptyset$, and so K is not proximinal.

In general it is difficult to deduce any structure concerning the behaviour of the metric projection mapping. However, one exception to this is when the set K is a subspace.

LEMMA 2.4. *Let K be a subspace of a normed linear space $(X, \|\cdot\|)$. Then for each $x \in X$, $k \in K$ and $\lambda \in \mathbb{R}$, $P_K(\lambda x + k) = \lambda P_K(x) + k$. Note: if $P_K(x) = \emptyset$, then $\lambda P_K(x) + k = \emptyset$.*

PROOF. Let $x \in X$, $k \in K$ and $\lambda \in \mathbb{R}$. Firstly, observe that

$$\begin{aligned} d(\lambda x + k, K) &= \inf_{y \in K} \|(\lambda x + k) - y\| \\ &= |\lambda| \inf_{z \in K} \|x - z\| \qquad \text{since } K \text{ is a subspace} \\ &= |\lambda|\, d(x, K). \end{aligned}$$

If $\lambda = 0$, then the result holds trivially. So suppose $\lambda \neq 0$. Therefore,

$$\begin{aligned} y \in P_K(\lambda x + k) &\Leftrightarrow \|y - (\lambda x + k)\| = d(\lambda x + k, K) \\ &\Leftrightarrow |\lambda| \left\| \left(\frac{y-k}{\lambda} \right) - x \right\| = |\lambda|\, d(x, K) \\ &\Leftrightarrow \left\| \left(\frac{y-k}{\lambda} \right) - x \right\| = d(x, K) \\ &\Leftrightarrow \frac{y-k}{\lambda} \in P_K(x) \\ &\Leftrightarrow y \in \lambda P_K(x) + k. \end{aligned}$$

□

EXAMPLE 4 ([3]). Let $(X, \|\cdot\|)$ be a normed linear space and $x^* \in X^* \setminus \{0\}$. Then $\ker(x^*)$ is a proximinal set if, and only if, x^* attains its norm (i.e., $\|x^*\| = |x^*(x)|$ for some $x \in S_X$).

PROOF. Suppose that $M := \ker(x^*)$ is a proximinal set. Then there exists an $x \in X$ such that $d(x, M) = \|x\| = 1$. To see this, choose $z \in X \setminus M$. This is possible since $x^* \neq 0$. Since M is proximinal, we can find $m \in M$ such that $\|z - m\| = d(z, M)$. Let

$$x := \frac{z-m}{d(z, M)} = \frac{z-m}{\|z-m\|}.$$

Clearly, $\|x\| = 1$ and from Lemma 2.4, it follows that

$$d(x, M) = d\left(\frac{z-m}{d(z,M)}, M \right) = d\left(\frac{z}{d(z,M)} + \frac{-m}{d(z,M)}, M \right) = \frac{d(z,M)}{d(z,M)} = 1.$$

Let $y \in X$. Then $y = \dfrac{x^*(y)}{x^*(x)} x + m$ for some $m \in M$. Now,

$$\begin{aligned} \frac{|x^*(y)|}{|x^*(x)|} &= \frac{|x^*(y)|}{|x^*(x)|} d(x, M) \\ &= \frac{|x^*(y)|}{|x^*(x)|} \inf_{z \in M} \|x - z\| \\ &= \inf_{z \in M} \left\| \frac{x^*(y)}{x^*(x)} x + m - z \right\| \quad \text{since } M \text{ is a subspace and } m \in M \\ &= \inf_{z \in M} \|y - z\| \\ &= d(y, M) \\ &\leq \|y\| \quad \text{since } 0 \in M. \end{aligned}$$

Thus, $|x^*(y)| \leq |x^*(x)| \|y\|$ for all $y \in X$. Hence, $\|x^*\| \leq |x^*(x)| \leq \|x^*\|$ since $\|x\| = 1$, and so $\|x^*\| = |x^*(x)|$.

Conversely, suppose $x^* \in X^* \setminus \{0\}$ attains its norm at $x \in S_X$, i.e., $\|x^*\| = x^*(x)$. Let $M := \ker(x^*)$ and $m \in M$. Then

$$1 = \frac{\|x^*\|}{\|x^*\|} = \frac{x^*(x)}{\|x^*\|} = \frac{x^*(x-m)}{\|x^*\|} \leq \frac{\|x^*\|\,\|x-m\|}{\|x^*\|} = \|x-m\|\,.$$

Therefore, $1 \leq d(x, M) \leq \|x - 0\| = \|x\| = 1$. Thus, $0 \in P_M(x)$. Now let $y \in X$. We can write $y = \lambda x + m$ for some $\lambda \in \mathbb{R}$ and $m \in M$. By Lemma 2.4, it follows that $P_M(y) = P_M(\lambda x + m) = \lambda P_M(x) + m \neq \emptyset$. Hence, M is proximinal. $\square$

We are particularly interested in finding necessary and sufficient conditions for a set to be a proximinal set or a Chebyshev set. Our first necessary condition, as suggested by Example 3, is that a proximinal set must be closed.

PROPOSITION 2.5. *Let K be a proximinal set in a normed linear space $(X, \|\cdot\|)$. Then K is nonempty and closed.*

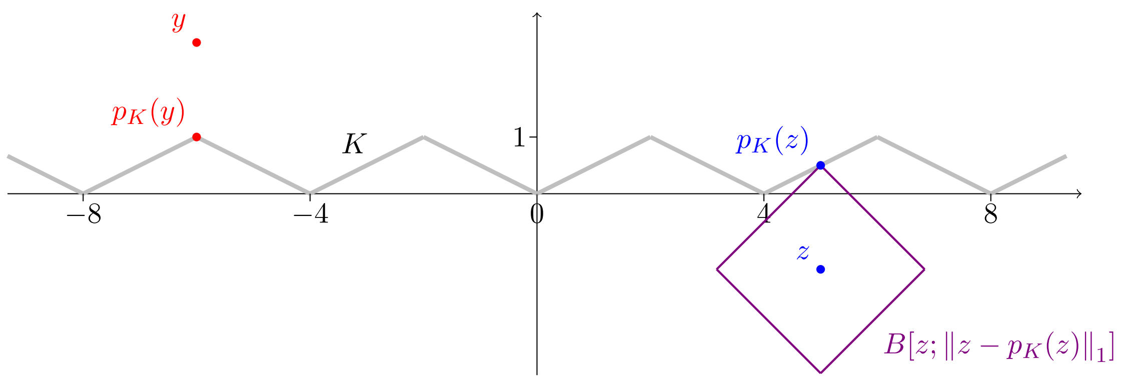
PROOF. To see that K is nonempty we simply note that $\emptyset \neq P_K(0) \subseteq K$. To show that K is closed it is sufficient to show that $\overline{K} \subseteq K$. To this end, let $x \in \overline{K}$. Since K is proximinal, there exists $k \in K$ such that $\|x - k\| = d(x, K) = 0$. Therefore, $x = k \in K$. $\square$

Of particular importance in the study of Chebyshev sets is the notion of convexity.

DEFINITION 2.6. Let X be a vector space and C be a subset of X. We shall say that C is **convex** if, for any $a, b \in C$ and any $\lambda \in [0, 1]$, we have $[a, b] \in C$. We say that C is **concave** if $X \setminus C$ is convex. Furthermore, we say that C is **midpoint convex** if for any $a, b \in C$, $\frac{a+b}{2} \in C$.

Motivated by Example 1, one might be tempted to conjecture that all Chebyshev sets are convex. However, the following example demonstrates that this is not the case.

EXAMPLE 5. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \frac{1}{2} d(x, 4\mathbb{Z})$ for all $x \in \mathbb{R}$.



Then $K := \mathrm{Graph}(f)$, viewed as a subset of $(\mathbb{R}^2, \|\cdot\|_1)$, is a (nonconvex) Chebyshev set. Indeed, for any $z := (a, b) \in \mathbb{R}^2$, $p_K(z) = (a, f(a))$.

The following result, in combination with Proposition 2.5, shows that when considering the convexity of a proximinal set (or a Chebyshev set) we need only check that it is midpoint convex.

PROPOSITION 2.7. *A closed subset of a normed linear space* $(X, \|\cdot\|)$ *is convex if, and only if, it is midpoint convex.*

PROOF. Clearly, a convex set is midpoint convex, so suppose $C \subseteq X$ is a closed midpoint convex set. Let $x, y \in C$. Consider the mapping $T : [0,1] \to X$ defined by $T(\lambda) := \lambda x + (1 - \lambda)y$. Let $U := \{\lambda \in [0,1] : T(\lambda) \notin C\}$. Since T is continuous and C is closed, it follows that U is open. If $U = \emptyset$, then $[x, y] \subseteq C$ and we're done, so suppose otherwise. Thus, there exist distinct $\lambda_1, \lambda_2 \in [0,1] \setminus U$ such that $(\lambda_1, \lambda_2) \subseteq U$. Therefore, $\frac{\lambda_1+\lambda_2}{2} \in (\lambda_1, \lambda_2) \subseteq U$, and so

$$\frac{T(\lambda_1) + T(\lambda_2)}{2} = T\left(\frac{\lambda_1 + \lambda_2}{2}\right) \notin C;$$

but this is impossible since $T(\lambda_1), T(\lambda_2) \in C$ and C is midpoint convex. Hence, it must be the case that $U = \emptyset$; which implies that C is convex. $\square$

REMARK. Having closedness in the hypotheses of Proposition 2.7 is essential as $\mathbb{Q} \subseteq \mathbb{R}$ is midpoint convex, but not convex.

There is also a corresponding notion of a function being convex and/or concave:

DEFINITION 2.8. Let K be a nonempty convex subset of a vector space X. We say that a function $f : K \to \mathbb{R}$ is **convex** if, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in K$ and all $\lambda \in [0,1]$, and **concave** if, $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in K$ and all $\lambda \in [0,1]$.

The following results show that the convexity (concavity) of a set is closely related to the convexity (concavity) of its distance function.

PROPOSITION 2.9. *Let* K *be a nonempty closed subset of a normed linear space* $(X, \|\cdot\|)$. *Then* K *is convex if, and only if, the distance function for* K *is convex.*

PROOF. Firstly, suppose that K is convex. Fix $x, y \in X$, $\lambda \in [0,1]$ and $\varepsilon > 0$. By the definition of the distance function for K, there exist $k_x, k_y \in K$ such that

$$\|x - k_x\| < d(x, K) + \varepsilon \quad \text{and} \quad \|y - k_y\| < d(y, K) + \varepsilon.$$

Since K is convex, it follows that

$$\begin{aligned} d(\lambda x + (1-\lambda)y, K) &\leq \|\lambda x + (1-\lambda)y - (\lambda k_x + (1-\lambda)k_y)\| \\ &= \|\lambda(x - k_x) + (1-\lambda)(y - k_y)\| \\ &\leq \lambda \|x - k_x\| + (1-\lambda)\|y - k_y\| \\ &< \lambda d(x, K) + (1-\lambda)d(y, K) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that the distance function for K is convex.
Conversely, suppose that the distance function for K is convex. Let $x, y \in K$ and $\lambda \in [0, 1]$. Then

$$0 \leq d(\lambda x + (1 - \lambda)y, K) \leq \lambda d(x, K) + (1 - \lambda)d(y, K) = 0.$$

As K is closed, this forces $\lambda x + (1 - \lambda)y \in K$ and so K is convex. $\square$

In order to expedite the proof of the next result, let us recall the following easily provable fact. If $(X, \|\cdot\|)$ is a normed linear space, $x, y \in X$, $0 < r, R$ and $\lambda \in [0, 1]$, then

$$B(\lambda x + (1 - \lambda)y; \lambda r + (1 - \lambda)R) = \lambda B(x; r) + (1 - \lambda)B(y; R)\,.$$

PROPOSITION 2.10 ([12, Lemma 3.1]). *Let K be a nonempty proper closed subset of a normed linear space $(X, \|\cdot\|)$. Then K is concave if, and only if, the distance function for K, restricted to co $(X \setminus K)$, is concave.*

PROOF. Suppose that K is concave. Then $\mathrm{co}\,(X \setminus K) = X \setminus K$. Let $x, y \in X \setminus K$ and $\lambda \in [0, 1]$. Clearly,

$$B(x; d(x, K)),\ B(y; d(y, K)) \subseteq X \setminus K.$$

Since $X \setminus K$ is convex, it follows that

$$\begin{aligned} B(\lambda x + (1 - \lambda)y; \lambda d(x, K) + (1 - \lambda)d(y, K)) &= \lambda B(x; d(x, K)) + (1 - \lambda)B(y; d(y, K)) \\ &\subseteq X \setminus K. \end{aligned}$$

Therefore, $\lambda d(x, K) + (1 - \lambda)d(y, K) \leq d(\lambda x + (1 - \lambda)y, K)$. Conversely, suppose that the distance function for K, restricted to co $(X \setminus K)$, is concave. Let $x, y \in X \setminus K$ and $\lambda \in [0, 1]$. Then, since K is closed, $d(\lambda x + (1 - \lambda)y, K) \geq \lambda d(x, K) + (1 - \lambda)d(y, K) > 0$. Thus, $\lambda x + (1 - \lambda)y \in X \setminus K$ as required. $\square$

The next result is useful when working with nearest points. It will be used repeatedly throughout the remainder of the article.

PROPOSITION 2.11 ([26, Lemma 12.1]). *Let K be a nonempty subset of a normed linear space $(X, \|\cdot\|)$. Suppose that $x \in X \setminus K$ and $z \in P_K(x)$, then for any $y \in [x, z)$, $z \in P_K(y)$.*

PROOF. Since $y \in [x, z)$, $\|x - y\| + \|y - z\| = \|x - z\|$. Now, $z \in K \cap B[y, \|y - z\|]$ and so $d(y, K) \leq \|y - z\|$. On the other hand, by the triangle inequality,

$$B(y, \|y - z\|) \subseteq B(x, \|x - y\| + \|y - z\|) = B(x, \|x - z\|) \subseteq X \setminus K$$

and so $\|y - z\| \leq d(y, K)$. Thus, $\|y - z\| = d(y, K)$; which shows that $z \in P_K(y)$. $\square$

It is perhaps natural to speculate that the metric projection mapping of a Chebyshev set is always continuous. This possibility is refuted by Example 7. However, the following weaker property is possessed by all metric projection mappings.

LEMMA 2.12. *Let K be a nonempty closed subset of a normed linear space $(X, \|\cdot\|)$. If $(x_n, y_n)_{n=1}^{\infty}$ is a sequence in $X \times X$, with $y_n \in P_K(x_n)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}(x_n, y_n) = (x, y)$, then $y \in P_K(x)$.*

PROOF. Let $(x_n, y_n)_{n=1}^{\infty}$ be a sequence in $X \times X$ converging to some (x, y), with $y_n \in P_K(x_n)$ for all $n \in \mathbb{N}$. Firstly, $y \in K$, since K is closed. Furthermore,

$$\|x - y\| = \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} d(x_n, K) = d(x, K),$$

since the distance function for K is continuous by Proposition 2.2. Thus, $y \in P_K(x)$. $\square$

COROLLARY 2.13. *If K is a Chebyshev set in a normed linear space $(X, \|\cdot\|)$, then the metric projection mapping for K has a closed graph.*

The following curious result can be, and will be, used to establish the convexity of Chebyshev sets in certain normed linear spaces.

PROPOSITION 2.14. *Let K be a nonempty closed subset of a normed linear space $(X, \|\cdot\|)$. Suppose $x \in X \setminus K$ and $z \in P_K(x)$. Define $y(\lambda) := z + \lambda(x - z)$ for each $\lambda \in \mathbb{R}$. Then $I := \{\lambda \geq 0 : z \in P_K(y(\lambda))\}$ is a nonempty closed interval.*

PROOF. By Proposition 2.11, if $\alpha \in I$ and $\beta \in [0, \alpha]$, then $\beta \in I$, which establishes that I is an interval. Obviously, $1 \in I$, and so I is nonempty. Finally, suppose that $(\lambda_n)_{n=1}^{\infty}$ is a sequence in I converging to some $\lambda \geq 0$. Clearly, $\lim_{n\to\infty} y(\lambda_n) = y(\lambda)$. Since $z \in P_K(y(\lambda_n))$ for all $n \in \mathbb{N}$, Lemma 2.12 says that $z \in P_K(y(\lambda))$. Therefore, $\lambda \in I$, and so I is closed. $\square$

The next definition and lemma concern a boundedness property possessed by all metric projection mappings.

DEFINITION 2.15. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be normed linear spaces and $\Phi : X \to \mathcal{P}(Y)$ a function. We say that Φ is **locally bounded on X** if for every $x_0 \in X$ there exist $r, M > 0$ such that, $\Phi(B(x_0; r)) \subseteq MB_Y$.

LEMMA 2.16. *Let K be a nonempty subset of a normed linear space $(X, \|\cdot\|)$. Then the metric projection mapping $x \mapsto P_K(x)$ is locally bounded on X.*

PROOF. Fix $x_0 \in X$ and let $r := 1$ and $M := d(x_0, K) + \|x_0\| + 2$. Suppose $y \in P_K(B(x_0; r))$, i.e., $y \in P_K(x)$ for some $x \in B(x_0; 1)$. Using the triangle inequality (twice) and the fact that the distance function for K is nonexpansive, we have that

$$\begin{aligned}
\|y\| = \|y - 0\| &\leq \|y - x_0\| + \|x_0 - 0\| \\
&\leq \|y - x\| + \|x - x_0\| + \|x_0 - 0\| \\
&= d(x, K) + \|x - x_0\| + \|x_0\| \\
&\leq (d(x_0, K) + \|x - x_0\|) + \|x - x_0\| + \|x_0\| \\
&< d(x_0, K) + 2 + \|x_0\| \\
&= M.
\end{aligned}$$

Hence, $P_K(B(x_0;r)) \subseteq MB_Y$, and so the metric projection mapping for K is locally bounded on X. $\square$

We now proceed to establish some more properties of the metric projection mapping. After defining what it means for a multivalued map to be continuous, we derive sufficient conditions for the continuity of the metric projection mapping. As an immediate corollary, we deduce a fundamental result - in finite dimensional spaces the metric projection mapping for a Chebyshev set is continuous.

DEFINITION 2.17. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be normed linear spaces and $\Phi : X \to \mathcal{P}(Y)$ a function. We say that Φ is **continuous at $\boldsymbol{x \in X}$** if $\Phi(x)$ is a singleton and for any sequence $(x_n, y_n)_{n=1}^{\infty}$ in the graph of Φ, with $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} y_n = z$, where $\Phi(x) = \{z\}$.

REMARK. It is clear that when $\Phi(x)$ is a singleton for each $x \in X$ this definition agrees with the standard definition of continuity when we view Φ as a function into Y.

DEFINITION 2.18. Let K be a subset of a normed linear space $(X, \|\cdot\|)$. We say that K is **boundedly compact** if for any $r > 0$, $rB_X \cap K$ is compact.

It follows almost immediately from this definition that all nonempty boundedly compact sets are proximinal and hence closed, see Proposition 2.5.

Our reason for considering boundedly compact sets is revealed in the next theorem.

THEOREM 2.19 ([52, Proposition 2.3]). *Let K be a boundedly compact subset of a normed linear space $(X, \|\cdot\|)$. If $P_K(x)$ is a singleton for some $x \in X$, then the metric projection mapping, $y \mapsto P_K(y)$, is continuous at x.*

PROOF. Let $x \in X$ and suppose $P_K(x) = \{z\}$ for some $z \in K$. To show that P_K is continuous at x, it suffices to show that for each sequence $(x_n, y_n)_{n=1}^{\infty}$ in the graph of P_K with $\lim_{n\to\infty} x_n = x$, there exists a subsequence $(x_{n_k}, y_{n_k})_{k=1}^{\infty}$ of $(x_n, y_n)_{n=1}^{\infty}$ such that $\lim_{k\to\infty} y_{n_k} = z$. To this end, let $(x_n, y_n)_{n=1}^{\infty}$ be a sequence in the graph of P_K with $\lim_{n\to\infty} x_n = x$. Since P_K is locally bounded (see Lemma 2.16) and K is boundedly compact, there exists a subsequence $(x_{n_k}, y_{n_k})_{k=1}^{\infty}$ of $(x_n, y_n)_{n=1}^{\infty}$ such that $\lim_{k\to\infty} y_{n_k} = y$ for some $y \in K$. Since $\lim_{k\to\infty} (x_{n_k}, y_{n_k}) = (x, y)$, Lemma 2.12 tells us that $y \in P_K(x)$. Thus, $y = z$ as required. $\square$

A special case of the previous theorem is the following folklore result.

COROLLARY 2.20 ([81, p.251]). *The metric projection mapping for a Chebyshev set in a finite dimensional normed linear space is continuous.*

2.2. Strictly convex and reflexive Banach spaces In this subsection we will define strict convexity of the norm and present some equivalent formulations of this notion. We will then examine the single-valuedness implications of strict convexity to the metric projection mapping. In the latter part of this subsection we will look at the implications of reflexivity to the study of proximinal sets.

DEFINITION 2.21 ([23]). A normed linear space $(X, \|\cdot\|)$ is said to be **strictly convex** (or, **rotund**) if for any
$x, y \in S_X$,

$$\left\| \frac{x+y}{2} \right\| = 1 \quad \text{if, and only if, } \; x = y.$$

REMARK. From this definition it follows that in a strictly convex normed linear space $(X, \|\cdot\|)$:

(i) if $x \neq y$ and $\|x\| = \|y\|$, then $\left\| \frac{x+y}{2} \right\| < \frac{\|x\| + \|y\|}{2}$;

(ii) the only nonempty convex subsets of S_X are singletons.

Many of the classical Banach spaces are strictly convex, for example all the L_p and ℓ_p spaces are strictly convex for any $1 < p < \infty$, [23, 40]. It is also known that every separable [23] or reflexive space [85] admits an equivalent norm that is strictly convex. On the other hand, $C(K)$-spaces are rarely strictly convex.

EXAMPLE 6. Let K be a compact Hausdorff space. Then $(C(K), \|\cdot\|_\infty)$ is strictly convex if, and only if, K is a singleton set.

PROOF. Clearly, if K is a singleton set, then $(C(K), \|\cdot\|_\infty)$ is strictly convex. On the other hand, if K is not a singleton set, then there exist distinct elements $x, y \in K$. Since K is completely regular, there exists a continuous function $f : K \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$. Let $g : K \to [0, 1]$ be defined by $g(k) := 1$ for all $k \in K$. Clearly, $f, g \in S_{C(K)}$ and $f \neq g$. However, $(f + g)(x) = 2$, and so $\|f + g\|_\infty = 2$; which shows that $(C(K), \|\cdot\|_\infty)$ is not strictly convex. $\square$

In practice, it is often convenient to apply the following equivalent characterisations of strict convexity.

LEMMA 2.22. *A normed linear space $(X, \|\cdot\|)$ is strictly convex if, and only if, for all $x, y \in X$ and all $r, R > 0$ such that $\|x - y\| = r + R$, we have $B[x; r] \cap B[y; R] = \{z\}$, where*

$$z := \left(\frac{R}{r+R} \right) x + \left(\frac{r}{r+R} \right) y.$$

PROOF. Suppose $(X, \|\cdot\|)$ is strictly convex, $x, y \in X$, $r, R > 0$ and $\|x - y\| = r + R$. Since

$$\|z - x\| = \left\| \left(\frac{R}{r+R} \right) x + \left(\frac{r}{r+R} \right) y - x \right\| = \left(\frac{r}{r+R} \right) \|x - y\| = r$$

and

$$\|z-y\| = \left\|\left(\frac{R}{r+R}\right)x + \left(\frac{r}{r+R}\right)y - y\right\| = \left(\frac{R}{r+R}\right)\|x-y\| = R,$$

we have $z \in B[x;r] \cap B[y;R]$. Let w be an arbitrary element of $B[x;r] \cap B[y;R]$. Since $\|x-w\| \leq r$ and $\|y-w\| \leq R$ and $r+R = \|x-y\| \leq \|x-w\| + \|y-w\|$, we must have that $\|x-w\| = r$ and $\|y-w\| = R$. However, if $u, v \in B[x;r] \cap B[y;R]$, then

$$\frac{u+v}{2} \in B[x;r] \cap B[y;R]$$

since $B[x;r] \cap B[y;R]$ is convex. On the other hand, if $u \neq v$, i.e., $(x-u) \neq (x-v)$, then

$$\left\|x - \left(\frac{u+v}{2}\right)\right\| = \left\|\frac{(x-u)+(x-v)}{2}\right\| < \frac{\|x-u\| + \|x-v\|}{2} = r,$$

which is impossible. Therefore, $B[x;r] \cap B[y;R] = \{z\}$.
For the converse, suppose $(X, \|\cdot\|)$ is not strictly convex. Hence, there exist distinct $x, y \in X$ such that

$$\|x\| = \|y\| = \left\|\frac{x+y}{2}\right\| = 1.$$

Let $r := 1 =: R$. Then $x, y \in B[0;r] \cap B[x+y;R]$ and $\|(x+y) - 0\| = r + R$. □

From Lemma 2.22 we can obtain another useful characterisation of strict convexity.

LEMMA 2.23. *A normed linear space $(X, \|\cdot\|)$ is strictly convex if, and only if, for any nonzero $x, y \in X$, if $\|x+y\| = \|x\| + \|y\|$, then $x = \alpha y$ for some $\alpha > 0$.*

PROOF. Suppose that $(X, \|\cdot\|)$ has the property given above. Let $x, y \in X$ with

$$\|x\| = \|y\| = \left\|\frac{x+y}{2}\right\| = 1.$$

Then $\|x+y\| = 2 = \|x\| + \|y\|$, so $x = \alpha y$ for some $\alpha > 0$. Since $\|x\| = \|y\| = 1$, $\alpha = 1$, and so $x = y$. Thus, $(X, \|\cdot\|)$ is strictly convex.
Conversely, suppose $(X, \|\cdot\|)$ is strictly convex and let $x, y \in X \setminus \{0\}$ be such that

$$\|x+y\| = \|x\| + \|y\|.$$

Since $x \neq -y$ and $\|x - (-y)\| = \|x\| + \|y\|$, it follows by Lemma 2.22 that

$$B[x;\|x\|] \cap B[-y;\|y\|] = \left\{\left(\frac{\|y\|}{\|x\| + \|y\|}\right)x - \left(\frac{\|x\|}{\|x\| + \|y\|}\right)y\right\}.$$

However, it is also clear that $0 \in B[x;\|x\|] \cap B[-y;\|y\|]$. Therefore,

$$\left(\frac{\|y\|}{\|x\| + \|y\|}\right)x - \left(\frac{\|x\|}{\|x\| + \|y\|}\right)y = 0,$$

which rearranges to give $x = (\|x\|/\|y\|)y$. □

Next we present our first application of the notion of strict convexity to the study of Chebyshev sets.

PROPOSITION 2.24. *Let K be a nonempty closed and convex subset of a strictly convex normed linear space $(X, \|\cdot\|)$. Then for each $x \in X$, $P_K(x)$ contains at most one element.*

PROOF. The result is clearly true if $x \in K$, so let us suppose that $x \in X \setminus K$. Looking for a contradiction, suppose that there exist distinct $y_1, y_2 \in P_K(x)$. Then

$$\|x - y_1\| = \|x - y_2\| = d(x, K) > 0.$$

Now, since K is convex, $\frac{y_1+y_2}{2} \in K$, and so

$$\begin{aligned} d(x,K) \leq \left\| x - \left(\frac{y_1+y_2}{2}\right)\right\| &= \left\| \frac{(x-y_1)+(x-y_2)}{2} \right\| \\ &< \frac{\|x-y_1\| + \|x-y_2\|}{2} \quad \text{since } \|\cdot\| \text{ is strictly convex} \\ &= d(x,K), \end{aligned}$$

which is impossible. Hence, $y_1 = y_2$, and so $P_K(x)$ is at most a singleton. □

COROLLARY 2.25. *Every convex proximinal set in a strictly convex normed linear space $(X, \|\cdot\|)$ is a Chebyshev set.*

REMARK. It is easy to show that in every normed linear space that is not strictly convex, there exists a convex proximinal set that is not a Chebyshev set.

THEOREM 2.26 (Support Theorem, [3], [29, Theorem V.9.5]). *Let C be a closed and convex subset of a normed linear space $(X, \|\cdot\|)$ with nonempty interior. Then for every $x \in Bd(C)$ there exists $x^* \in X^* \setminus \{0\}$ such that $x \in \operatorname{argmax}(x^*|_C)$.*

DEFINITION 2.27. Let K be a subset of a normed linear space $(X, \|\cdot\|)$ and let $x \in K$. We say that $\boldsymbol{x^* \in X^* \setminus \{0\}}$ **supports** $\boldsymbol{K}$ if $\operatorname{argmax}(x^*|_K) \neq \emptyset$ and we say that $\boldsymbol{x^* \in X^* \setminus \{0\}}$ **supports $\boldsymbol{K}$ at $\boldsymbol{x}$** if $x \in \operatorname{argmax}(x^*|_K)$.

LEMMA 2.28. *Let $(X, \|\cdot\|)$ be a strictly convex normed linear space and $x^* \in X^* \setminus \{0\}$. If $x \in X$ and $r > 0$, then $\operatorname{argmax}\left(x^*|_{B[x;r]}\right)$ is at most a singleton set.*

PROOF. Since $\operatorname{argmax}\left(x^*|_{B[x;r]}\right)$ is the intersection of a hyperplane and a closed ball, it is convex. On the other hand, $\operatorname{argmax}\left(x^*|_{B[x;r]}\right)$ must be a subset of $S[x;r]$. Therefore, from the strict convexity of the norm, $\operatorname{argmax}\left(x^*|_{B[x;r]}\right)$ must be at most a singleton. □

Using the previous definitions and results we can prove the following interesting result which will be used in Section 4.

LEMMA 2.29. *Let $(X, \|\cdot\|)$ be a strictly convex normed linear space and let K be a closed concave subset of X. Suppose that $x \in P_K(y) \cap P_K(z)$ for some $y, z \in X$. Then x, y, z are collinear.*

PROOF. If either y or z is a member of K, then the result holds trivially, so suppose otherwise. To show that x, y, z are collinear it is sufficient to show that

$$\frac{x-y}{\|x-y\|} = \frac{x-z}{\|x-z\|}.$$

Let $C := \overline{X \setminus K}$. Then, by Corollary 3.38, $\mathrm{Bd}(C) = \mathrm{Bd}(X \setminus K) = \mathrm{Bd}(K)$. Thus, $x \in \mathrm{Bd}(C)$. Therefore, by Theorem 2.26, there exists $x^* \in X^* \setminus \{0\}$ that supports C at x. Now since $x \in B[y; d(y,K)] \cap B[z; d(z,K)]$ and $B[y; d(y,K)] \cup B[z; d(z,K)] \subseteq C$, x^* supports both $B[y; d(y,K)]$ and $B[z; d(z,K)]$ at x. It then follows, after translation and dilation, that x^* supports B_X at both

$$\frac{x-y}{\|x-y\|} \quad \text{and} \quad \frac{x-z}{\|x-z\|}.$$

However, by Lemma 2.28, $\dfrac{x-y}{\|x-y\|} = \dfrac{x-z}{\|x-z\|}$ and we are done. □

In order to obtain some further single-valuedness implications of the notion of strict convexity to the metric projection mapping we will need to consider some notions of differentiability.

DEFINITION 2.30. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be normed linear spaces and U be a nonempty open subset of X. We say a function $f : U \to Y$ is **Gâteaux differentiable at $x \in U$** if there exists a bounded linear operator $T_x : X \to Y$ such that

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = T_x(h) \qquad (*)$$

for all $h \in S_X$. The operator T_x is called the **Gâteaux derivative of f at x**. If f is Gâteaux differentiable at each point of U, then we simply say that f is **Gâteaux differentiable**. If the limit in equation (*) is uniform with respect to all $h \in S_X$, then we say that f is **Fréchet differentiable at $x \in U$** and we call T_x the **Fréchet derivative of f at x**. Finally, if f is Fréchet differentiable at each point of U, then we simply say that f is **Fréchet differentiable**.

LEMMA 2.31. *Let K be a nonempty closed subset of a normed linear space $(X, \|\cdot\|)$. Suppose $x \in X \setminus K$ is a point of Gâteaux differentiability of the distance function for K, with Gâteaux derivative $f \in X^*$ and $y \in P_K(x)$. Then $\|f\| = 1$ and $f(x-y) = \|x-y\| = d(x,K)$.*

PROOF. Since $x \notin K$, $d(x,K) = \|x-y\| > 0$. Let $0 \leq t \leq 1$. Since

$$x + t(y-x) \in [x, y],$$

Proposition 2.11 tells us that $y \in P_K\left(x + t(y-x)\right)$ and so

$$d(x + t(y-x), K) = \|x + t(y-x) - y\| = (1-t)\|x-y\|.$$

Therefore,

$$\begin{aligned} f(y-x) &= \lim_{t\to 0^+} \frac{d(x+t(y-x),K)-d(x,K)}{t} \\ &= \lim_{t\to 0^+} \frac{(1-t)\|x-y\|-\|x-y\|}{t} \\ &= \lim_{t\to 0^+} \frac{-t\|x-y\|}{t} = -\|x-y\|. \end{aligned}$$

Thus, $f(x-y)=\|x-y\|=d(x,K)$. Proposition 2.2 then implies that, for any $z\in X$,

$$\begin{aligned} |f(z)| &= \left|\lim_{t\to 0} \frac{d(x+tz,K)-d(x,M)}{t}\right| \\ &= \lim_{t\to 0}\left|\frac{d(x+tz,K)-d(x,K)}{t}\right| \\ &\le \lim_{t\to 0} \frac{\|x+tz-x\|}{|t|} = \|z\|. \end{aligned}$$

Thus, $\|f\|=1$. □

THEOREM 2.32 ([14, Theorem 5.1]). *Let K be a nonempty closed subset of a strictly convex normed linear space $(X,\|\cdot\|)$. Suppose that $x\in X$ is a point of Gâteaux differentiability of the distance function for K. Then $P_K(x)$ contains at most one element.*

PROOF. If $x\in K$, then $P_K(x)=\{x\}$. Therefore, we may assume $x\notin K$. Suppose that $y,z\in P_K(x)$ and let $f\in X^*$ be the Gâteaux derivative of the distance function for K at x. Applying Lemma 2.31 gives

$$f\left(\frac{x-y}{\|x-y\|}\right)=\|f\|=f\left(\frac{x-z}{\|x-z\|}\right).$$

As $(X,\|\cdot\|)$ is strictly convex, f can attain its norm at, at most one point of S_X. Therefore,

$$\frac{x-y}{\|x-y\|}=\frac{x-z}{\|x-z\|}.$$

Since $\|x-y\|=\|x-z\|=d(x,K)$, we have $y=z$, which completes the proof. □

In order to exploit Theorem 2.32 we need to know something about the differentiability of distance functions. The next classical result provides some general information in this direction.

THEOREM 2.33 ([67]). *Every real-valued Lipschitz function defined on a nonempty open subset of a finite dimensional normed linear space is Gâteaux differentiable almost everywhere (with respect to the Lebesgue measure).*

COROLLARY 2.34. *Let K be a nonempty closed subset of a strictly convex finite dimensional normed linear space $(X,\|\cdot\|)$. Then $P_K(x)$ is a singleton for almost all $x\in X$ (with respect to the Lebesgue measure).*

In order to generalise this result beyond finite dimensional spaces one needs to know about the differentiability of real-valued Lipschitz functions defined on infinite dimensional normed linear spaces, see [14, 56, 66].

Next we present some sufficient conditions for a subset of a normed linear space to be a proximinal set. We start by considering some general facts concerning lower semi-continuous functions.

DEFINITION 2.35. Let (X, τ) be a topological space. We say a function $f : X \to \mathbb{R} \cup \{\infty\}$ is **lower semi- continuous** if for every $\alpha \in \mathbb{R}$, $\{x \in X : f(x) \leq \alpha\}$ is a closed set.

REMARK. It follows from the definition of lower semi-continuity that if $f : X \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous and $x = \lim_{n\to\infty} x_n$, then $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

PROPOSITION 2.36. *Let C be a closed convex subset of a normed linear space $(X, \|\cdot\|)$. Then C is weakly closed.*

PROOF. If C is empty or the whole space, then C is weakly closed, so let us suppose otherwise. Let $x_0 \in X \setminus C$. Since C is closed and convex, we have, by the Hahn-Banach Theorem, the existence of an $f_{x_0} \in X^*$ such that $f_{x_0}(x_0) > \sup_{x\in C} f_{x_0}(x)$. Thus, $x_0 \in f_{x_0}^{-1}\big((\sup_{x\in C} f_{x_0}(x), \infty)\big)$, which, being the inverse image of an open set, is weakly open. It is then straightforward to check that $X \setminus C = \bigcup_{x_0 \in X\setminus C} f_{x_0}^{-1}\big((\sup_{x\in C} f_{x_0}(x), \infty)\big)$. Hence, $X \setminus C$, being the union of weakly open sets, is weakly open. Thus, C is weakly closed. $\square$

LEMMA 2.37. *Let $(X, \|\cdot\|)$ be a normed linear space and let $f : X \to \mathbb{R}$ be a continuous convex function. Then f is lower semi-continuous with respect to the weak topology on X.*

PROOF. Let $\alpha \in \mathbb{R}$. We need to show that the set $A := \{x \in X : f(x) \leq \alpha\}$ is closed with respect to the weak topology. Since f is convex and continuous, $A = f^{-1}((-\infty, \alpha])$ is convex and closed with respect to the norm topology. Hence, by Propostion 2.36, we conclude that A is weakly closed. $\square$

Next we see that, as with continuous real-valued functions, lower semi-continuous functions defined on compact spaces attain their minimum value and are hence, bounded from below.

PROPOSITION 2.38. *Let $f : X \to \mathbb{R}\cup\{\infty\}$ be a proper lower semi-continuous function defined on a compact space X. Then* $\operatorname{argmin}(f) \neq \emptyset$.

PROOF. Let $(r_n)_{n=1}^{\infty}$ be a strictly decreasing sequence of real numbers such that converge to $\inf_{x\in X} f(x)$. Since X is compact, $\bigcap_{n\in\mathbb{N}} f^{-1}((-\infty, r_n]) \neq \emptyset$ because $\{f^{-1}((-\infty, r_n]) : n \in \mathbb{N}\}$ is a decreasing sequence of nonempty closed subsets of X. Furthermore, $\bigcap_{n\in\mathbb{N}} f^{-1}((-\infty, r_n])$ is contained in $\operatorname{argmin}(f)$. Indeed, if $x \in \bigcap_{n\in\mathbb{N}} f^{-1}((-\infty, r_n])$, then $f(x) \leq r_n$ for all $n \in \mathbb{N}$, and so $\inf_{y\in X} f(y) \leq f(x) \leq \lim_{n\to\infty} r_n = \inf_{y\in X} f(y)$; which completes the proof. $\square$

THEOREM 2.39. *Every nonempty weakly closed subset of a reflexive Banach space is proximinal. In particular, every nonempty closed and convex subset of a reflexive Banach space is proximinal.*

PROOF. Let K be a nonempty weakly closed subset of a reflexive Banach space $(X, \|\cdot\|)$. Fix, $x \in X \setminus K$ and let $f : K \to [0, \infty)$ be defined by $f(k) := \|x - k\|$. Then

$$\emptyset \neq \operatorname{argmin}(f|_{K \cap B[x; d(x,K)+1]}) = \operatorname{argmin}(f)$$

as $f|_{K \cap B[x;d(x,K)+1]}$ is weakly lower semi-continuous and $K \cap B[x; d(x, K) + 1]$ is weakly compact; since it is weakly closed and bounded. □

COROLLARY 2.40. *Every nonempty closed and convex subset of a strictly convex reflexive Banach space is a Chebyshev set.*

PROOF. The proof follows directly from Theorem 2.39 and Corollary 2.25. □

It is interesting to see that the result from Theorem 2.39 is optimal in the sense that if every nonempty closed and convex set is proximinal, then the underlying space must be reflexive.

THEOREM 2.41 ([81, Corollary 2.4], [24, Corollary 2.12]). *Let $(X, \|\cdot\|)$ be a Banach space. The following are equivalent:*

(i) *X is reflexive,*

(ii) *every nonempty weakly closed subset of X is proximinal,*

(iii) *every nonempty closed convex subset of X is proximinal,*

(iv) *every closed subspace of X is proximinal,*

(v) *every closed hyperplane in X is proximinal.*

PROOF. Theorem 2.39 says that (i) $\Rightarrow$ (ii). It is obvious that (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v). Finally, (v) $\Rightarrow$ (i) follows from Example 4 and James' Theorem [44, 58]. □

As an application of some of the results that we have obtained so far, let us show how Schauder's Fixed Point Theorem can be deduced from Brouwer's Fixed Point Theorem.

THEOREM 2.42 (Schauder's Fixed Point Theorem [80]). *Let K be a nonempty compact convex subset of a normed linear space $(X, \|\cdot\|)$. If $f : K \to K$ is a continuous function, then f has a fixed point, i.e., there exists a point $k \in K$ such that $f(k) = k$.*

PROOF. As the closed span of K is a separable normed linear space, we may assume, without loss of generality, that X is separable. Hence, by Appendix D, we can assume, after possibly renorming, that the norm on X is strictly convex. For each $n \in \mathbb{N}$, let F_n be a finite subset of K such that $K \subseteq F_n + (1/n)B_X$; note that this is possible since K is totally bounded. Let K_n be the convex hull of F_n. Then K_n is a compact convex subset of K. Thus, by Corollary 2.25

and Corollary 2.20, p_{K_n} is continuous. Define $f_n : K_n \to K_n$ by $f_n(x) := p_{K_n}(f(x))$. By Brouwer's Fixed Point Theorem [18], f_n has a fixed point $x_n \in K_n$. Now,

$$\|f(x) - f_n(x)\| \leq (1/n) \text{ for all } x \in K.$$

Therefore, $\|f(x_n) - x_n\| \leq (1/n)$. It is now routine to check that every cluster point of the sequence $(x_n)_{n=1}^{\infty}$ is a fixed point of f. $\square$

2.3. Continuity of the metric projection In Section 3 it is shown that continuity of the metric projection mapping plays a key role in deriving the convexity of a Chebyshev set. Therefore, it is natural to ask whether every Chebyshev set has a continuous metric projection mapping. Unfortunately, finding an example of a Chebyshev set with a discontinuous metric projection mapping is not as straightforward as one might first imagine. So we begin this section by presenting an example, due to B. Kripke, of such a set.

EXAMPLE 7 ([54], [41, p. 246]). There exists a reflexive space containing a convex Chebyshev set with a discontinuous metric projection mapping.

PROOF. Consider the Hilbert space $\ell_2(\mathbb{Z})$. We define a linear map $T : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ componentwise by

$$T(x)_n := \begin{cases} x_n, & \text{if } n \leq 0 \\ x_{-n} + \frac{1}{n}x_n, & \text{if } 0 < n \end{cases}$$

where $(\ldots, x_{-1}, x_0, x_1, \ldots) := x \in \ell_2(\mathbb{Z})$ and $n \in \mathbb{Z}$. It is straightforward to check that T is injective. Furthermore, T is continuous. To see this, suppose that $x \in B_{\ell_2(\mathbb{Z})}$, i.e., suppose that $\|x\|_2^2 = \sum_{n\in\mathbb{Z}} x_n^2 \leq 1$. Making use of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|T(x)\|_2^2 &= \sum_{n\leq 0} x_n^2 + \sum_{n>0}\left(x_{-n} + \frac{x_n}{n}\right)^2 \\
&= \sum_{n\leq 0} x_n^2 + \sum_{n>0} x_{-n}^2 + \sum_{n>0}\left(\frac{x_n}{n}\right)^2 + 2\sum_{n>0}\frac{x_{-n}x_n}{n} \\
&\leq \sum_{n\leq 0} x_n^2 + \sum_{n>0} x_{-n}^2 + \sum_{n>0}\left(\frac{x_n}{n}\right)^2 + 2\sqrt{\left(\sum_{n>0} x_{-n}^2\right)\left(\sum_{n>0}\left(\frac{x_n}{n}\right)^2\right)} \\
&\leq 5,
\end{aligned}$$

and so $\|T\| \leq \sqrt{5}$.

Next we define a new norm $\|\cdot\|'$ on $\ell_2(\mathbb{Z})$ by $\|x\|' := \max\{\|x\|_2, 2|x_0|\} + \|T(x)\|_2$. It is straightforward to check that this is an equivalent norm, since for any $x \in \ell_2(\mathbb{Z})$,

$$\|x\|_2 \leq \|x\|' \leq (2 + \|T\|)\|x\|_2 .$$

Therefore, $\left(\ell_2(\mathbb{Z}), \|\cdot\|'\right)$ is reflexive and complete. Since $\|\cdot\|_2$ is strictly convex (it is induced by an inner product) and T is injective, it follows that $\|\cdot\|'$ is strictly convex. Also, define

$K := \overline{\text{span}}\{e_n : 1 \leq n\}$, where e_n is the element of $\ell_2(\mathbb{Z})$ with 1 as its n^{th} coordinate and zeros everywhere else. By Corollary 2.40, K, viewed as a subset of $\left(\ell_2(\mathbb{Z}), \|\cdot\|'\right)$, is a Chebyshev set. We now show that K has a discontinuous metric projection mapping.
Let $y := \sum_{n=1}^k \alpha_n e_n \in \text{span}\{e_n : 1 \leq n\}$ for some $1 \leq k$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Then

$$\begin{aligned}\|e_0 - y\|' &= \left\|e_0 - \sum_{n=1}^k \alpha_n e_n\right\|' \\ &= \max\left\{\left\|e_0 - \sum_{n=1}^k \alpha_n e_n\right\|_2, 2\right\} + \left\|T\left(e_0 - \sum_{n=1}^k \alpha_n e_n\right)\right\|_2 \\ &= \max\left\{2, \sqrt{1 + \sum_{n=1}^k \alpha_n^2}\right\} + \sqrt{1 + \sum_{n=1}^k \left(\frac{\alpha_n}{n}\right)^2}.\end{aligned}$$

This quantity can be minimised by setting $\alpha_n = 0$ for all $n \in \{1, \ldots, k\}$. Thus,

$$d\left(e_0, \text{span}\{e_n : 1 \leq n\}\right) = \|e_0 - 0\|' = 3.$$

Since $d\left(e_0, \text{span}\{e_n : 1 \leq n\}\right) = d\left(e_0, \overline{\text{span}}\{e_n : 1 \leq n\}\right)$ and K is a Chebyshev set, it follows that $p_K\left(e_0\right) = 0$. Fix $j > 0$. We may assume, without loss of generality, that $j \leq k$. Then

$$\begin{aligned}\left\|\left(e_0 + \frac{e_{-j}}{j}\right) - y\right\|' &= \left\|e_0 + \frac{e_{-j}}{j} - \sum_{n=1}^k \alpha_n e_n\right\|' \\ &= \max\left\{\left\|e_0 + \frac{e_{-j}}{j} - \sum_{n=1}^k \alpha_n e_n\right\|_2, 2\right\} + \left\|T\left(e_0 + \frac{e_{-j}}{j} - \sum_{n=1}^k \alpha_n e_n\right)\right\|_2 \\ &= \max\left\{2, \sqrt{1 + \frac{1}{j^2} + \sum_{n=1}^k \alpha_n^2}\right\} + \sqrt{1 + \frac{1}{j^2} + \left(\frac{1 - \alpha_j}{j}\right)^2 + \sum_{\substack{n=1 \\ n \neq j}}^k \left(\frac{\alpha_n}{n}\right)^2}.\end{aligned}$$

This quantity can be minimised by setting $\alpha_j = 1$ and $\alpha_n = 0$ for all $n \neq j$. Thus,

$$d\left(e_0 + \frac{e_{-j}}{j}, \text{span}\{e_n : 1 \leq n\}\right) = \left\|e_0 + \frac{e_{-j}}{j} - e_j\right\|' = 2 + \sqrt{1 + \frac{1}{j^2}}.$$

Since $d\left(e_0 + \frac{e_{-j}}{j}, \text{span}\{e_n : 1 \leq n\}\right) = d\left(e_0 + \frac{e_{-j}}{j}, \overline{\text{span}}\{e_n : 1 \leq n\}\right)$ and K is a Chebyshev set, it follows that $p_K\left(e_0 + \frac{e_{-j}}{j}\right) = e_j$. Finally, observe that

$$\lim_{j \to \infty}\left(e_0 + \frac{e_{-j}}{j}\right) = e_0 \quad \text{whilst} \quad \lim_{j \to \infty} p_K\left(e_0 + \frac{e_{-j}}{j}\right) = \lim_{j \to \infty} e_j \neq 0 = p_K(e_0),$$

showing that the metric projection mapping $x \mapsto p_K(x)$ is not continuous. $\square$

Another example of a Chebyshev set in a strictly convex reflexive space with a discontinuous metric projection mapping is given in [19].

We now give some hypotheses that guarantee that the metric projection mapping is continuous. To do this we consider the following property of a norm which was first considered in [68,72,73].

DEFINITION 2.43 ([48,51]). Let $(X, \|\cdot\|)$ be a normed linear space. We say $\|\cdot\|$ is a **Kadec-Klee norm** if for every sequence $(x_n)_{n=1}^{\infty}$ in X, if $\lim_{n\to\infty} x_n = x$ with respect to the weak topology and $\lim_{n\to\infty} \|x_n\| = \|x\|$, then $\lim_{n\to\infty} x_n = x$ with respect to the norm topology.

Many norms that we are familiar with are Kadec-Klee norms.

EXAMPLE 8. Every finite dimensional space has a Kadec-Klee norm as does every Hilbert space.

PROOF. In every finite dimensional normed linear space the weak and norm topologies coincide and so it directly follows that every norm on a finite dimensional normed linear space is a Kadec-Klee norm. In any Hilbert space $(H, \langle\cdot,\cdot\rangle)$, the norm satisfies the identity

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle.$$

Hence, if $(x_n)_{n=1}^{\infty}$ is a sequence in H and $\lim_{n\to\infty} x_n = x$ with respect to the weak topology and $\lim_{n\to\infty} \|x_n\| = \|x\|$, then $\lim_{n\to\infty} x_n = x$ with respect to the norm topology since the functional $y \mapsto \langle y, x\rangle$ is weakly continuous and $\langle x, x\rangle = \|x\|^2$. $\square$

Note: it follows from Clarkson's inequalities [23,40] that for any $1 < p < \infty$, L_p has a Kadec-Klee norm.

THEOREM 2.44 ([24, Theorem 2.7 and Corollary 2.16]). *Let K be a weakly closed Chebyshev set in a reflexive normed linear space $(X, \|\cdot\|)$ with a Kadec-Klee norm. Then K has a continuous metric projection mapping.*

PROOF. Let $x \in X$. To show that p_K is continuous at x it suffices to show that for each sequence $(x_n)_{n=1}^{\infty}$ with $x = \lim_{n\to\infty} x_n$ there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $\lim_{k\to\infty} p_K(x_{n_k}) = p_K(x)$. To this end, let $(x_n)_{n=1}^{\infty}$ be a sequence in X with $x = \lim_{n\to\infty} x_n$. Since p_K is locally bounded (see Lemma 2.16) and X is reflexive, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and an element $y \in X$ such that $(p_K(x_{n_k}))_{k=1}^{\infty}$ converges to y with respect to the weak topology on X. Since K is weakly closed, $y \in K$. Therefore,

$$\begin{aligned} d(x, K) \le \|x - y\| &\le \liminf_{k\to\infty} \|x_{n_k} - p_K(x_{n_k})\| \\ &\le \limsup_{k\to\infty} \|x_{n_k} - p_K(x_{n_k})\| = \lim_{k\to\infty} d(x_{n_k}, K) = d(x, K) \end{aligned}$$

since $(x_{n_k} - p_K(x_{n_k}))_{k=1}^{\infty}$ converges to $(x-y)$ with respect to the weak topology on X and the norm is lower semi-continuous with respect to the weak topology on X.

Now, because K is a Chebyshev set it follows that $y = p_K(x)$. Furthermore, since $\|\cdot\|$ is a Kadec-Klee norm, $(x_{n_k} - p_K(x_{n_k}))_{k=1}^{\infty}$ converges in norm to $(x-y) = (x - p_K(x))$. Therefore, $(p_K(x_{n_k}))_{k=1}^{\infty}$ converges in norm to $p_K(x)$; which completes the proof. $\square$

We finish this subsection by presenting a classical result of Robert R. Phelps, which shows that the convexity of a Chebyshev set in an inner product space is characterised by the metric projection mapping being nonexpansive. To prove this fact we first need a result that characterises best approximations for convex sets in inner product spaces.

PROPOSITION 2.45 ([2]). *Let K be a convex subset of an inner product space $(X, \langle\cdot,\cdot\rangle)$ and let $x \in X$, $y \in K$. Then $y \in P_K(x)$ if, and only if, $\langle x-y, z-y\rangle \leq 0$ for all $z \in K$.*

PROOF. Let $x \in X, y \in K$ and suppose the above inequality holds. If $x = y$, then there is nothing to prove so assume otherwise. For any $z \in K$,

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle \\ &= \langle x-y, x-z\rangle + \langle x-y, z-y\rangle \\ &\leq \langle x-y, x-z\rangle \\ &\leq \|x-y\|\,\|x-z\| \quad \text{by the Cauchy-Schwarz inequality}.\end{aligned}$$

Thus, $\|x-y\| \leq \|x-z\|$ for all $z \in K$, and so $y \in P_K(x)$.
Conversely, suppose that for some $z \in K$, $\langle x-y, z-y\rangle > 0$. Clearly, $z \neq y$. Choose

$$0 < \lambda < \min\left\{1, \frac{2\langle x-y, z-y\rangle}{\|z-y\|^2}\right\},$$

which guarantees that $2\langle x-y, z-y\rangle - \lambda\|z-y\|^2 > 0$. Since K is convex, $y_\lambda := \lambda z + (1-\lambda)y$ in a member of K. Therefore,

$$\begin{aligned}\|x-y_\lambda\|^2 &= \langle x-y_\lambda, x-y_\lambda\rangle \\ &= \langle x-y-\lambda(z-y), x-y-\lambda(z-y)\rangle \\ &= \|x-y\|^2 - \lambda\left(2\langle x-y, z-y\rangle - \lambda\|z-y\|^2\right) \\ &< \|x-y\|^2.\end{aligned}$$

Thus, $\|x-y_\lambda\| < \|x-y\|$, and so $y \notin P_K(x)$. $\square$

THEOREM 2.46 ([64]). *Let K be a Chebyshev set in an inner product space $(X, \langle\cdot,\cdot\rangle)$. Then K is convex if, and only if, $\|p_K(x) - p_K(y)\| \leq \|x-y\|$ for all $x, y \in X$, that is, the metric projection mapping is nonexpansive.*

PROOF. Suppose that K is convex and $x, y \in X$. Clearly, the required inequality holds if $p_K(x) = p_K(y)$ so we will assume otherwise. By Proposition 2.45,

$$\langle x - p_K(x), p_K(y) - p_K(x)\rangle \leq 0 \quad \text{and} \quad \langle y - p_K(y), p_K(x) - p_K(y)\rangle \leq 0$$

or equivalently,

$$\langle x - p_K(x), p_K(y) - p_K(x)\rangle \leq 0 \quad \text{and} \quad \langle p_K(y) - y, p_K(y) - p_K(x)\rangle \leq 0.$$

Adding these two inequalities gives $\langle x - y + p_K(y) - p_K(x), p_K(y) - p_K(x)\rangle \leq 0$, and so

$$\langle p_K(y) - p_K(x), p_K(y) - p_K(x)\rangle \leq \langle y - x, p_K(y) - p_K(x)\rangle.$$

Therefore,

$$\begin{aligned}\|p_K(y) - p_K(x)\|^2 &= \langle p_K(y) - p_K(x), p_K(y) - p_K(x)\rangle \\ &\leq \langle y - x, p_K(y) - p_K(x)\rangle \\ &\leq \|y - x\| \, \|p_K(y) - p_K(x)\|.\end{aligned}$$

Thus, $\|p_K(y) - p_K(x)\| \leq \|x - y\|$ as required.
For the converse, suppose that the metric projection mapping is nonexpansive but K is not convex. Since K is closed, it is not midpoint convex, and so we can find $x, y \in K$ such that $z := \frac{x+y}{2} \notin K$. Clearly, $x \neq y$. Set $r := \frac{\|x-y\|}{2} > 0$. We claim that $p_K(z) \in B[x; r] \cap B[y; r]$. This follows since

$$\|p_K(z) - x\| = \|p_K(z) - p_K(x)\| \leq \|z - x\| = r$$

and

$$\|p_K(z) - y\| = \|p_K(z) - p_K(y)\| \leq \|z - y\| = r.$$

Since $(X, \langle\cdot,\cdot\rangle)$ is strictly convex (as it is an inner product space) and $\|x - y\| = 2r$, Lemma 2.22 tells us that $B[x; r] \cap B[y; r] = \left\{\frac{x+y}{2}\right\} = \{z\}$. This forces $z = p_K(z) \in K$, which is impossible. Hence, K is convex. $\square$

Let us end this section with a few remarks concerning the single-valuedness of the metric projection mapping. In 1963, S. Stečkin [84] proved that in a strictly convex normed linear space $(X, \|\cdot\|)$, for every nonempty subset K of X, $\{x \in X : P_K(x)$ is at most a singleton$\}$ is dense in X. This result can be deduced from the following geometric fact. In a strictly convex normed linear space $(X, \|\cdot\|)$, if $x \notin K$ and $z \in P_K(x)$, then $P_K(y) = \{z\}$ for all $y \in (x, z)$. To see why this is true we simply note that by Lemma 2.23,

$$B[y; \|z - y\|]\backslash\{z\} \subseteq B(x; \|z - x\|) \subseteq X \setminus K \quad \text{and so} \quad B[y; \|z - y\|] \cap K = \{z\}$$

for all $y \in (z, x)$. Stečkin also proved that if $(X, \|\cdot\|)$ is complete (i.e., a Banach space) and the norm is "locally uniformly rotund" (see [85] for the definition), then for each nonempty subset K of X, the set

$$\{x \in X : P_K(x) \text{ is at most a singleton}\}$$

is "residual" (i.e., contains the intersection of a countably family of dense open subsets of X). Stečkin also conjectured that in any strictly convex space $(X, \|\cdot\|)$,

$$\{x \in X : P_K(x) \text{ is at most a singleton}\}$$

is residual in X, for every nonempty subset K of X. Stečkin's conjecture is still open. For the latest state of play, see [74].

3. The convexity of Chebyshev sets

As shown by Example 5, a Chebyshev set need not be convex. However, in this section we present four distinct proofs which show that, under various additional hypotheses (all of which include the assumption that the metric projection mapping is continuous), a Chebyshev set is convex. In this way, this section follows the pattern of the paper [11].

3.1. Proof by fixed-point theorem In this subsection we make use of Schauder's Fixed Point Theorem to prove that a boundedly compact Chebyshev set in a smooth Banach space is necessarily convex.

Let us begin with the following simple observation.

LEMMA 3.1. *Let J be an open halfspace of a normed linear space $(X, \|\cdot\|)$ and let $x, y \in X$. If $\frac{1}{2}(x+y) \in J$, then either $x \in J$ or $y \in J$.*

We shall also require the following geometric consequence of smoothness.

LEMMA 3.2 ([1]). *Let $(X, \|\cdot\|)$ be a normed linear space and let $x_0 \in X \setminus \{0\}$. If the norm is smooth at x_0, then $\{y \in X : 0 < x^*(y)\} = \bigcup_{n\in\mathbb{N}} B(nx_0; n\|x_0\|)$, where x^* is the Gâteaux derivative of the norm at x_0. Furthermore, if $z \in X$, then $\{y \in X : x^*(z) < x^*(y)\}$ equals $\bigcup_{n\in\mathbb{N}} B(z + nx_0; n\|x_0\|)$.*

PROOF. We first show that $\bigcup_{n\in\mathbb{N}} B(nx_0; n\|x_0\|) \subseteq \{y \in X : 0 < x^*(y)\}$. To this end, let $x \in \bigcup_{n\in\mathbb{N}} B(nx_0; n\|x_0\|)$. Then there exists an $n \in \mathbb{N}$ such that $x \in B(nx_0; n\|x_0\|)$, i.e., $\|x - nx_0\| < n\|x_0\|$. Since x^* is the Gâteaux derivative of the norm at x_0, it follows that $x^*(x_0) = \|x_0\|$ and $\|x^*\| \leq 1$. Therefore,

$$\begin{aligned} x^*(x) &= x^*(nx_0 + (x - nx_0)) \\ &= n\|x_0\| + x^*(x - nx_0) \\ &\geq n\|x_0\| - |x^*(x - nx_0)| \\ &\geq n\|x_0\| - \|x - nx_0\| \quad \text{since } \|x^*\| \leq 1 \\ &> 0 \quad \text{since } x \in B(nx_0; n\|x_0\|). \end{aligned}$$

Thus, $\bigcup_{n\in\mathbb{N}} B(nx_0; n\|x_0\|) \subseteq \{y \in X : 0 < x^*(y)\}$.

Suppose that $x \in \{y \in X : 0 < x^*(y)\}$. Then

$$\lim_{n\to\infty} \frac{\|x_0 + (1/n)(-x)\| - \|x_0\|}{(1/n)} = x^*(-x) = -x^*(x) < 0.$$

Therefore, there exists an $n \in \mathbb{N}$ such that

$$\frac{\|x_0 + (1/n)(-x)\| - \|x_0\|}{(1/n)} < 0.$$

Thus, $\|nx_0 - x\| < n\|x_0\|$, and so $x \in B(nx_0; n\|x_0\|) \subseteq \bigcup_{k\in\mathbb{N}} B(kx_0; k\|x_0\|)$.

Finally, if $z \in X$, then

$$\begin{aligned}\{y \in X : x^*(z) < x^*(y)\} &= \{y \in X : 0 < x^*(y-z)\} \\ &= \{w+z \in X : 0 < x^*(w)\} \\ &= z + \{w \in X : 0 < x^*(w)\} \\ &= z + \bigcup_{n\in\mathbb{N}} B(nx_0; n\|x_0\|) = \bigcup_{n\in\mathbb{N}} B(z+nx_0; n\|x_0\|)\,.\end{aligned}$$

This completes the proof. $\square$

DEFINITION 3.3 ([33]). We say that a Chebyshev set K in a normed linear space $(X, \|\cdot\|)$ is a **sun** if for every $x \in X \setminus K$ and $0 < \lambda$, $p_K(x_\lambda) = p_K(x)$, where $x_\lambda := p_K(x) + \lambda(x - p_K(x))$.

Note: in light of Proposition 2.14 and Proposition 2.11 we know that

$$I := \{\lambda \geq 0 : p_K(x_\lambda) = p_K(x)\}$$

is a closed interval containing $[0,1]$. Thus, for K to be a sun we only require that $(1,\infty) \subseteq I$.

The following two results (the first of which is a folklore result) demonstrate the relationship between a Chebyshev set being a sun and being convex.

THEOREM 3.4. *In any normed linear space, every convex Chebyshev set is a sun.*

PROOF. Let K be a convex Chebyshev subset of a normed linear space $(X, \|\cdot\|)$ and suppose that $x \in X \setminus K$. Then $p_K(x) \in K \cap B[x; \|x - p_K(x)\|]$ and so $0 \in B[x; \|x - p_K(x)\|] - K$. However,

$$0 \notin B(x; \|x - p_K(x)\|) - K \quad \text{since} \quad B(x; \|x - p_K(x)\|) \cap K = \emptyset.$$

Therefore,

$$0 \in B[x; \|x - p_K(x)\|] - K \subseteq \overline{B(x; \|x - p_K(x)\|) - K}$$

but

$$0 \notin B(x; \|x - p_K(x)\|) - K = \text{int}(\overline{B(x; \|x - p_K(x)\|) - K}), \quad \text{by, Corollary 3.38.}$$

Thus, $0 \in \text{Bd}(\overline{B(x; \|x - p_K(x)\|) - K})$. Therefore, by the Support Theorem, see Theorem 2.26, there exists an $f \in S_{X^*}$ such that $\sup\{f(y) : y \in B[x; \|x - p_K(x)\|] - K\} \leq f(0) = 0$. Thus,

$$f(y) \leq f(k) \quad \text{for all } y \in B[x; \|x - p_K(x)\|] \quad \text{and all } k \in K.$$

In particular,

$$\max_{y \in B[x;\|x-p_K(x)\|]} f(y) = f(p_K(x)) \quad \text{since} \quad p_K(x) \in B[x; \|x - p_K(x)\|] \cap K \quad \text{and}$$

$$\min_{k\in K} f(k) = f(p_K(x)) \quad \text{since} \quad p_K(x) \in B[x; \|x - p_K(x)\|] \cap K.$$

Let $\alpha := f(p_K(x))$, then $B[x; \|x - p_K(x)\|] \subseteq f^{-1}((-\infty, \alpha])$ and $K \subseteq f^{-1}([\alpha, \infty))$. It also follows that

$$f(p_K(x) - x) = \|x - p_K(x)\| \quad \text{or, equivalently,} \quad f(x - p_K(x)) = -\|x - p_K(x)\| \qquad (*)$$

Let $\lambda > 0$ and let $x_\lambda := p_K(x) + \lambda(x - p_K(x))$. Then $p_K(x) \in K \cap B[x_\lambda; \|x_\lambda - p_K(x)\|]$ and so $d(x_\lambda, K) \leq \|x_\lambda - p_K(x)\|$. On the other hand, $B(x_\lambda; \|x_\lambda - p_K(x)\|) \subseteq f^{-1}((-\infty, \alpha)) \subseteq X \setminus K$ since if $y \in B(x_\lambda; \|x_\lambda - p_K(x)\|)$ then

$$\begin{aligned} f(y) &= f(y - x_\lambda) + f(x_\lambda) \\ &\leq \|y - x_\lambda\| + f(x_\lambda) \quad \text{since } \|f\| \leq 1. \\ &< \|p_K(x) - x_\lambda\| + f(x_\lambda) \quad \text{since } y \in B(x_\lambda; \|x_\lambda - p_K(x)\|). \\ &= \lambda\|x - p_K(x)\| + \lambda f(x - p_K(x)) + f(p_K(x)) \\ &= f(p_K(x)) = \alpha. \quad \text{by } (*) \end{aligned}$$

Thus, $\|x_\lambda - p_K(x)\| \leq d(x_\lambda, K)$ and so $\|x_\lambda - p_K(x)\| = d(x_\lambda, K)$. This shows that $p_K(x_\lambda)$ equals $p_K(x)$. ☐

REMARK. We should note that there is also a short indirect proof of the previous theorem. Indeed, suppose to the contrary that K is a convex Chebyshev set that is not a sun. Then there exists an $x \in X \setminus K$ and $\lambda > 1$ such that $p_K(x_\lambda) \neq p_K(x)$, where $x_\lambda := p_K(x) + \lambda(x - p_K(x))$. Let $\mu := 1/\lambda$, then $0 < \mu < 1$ and $x = \mu x_\lambda + (1 - \mu)p_K(x)$. Now, since K is convex, $\mu p_K(x_\lambda) + (1 - \mu)p_K(x) \in K$. However,

$$\begin{aligned} \|x - [\mu p_K(x_\lambda) + (1 - \mu)p_K(x)]\| &= \|[\mu x_\lambda + (1 - \mu)p_K(x)] - [\mu p_K(x_\lambda) + (1 - \mu)p_K(x)]\| \\ &= \mu\|x_\lambda - p_K(x_\lambda)\| < \mu\|x_\lambda - p_K(x)\| = \|x - p_K(x)\|; \end{aligned}$$

which contradicts the fact that $p_K(x)$ is the nearest point in K to x. This contradiction completes the proof.

THEOREM 3.5 ([1, 87]). *Let K be a Chebyshev set in a smooth normed linear space $(X, \|\cdot\|)$. If K is a sun, then K is convex.*

PROOF. Looking for a contradiction, suppose that K is not convex. Since K is closed, Proposition 2.7 tells us that there exist $x, y \in K$ such that $\frac{1}{2}(x + y) \notin K$. Let

$$z := p_K\left(\frac{x + y}{2}\right) \quad \text{and} \quad x_0 := \frac{x + y}{2} - z.$$

By Lemma 3.2,

$$z + x_0 \in B(z + x_0; \|x_0\|) \subseteq J := \{w \in X : x^*(z) < x^*(w)\} \subseteq \bigcup_{n \in \mathbb{N}} B(z + nx_0; n\|x_0\|),$$

where x^* is the Gâteaux derivative of the norm at x_0. We claim that $\bigcup_{n \in \mathbb{N}} B(z + nx_0; n\|x_0\|)$ is contained in $X \setminus K$. For $1 < n \in \mathbb{N}$, $z + x_0 = [1 - (1/n)]z + [1/n](z + nx_0) \in (z, z + nx_0)$. Since K is a sun,

$$z = p_K(z + x_0) = p_K(z + nx_0) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $d(z+nx_0, K) = \|(z + nx_0) - z\| = n\|x_0\|$ for all $n \in \mathbb{N}$. Thus, $B(z + nx_0; n\|x_0\|)$ is contained in $X \setminus K$ for all $n \in \mathbb{N}$. From Lemma 3.1 it follows that either x or y is in $J \subseteq \bigcup_{n\in\mathbb{N}} B(z + nx_0; n\|x_0\|) \subseteq X \setminus K$. However, this is impossible, since both x and y are members of K. $\square$

To see that smoothness of the normed linear space is required, consider again Example 5. The key point is that since the unit ball in $\left(\mathbb{R}^2, \|\cdot\|_1\right)$ has sharp corners, i.e., the norm is not smooth, the union of the balls discussed in Lemma 3.2 does not have to fill out a whole half space.

In order to present our first proof concerning the convexity of Chebyshev sets we need to apply the following modification of Schauder's Fixed Point Theorem. To obtain this modification we use the fact that the closed convex hull of a compact subset of a Banach space is compact, see [57] or [29, p. 416].

THEOREM 3.6 ([37, Theorem 7.6]). *Let K be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$. If $f : K \to K$ is a continuous function and $\overline{f(K)}$ is compact, then f has a fixed point.*

THEOREM 3.7 ([87]). *Let $(X, \|\cdot\|)$ be a smooth Banach space. Then every boundedly compact Chebyshev set in X is convex. In particular, every Chebyshev subset of a smooth finite dimensional normed linear space is convex.*

PROOF. Let K be a Chebyshev set in X. In light of Theorem 3.5, it suffices to show that K is a sun. To this end, let $x \in X \setminus K$ and let $I_x := \{\lambda \geq 0 : p_K(x) = p_K\left(p_K(x) + \lambda(x - p_K(x))\right)\}$. By Proposition 2.14, I_x is a nonempty closed inteval. Looking for a contradiction, suppose K is not a sun, i.e., $I_x \neq [0, \infty)$. Then there exists $\lambda_0 \geq 1$ such that $I_x = [0, \lambda_0]$. Let $x_0 := p_K(x) + \lambda_0 (x - p_K(x))$. Note that $x_0 \notin K$ since $p_K(x_0) = p_K(x) \neq p_K(x) + \lambda_0 (x - p_K(x)) = x_0$, where we made use of the fact that K is closed and $x \notin K$. Choose $\varepsilon \in (0, d(x_0, K))$ so that $p_K(B[x_0; \varepsilon])$ is bounded and consider the function $f : B[x_0; \varepsilon] \to B[x_0; \varepsilon]$ defined by

$$f(x) := x_0 + \varepsilon \frac{x_0 - p_K(x)}{\|x_0 - p_K(x)\|}.$$

Since K is boundedly compact, we have by Theorem 2.19 that p_K is continuous. It then follows that (i) f is continuous and (ii) $\overline{f(B[x_0; \varepsilon])} \subseteq B[x_0; \varepsilon]$ is compact, since $f(B[x_0; \varepsilon])$ is contained in $g(\overline{p_K(B[x_0; \varepsilon])})$, where $g : X \setminus \{x_0\} \to X$ is the continuous function defined by

$$g(y) := x_0 + \varepsilon \frac{x_0 - y}{\|x_0 - y\|},$$

and $\overline{p_K(B[x_0; \varepsilon])}$ is compact. Therefore, by Theorem 3.6, there exists an element $x_\infty \in B[x_0; \varepsilon]$ such that $f(x_\infty) = x_\infty$. Rearranging the equation $f(x_\infty) = x_\infty$, we see that

$$x_0 = \left(\frac{\|x_0 - p_K(x_\infty)\|}{\|x_0 - p_K(x_\infty)\| + \varepsilon}\right) x_\infty + \left(\frac{\varepsilon}{\|x_0 - p_K(x_\infty)\| + \varepsilon}\right) p_K(x_\infty).$$

Hence, $x_0 \in (x_\infty, p_K(x_\infty))$, and so by Proposition 2.11, $p_K(x_\infty) = p_K(x_0) = p_K(x)$. On the other hand,

$$\begin{aligned} x_\infty &= x_0 + \varepsilon \frac{x_0 - p_K(x_\infty)}{\|x_0 - p_K(x_\infty)\|} \\ &= x_0 + \varepsilon \frac{x_0 - p_K(x)}{\|x_0 - p_K(x)\|} \quad \text{since } p_K(x_\infty) = p_K(x) \\ &= p_K(x) + \lambda_0 (x - p_K(x)) + \varepsilon \frac{x - p_K(x)}{\|x - p_K(x)\|} \\ &= p_K(x) + \left(\lambda_0 + \frac{\varepsilon}{\|x - p_K(x)\|} \right) (x - p_K(x)). \end{aligned}$$

Thus, we have that $\lambda_0 + \dfrac{\varepsilon}{\|x - p_K(x)\|} \in I_x$. However, this is impossible since λ_0 is strictly smaller than $\lambda_0 + \dfrac{\varepsilon}{\|x - p_K(x)\|}$. Therefore, K is a sun. □

COROLLARY 3.8 ([34]). *Let $(X, \|\cdot\|)$ be a strictly convex smooth finite dimensional normed linear space. Then a subset of X is a Chebyshev set if, and only if, it is nonempty, closed and convex.*

PROOF. This result follows directly from Corollary 2.40 and Theorem 3.7. □

3.2. Proof using inversion in the unit sphere In this subsection we limit our discussion to Hilbert spaces. In particular we use "inversion in the unit sphere" to relate nearest points to farthest points. In this way, we prove that a Chebyshev set in a Hilbert space with a continuous metric projection mapping is convex.

DEFINITION 3.9. Let $(X, \|\cdot\|)$ be a normed linear space and K be a nonempty bounded subset of X. For any point $x \in X$, we define $r(x, K) := \sup_{y \in K} \|x - y\|$. We refer to the map $x \mapsto r(x, K)$ as the **radial function for K**.

PROPOSITION 3.10. *Let K be a nonempty bounded subset of a normed linear space $(X, \|\cdot\|)$. Then the radial function for K is convex and nonexpansive (and hence continuous).*

PROOF. For each $k \in K$, let $g_k : X \to [0, \infty)$ be defined by $g_k(x) := \|x - k\|$. Then each g_k is convex and nonexpansive. Now, for each $x \in X$, $r(x, K) = \sup\{g_k(x) : k \in K\}$. Thus, as the pointwise supremum of a family of convex nonexpansive mappings, the radial function is itself convex and nonexpansive. □

DEFINITION 3.11. Let $(X, \|\cdot\|)$ be a normed linear space and let K be a subset of X. We define a set valued mapping $F_K : X \to \mathcal{P}(K)$ by $F_K(x) := \{y \in K : \|x - y\| = r(x, K)\}$, if K is nonempty and bounded and by $F_K(x) = \emptyset$ otherwise. We refer to the elements of $F_K(x)$ as the **farthest points from x in K**.

We say that K is a **remotal set** if $F_K(x)$ is a nonempty for each $x \in X$. Furthermore, we say that K is a **uniquely remotal set** if $F_K(x)$ is a singleton for each $x \in X$.

For a uniquely remotal set we define a mapping $f_K : X \to K$ as the mapping that assigns to each $x \in X$ the unique element of $F_K(x)$.

We will refer to both F_K and f_K as the **farthest point map (for K)**.

REMARK. It follows from the definition that every remotal set is nonempty and bounded.

To begin, we need the following technical result concerning the farthest point mapping.

LEMMA 3.12. *Let K be a remotal set in a normed linear space $(X, \|\cdot\|)$. Let $x \in X$, $z \in F_K(x)$ and suppose that $x \notin F_K(x)$. If for each $\lambda \in (0,1)$, $x_\lambda := x + \lambda(z - x)$, then*

$$r(x_\lambda, K) \le r(x, K) - \|x_\lambda - x\| \left(1 - \frac{\|z - z_\lambda\|}{\|x - z\|}\right),$$

where $z_\lambda \in F_K(x_\lambda)$ for each $\lambda \in (0,1)$.

PROOF. Let $w_\lambda := \lambda z_\lambda + (1-\lambda)x \in (x, z_\lambda)$. Then

$$\begin{aligned} r(x_\lambda, K) &= \|x_\lambda - z_\lambda\| \\ &\le \|x_\lambda - w_\lambda\| + \|w_\lambda - z_\lambda\| \quad \text{by the triangle inequality} \\ &= \lambda \|z - z_\lambda\| + (1-\lambda)\|x - z_\lambda\| \\ &\le \lambda \|z - z_\lambda\| + (1-\lambda)\|x - z\| \quad \text{since } z \in F_K(x) \text{ and } z_\lambda \in K. \end{aligned}$$

Therefore, $r(x_\lambda, K) \le r(x, K) - \lambda(\|x - z\| - \|z - z_\lambda\|)$. Now, from the definition of x_λ, we have that $\lambda = \dfrac{\|x_\lambda - x\|}{\|x - z\|}$. Thus, $r(x_\lambda, K) \le r(x, K) - \|x_\lambda - x\| \left(1 - \dfrac{\|z - z_\lambda\|}{\|x - z\|}\right)$. ☐

COROLLARY 3.13. *Let K be a remotal set in a normed linear space $(X, \|\cdot\|)$. Suppose $x \in X$ and $x \notin F_K(x)$. If the farthest point map for K is continuous at $x \in X$, then*

$$\lim_{\lambda \to 0^+} \frac{\|x_\lambda - z_\lambda\| - \|x - z\|}{\|x_\lambda - x\|} = \lim_{\lambda \to 0^+} \frac{r(x_\lambda, K) - r(x, K)}{\|x_\lambda - x\|} = -1,$$

where $\{z\} = F_K(x)$, $x_\lambda := x + \lambda(z - x)$ and $z_\lambda \in F_K(x_\lambda)$ for each $\lambda \in (0,1)$.

PROOF. We see that for any $\lambda \in (0,1)$,

$$\begin{aligned} -1 = -\frac{\|x_\lambda - x\|}{\|x_\lambda - x\|} &\le \frac{r(x_\lambda, K) - r(x, K)}{\|x_\lambda - x\|} \quad \text{since the radial function for } K \text{ is nonexpansive} \\ &\le -\left(1 - \frac{\|z - z_\lambda\|}{\|x - z\|}\right) \quad \text{by Lemma 3.12.} \end{aligned}$$

Since $\lim_{\lambda \to 0^+} x_\lambda = x$ and the farthest point map for K is continuous at x, the right hand side of the above inequality converges to -1 as $\lambda \to 0^+$. Hence, $\lim_{\lambda \to 0^+} \dfrac{r(x_\lambda, M) - r(x, M)}{\|x_\lambda - x\|} = -1$. ☐

The next result that we require is well-known in optimisation theory.

PROPOSITION 3.14. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $f : X \to [0, \infty)$ be a continuous convex function such that $\liminf_{\|x\|\to\infty} f(x) = \infty$. Then argmin$(f) \neq \emptyset$, that is, f attains its global minimum.*

PROOF. Since f is continuous and convex, f is lower semi-continuous with respect to the weak topology on X (see Lemma 2.37). Let $K := \{x \in X : f(x) \leq f(0)\}$. Then

(i) $K \neq \emptyset$ since $0 \in K$,

(ii) K is bounded since $\liminf_{\|x\|\to\infty} f(x) = \infty$,

(iii) K is weakly compact, since it is closed and convex (hence weakly closed) and bounded.

Therefore, by Proposition 2.38, $\emptyset \neq \text{argmin}(f|_K)$. Thus, $\emptyset \neq \text{argmin}(f|_K) = \text{argmin}(f)$. $\square$

Our next result shows that within the context of reflexive Banach spaces there are very few uniquely remotal sets that possess a continuous farthest point mapping.

PROPOSITION 3.15 ([10]). *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let K be a uniquely remotal set in X. If the farthest point mapping, $x \mapsto f_K(x)$, is continuous on X, then K is a singleton.*

PROOF. It is easy to see that $\liminf_{\|x\|\to\infty} r(x, K) = \infty$. Thus, by Proposition 3.14, there exists $x_0 \in X$ such that $r(x_0, K) \leq r(x, K)$ for all $x \in X$. However, it follows from the continuity of f_K and Corollary 3.13, that unless $x_0 = f_K(x_0)$ we have a contradiction. Thus, $x_0 = f_K(x_0)$, which implies that K is a singleton. $\square$

One can show, though we will not do so here, that as with the metric projection map, the farthest point map has a closed graph. Hence, in finite dimensional spaces the farthest point mapping of every uniquely remotal set is continuous. For more information on remotal/uniquely remotal sets, see [9, 63].

COROLLARY 3.16 ([52]). *Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space. Then every uniquely remotal subset of X is a singleton.*

DEFINITION 3.17. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. We define $i : X \setminus \{0\} \to X \setminus \{0\}$ by $i(x) := \dfrac{x}{\|x\|^2}$ and we call this map, **inversion in the unit sphere**.

Note: it is clear that i is continuous. Furthermore, since i is invertible, with $i^{-1} = i$, we see that i is a homeomorphism.

LEMMA 3.18. *For any inner product space $(X, \langle\cdot,\cdot\rangle)$, and any $x \in X \setminus \{0\}$,*

$$i\left(B[x; \|x\|] \setminus \{0\}\right) \subseteq \left\{y \in X : \frac{1}{2} \leq \langle y, x\rangle\right\}.$$

PROOF. Let $w \in i\left(B[x;\|x\|]\backslash\{0\}\right)$. Then $w = i(z)$ for some $z \in B[x;\|x\|]\backslash\{0\}$. Now,

$$\begin{aligned}
z \in B[x;\|x\|]\backslash\{0\} &\Leftrightarrow z \neq 0 \text{ and } \|z-x\| \leq \|x\| \\
&\Leftrightarrow z \neq 0 \text{ and } \|z-x\|^2 \leq \|x\|^2 \\
&\Leftrightarrow z \neq 0 \text{ and } \|z\|^2 - 2\langle z, x\rangle + \|x\|^2 \leq \|x\|^2 \\
&\Leftrightarrow z \neq 0 \text{ and } \|z\|^2 \leq 2\langle z, x\rangle \\
&\Leftrightarrow \frac{1}{2} \leq \left\langle \frac{z}{\|z\|^2}, x \right\rangle = \langle w, x\rangle \\
&\Rightarrow w \in \left\{ y \in X : \frac{1}{2} \leq \langle y, x\rangle \right\}.
\end{aligned}$$

□

PROPOSITION 3.19. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Let $x \in X$ and let $\delta > \|x\|$. Then*

$$i\left(B[x;\delta]\backslash\{0\}\right) = X \setminus B\big(-\delta' x; \delta'\delta\big), \qquad \textit{where } \delta' := \tfrac{1}{\delta^2 - \|x\|^2}.$$

In particular, if $y \notin B[x;\delta]$, then $i(y) \in B(-\delta' x; \delta'\delta)$. Furthermore, $i\left(S[x;\delta]\right) = S[-\delta' x; \delta\delta']$.

PROOF. Let $w \in X \setminus B(-\delta' x; \delta'\delta)$. Then $w = i(y)$ for some $y \in X\backslash\{0\}$. Therefore,

$$\begin{aligned}
& w \in X \setminus B\big(-\delta' x; \delta'\delta\big) \\
\Leftrightarrow\ & y \in X \setminus \{0\} \text{ and } \left\| \frac{y}{\|y\|^2} - \frac{-x}{\delta^2 - \|x\|^2} \right\| \geq \frac{\delta}{\delta^2 - \|x\|^2} \\
\Leftrightarrow\ & y \in X \setminus \{0\} \text{ and } \left\| y\left(\delta^2 - \|x\|^2\right) + x\,\|y\|^2 \right\| \geq \delta\,\|y\|^2 \\
\Leftrightarrow\ & y \in X \setminus \{0\} \text{ and } \left\| y\left(\delta^2 - \|x\|^2\right) + x\,\|y\|^2 \right\|^2 \geq \delta^2\,\|y\|^4 \\
\Leftrightarrow\ & y \in X \setminus \{0\} \text{ and } \|y\|^2\left(\delta^2 - \|x\|^2\right)^2 + 2\left(\delta^2 - \|x\|^2\right)\|y\|^2\,\langle y, x\rangle + \|x\|^2\,\|y\|^4 \geq \delta^2\,\|y\|^4 \\
\Leftrightarrow\ & y \in X \setminus \{0\} \text{ and } \left(\delta^2 - \|x\|^2\right)^2 + 2\left(\delta^2 - \|x\|^2\right)\langle y, x\rangle + \|x\|^2\,\|y\|^2 \geq \delta^2\,\|y\|^2 \\
\Leftrightarrow\ & y \in X \setminus \{0\} \text{ and } \left(\delta^2 - \|x\|^2\right) + 2\langle y, x\rangle \geq \|y\|^2 \\
\Leftrightarrow\ & y \in X \setminus \{0\} \text{ and } \|y\|^2 - 2\langle y, x\rangle + \|x\|^2 \leq \delta^2 \\
\Leftrightarrow\ & y \in X \setminus \{0\} \text{ and } \|y - x\|^2 \leq \delta^2 \\
\Leftrightarrow\ & y \in B[x;\delta] \setminus \{0\} \\
\Leftrightarrow\ & w \in i\left(B[x;\delta]\backslash\{0\}\right).
\end{aligned}$$

Thus, $i\left(B[x;\delta]\backslash\{0\}\right) = X \setminus B(-\delta' x; \delta'\delta)$. Finally, to see that $i\left(S[x;\delta]\right) = S[-\delta' x; \delta\delta']$, observe that all the implications (apart from the first one and the last two) in the above working still hold if we replace each inequality by an equality. □

THEOREM 3.20 ([4, p. 238]). *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and A be a Chebyshev set in X. If the metric projection mapping p_A is continuous on X, then A is convex.*

PROOF. Suppose, looking for a contradiction, that A is not convex. Let $x_0 \in \mathrm{co}(A) \setminus A$ and define $K := A - x_0$. It is clear that $0 \in \mathrm{co}(K) \setminus K$. Furthermore, K is a Chebyshev set with a continuous metric projection map. Indeed, for each $x \in X$, $p_K(x) = p_A(x + x_0) - x_0$. To see this, first note that $p_K(x) \in K$ and for any $k \in K \setminus \{p_K(x)\}$,

$$\begin{aligned}\|p_K(x) - x\| &= \|p_A(x + x_0) - (x + x_0)\|\\ &< \|(k + x_0) - (x + x_0)\| \quad \text{since } k + x_0 \in A \text{ and } k + x_0 \neq p_A(x + x_0)\\ &= \|x - k\|.\end{aligned}$$

Since p_A is continuous, we see that p_K is continuous.

Let $G := i(K)$. Since K is a Chebyshev set (and so nonempty and closed) and $0 \notin K$, it follows that G is a nonempty closed and bounded subset of $X \setminus \{0\}$. We claim that G is a uniquely remotal set in X whose farthest point mapping is continuous. Let $x \in X$. It follows from the definition of $r(x, G)$ that $G \subseteq B[x; r(x, G)] \setminus \{0\}$, and moreover, $G \not\subseteq B[x; \delta]$ for any $\delta \in (0, r(x, G))$. We now show that $\|x\| < r(x, G)$. Looking for a contradiction, suppose $\|x\| \geq r(x, G)$. By Lemma 3.18,

$$K = i(i(K)) = i(G) \subseteq i\left(B[x; r(x, G)] \setminus \{0\}\right) \subseteq i\left(B[x; \|x\|] \setminus \{0\}\right) \subseteq \left\{z \in X : \frac{1}{2} \leq \langle z, x\rangle\right\}.$$

Thus, $0 \in \mathrm{co}(K) \subseteq \left\{z \in X : \frac{1}{2} \leq \langle z, x\rangle\right\}$, which is impossible. Therefore, we must have that $\|x\| < r(x, G)$. Next we claim that

$$d\left(\frac{-x}{r^2(x, G) - \|x\|^2}, K\right) = \frac{r(x, G)}{r^2(x, G) - \|x\|^2}.$$

From Proposition 3.19,

$$K = i(G) \subseteq i\left(B[x; r(x, G)] \setminus \{0\}\right) \subseteq X \setminus B\left(\frac{-x}{r^2(x, G) - \|x\|^2}; \frac{r(x, G)}{r^2(x, G) - \|x\|^2}\right).$$

Therefore,

$$\frac{r(x, G)}{r^2(x, G) - \|x\|^2} \leq d\left(\frac{-x}{r^2(x, G) - \|x\|^2}, K\right).$$

Next, let $\delta \in (\|x\|, r(x, G))$. By our earlier remark, we can find $g_\delta \in G \setminus B[x; \delta]$. Applying Proposition 3.19 again, we have that

$$i(g_\delta) \in K \cap B\left(\frac{-x}{\delta^2 - \|x\|^2}; \frac{\delta}{\delta^2 - \|x\|^2}\right).$$

Hence,

$$\left\| i(g_\delta) - \frac{-x}{r^2(x,G) - \|x\|^2} \right\| \leq \left\| i(g_\delta) - \frac{-x}{\delta^2 - \|x\|^2} \right\| + \left\| \frac{-x}{\delta^2 - \|x\|^2} - \frac{-x}{r^2(x,G) - \|x\|^2} \right\|$$
$$< \frac{\delta}{\delta^2 - \|x\|^2} + \frac{\|x\| \left(r^2(x,G) - \delta^2 \right)}{\left(\delta^2 - \|x\|^2 \right) \left(r^2(x,G) - \|x\|^2 \right)}.$$

Therefore,

$$d\left(\frac{-x}{r^2(x,G) - \|x\|^2}, K \right) \leq \left\| i(g_\delta) - \frac{-x}{r^2(x,G) - \|x\|^2} \right\|$$
$$< \frac{\delta}{\delta^2 - \|x\|^2} + \frac{\|x\| \left(r^2(x,G) - \delta^2 \right)}{\left(\delta^2 - \|x\|^2 \right) \left(r^2(x,G) - \|x\|^2 \right)}.$$

Taking the limit as $\delta \to r(x,G)$, we obtain

$$d\left(\frac{-x}{r^2(x,G) - \|x\|^2}, K \right) \leq \frac{r(x,G)}{r^2(x,G) - \|x\|^2}.$$

Thus,

$$d\left(\frac{-x}{r^2(x,G) - \|x\|^2}, K \right) = \frac{r(x,G)}{r^2(x,G) - \|x\|^2}$$

as claimed. Since

$$p_K\left(\frac{-x}{r^2(x,G) - \|x\|^2} \right) \in K \cap S\left[\frac{-x}{r^2(x,G) - \|x\|^2}; \frac{r(x,G)}{r^2(x,G) - \|x\|^2} \right]$$
$$= K \cap i\left(S[x; r(x,G)] \right) = i(G) \cap i\left(S[x; r(x,G)] \right),$$

we have that

$$i\left(p_K\left(\frac{-x}{r^2(x,G) - \|x\|^2} \right) \right) \in G \cap S[x; r(x,G)].$$

Thus,

$$i\left(p_K\left(\frac{-x}{r^2(x,G) - \|x\|^2} \right) \right) \in F_G(x).$$

Moreover, since i is a bijection and K is a Chebyshev set, it follows that

$$F_G(x) = \left\{ i\left(p_K\left(\frac{-x}{r^2(x,G) - \|x\|^2} \right) \right) \right\},$$

and so G is a uniquely remotal set. Since p_K is continuous on X, it follows that f_G is continuous on X. Finally, it follows from Proposition 3.15 that G is a singleton, and so K is a singleton. However, this contradicts the fact that, by construction, K is nonconvex. $\square$

3.3. A proof using convex analysis Again we limit ourselves to Hilbert spaces, but this time we make use of tools from convex analysis.

DEFINITION 3.21. Let $(X, \|\cdot\|)$ be a normed linear space and $f : X \to \mathbb{R} \cup \{\infty\}$. We say that a sequence $(x_n)_{n=1}^{\infty}$ in X is **minimising** if $\lim_{n\to\infty} f(x_n) = \inf_{x\in X} f(x)$.

DEFINITION 3.22. Let $(X, \|\cdot\|)$ be a normed linear space and $f : X \to \mathbb{R} \cup \{\infty\}$. We say that f has a **strong minimum at $x \in X$** if f has a global minimum at x and each minimising sequence $(x_n)_{n=1}^{\infty}$ in X converges to x.

DEFINITION 3.23. Let $(X, \|\cdot\|)$ be a normed linear space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper function that is bounded below by a continuous linear functional. We define a function $\overline{\mathbf{co}}(f)$: $X \to \mathbb{R} \cup \{\infty\}$ by

$$\overline{\text{co}}(f)(x) := \sup\{\psi(x) : \psi : X \to \mathbb{R} \text{ is continuous, convex and } \psi(y) \leq f(y) \text{ for all } y \in X\}.$$

It is immediately clear from this definition that $\overline{\text{co}}(f)$ is convex, lower semicontinuous, and $\overline{\text{co}}(f)(x) \leq f(x)$ for all $x \in X$. Moreover, if $x^* \in X^*$, then $\overline{\text{co}}(f - x^*) = \overline{\text{co}}(f) - x^*$.

THEOREM 3.24 ([79]). *Let $(X, \|\cdot\|)$ be a normed linear space. Suppose that $f : X \to \mathbb{R}\cup\{\infty\}$ and $\displaystyle\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|} > 0$. If f has a strong minimum at $x_0 \in X$, then argmin $(\overline{co}(f)) =$ argmin(f).*

PROOF. Clearly, $\{x_0\} = \text{argmin}(f) \subseteq \text{argmin}\,(\overline{\text{co}}(f))$ and $f(x_0) = \overline{\text{co}}(f)(x_0)$. So it suffices to show that

$$\text{argmin}\,(\overline{\text{co}}(f)) \subseteq \text{argmin}(f) = \{x_0\}.$$

To do this, we show that $\text{argmin}\,(\overline{\text{co}}(f)) \subseteq B[x_0; \varepsilon]$ for each $\varepsilon > 0$. To this end, let $\varepsilon > 0$. We claim that for some $n \in \mathbb{N}$, the continuous convex function $c_n : X \to \mathbb{R}$, defined by

$$c_n(x) := f(x_0) + \frac{1}{n} d(x, B[x_0; \varepsilon]),$$

satisfies the property that $c_n(x) \leq f(x)$ for all $x \in X$. Suppose, looking for a contradiction, that this is not the case. Then, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $f(x_n) < c_n(x_n)$.

Since $\displaystyle\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|} > 0$, the sequence $(x_n)_{n=1}^{\infty}$ is bounded. Therefore, $\lim_{n\to\infty} f(x_n) = f(x_0)$. However, since x_0 is a strong minimum of f, we must have that $\lim_{n\to\infty} x_n = x_0$. Hence, for n sufficiently large, $x_n \in B[x_0; \varepsilon]$, and so $f(x_n) < c_n(x_n) = f(x_0) + \dfrac{1}{n} d(x_n, B[x_0; \varepsilon]) = f(x_0)$, which is clearly impossible since $f(x_0) = \inf_{x\in X} f(x)$. This proves the claim. Finally, since c_n is continuous and convex, $c_n(x) \leq \overline{\text{co}}(f)(x)$ for all $x \in X$. This implies that $\text{argmin}\,(\overline{\text{co}}(f))$ is contained in $B[x_0; \varepsilon]$, since $\overline{\text{co}}(f)(x_0) = f(x_0) < c_n(x) \leq \overline{\text{co}}(f)(x)$ for all $x \notin B[x_0; \varepsilon]$. $\square$

THEOREM 3.25 ([4, 79]). *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let K be a Chebyshev set in X. If for each $x \in X \setminus K$, the function $k \mapsto \|x - k\|$, defined on K, has a strong minimum at $p_K(x)$, then K is convex.*

PROOF. Define $f : X \to [0, \infty]$ by

$$f(x) = \begin{cases} \|x\|^2, & \text{if } x \in K \\ \infty, & \text{if } x \notin K. \end{cases}$$

Looking for a contradiction, suppose that K is not convex. Then there exists

$$z \in \text{co}(K) \setminus K \subseteq \text{Dom}\,(\overline{\text{co}}(f)) \setminus K,$$

as $\text{Dom}\,(\overline{\text{co}}(f))$ is convex and contains $\text{Dom}(f) = K$. Now, by the Brøndsted-Rockafellar Theorem, see [17] or [65, Theorem 3.17], there exists a point $z_0 \in \text{Dom}\,(\partial\,(\overline{\text{co}}(f))) \setminus K$. Let $x^* \in \partial\,(\overline{\text{co}}(f))\,(z_0)$. Then $z_0 \in \text{argmin}(\overline{\text{co}}(f) - x^*)$ and

$$\liminf_{\|x\| \to \infty} \frac{(f - x^*)(x)}{\|x\|} = \infty.$$

By Riesz's Representation Theorem, (see [70], or more recently, [77, p. 248]), there exists $x \in X$ such that $x^*(y) = \langle y, x \rangle$ for all $y \in X$. Therefore,

$$(f - x^*)(k) = f(k) - \langle k, x \rangle = \left\| k - \frac{x}{2} \right\|^2 - \frac{\|x\|^2}{4}$$

for all $k \in K = \text{Dom}(f - x^*)$. Hence, it follows that $f - x^*$ has a strong minimum at $p_K\left(\frac{x}{2}\right)$. Putting all of this together, we obtain the following:

$$\begin{aligned} z_0 \in \text{argmin}\,(\overline{\text{co}}(f) - x^*) &= \text{argmin}\,(\overline{\text{co}}(f - x^*)) \\ &= \text{argmin}\,(f - x^*) \quad \text{by Theorem 3.24} \\ &= \left\{ p_K\!\left(\frac{x}{2}\right) \right\} \subseteq K. \end{aligned}$$

However, this is impossible, since $z_0 \notin K$. Hence, K is convex. ☐

The short-coming of the previous result is that it is difficult to determine whether or not the function $k \mapsto \|x - k\|$ has a strong minimum at $p_K(x)$. So next we try to alleviate this problem.

COROLLARY 3.26. *Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let K be a Chebyshev set in X. If $x \mapsto d(x, K)$ is Fréchet differentiable on $X \setminus K$, then K is convex.*

PROOF. Let $x \in X \setminus K$. We will show that the function $k \mapsto \|x - k\|$, defined on K, has a strong minimum at $p_K(x)$. Since every subsequence of a minimising sequence is again a minimising sequence, it is sufficient to show that for every minimising sequence $(z_n)_{n=1}^\infty$ in K there exists a subsequence $(z_{n_k})_{k=1}^\infty$ of $(z_n)_{n=1}^\infty$ such that $\lim_{k \to \infty} z_{n_k} = p_K(x)$. To this end, let $(z_n)_{n=1}^\infty$ be a sequence in K with $\lim_{n \to \infty} \|x - z_n\| = d(x, K)$. Since $(z_n)_{n=1}^\infty$ is bounded and X is reflexive, there exists a subsequence $(z_{n_k})_{k=1}^\infty$ of $(z_n)_{n=1}^\infty$ and an element $z \in X$ such that $(z_{n_k})_{k=1}^\infty$ converges to z with respect to the weak topology on X. Let x^* be the Fréchet derivative of the distance function for K at x. By Lemma 2.31, $\|x^*\| \leq 1$. For each $n \in \mathbb{N}$, let $\lambda_n := \sqrt{\|x - z_n\| - d(x,K) + (1/n)}$. Then $\lim_{k\to\infty} \dfrac{\|x - z_{n_k}\| - d(x,K)}{\lambda_{n_k}} = 0$ and so

$$\begin{aligned}
x^*(z-x) &= \lim_{k\to\infty} x^*(z_{n_k} - x) \\
&= \lim_{k\to\infty} \frac{d(x + \lambda_{n_k}(z_{n_k} - x), K) - d(x,K)}{\lambda_{n_k}} \\
&\leq \lim_{k\to\infty} \frac{\|x + \lambda_{n_k}(z_{n_k} - x) - z_{n_k}\| - d(x,K)}{\lambda_{n_k}} \\
&= \lim_{k\to\infty} \frac{(1-\lambda_{n_k})\,\|x - z_{n_k}\| - d(x,K)}{\lambda_{n_k}} \\
&= \lim_{k\to\infty} \left[-\|x - z_{n_k}\| + \frac{\|x - z_{n_k}\| - d(x,K)}{\lambda_{n_k}} \right] = -d(x,K).
\end{aligned}$$

Therefore, $d(x,K) \leq x^*(x-z)$; which implies that

$$d(x,K) \leq x^*(x-z) \leq \|x^*\|\,\|x-z\| \leq \|x-z\| \leq \liminf_{k\to\infty} \|x - z_{n_k}\| = d(x,K)$$

since $(x - z_{n_k})_{k=1}^{\infty}$ converges to $(x-z)$ with respect to the weak topology on X and the norm is lower semi-continuous with respect to the weak topology on X. Thus, $(x - z_{n_k})_{k=1}^{\infty}$ converges to $(x-z)$ with respect to the weak topology on X and $\lim_{k\to\infty} \|x - z_{n_k}\| = \|x - z\| = d(x,K)$. Hence, since the norm is a Kadec-Klee norm, $(x - z_{n_k})_{k=1}^{\infty}$ converges in norm to $(x-z)$, and so $z = \lim_{k\to\infty} z_{n_k}$. Therefore, $z \in K$. However, since K is a Chebyshev set and $\|x - z\|$ equals $d(x,K)$, it must be the case that $z = p_K(x)$. □

Of course we are now left wondering when the distance function $x \mapsto d(x,K)$ is Fréchet differentiable on $X \setminus K$. To solve this we consider the following.

COROLLARY 3.27. *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let K be a Chebyshev set in X. If the metric projection map, $x \mapsto p_K(x)$, is continuous on $X \setminus K$, then the distance function, $x \mapsto d(x,K)$, is Fréchet differentiable on $X \setminus K$, and so K is convex.*

PROOF. This follows directly from [4, Proposition]. □

REMARK. Several authors have approached the question of the convexity of Chebshev sets by considering the differentiability of the associated distance function, see [38, 42, 92].

3.4. Vlasov's theorem In this subsection we prove Vlasov's theorem, which has some of the weakest assumptions for a Chebyshev set to be convex; requiring only completeness, continuity of the metric projection map and strict convexity of the dual space. First we require a number of preliminary results.

LEMMA 3.28 ([41, p. 238]). *Let K be a proximinal set in a normed linear space $(X, \|\cdot\|)$. Let $x \in X \setminus K$ and $z \in P_K(x)$. If for each $\lambda > 0$, $x_\lambda := x + \lambda(x - z)$, then*

$$d(x, K) + \|x_\lambda - x\| \left(1 - \frac{\|z - z_\lambda\|}{\|x - z\|}\right) \leq d(x_\lambda, K),$$

where $z_\lambda \in P_K(x_\lambda)$ for each $\lambda > 0$.

PROOF. Let $w_\lambda := \left(\frac{1}{1+\lambda}\right) x_\lambda + \left(\frac{\lambda}{1+\lambda}\right) z_\lambda \in (x_\lambda, z_\lambda)$. Then

$$\begin{aligned} d(x, K) &= \|x - z\| \\ &\leq \|x - z_\lambda\| \quad \text{since } z \in P_K(x) \text{ and } z_\lambda \in K \\ &\leq \|x - w_\lambda\| + \|w_\lambda - z_\lambda\| \quad \text{by the triangle inequality} \\ &= \left(\frac{\lambda}{1+\lambda}\right) \|z_\lambda - z\| + \left(\frac{1}{1+\lambda}\right) \|x_\lambda - z_\lambda\|. \end{aligned}$$

Therefore, $d(x, K) + \lambda\left(d(x, K) - \|z_\lambda - z\|\right) \leq d(x_\lambda, K)$. Now, from the definition of x_λ, we have that $\lambda = \dfrac{\|x_\lambda - x\|}{\|x - z\|}$. Thus, $d(x, K) + \|x_\lambda - x\| \left(1 - \dfrac{\|z - z_\lambda\|}{\|x - z\|}\right) \leq d(x_\lambda, K)$. □

COROLLARY 3.29 ([41, p.238]). *Let K be a proximinal set in a normed linear space $(X, \|\cdot\|)$. If the metric projection map is continuous at $x \in X \setminus K$, then*

$$\lim_{\lambda \to 0^+} \frac{\|x_\lambda - z_\lambda\| - \|x - z\|}{\|x_\lambda - x\|} = \lim_{\lambda \to 0^+} \frac{d(x_\lambda, K) - d(x, K)}{\|x_\lambda - x\|} = 1,$$

where $\{z\} = P_K(x)$, $x_\lambda := x + \lambda(x - z)$ and $z_\lambda \in P_K(x_\lambda)$ for each $\lambda > 0$.

PROOF. We see that for any $\lambda > 0$,

$$\begin{aligned} 1 = \frac{\|x_\lambda - x\|}{\|x_\lambda - x\|} &\geq \frac{d(x_\lambda, K) - d(x, K)}{\|x_\lambda - x\|} \quad \text{since the distance function for } K \text{ is nonexpansive} \\ &\geq 1 - \frac{\|z - z_\lambda\|}{\|x - z\|} \quad \text{by Lemma 3.28.} \end{aligned}$$

Since $\lim_{\lambda \to 0^+} x_\lambda = x$ and the metric projection map is continuous at x, the right hand side of the above inequality converges to 1 as $\lambda \to 0^+$. Hence, $\lim_{\lambda \to 0^+} \dfrac{d(x_\lambda, M) - d(x, M)}{\|x_\lambda - x\|} = 1$. □

We require the following variational result from optimisation theory.

THEOREM 3.30 (Primitive Ekeland Theorem [36]). *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a bounded below, lower semi-continuous function on X. If $\varepsilon > 0$ and $x_0 \in X$, then there exists $x_\infty \in X$ such that:*

(i) $f(x_\infty) \leq f(x_0) - \varepsilon d(x_\infty, x_0)$ *and*

(ii) $f(x_\infty) - \varepsilon d(x_\infty, x) < f(x)$ *for all* $x \in X \setminus \{x_\infty\}$.

PROOF. See Appendix A. $\square$

COROLLARY 3.31 ([88]). *Let* (X, d) *be a complete metric space and let* $f : X \to \mathbb{R}$ *be a bounded above continuous function on* X*. If* $\varepsilon > 0$ *and* $x_0 \in X$*, then there exists* $x_\infty \in X$ *such that:*

(i) $f(x_\infty) \geq f(x_0) + \varepsilon d(x_\infty, x_0)$ *and*

(ii) $f(x) < f(x_\infty) + \varepsilon d(x_\infty, x)$ *for all* $x \in X \setminus \{x_\infty\}$.

In particular,

$$\limsup_{x \to x_\infty} \frac{f(x) - f(x_\infty)}{d(x_\infty, x)} \leq \varepsilon.$$

PROOF. The result follows from applying Theorem 3.30 to $-f$. $\square$

Rather than looking directly at convexity, we shall investigate the weaker property of almost convexity.

DEFINITION 3.32 ([89]). We will say that a closed subset A of a normed linear space $(X, \|\cdot\|)$ is **almost convex** if, for any closed ball $B[x; r] \subseteq X \setminus A$ and $N > 0$, there exist $x' \in X$ and $r' > N$ such that

$$B[x; r] \subseteq B\left[x'; r'\right] \subseteq X \setminus A.$$

THEOREM 3.33 ([89]). *Let* K *be a Chebyshev set in a Banach space* $(X, \|\cdot\|)$*. If the metric projection map for* K *is continuous, then* K *is almost convex.*

PROOF. Suppose that $B[x_0; \beta] \subseteq X \setminus K$ for some $x_0 \in X$ and $0 < \beta$. Let $N > 0$ be arbitrary. Choose $\varepsilon \in (0, 1)$ such that $d(x_0, K) + \varepsilon N > \beta + N$ (note: this is possible since $d(x_0, K) > \beta$). Consider the function $f : B[x_0, N] \to \mathbb{R}$ defined by $f(x) := d(x, K)$. Since $B[x_0; N]$ is a complete metric space and the distance function for K is continuous on $B[x_0; N]$, Corollary 3.31 gives the existence of an $x_\infty \in B[x_0; N]$ such that

(i) $f(x_\infty) \geq f(x_0) + \varepsilon d(x_\infty, x_0)$ and

(ii) $\displaystyle\limsup_{x \to x_\infty} \frac{f(x) - f(x_\infty)}{d(x_\infty, x)} \leq \varepsilon < 1$.

Now, if $\|x_\infty - x\| < N$, then it follows from Corollary 3.29 that $\displaystyle\limsup_{x \to x_\infty} \frac{f(x) - f(x_\infty)}{d(x_\infty, x)} = 1$, which is impossible by (ii). Thus, $\|x_\infty - x\| = N$. Hence, from (i) we obtain that

$$d(x_\infty, K) \geq d(x_0, K) + \varepsilon N > \beta + N.$$

Therefore, $B[x_0; \beta] \subseteq B[x_\infty; \beta + N] \subseteq X \setminus K$. Since $N > 0$ was arbitrary, we conclude that K is almost convex. $\square$

We now establish a condition that guarantees that an almost convex set is convex. To achieve this we need to prove some preliminary results.

LEMMA 3.34 ([41, p. 241]). *Let $(X, \|\cdot\|)$ be a normed linear space with a strictly convex dual space and let $f \in S_{X^*}$. If $(B[z_n; r_n])_{n=1}^{\infty}$ is an increasing sequence of closed balls in X such that*

(i) $B[z_n; r_n] \subseteq \{x \in X : f(x) \leq 1\}$ *for all* $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} r_n = \infty$,

then there exists some $r \leq 1$ such that $\overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]} = \{z \in X : f(z) \leq r\}$, i.e., $\overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]}$ is a closed half space.

PROOF. Let $r := \sup \left\{ f(x) : x \in \overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]} \right\} \leq 1$. Suppose, looking for a contradiction, that

$$\overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]} \neq \{x \in X : f(x) \leq r\}.$$

Then there exists a $z \in \{x \in X : f(x) \leq r\} \backslash \overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]}$. Since $\overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]}$ is nonempty, closed and convex, there exists some $g \in S_{X^*}$ such that

$$\sup \left\{ g(x) : x \in \overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]} \right\} < g(z).$$

Clearly, $f \neq g$. For each $n \in \mathbb{N}$, let $x_n := \dfrac{z_1 - z_n}{r_n}$. Since $z_1 \in B[z_1; r_1] \subseteq B[z_n; r_n]$ for all $n \in \mathbb{N}$, it follows that $\|x_n\| = \dfrac{\|z_1 - z_n\|}{r_n} \leq 1$. Let $\varepsilon := r - f(z_1) > 0$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned} f(z_1) &= r - \varepsilon \\ &\geq \sup \{f(x) : x \in B[z_n; r_n]\} - \varepsilon \quad \text{since } B[z_n; r_n] \subseteq \overline{\bigcup_{k\in\mathbb{N}} B[z_k; r_k]} \\ &= (f(z_n) + r_n) - \varepsilon \quad \text{since } B[z_n; r_n] = z_n + r_n B_X \text{ and } \|f\| = 1. \end{aligned}$$

Therefore, $1 \geq f(x_n) = \dfrac{f(z_1) - f(z_n)}{r_n} \geq 1 - \dfrac{\varepsilon}{r_n}$ for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} f(x_n) = 1$. By a similar argument, $\lim_{n\to\infty} g(x_n) = 1$. Thus, $\lim_{n\to\infty} \left(\dfrac{f+g}{2}\right)(x_n) = 1$, and so $\left\|\dfrac{f+g}{2}\right\| = 1$. However, this contradicts the strict convexity of X^*. $\square$

LEMMA 3.35. *Let C be a closed convex subset of a normed linear space $(X, \|\cdot\|)$. Then either, $C = X$, or else, C is contained in a half space.*

PROOF. Suppose $C \neq X$. Let $x \in X \setminus C$. By the Hahn-Banach Theorem, there exists a hyperplane separating x and C. Thus, C is contained in one of the half spaces determined by this hyperplane. $\square$

LEMMA 3.36. *Let J be a closed halfspace of a normed linear space $(X, \|\cdot\|)$. If for some $x, y \in X$, $\frac{x+y}{2} \in \operatorname{int}(J)$, then either $x \in \operatorname{int}(J)$ or $y \in \operatorname{int}(J)$.*

PROOF. This follows directly from Lemma 3.1. $\square$

LEMMA 3.37. *Let $(C_n)_{n=1}^{\infty}$ be an increasing sequence of convex sets in a normed linear space $(X, \|\cdot\|)$. Suppose that $\operatorname{int}(C_k) \neq \emptyset$ for some $k \in \mathbb{N}$. Then $\operatorname{int}\left(\overline{\bigcup_{n=1}^{\infty} C_n}\right) = \bigcup_{n=1}^{\infty} \operatorname{int}(C_n)$.*

PROOF. Clearly, $\bigcup_{n=1}^{\infty} \operatorname{int}(C_n) \subseteq \operatorname{int}\left(\overline{\bigcup_{n=1}^{\infty} C_n}\right)$, so it suffices to show the reverse set inclusion. Let $x \in \operatorname{int}\left(\overline{\bigcup_{n=1}^{\infty} C_n}\right)$. Then there exists $r > 0$ such that $B(x; r) \subseteq \operatorname{int}\left(\overline{\bigcup_{n=1}^{\infty} C_n}\right)$. Moreover, there exist $k \in \mathbb{N}$, $s > 0$, and $y \in C_k$ such that $B(y; s) \subseteq C_k$. Now choose $\lambda > 0$ sufficiently small so that

$$B(x + \lambda(x - y); \lambda s) \subseteq B(x; r) \subseteq \overline{\bigcup_{n=1}^{\infty} C_n}.$$

Hence, there exist $z \in B(x + \lambda(x - y); \lambda s)$ and $m \geq k$ such that $z \in C_m$. It is straightforward to verify that

$$x \in B\left(\frac{z + \lambda y}{1 + \lambda}; \frac{\lambda s}{1 + \lambda}\right) = \left(\frac{1}{1 + \lambda}\right) z + \left(\frac{\lambda}{1 + \lambda}\right) B(y; s).$$

Since $z \in C_m$ and $B(y; s) \subseteq C_k \subseteq C_m$, we have from the convexity of C_m that $x \in \operatorname{int}(C_m)$. Therefore, $x \in \bigcup_{n=1}^{\infty} \operatorname{int}(C_n)$. $\square$

COROLLARY 3.38. *Let C be a convex subset of a normed linear space $(X, \|\cdot\|)$. If $\operatorname{int}(C) \neq \emptyset$, then $\operatorname{int}(\overline{C}) = \operatorname{int}(C)$.*

THEOREM 3.39 ([89]). *Let $(X, \|\cdot\|)$ be a normed linear space with strictly convex dual space. Then any closed almost convex subset of X is convex.*

PROOF. Looking for a contradiction, suppose $C \subseteq X$ is a closed almost convex set that is not convex. By Proposition 2.7, C is not midpoint convex. Therefore, there exist $x, y \in C$ such that $c := \frac{x+y}{2} \notin C$. As C is closed, $d(c, M) > 0$. Since

$$B\left[c; \frac{d(c, M)}{2}\right] \subseteq X \setminus C$$

and C is almost convex, there exist sequences $(z_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$, in X and $[0, \infty)$ respectively, such that

$$B\left[c; \frac{d(c, M)}{2}\right] \subseteq B[z_n; r_n] \subseteq B[z_{n+1}; r_{n+1}] \quad \text{and} \quad B[z_n; r_n] \subseteq X \setminus C$$

for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} r_n = \infty$. If $\overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]} = X$, Lemma 3.37 tells us that

$$\bigcup_{n\in\mathbb{N}} B(z_n; r_n) = \operatorname{int}\left(\overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]}\right) = \operatorname{int}(X) = X.$$

Thus, $x \in B(z_k; r_k)$ for some $k \in \mathbb{N}$, which is impossible since $x \in C$. Therefore, $\overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]}$ is not the whole space, and so by Lemma 3.35, is contained in a half space. Hence, by Lemma 3.34, $\overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]}$ is a closed half space. Since $c \in \operatorname{int}\left(B\left[c; \frac{d(c,M)}{2}\right]\right) \subseteq \operatorname{int}\left(\overline{\cup_{n\in\mathbb{N}} B[z_n; r_n]}\right)$, it follows by Lemma 3.36 and Lemma 3.37 that either x or y is in

$$\operatorname{int}\left(\overline{\bigcup_{n\in\mathbb{N}} B[z_n; r_n]}\right) = \bigcup_{n\in\mathbb{N}} B(z_n; r_n),$$

which is impossible since $x, y \in C$. Hence, C is convex. $\square$

THEOREM 3.40 ([89]). *In any Banach space with a strictly convex dual space, every Chebyshev set with a continuous metric projection mapping is convex.*

PROOF. Since K has a continuous metric projection mapping, it is almost convex, by Theorem 3.33. Since any Chebyshev set is necessarily closed and X^* is strictly convex, K is convex by Theorem 3.39. $\square$

The strict convexity of the dual is essential here. Consider Example 5. By Corollary 2.20, the metric projection mapping for K is continuous and it is straightforward to check that K is almost convex. However, K is clearly not convex.

COROLLARY 3.41. *For any $1 < p < \infty$, every Chebyshev set in L_p with a continuous metric projection mapping is convex.*

PROOF. Again we need only check that L_p has a strictly convex dual for $1 < p < \infty$. However, by [39,71] or more recently, [77, p. 284], the dual of L_p is L_q (where $q := \frac{p}{p-1}$), which is strictly convex, [23,40]. $\square$

COROLLARY 3.42 ([4]). *In any Hilbert space, every Chebyshev set with a continuous metric projection mapping is convex.*

PROOF. By the previous theorem, all we need to check is that a Hilbert space has a strictly convex dual. However, this follows since the dual of a Hilbert space is again a Hilbert space and so is strictly convex [23,40]. $\square$

Let us end this section with a brief historical review and some comments about future directions.

A little history: The problem of the convexity of Chebyshev sets in infinite dimensional spaces was considered by N. V. Efimov and S. B. Stečkin in a series of papers published between 1958 and 1961 in Doklady [32–35]. In these papers the term "Chebyshev set" was first coined.

However, earlier and independently, the "Chebyshev set problem" had been considered by V. Klee [50]. Moreover, Klee also considered the problem of the existence, in a Hilbert space, of a Chebyshev set with a convex complement (nowadays called a **Klee cavern**). Note however, that later in [4], Asplund showed that the existence of a Klee cavern is equivalent to the existence of a non-convex Chebyshev set. Going back further in time, we note that according to the paper [52], Frederick A. Ficken had earlier, in 1951, shown that in a Hilbert space, every compact Chebyshev set is convex, by using the method of inversion in the unit sphere. So this appears to be the first result concerning the "Chebyshev set problem" in infinite dimensional normed linear spaces. Not surprisingly though, convexity of sets in finite dimensional normed linear spaces that admit unique best approximations had been considered earlier. As far as the authors of this paper know, the first results concerning the convexity of sets that admit unique best approximations are due to Bunt [20] in 1934, Motzkin [60, 61] in 1935 and Kritikos [55] in 1938. All of these authors independently showed that in Euclidean space (of various dimensions) every Chebyshev set is convex. In 1940 Jessen [45], aware of Kritikos' proof, gave yet another proof in $\mathbb{R}^n$. Busemann in 1947 [21] noted that Jessen's proof could be extended to "straight-line spaces" and in 1955 [22] showed how this could be done. Since a finite dimensional normed linear space is a "straight-line space" if, and only if, it is strictly convex, Busemann's result implies that in a smooth, strictly convex finite dimensional normed linear space, every Chebyshev set is convex. Valentine [86] independently in 1964 gave essentially the same proof as Busemann.

In 1953, Klee [50] stated that in a finite dimensional normed linear space, every Chebyshev set is a "sun" and gave a characteristion of Chebyshev sets in smooth, strictly convex finite dimensional normed linear spaces. However, as he noted in 1961 [52], the argument in [50] was garbled and he proceeded to prove a stronger result, which in finite dimensions, shows that the requirement of strict convexity could be dropped. Thus, Klee was the first to show that in a smooth finite dimensional normed linear space every Chebyshev set is convex.

Future directions: As can be seen from this section of the paper, the main impediment to proving that every Chebyshev subset of a Hilbert space is convex, is removing the continuity assumptions on the metric projection mapping. So we point out here, that some progress has already been made in this direction, see [6, 7, 28, 91].

We should also point out that there are many other approaches to showing the convexity of Chebyshev sets in Hilbert spaces that were not pursued here. For example, by exploiting the theory of maximal monotone operators, one can obtain results that are very similar to those presented in this section. Indeed, it follows from [4], that if K is a proximinal subset of a Hilbert space H, then $P_K(x) \subseteq \partial f(x)$ for all $x \in H$, where $f(x) := \sup\{\langle x, y\rangle - \|y\|^2 : y \in K\}$ is a continuous convex function and $\partial f(x)$ is the set of all subgradients of f at x, see [65, p. 6]. Now, by [76], the mapping $x \mapsto \partial f(x)$ is a maximal monotone operator. On the other hand, by [75, Theorem 2], the norm closure of the range of a maximal monotone operator on a reflexive Banach space is convex, and so the norm closure of the range of the subdifferential mapping, $x \mapsto \partial f(x)$, is convex. Let us now recall that the subdifferential mapping, $x \mapsto \partial f(x)$, is a minimal weak cusco, see [65, Theorem 7.9]. Hence, if $x \mapsto P_K(x)$ is a weak cusco, then $\partial f(x) = P_K(x)$ for all $x \in H$ (here we are identifying H with its dual H^*). This then tell us that K, which is the range of P_K, is convex. So the only question that remains is: "when is

$x \mapsto P_K(x)$ a weak cusco ?". One answer to this is: when K is weakly closed and $P_K(x)$ is nonempty and convex for each $x \in H$.

4. A non-convex Chebyshev set in an inner product space

4.1. Background and motivation In this chapter we will construct a non-convex Chebyshev set in the inner product space consisting of all real sequences with only finitely many non-zero terms, equipped with the Euclidean inner product. This construction was first proposed by Johnson [47]. Whilst the motivation behind this construction is certainly geometric, Johnson's proof was predominantly algebraic and analytic and contained at least two errors, two of which were later corrected by Jiang [46]. The proof presented in this article is more geometric in nature, based upon a proof of Balaganskii and Vlasov [7]. We hope that the more detailed proof presented here assists the reader in understanding the numerous technical details of the construction, as well as, appreciating the general results that contribute to the construction.

Throughout the remainder of the chapter we will be working in Euclidean space, which we denote by E (or E_n if we wish to make the dimension explicit). That is, $\mathbb{R}^n$ for some n in $\mathbb{N}\cup\{0\}$, equipped with the usual Euclidean inner product. We will also make use of the following notation: for any $x \in E$ we will write x' for $(x, 0) \in E \times \mathbb{R}$.

The idea behind the construction is as follows. We start with $M_1 := E_1 \backslash (-2, 1)$, which is clearly not a Chebyshev set in E_1 since $P_{M_1}(\frac{-1}{2}) = \{-2, 1\}$. On the other hand, $P_{M_1}(0) = \{1\}$ is a singleton. We now construct a non-convex set $M_2 \subseteq E_2$ such that $M_1 \times \{0\} = M_2 \cap (E_1 \times \{0\})$, every point in $E_1 \times \{0\}$ has a unique nearest point in M_2, and $P_{M_1}(0) \times \{0\} = P_{M_2}(0)$. Since M_2 is non-convex, by Theorem 3.7, it is not a Chebyshev set in E_2.
This gives us an idea of how to proceed. Construct a non-convex set $M_3 \subseteq E_3$ so that $M_2 \times \{0\} = M_3 \cap (E_2 \times \{0\})$, every point in $E_2 \times \{0\}$ has a unique nearest point in M_3, and $P_{M_2}(x) \times \{0\} = P_{M_3}(x')$ for all $x \in E_1 \times \{0\}$. Repeating this process indefinitely (formally we will use induction) produces a sequence of non-convex sets $(M_n)_{n=1}^{\infty}$ such that $M_n \subseteq E_n$, $M_n \times \{0\} = M_{n+1} \cap (E_n \times \{0\})$, each point in $E_n \times \{0\}$ has a unique nearest point in M_{n+1}, and $P_{M_n}(x) \times \{0\} = P_{M_{n+1}}(x')$ for all $x \in E_{n-1} \times \{0\}$, for all $n \in \mathbb{N}$.

The problem is, how do we actually construct M_{n+1} given M_n, for some $n \in \mathbb{N}$, such that it has the aforementioned properties? Whilst much of the arguments are quite technical, the key idea is given in Theorem 2.32, which states that if K is a nonempty subset of a strictly convex normed linear space $(X, \|\cdot\|)$ and $x \in X$ is a point of Gâteaux differentiability of the distance function for K, then $P_K(x)$ has at most one element. Of course this begs the question, how do we prove that a given set has a differentiable distance function? To get around this problem, rather than starting with a set and trying to show that its distance function is differentiable, we will start with a differentiable function, and using a clever trick [7, p.1175–1180], construct a set that has this function as its distance function.

4.2. The construction

4.2.1. Preliminaries We now define various objects that will be used throughout the rest of the chapter.

DEFINITION 4.1. Let $M \subseteq E$ and let $b > 0$. We say ***M* has the *b*-boundary property** if, for any $y \in \mathrm{Bd}(M)$, there exists $x \in E$ such that $\|x - y\| = d(x, M) = b$.

DEFINITION 4.2. Given a Euclidean space E (with dimension n), we define the subspace $E^{(0)}$ by

$$E^{(0)} := \{(x_1, \ldots, x_n) \in E : x_n = 0\}.$$

DEFINITION 4.3. Let M be a nonempty, closed subset of E. Suppose that M has the b-boundary property for some $b > 0$, $E \setminus M$ is convex and bounded, $0 \notin M$, and every point in $E^{(0)}$ has a unique nearest point in M. We define the following objects:

$$C := \{(x, r) \in E \times \mathbb{R} : x \in \overline{E \setminus M}, |r| \leq d(x, M)\}, \quad A := \{(x, r) \in C : r \geq 0\},$$

$$R : C \to \mathbb{R}, \text{ by } R(w) := \frac{d(x, M) - r}{\sqrt{2}} \text{ for all } w := (x, r) \in C,$$

$$\xi := \frac{d\sqrt{8}}{b}, \text{ where } d := \mathrm{diam}(E \setminus M) = \sup\{\|x - y\| : x, y \in E \setminus M\},$$

$$Q := \{w \in C : d(w, E^{(0)} \times \mathbb{R}) \leq \xi R(w)\},$$

$$D := (E \times [0, \infty)) \bigcap \left(\bigcup_{w \in Q} B(w; R(w)) \right).$$

REMARK. It is clear that $b \leq \frac{d}{2}$.

Before establishing some further properties of these objects, we briefly describe the motivation behind their definitions. The set C is simply the volume contained between the graph of the distance function for M, restricted to $\overline{E \setminus M}$, and its reflection in $E \times 0$. The function R measures the distance from a point $w \in C$ to the graph of the distance function for M, restricted to $\overline{E \setminus M}$. Rewriting the definition of Q gives

$$Q = \left\{ (x, r) \in C : r \leq d(x, M) - \frac{\sqrt{2}}{\xi} d\left(x, E^{(0)}\right) \right\},$$

which shows that Q is just C with points removed in a linear manner as we move away from $E^{(0)}$. The inclusion of the $d(\cdot, E^{(0)})$ term in the definition of Q will later be used to show that our metric projections coincide on $E^{(0)}$. Now that we have a proper subset of C, we are able to construct the 'smooth' set D inside A (it looks like A without a sharp point at the top) by taking the union of balls of an appropriate radius (given by the function R). This 'smooth' set is then used to construct a Gâteaux differentiable function, $\rho : \overline{E \setminus M} \to \mathbb{R}$, defined by

$$\rho(x) := \max\{r : (x, r) \in \overline{D}\}$$

(we will prove later that this is well defined). From this we construct a set M'' in $E \times \mathbb{R}$ that has ρ as its distance function (for at least some of its points). It is essential that our smoothed set D lies inside A, since we require $M \times \{0\} = M'' \cap (E \times \{0\})$. The differentiability of this distance function then enables us to prove the uniqueness of nearest points.

PROPOSITION 4.4. *C and A are closed, convex, and bounded (and hence compact).*

PROOF. Since the distance function for M is continuous, it is straightforward to check that C and A are closed. To see the convexity of C, suppose $(x, r), (y, s) \in C$ and $0 \leq \lambda \leq 1$. Since $\overline{E \setminus M}$ is convex by assumption, we have $\lambda x + (1-\lambda)y \in \overline{E \setminus M}$. Furthermore,

$$\begin{aligned} |\lambda r + (1-\lambda)s| &\leq \lambda |r| + (1-\lambda)|s| \\ &\leq \lambda d(x, M) + (1-\lambda)d(y, M) \\ &\leq d(\lambda x + (1-\lambda)y, M), \end{aligned}$$

since the distance function for M restricted to $\overline{E \setminus M}$ is concave by Proposition 2.10. Thus, $\lambda(x, r) + (1-\lambda)(y, s) \in C$, and so C is convex. An almost identical proof shows that A is convex. Finally, to see that C (and hence $A \subseteq C$) is bounded, observe that since $\overline{E \setminus M}$ is bounded and the distance function for M is continuous, we have $r := \max\{d(x, M) : x \in \overline{E \setminus M}\} < \infty$, and so $\overline{E \setminus M} \subseteq B(y; s)$ for some $y \in E$, $s > 0$. Therefore, $C \subseteq B\left(y'; \sqrt{r^2 + s^2}\right)$, and so C is bounded. □

PROPOSITION 4.5. *The map $R : C \to \mathbb{R}$ is continuous and concave.*

PROOF. Suppose $w := (x, r), v := (y, s) \in C$ and $0 \leq \lambda \leq 1$. Since $x, y \in \overline{E \setminus M}$, Proposition 2.10 gives

$$\begin{aligned} \lambda R(w) + (1-\lambda)R(v) &= \lambda \left(\frac{d(x, M) - r}{\sqrt{2}} \right) + (1-\lambda) \left(\frac{d(y, M) - s}{\sqrt{2}} \right) \\ &= \frac{\lambda d(x, M) + (1-\lambda)d(y, M) - (\lambda r + (1-\lambda)s)}{\sqrt{2}} \\ &\leq \frac{d(\lambda x + (1-\lambda)y, M) - (\lambda r + (1-\lambda)s)}{\sqrt{2}} \\ &= R(\lambda w + (1-\lambda)v). \end{aligned}$$

Thus, R is concave. Continuity of R follows by the continuity of the distance function for M. □

PROPOSITION 4.6. *Q is closed and bounded (and hence compact).*

PROOF. Let $(w_n)_{n=1}^{\infty}$ be a sequence in Q converging to some $w \in E \times \mathbb{R}$. Since C is closed and $Q \subseteq C$, $w \in C$. By the continuity of both R and the distance function for $E^{(0)} \times \mathbb{R}$, we have

$$d(w, E^{(0)} \times \mathbb{R}) = \lim_{n \to \infty} d(w_n, E^{(0)} \times \mathbb{R}) \leq \lim_{n \to \infty} \xi R(w_n) = \xi R(w).$$

Thus, $w \in Q$, and so Q is closed. Moreover, since C is bounded, so is Q. □

PROPOSITION 4.7. *The closure of D is given by* $\overline{D} = (E \times [0,\infty)) \bigcap \left(\bigcup_{w \in Q} B[w; R(w)] \right)$.

PROOF. It is clear that set inclusion holds in one direction so let $x \in \overline{D}$ and $(x_n)_{n=1}^{\infty}$ be a sequence in D converging to x. Clearly, $x \in E \times [0,\infty)$. By the definition of D, there exists a sequence $(w_n)_{n=1}^{\infty}$ in Q such that $\|x_n - w_n\| < R(w_n)$ for all $n \in \mathbb{N}$. Since Q is compact, we may assume, without loss of generality, that $\lim_{n\to\infty} w_n = w$, for some $w \in Q$. As R is continuous, it follows that

$$\|x - w\| = \left\| \lim_{n\to\infty} x_n - \lim_{n\to\infty} w_n \right\| = \lim_{n\to\infty} \|x_n - w_n\| \le \lim_{n\to\infty} R(w_n) = R(w).$$

Hence, $x \in B[w; R(w)]$ and we're done. □

LEMMA 4.8. *For any $x \in E \setminus M$, we have $d(x', E^{(0)} \times \mathbb{R}) \le d$.*

PROOF. Let $x \in E \setminus M$. Since $0 \in E^{(0)} \times \mathbb{R}$ and $0 \in E \setminus M$, we have

$$d(x', E^{(0)} \times \mathbb{R}) \le \left\| x' - 0 \right\| = \|x - 0\| \le \operatorname{diam}(E \setminus M) =: d$$

□

PROPOSITION 4.9. $(E \setminus M) \times \{0\} \subseteq D$.

PROOF. Let $x' \in (E \setminus M) \times \{0\}$. Clearly, $x' \in E \times [0,\infty)$ and $x' \in C$. Firstly, suppose that $d(x, M) \ge \frac{b}{2}$, so that $R(x') = \frac{d(x,M)}{\sqrt{2}} \ge \frac{b}{\sqrt{8}} > 0$. Using Lemma 4.8, we have

$$d(x', E^{(0)} \times \mathbb{R}) \le d = \frac{d\sqrt{8}}{b} \cdot \frac{b}{\sqrt{8}} = \xi \frac{b}{\sqrt{8}} \le \xi R(x').$$

Thus, $x' \in Q$. Since $x' \in B(x'; R(x'))$, it follows that $x' \in D$.

Alternatively, suppose that $d(x, M) < \frac{b}{2}$. Let $y \in P_M(x)$. Since M has the b-boundary property, there exists $z \in E$ such that $d(z, M) = \|z - y\| = b$. Thus, $z \in P_M(y)$, and so by Lemma 2.29, x, y, z are collinear. Since $0 < \|x - y\| < \frac{b}{2} < b = \|y - z\|$, we have $x \in (y, z)$, and so $\|x - z\| < \|y - z\| = b$. Let $w := (z, -b)$. We will now show that $w \in Q$. Firstly, it is clear that $w \in C$. Then

$$R(w) = \frac{d(z, M) - (-b)}{\sqrt{2}} = b\sqrt{2},$$

and so $d(w, E^{(0)} \times \mathbb{R}) \le d \le \frac{d\sqrt{8}}{b} \cdot b\sqrt{2} = \xi R(w)$, as required. Finally, we have

$$\left\| x' - w \right\|^2 = \|(x, 0) - (z, -b)\|^2 = \|x - z\|^2 + b^2 < 2b^2 = R(w)^2.$$

Therefore, $x' \in B(w; R(w))$, and so $x' \in D$. □

PROPOSITION 4.10. $\overline{D} \subseteq A$.

PROOF. Let $z := (y,t) \in \overline{D} = (E \times [0,\infty)) \bigcap \left(\bigcup_{w \in Q} B[w; R(w)]\right)$. Therefore, $t \geq 0$ and there exists some $w := (x,r) \in Q \subseteq C$ such that $\|z - w\| \leq R(w)$. Suppose, looking for a contradiction, that $d(x,M) < t$. As $w \in C$, it follows that $r \leq |r| \leq d(x,M) < t$. Then since

$$\|z - w\|^2 = \|y - x\|^2 + (t - r)^2 \leq R(w)^2,$$

we have

$$t \leq r + R(w) = r + \frac{d(x,M) - r}{\sqrt{2}} \leq r + d(x,M) - r = d(x,M),$$

which is impossible. Thus, $d(x,M) \geq t$. Observe that

$$\begin{aligned}
&(d(x,M) - t)^2 - \left(\left(\frac{d(x,M) - r}{\sqrt{2}}\right)^2 - (t - r)^2\right) \\
&= (d(x,M) - r - (t - r))^2 - \left(\left(\frac{d(x,M) - r}{\sqrt{2}}\right)^2 - (t - r)^2\right) \\
&= \left(\frac{d(x,M) - r}{\sqrt{2}}\right)^2 + 2(t - r)^2 - 2\left(d(x,M) - r\right)(t - r) \\
&= \left(\frac{d(x,M) - r}{\sqrt{2}} - \sqrt{2}(t - r)\right)^2 \geq 0.
\end{aligned}$$

Rearranging and taking the square root of both sides gives

$$d(x,M) - t \geq \sqrt{\left(\frac{d(x,M) - r}{\sqrt{2}}\right)^2 - (t - r)^2}.$$

Using this and the non-expansivity of the distance function for M gives

$$\begin{aligned}
d(y,M) &\geq d(x,M) - \|y - x\| \\
&\geq d(x,M) - \sqrt{R(w)^2 - (r - t)^2} \\
&= d(x,M) - \sqrt{\left(\frac{d(x,M) - r}{\sqrt{2}}\right)^2 - (r - t)^2} \\
&\geq t \geq 0.
\end{aligned}$$

To complete the proof we need to show that $y \in \overline{E \setminus M}$. By the above working, if $t > 0$, then $d(y,M) > 0$, so we may as well suppose $t = 0$. Since

$$d(x,M)^2 - R(w)^2 + r^2 = \frac{d(x,M)^2}{2} + d(x,M)r + \frac{r^2}{2} = \left(\frac{d(x,M) + r}{\sqrt{2}}\right)^2 \geq 0,$$

we have that $[\|x - y\|^2 \leq R(w)^2 - r^2 \leq d(x,M)^2$. As $w \in C$, $x \in \overline{E \setminus M}$, and so we conclude that $y \in \overline{E \setminus M}$. Hence, $z := (y,t) \in A$, and so $\overline{D} \subseteq A$. $\square$

PROPOSITION 4.11. *Q and D are convex.*

PROOF. Let $\lambda \in [0,1]$ and suppose $w := (x,r), v := (y,s) \in Q$. Since $E^{(0)} \times \mathbb{R}$ is convex, by Proposition 2.9 and Proposition 4.5, we have

$$\begin{aligned} d(\lambda w + (1-\lambda)v, E^{(0)} \times \mathbb{R}) &\leq \lambda d(w, E^{(0)} \times \mathbb{R}) + (1-\lambda) d(v, E^{(0)} \times \mathbb{R}) \\ &\leq \lambda \xi R(w) + (1-\lambda)\xi R(v) \\ &\leq \xi R(\lambda w + (1-\lambda)v). \end{aligned}$$

Since C is convex, $\lambda w + (1-\lambda)v \in Q$, and so Q is convex. Now let $x, y \in D$. Therefore, $x, y \in E \times [0,\infty)$ and there exist $w, v \in Q$ such that $\|x - w\| < R(w)$ and $\|y - v\| < R(v)$. Making use of the concavity of R again, we see that

$$\begin{aligned} \|\lambda x + (1-\lambda)y - (\lambda w + (1-\lambda)v)\| &= \|\lambda(x-w) + (1-\lambda)(y-v)\| \\ &\leq \lambda \|x - w\| + (1-\lambda)\|y - v\| \\ &< \lambda R(w) + (1-\lambda) R(v) \\ &\leq R(\lambda w + (1-\lambda)v). \end{aligned}$$

Since Q and $E \times [0,\infty)$ are convex, we conclude that D is convex. ☐

4.2.2. Smoothness and unique nearest points In Chapter 2 we introduced the related concepts of Gâteaux differentiability and smoothness of normed linear spaces. We now define what it means for a subset of a normed linear space to be smooth.

DEFINITION 4.12. Let K be a subset of a normed linear space $(X, \|\cdot\|)$ and let $x \in K$. We say that K is **smooth at x** if there exists a unique $f \in S_{X^*}$ such that f supports K at x.

REMARK. It is well-known, see [83], [37, Lemma 8.4] or [41, p.124], that a normed linear space is smooth, in the sense that the norm is Gâteaux differentiable everywhere except at 0 if, and only if, the closed unit ball is smooth at every point of S_X.

We now present a number of results that will later be used to prove a smoothness condition for $\overline{D}$. First we introduce the concept of a cone in Euclidean space and derive some of its properties.

DEFINITION 4.13. Let $x \in E$ and $r \geq 0$. Define the cone of height and base radius r as

$$K[x;r] := \{(y,t) \in E \times [0,\infty) : \|y - x\| + t \leq r\} \subseteq E \times \mathbb{R}.$$

That is, $K[x;r]$ is the right-circular cone with vertex (x,r) and base $B[x;r] \times \{0\}$.

The following result gives a different way of describing a cone, and will be used to prove a smoothness condition for cones.

PROPOSITION 4.14. *For any $x \in E$ and $r > 0$,*

$$K[x;r] = E \times [0,\infty) \cap \bigcup_{\lambda \in [0,1]} B\Big[(x,(1-2\lambda)r);\lambda\sqrt{2}r\Big].$$

PROOF. See Appendix B. $\square$

COROLLARY 4.15. *Let $x \in E$ and $r > 0$. Then $K[x;r]$ is smooth at any $(y,t) \in E \times (0,r)$, such that $\|y-x\| + t = r$.*

PROOF. See Appendix B. $\square$

PROPOSITION 4.16. *Let $x \in E \backslash M$ and suppose $(x, d(x,M)) \in \overline{D}$. Then $K[x; d(x,M)] \subseteq \overline{D}$.*

PROOF. See Appendix B. $\square$

We now recall some results regarding the Gâteaux differentiability of the distance function.

THEOREM 4.17 ([7, Theorem 1.4]). *Let K be a nonempty closed subset of a Fréchet smooth normed linear space $(X, \|\cdot\|)$ and let $x_0 \in X \setminus K$. If $P_K(x_0)$ is nonempty and*

$$\lim_{t \to 0^+} \frac{d(x_0 + tv, K) - d(x_0, K)}{t} = 1$$

for some $v \in S_X$, then the distance function for K is Gâteaux differentiable at x_0.

PROOF. See Appendix C $\square$

COROLLARY 4.18. *Let K be a nonempty, proximinal subset of a smooth normed linear space $(X, \|\cdot\|)$ and $x \in X \setminus K$ be such that the metric projection mapping for K is continuous at x. Then the distance function for K is Gâteaux differentiable at x.*

PROOF. Let $v := \frac{x-z}{\|x-z\|} \in S_X$, where $\{z\} = P_K(x)$. By Corollary 3.29, setting $\lambda := \frac{t}{\|x-z\|}$ gives

$$\lim_{t \to 0^+} \frac{d(x+tv, K) - d(x,M)}{t} = \lim_{\lambda \to 0^+} \frac{d(x_\lambda, K) - d(x,K)}{\|x_\lambda - x\|} = 1,$$

where $x_\lambda = x + \lambda(x-z)$. Theorem 4.17 then shows that the distance function for K is Gâteaux differentiable at x. $\square$

Finally, we are ready to prove the smoothness condition on $\overline{D}$. This result will be required to prove the uniqueness of nearest points in our construction.

PROPOSITION 4.19. *$\overline{D}$ is smooth at any $(x, r) \in \mathrm{Bd}(D)$ such that $r > 0$.*

PROOF. Let $w := (x, r) \in \mathrm{Bd}(D)$ with $r > 0$. By Proposition 4.7, there exists a $w_0 := (x_0, r_0)$ in Q such that $\|w - w_0\| = R(w_0)$. If $R(w_0) > 0$, then any hyperplane supporting $\overline{D}$ at w will also support the closed ball $B[w_0; R(w_0)] \subseteq \overline{D}$ at w. Since $(E, \langle \cdot, \cdot \rangle)$ is an inner product space, it is smooth, and so this hyperplane must be unique. Alternatively, $R(w_0) = 0$, which implies $w = w_0 \in Q$ and $d(x, M) = r > 0$. Hence, $x \notin M$ and $w \in \mathrm{Bd}(A)$. By the definition of Q, $d(w, E^{(0)} \times \mathbb{R}) \leq \xi R(w) = 0$, and since $E^{(0)} \times \mathbb{R}$ is closed, we must have $w \in E^{(0)} \times \mathbb{R}$. Hence, $x \in E^{(0)}$, and so $P_M(x) = \{\hat{x}\}$ for some $\hat{x} \in M$. Theorem 2.19 tells us that the metric projection mapping for M is continuous at x, and so, by Corollary 4.18, the distance function for M is Gâteaux differentiable at x. Since

$$A = \left\{ (y, s) : y \in \overline{E \setminus M}, 0 \leq s \leq d(y, M) \right\},$$

it follows that $w \in \mathrm{Bd}(A)$ is a smooth point for A. Let H be the unique supporting hyperplane for A at w. Since $\overline{D} \subseteq A$, H also supports $\overline{D}$ at w. Suppose, looking for a contradiction, that $\overline{D}$ is not smooth at w. Hence, there exists a hyperplane $H_1 \neq H$, that also supports $\overline{D}$ at w. Next we implement a neat trick, which enables us to view the problem in just two dimensions. Consider the two dimensional plane Z in $E \times \mathbb{R}$ containing the points $x', \hat{x}'$, and w (it is clear that these points are not collinear so we do indeed have a two dimensional plane). Suppose, looking for a contradiction, that $H \cap Z = H_1 \cap Z$. Using Corollary 4.18 again, we see that H has slope 1 in the direction $x - \hat{x}$, and so $\frac{w + \hat{x}'}{2} \in H$. Since $\frac{w + \hat{x}'}{2} \in Z$ also, $\frac{w + \hat{x}'}{2} \in H_1$. As H_1 supports $\overline{D}$ at w and, by Proposition 4.16, $K[x; d(x, M)] \subseteq \overline{D}$, it follows that H_1 supports $K[x; d(x, M)]$ at $\frac{w + \hat{x}'}{2}$. Similarly, since H supports A at w and

$$K[x; d(x, M)] \subseteq \overline{D} \subseteq A,$$

H supports $K[x; d(x, M)]$ at $\frac{w + \hat{x}'}{2}$ as well. However, $\frac{w + \hat{x}'}{2} = \left(\frac{x + \hat{x}}{2}, \frac{d(x, M)}{2} \right)$, and since $0 < \frac{d(x, M)}{2} < r$, Corollary 4.15 implies that the cone $K[x; d(x, M)]$ is smooth at $\frac{w + \hat{x}'}{2}$. Thus, $H = H_1$, which is impossible. Therefore, $H \cap Z \neq H_1 \cap Z$.

Next consider the sequences $(x_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ defined by

$$x_n := x + \frac{1}{n}(x - \hat{x}) \quad \text{and} \quad r_n := \left(1 - \frac{1}{n}\right) d(x, M) \quad \text{for all } n \in \mathbb{N},$$

respectively. Define the sequence $(w_n)_{n=1}^{\infty}$ by $w_n := (x_n, r_n)$ for all $n \in \mathbb{N}$. Clearly, we have $\lim_{n \to \infty} w_n = w$. Let $0 < \alpha \leq \frac{\pi}{2}$ be the angle between $H \cap Z$ and $H_1 \cap Z$. For each $n \in \mathbb{N}$, let u_n be the nearest point on $H_1 \cap Z$ to w_n. Thus, $\langle w - u_n, w_n - u_n \rangle = 0$ for all $n \in \mathbb{N}$, and taking the limit of this expression shows $\lim_{n \to \infty} u_n = w$. For each $n \in \mathbb{N}$, define $w_n^{\dagger} := (x, r_n)$. Clearly, $\lim_{n \to \infty} w_n^{\dagger} = w$.

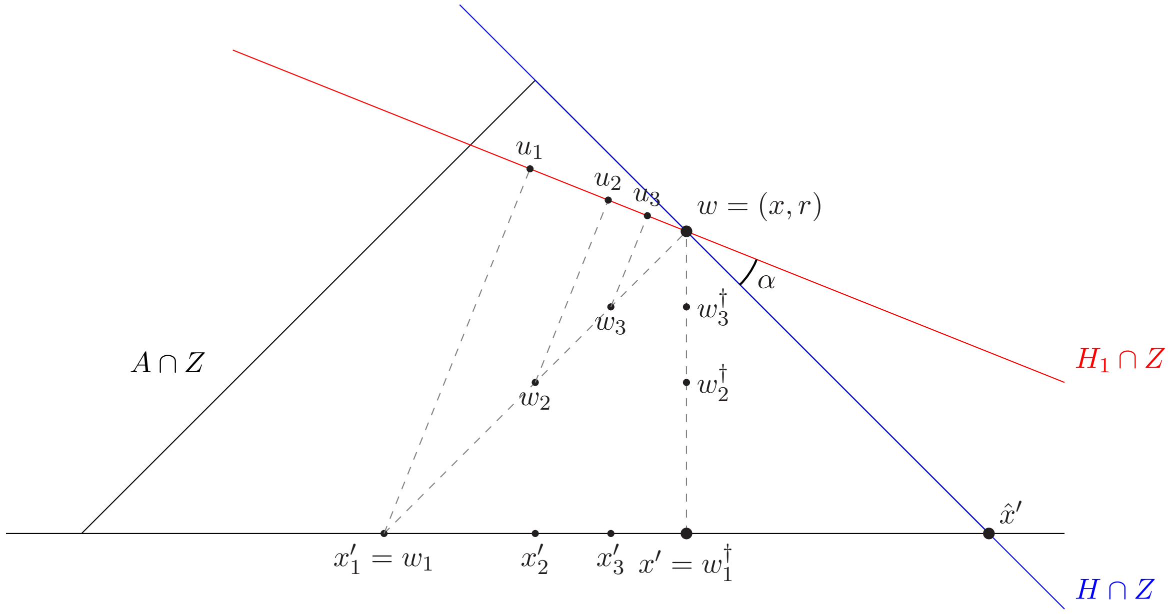


FIGURE 1. Diagram showing the first few terms of each sequence.

Therefore, for all $n \in \mathbb{N}$,

$$\angle w w_n^\dagger w_n = \frac{\pi}{2},$$

$$\left\| w_n^\dagger - w_n \right\| = \|x - x_n\| = d(x, M) - r_n = \|w - x'\| - r_n = \left\| w - w_n^\dagger \right\|,$$

$$\angle w_n^\dagger w w_n = \frac{\pi}{4},$$

$$\angle x' w \hat{x}' = \frac{\pi}{4},$$

$$\angle \hat{x}' w w_n = \frac{\pi}{2}.$$

Thus, since $0 < \alpha \leq \frac{\pi}{2}$, $1 > \cos(\alpha) = \dfrac{\|u_n - w_n\|}{\|w - w_n\|}$ for all $n \in \mathbb{N}$. Furthermore, $\sqrt{2}\,\|x - x_n\| = \|w - w_n\|$ and $r_n = d(x, M) - \|x - x_n\|$ for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$

$$R(w_n) = \frac{d(x_n, M) - r_n}{\sqrt{2}} = \frac{d(x_n, M) - d(x, M) + \|x - x_n\|}{\sqrt{2}}.$$

By Theorem 2.19 and Corollary 3.29, $\lim_{n\to\infty} \dfrac{d(x_n, M) - d(x, M)}{\|x_n - x\|} = 1$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \frac{R(w_n)}{\|w - w_n\|} &= \lim_{n\to\infty} \frac{d(x_n, M) - d(x, M) + \|x - x_n\|}{\sqrt{2}\,\|w - w_n\|} \\
&= \lim_{n\to\infty} \frac{d(x_n, M) - d(x, M) + \|x - x_n\|}{2\,\|x - x_n\|} \\
&= \lim_{n\to\infty} \frac{d(x_n, M) - d(x, M)}{2\,\|x - x_n\|} + \frac{1}{2} = 1.
\end{aligned}$$

As $x \in E^{(0)}$, $w_n^\dagger \in E^{(0)} \times \mathbb{R}$ for all $n \in \mathbb{N}$, and so

$$\frac{d(w_n, E^{(0)} \times \mathbb{R})}{R(w_n)} \leq \frac{\left\|w_n - w_n^\dagger\right\|}{R(w_n)} \leq \frac{\|w - w_n\|}{\sqrt{2}R(w_n)}$$

for all $n \in \mathbb{N}$. Since $b \leq \frac{d}{2}$, our earlier working gives

$$\lim_{n\to\infty} \frac{\|w - w_n\|}{\sqrt{2}R(w_n)} = \frac{1}{\sqrt{2}} < \frac{d\sqrt{8}}{b} = \xi.$$

Hence, there exists some $N \in \mathbb{N}$, such that for all $n > N$, $d(w_n, E^{(0)} \times \mathbb{R}) \leq \xi R(w_n)$. Since

$$d(x_n, M) \geq d(x, M) - \|x - x_n\| = \left(1 - \frac{1}{n}\right) d(x, M) = r_n = |r_n|,$$

$w_n \in C$ for all $n \in \mathbb{N}$, and so $w_n \in Q$ for all $n > N$. Hence, by Proposition 4.7, $B[w_n; R(w_n)]$ is contained in $\overline{D}$ for all $n > N$. Since u_n is the nearest point on $H_1 \cap Z$ to w_n for any $n \in \mathbb{N}$, and H_1 supports $\overline{D}$, we have

$$\frac{R(w_n)}{\|w - w_n\|} \leq \frac{\|u_n - w_n\|}{\|w - w_n\|} = \cos(\alpha) < 1$$

for all $n > N$, which contradicts $\lim_{n\to\infty} \frac{R(w_n)}{\|w - w_n\|} = 1$. Hence, $\overline{D}$ is smooth at w. $\square$

We now work toward the key theorem for this chapter.

LEMMA 4.20. *Suppose* $u := (u_1, \ldots u_{n+1}), v := (v_1, \ldots v_{n+1}) \in E_{n+1}, \|u\| = \|v\|$, *and* $(u_1, \ldots, u_n) = (v_1, \ldots, v_n)$. *If* $u_{n+1}, v_{n+1} \geq 0$ *or* $u_{n+1}, v_{n+1} \leq 0$, *then* $u = v$.

PROOF. We have $u_{n+1}^2 = \|u\|^2 - \sum_{k=1}^n u_k^2 = \|v\|^2 - \sum_{k=1}^n v_k^2 = v_{n+1}^2$. Since u_{n+1} and v_{n+1} have the same sign, it follows that $u_{n+1} = v_{n+1}$, and so $u = v$. $\square$

The following result is a slight modification of Theorem 2.32. We we will make use of it later to prove the uniqueness of nearest points.

PROPOSITION 4.21. *Let U be a nonempty open subset of E and let K' be a nonempty subset of $E \times [0, \infty)$. If $\phi : U \to \mathbb{R}$ is defined by $\phi(y) := d(y', K')$ and $x \in U$ is a point of Gâteaux differentiability of ϕ, then $P_{K'}(x')$ contains at most one element.*

PROOF. Clearly, if $P_{K'}(x')$ is empty or $\{x'\}$ we're done, so we might as well assume that $y \in P_{K'}(x')$ and $y \neq x'$. For any $z \in E$ and $\lambda > 0$ we have

$$\|x' + \lambda z' - y\| \geq d(x' + \lambda z', K').$$

Since U is open, for sufficiently small $\lambda > 0$, $x + \lambda z \in E \setminus K$. Thus, for sufficiently small $\lambda > 0$,

$$\frac{\|x' + \lambda z' - y\| - \|x' - y\|}{\lambda} \geq \frac{d(x' + \lambda z', K') - d(x', K')}{\lambda} = \frac{\phi(x + \lambda z) - \phi(x)}{\lambda}.$$

Taking the limit of both sides as $\lambda \to 0^+$ gives $g(z') \geq f(z)$, where $g \in (E \times \mathbb{R})^*$ is the Gâteaux derivative of the norm at $(x' - y) \neq 0$ (which exists since $E \times \mathbb{R}$ is smooth) and $f \in E^*$ is the Gâteaux derivative of ϕ at x. It is straightforward to show that $g(v) = \dfrac{x' - y}{\|x' - y\|} \cdot v$ for any v in $E \times \mathbb{R}$. Since all of the above inequalities still hold if we replace z by $-z$ and $z \in E$ was arbitrary, we have $g(z') = f(z)$ for all $z \in E$. Thus, if $w \in P_{K'}(x')$ is another best approximation to x' in K', we have $g(z') = f(z) = h(z')$ for all $z \in E$, where $h \in (E \times \mathbb{R})^*$ is the Gâteaux derivative of the norm at $x' - w \neq 0$. As before, $h(v) = \dfrac{x' - w}{\|x' - w\|} \cdot v$ for all $v \in E \times \mathbb{R}$. Therefore, $\left.\frac{x'-y}{\|x'-y\|}\right|_{E \times \{0\}} = \left.\frac{x'-w}{\|x'-w\|}\right|_{E \times \{0\}}$, i.e., these points agree apart from at possibly their last coordinate. Since $\left\|\frac{x'-y}{\|x'-y\|}\right\| = 1 = \left\|\frac{x'-w}{\|x'-w\|}\right\|$ and $x' - y, x' - w \in E \times [0, -\infty)$, it follows by Lemma 4.20 that $\dfrac{x' - y}{\|x' - y\|} = \dfrac{x' - w}{\|x' - w\|}$. As $\|x' - y\| = \|x' - w\| = d(x', K')$, we have $y = w$, which completes the proof.

$\square$

4.2.3. The inductive step Firstly, we define a function on $\overline{E \setminus M}$ using $\overline{D}$ and derive some of its properties.

DEFINITION 4.22. Define $\rho : \overline{E \setminus M} \to \mathbb{R}$ by $\rho(x) := \max\{r : (x, r) \in \overline{D}\}$ for all x in $\overline{E \setminus M}$.

REMARK. This function is well defined since $\overline{D}$ is closed and bounded and $(E \setminus M) \times \{0\} \subseteq D$. Furthermore, it is straightforward to show that for any nonnegative concave function $f : U \to \mathbb{R}$, where U is a convex subset of E, with nonempty interior, the **hypograph of f**,

$$\mathrm{hyp}(f) := \{(x, r) : x \in U, 0 \leq r \leq f(x)\},$$

is smooth at $(x, f(x))$ for some $x \in \mathrm{int}(U)$ if, and only if, $f|_{\mathrm{int}(U)}$ is Gâteaux differentiable at x.

PROPOSITION 4.23. *For all $x \in E \setminus M$, $0 < \rho(x) \leq d(x, M)$. Furthermore, ρ is concave, $\rho|_{\mathrm{Bd}(M)} = 0$, and $\rho|_{E\setminus M}$ is Gâteaux differentiable.*

PROOF. Let $x \in \overline{E \setminus M}$. By the definition of ρ and Proposition 4.10, $(x, \rho(x)) \in \overline{D} \subseteq A$. Hence, by the definition of A, $0 \leq \rho(x) \leq d(x, M)$. Thus, for any $x \in \mathrm{Bd}(M)$,

$$0 \leq \rho(x) \leq d(x, M) = 0,$$

and so $\rho|_{\mathrm{Bd}(M)} = 0$. To see the concavity of ρ, suppose $x, y \in \overline{E \setminus M}$ and $0 \leq \lambda \leq 1$. Since $(x, \rho(x)), (y, \rho(y)) \in \overline{D}$ and $\overline{D}$ is convex, we have

$$(\lambda x + (1-\lambda)y, \lambda\rho(x) + (1-\lambda)\rho(y)) = \lambda (x, \rho(x)) + (1-\lambda)(y, \rho(y)) \in \overline{D}.$$

Thus, $\lambda\rho(x) + (1-\lambda)\rho(y) \leq \rho(\lambda x + (1-\lambda)y)$. Suppose, looking for a contradiction, that $\rho(x) = 0$ for some $x \in E \setminus M$. Since ρ is concave and $\rho|_{\mathrm{Bd}(M)} = 0$, this implies that $\rho = 0$. We now show that $\rho(0) > 0$, which is of course a contradiction. As $0 \in E \setminus M$, we have $(0, d(0, M)) \in C$ and $d(0, M) > 0$. Furthermore, since $0 \in E^{(0)}$, $(0, d(0, M)) \in Q$. By Proposition 4.7, we have $(0, d(0, M)) \in \overline{D}$, and so $\rho(0) \geq d(0, M) > 0$. Thus, $\rho(x) > 0$ for all $x \in E \setminus M$.
Finally, we check that $\rho|_{E\setminus M}$ is Gâteaux differentiable. Let $x \in E \setminus M$. Since $\rho(x) > 0$, $(x, \rho(x)) \in \mathrm{Bd}(D)$, and $\overline{D} = \mathrm{hyp}(\rho)$, Proposition 4.19 and the previous remark tell us that ρ is Gâteaux differentiable at x. □

PROPOSITION 4.24. *Let $x \in \overline{E \setminus M}$.*

(i) *If $d(x, M) \leq \frac{b}{2}$, then $\rho(x) = d(x, M)$.*

(ii) *If $d(x, M) \geq \frac{b}{2}$, then $\rho(x) \geq \frac{b}{2}$.*

PROOF. (i) Let $x \in \overline{E \setminus M}$ be such that $d(x, M) \leq \frac{b}{2}$ and $y \in P_M(x)$. Since M has the b-boundary property, there exists some $z \in E \setminus M$ such that $\|y - z\| = d(z, M) = b$. Since $y \in P_M(z)$, Lemma 2.29 tells us that x, y, z are collinear. Let $w := \frac{y+z}{2}$. Since $w \in [y, z]$, $y \in P_M(w)$ and $d(w, M) = \|y - w\| = \frac{\|y-z\|}{2} = \frac{b}{2}$. We now show that $\rho(w) = \frac{b}{2}$. Since $\rho(w) \leq d(w, M)$, it is sufficient to show that $\left(w, \frac{b}{2}\right) \in \overline{D}$. Clearly, $z' \in C$ and $R(z') = \frac{d(z,M)}{\sqrt{2}} = \frac{b}{\sqrt{2}}$. As $E^{(0)} \cap (E \setminus M) \neq \emptyset$, we have that

$$\begin{aligned} d(z', E^{(0)} \times \mathbb{R}) &= d(z, E^{(0)}) \\ &\leq d(z, E^{(0)} \cap (E \setminus M)) \\ &\leq d \leq \frac{d\sqrt{8}}{b} \cdot \frac{b}{\sqrt{2}} = \xi R(z'), \end{aligned}$$

and so $z' \in Q$. Also, $\left\|\left(w, \frac{b}{2}\right) - z'\right\|^2 = \|w - z\|^2 + \left(\frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \frac{b^2}{2}$. Hence, $\left\|\left(w, \frac{b}{2}\right) - z'\right\| = \frac{b}{\sqrt{2}} = R(z')$, and so $\left(w, \frac{b}{2}\right) \in B[z; R(z')] \subseteq \overline{D}$ as required.

Finally, since $d(x, M) = \|x - y\| \leq \frac{b}{2} = \|y - w\|$ and x, y, z are collinear, there exists some $\lambda \in [0, 1]$ such that $x = \lambda y + (1 - \lambda)w$. As ρ is concave and $\rho(y) = 0$ (as $y \in \mathrm{Bd}(M)$), we have

$$\rho(x) \geq \lambda\rho(y) + (1 - \lambda)\rho(w) = (1 - \lambda)\rho(w) = (1 - \lambda)\frac{b}{2} = \|y - x\| = d(x, M).$$

Since $\rho(x) \leq d(x, M)$ by Proposition 4.23, $\rho(x) = d(x, M)$ as required.

(ii) Let $x \in \overline{E \setminus M}$ be such that $d(x, M) \geq \frac{b}{2}$. Take any straight line in E passing through x. Since $E \setminus M$ is bounded and convex, this line must intercept $\mathrm{Bd}(M)$ at two distinct points y_1 and y_2. As $d(y_1, M) = d(y_2, M) = 0$ and the distance function for M is continuous, the Intermediate Value Theorem tells us that there exist $z_1 \in [x, y_1)$ and $z_2 \in [x, y_2)$ such that $d(z_1, M) = d(z_2, M) = \frac{b}{2}$. By (i), $\rho(z_1) = \rho(z_2) = \frac{b}{2}$. Since ρ is concave and $x \in [z_1, z_2]$, it follows that $\rho(x) \geq \frac{b}{2}$.

$\square$

We now use a clever trick to construct a set in $E \times [0, \infty)$ such that the distance function for this set, restricted to $E \setminus M$, is given by the smooth function ρ. Making use of Proposition 4.21, we will be able to prove the uniqueness of nearest points.

DEFINITION 4.25. Define the set $M' := (E \times [0, \infty)) \setminus \bigcup_{x \in E \setminus M} B(x'; \rho(x))$.

PROPOSITION 4.26. *M' has the following properties.*

(i) *$M \times \{0\} = M' \cap (E \times \{0\})$, M' is closed in $E \times \mathbb{R}$, and $(E \times [0, \infty)) \setminus M'$ is nonempty, convex and bounded.*

(ii) *For every $x \in E$ there exists a unique $y \in M'$ such that $\|x' - y\| = d(x', M')$. Furthermore, for $x \in E \setminus M$, $d(x', M') = \rho(x)$.*

(iii) $0 \notin M'$.

(iv) *For every $y \in \mathrm{Bd}(M') \setminus (\mathrm{int}(M) \times \{0\})$ there exists $x \in E \setminus M$ such that $P_{M'}(x') = \{y\}$.*

PROOF. (i) Firstly, we show that $M \times \{0\} \subseteq M'$. Suppose, looking for a contradiction, that there exists $x \in M$ such that $x' \notin M'$. Since $x' \in E \times [0, \infty)$, it follows that

$$x' \in \bigcup_{y \in E \setminus M} B\big(y'; \rho(y)\big).$$

Hence, for some $y \in E \setminus M$ we have $\|x - y\| = \|x' - y'\| < \rho(y) \leq d(y, M)$, which is impossible since $x \in M$. Therefore, $M \times \{0\} \subseteq M'$. Secondly, we show that

$$(E \setminus M) \times \{0\} \subseteq (E \times [0, \infty)) \setminus M'.$$

It is straightforward to show that $(E \times [0,\infty)) \setminus M' = (E \times [0,\infty)) \cap \bigcup_{x \in E \setminus M} B\big(x'; \rho(x)\big)$. Let $x' \in (E \setminus M) \times \{0\}$. Clearly, $x' \in E \times [0,\infty)$. By Proposition 4.23, $\rho(x) > 0$, and so

$$x' \in B\big(x'; \rho(x)\big) \subseteq \bigcup_{z \in E \setminus M} B\big(z'; \rho(z)\big).$$

Thus, $x' \in E \times [0,\infty) \setminus M'$. Combining these two set inclusions gives the equation $M \times \{0\} = M' \cap (E \times \{0\})$ as required. Furthermore, since $0 \in E \setminus M$, $(E \times [0,\infty)) \setminus M'$ is nonempty.

To see that M' is closed in $E \times \mathbb{R}$, observe that $(E \times \mathbb{R}) \setminus M'$ is a union of two open sets, i.e., $(E \times \mathbb{R}) \setminus M' = (E \times (-\infty, 0)) \cup \bigcup_{x \in E \setminus M} B(x'; \rho(x))$.

The convexity of $(E \times [0,\infty)) \setminus M' = (E \times [0,\infty)) \cap \bigcup_{x \in E \setminus M} B(x'; \rho(x))$ follows from the concavity of ρ. Let $x, y \in (E \times [0,\infty)) \setminus M'$ and $0 \leq \lambda \leq 1$. Clearly, the convex combination, $\lambda x + (1-\lambda) y \in E \times [0,\infty)$. Moreover, there exist $w, z \in E \setminus M$ such that $\|x - w'\| < \rho(w)$ and $\|y - z'\| < \rho(z)$. Then

$$\begin{aligned}
\big\|\lambda x + (1-\lambda) y - (\lambda w + (1-\lambda) z)'\big\| &= \big\|\lambda(x - w') + (1-\lambda)(y - z')\big\| \\
&\leq \lambda \big\|x - w'\big\| + (1-\lambda) \big\|y - z'\big\| \\
&< \lambda \rho(w) + (1-\lambda) \rho(z) \\
&\leq \rho(\lambda w + (1-\lambda) z).
\end{aligned}$$

Therefore, $\lambda x + (1-\lambda) y \in B((\lambda w + (1-\lambda) z)'; \rho(\lambda w + (1-\lambda) z))$. Since $E \setminus M$ is convex, it follows that $\lambda w + (1-\lambda) z \in E \setminus M$, and so $\lambda x + (1-\lambda) y \in (E \times [0,\infty)) \setminus M'$ as required.

Finally, since $E \setminus M$ is bounded and $\rho(x) \leq d(x, M)$ for all $x \in \overline{E \setminus M}$, it follows that ρ is bounded, and so $(E \times [0,\infty)) \setminus M'$ is bounded.

(ii) Let $x \in E$. If $x \in M$, then $x' \in M'$, and so x' has itself as its unique nearest point in M'. Alternatively, $x \in E \setminus M$. We will now find a point $y \in M'$ such that $\|x' - y\| = \rho(x) = d(x', M')$. Observe that $d(x', M') \geq \rho(x)$ by the definition of M', so it is sufficient to find $y \in M'$ such that $\|x' - y\| = \rho(x)$. Let H be a hyperplane in $E \times \mathbb{R}$, supporting the convex set $\overline{D}$ at $(x, \rho(x)) \in \mathrm{Bd}(\overline{D})$. Firstly, suppose H is parallel to $E \times \{0\}$. Let $y := (x, \rho(x))$. Since ρ is concave and smooth on $E \setminus M$, ρ must have a global maximum at x. Clearly, $\|x' - y\| = \rho(x)$, so all we need show is that $y \in M'$. Suppose, looking for a contradiction, that $y \notin M'$. Since $\rho(x) > 0$, $y \in E \times [0,\infty)$, and so $y \in B(w'; \rho(w))$ for some $w \in E \setminus M$. Then

$$\begin{aligned}
\big\|y - w'\big\|^2 &= \|(x, \rho(x)) - (w, 0)\|^2 \\
&= \|w - x\|^2 + \rho(x)^2 \\
&\geq \rho(x)^2,
\end{aligned}$$

and so $\rho(x) \leq \|y - w'\| < \rho(w)$, which contradicts the maximality of $\rho(x)$. Hence, $y \in M'$, and so $y \in P_{M'}(x')$.

Alternatively, H and $E \times \{0\}$ are not parallel. The intersection of these two hyperplanes is a translate of a codimension 2 subspace of $E \times \mathbb{R}$, which we call N. Suppose, looking for a contradiction, that there exists some $y' \in N \cap ((E \setminus M) \times \{0\})$. Since $y' \in (E \setminus M) \times \{0\} \subseteq D$ and H supports $\overline{D}$, we have that $\rho(y) = 0$, which is impossible by Proposition 4.23. Thus, $N \cap ((E \setminus M) \times \{0\}) = \emptyset$. As $x \in E \setminus M$, this implies that H is not perpendicular to $E \times \{0\}$. Let H_1 be a hyperplane in $E \times \mathbb{R}$ passing through N and tangent to $B[x'; \rho(x)]$ such that the point of contact y is in $E \times [0, \infty)$. This is possible since $B[x'; \rho(x)] \cap (E \times \{0\}) \subseteq \overline{E \setminus M} \times \{0\}$. Clearly, $\|x' - y\| = \rho(x) > 0$, so again it is sufficient to show that $y \in M'$. Suppose H and $E \times \{0\}$ form an angle $0 < \alpha \leq \frac{\pi}{4}$, whilst H_1 and $E \times \{0\}$ form an angle α_1. Since H_1 and the vector $y - x'$ must be orthogonal, it follows that $\tan(\alpha) = \sin(\alpha_1)$. Therefore, the distance from any point $(w, r) \in E \times \mathbb{R}$ on H to $E \times \{0\}$, which is just r, is equal to the distance from w' to H_1. Looking for a contradiction, suppose that $y \notin M'$. By construction, $y \in E \times [0, \infty)$, so there exists $z \in E \setminus M$ such that $\|y - z'\| < \rho(z)$. Let v be the point directly above z' lying on H. Since H supports $\overline{D}$, $(z, \rho(z)) \in \overline{D}$ and $y \in H_1$, it follows by the previous observation that $\rho(z) \leq d\left(v, E \times \{0\}\right) = d(z', H_1) \leq \|y - z'\| < \rho(z)$, which is clearly impossible. Therefore, $y \in M'$ as required.

We have now established that $d(z', M') = \rho(z)$ for all $z \in E \setminus M$. As $\rho|_{E \setminus M}$ is smooth, Proposition 4.21 tells us that $P_{M'}(x)$ is a singleton.

(iii) By assumption, $0 \in E \setminus M$. By (ii) and Proposition 4.23, we have $d(0, M') = \rho(0) > 0$. Hence, $0 \notin M'$.

(iv) We begin by showing that

$$\overline{\bigcup_{x \in E \setminus M} B(x'; \rho(x))} = \bigcup_{x \in E \setminus M} B\left[x'; \rho(x)\right].$$

Let $y \in \overline{\bigcup_{x \in E \setminus M} B(x'; \rho(x))}$. Hence, there exist sequences $(y_n)_{n=1}^{\infty}$ and $(x_n)_{n=1}^{\infty}$ in $E \times \mathbb{R}$ and $E \setminus M$ respectively, such that $\lim_{n \to \infty} y_n = y$ and $\|y_n - x_n'\| < \rho(x_n)$ for all $n \in \mathbb{N}$. Since $\overline{E \setminus M}$ is compact, we may assume, without loss of generality, that $(x_n)_{n=1}^{\infty}$ converges to some $x \in \overline{E \setminus M}$. Firstly, suppose $x \in E \setminus M$. Then

$$\left\|y - x'\right\| = \left\|\lim_{n \to \infty} y_n - \lim_{n \to \infty} x_n'\right\| = \lim_{n \to \infty} \left\|y_n - x_n'\right\| \leq \lim_{n \to \infty} \rho(x_n) = \rho(x),$$

since ρ is smooth (and hence continuous) on $E \setminus M$. Thus, $y \in B[x'; \rho(x)]$. Alternatively, $x \in \mathrm{Bd}(M)$. Therefore,

$$\left\|y - x'\right\| \leq \lim_{n \to \infty} \rho(x_n) \leq \lim_{n \to \infty} d(x_n, M) = d(x, M) = 0,$$

and so $y = x' \in \mathrm{Bd}(M) \times \{0\}$. Since M has the b-boundary property, there exists $z \in \overline{E \setminus M}$ such that $\|y - z\| = d(z, M) = b$. Consider the point $w := \frac{y+z}{2}$. Since $y, z \in \overline{E \setminus M}$, which is convex, we have $w \in \overline{E \setminus M}$. As $w \in [y, z]$,

$$d(w, M) = \|y - w\| = \left\|\frac{y - z}{2}\right\| = \frac{b}{2} > 0$$

by Proposition 2.11, and so $w \in E \setminus M$. By Proposition 4.24, $\rho(w) = d(w, M) = \frac{b}{2} > 0$. Therefore, $y \in B[w; d(w, M)] \times \{0\} \subseteq B[w'; \rho(w)]$. Thus, $\overline{\bigcup_{x \in E \setminus M} B(x'; \rho(x))}$ is contained in $\bigcup_{x \in E \setminus M} B[x'; \rho(x)]$. Set inclusion obviously holds in the reverse direction, which establishes the required equality.

Now suppose $w \in \mathrm{Bd}(M') \setminus (\mathrm{int}(M) \times \{0\})$. By what we've just shown, there exists some $x \in E \setminus M$ such that $w \in B[x'; \rho(x)]$. Thus, $\|x' - w\| = \rho(x)$, since otherwise $w \notin M'$. By (ii), there exists a unique point in M' with distance $\rho(x) = d(x', M')$ from x'. Hence, $\{w\} = P_{M'}(x')$ as required.

□

PROPOSITION 4.27. *For any $x \in \mathrm{Bd}(M') \setminus (\mathrm{int}(M) \times \{0\})$ there exists $y \in E \times [0, \infty)$ such that*

$$\|x - y\| = d(y, M') = \frac{b}{2}.$$

PROOF. Let $x := (v, r) \in \mathrm{Bd}(M') \setminus (\mathrm{int}(M) \times \{0\})$. By Proposition 4.26, part (iv) there exists $w \in E \setminus M$ such that $P_{M'}(w') = \{x\}$, and by part (ii) of the same proposition, $\|w' - x\| = d(w', M') = \rho(w)$. Firstly, suppose $d(w', M') \geq \frac{b}{2}$. Then there exists $y \in [w', x]$ such that $\|y - x\| = d(y, M') = \frac{b}{2}$. Clearly, $y \in E \times [0, \infty)$, so we're done. Alternatively, suppose $d(w', M') = \rho(w) < \frac{b}{2}$. By Proposition 4.24 part (ii), $d(w, M) < \frac{b}{2}$, and hence, by part (i) of the same result, $d(w, M) = \rho(w) = d(w', M')$. Since $M \times \{0\} \subseteq M'$, M is proximinal, and $P_{M'}(w') = \{x\}$ we must have that $x := (v, 0) \in M \times \{0\}$. Clearly, $v \in \mathrm{Bd}(M)$, and so there exists $z \in E \setminus M$ such that $\|v - z\| = d(z, M) = b$, since M has the b-boundary property. Let $y := \left(\frac{v+z}{2}\right)' \in (E \setminus M) \times \{0\} \subseteq E \times [0, \infty)$. By Proposition 4.24 and Proposition 4.26 part(ii) $\frac{b}{2} = d\left(\frac{v+z}{2}, M\right) = \rho\left(\frac{v+z}{2}\right) = d\left(y', M'\right)$ since $d\left(\frac{v+z}{2}, M\right) = \left\|\frac{v+z}{2} - v\right\| = \frac{b}{2}$. □

Whilst we have shown that M' has convex complement in $E \times [0, \infty)$, its complement in $E \times \mathbb{R}$ is not convex. We use the following flip-stretch operator to extend M' to a set with convex complement in $E \times \mathbb{R}$.

DEFINITION 4.28. For $\theta > 1$, define $\psi_\theta : E \times \mathbb{R} \to E \times \mathbb{R}$ by $\psi_\theta(x, r) = (x, -\theta r)$ for all $(x, r) \in E \times \mathbb{R}$.

LEMMA 4.29. *The map ψ_θ is a bijective linear map, with inverse given by*

$$\psi_\theta^{-1}(x, r) = \left(x, \frac{-r}{\theta}\right) \quad \textit{for all } (x, r) \in E \times \mathbb{R}.$$

We also have the following inequalities $\|(x, r)\| \leq \|\psi_\theta(x, r)\| \leq \theta \|(x, r)\|$ for any (x, r) in $E \times \mathbb{R}$, with the first inequality being strict unless $r = 0$ and the second strict unless $x = 0$. Thus, both ψ_θ and its inverse are bounded and hence continuous.

PROOF. Linearity and bijectiveness are trivial. For the inequality, observe that for any (x, r) in $E \times \mathbb{R}$,

$$\|(x, r)\|^2 = \|x\|^2 + r^2 \leq \|x\|^2 + (\theta r)^2 = \|\psi_\theta(x, r)\|^2 \leq (\theta \|x\|)^2 + (\theta r)^2 = \theta^2 \|(x, r)\|^2.$$

Replacing (x,r) by $\psi_\theta^{-1}(x,r)$ in these inequalities shows that

$$\frac{1}{\theta}\|(x,r)\| \leq \left\|\psi_\theta^{-1}(x,r)\right\| \leq \|(x,r)\|$$

for all $(x,r) \in E \times \mathbb{R}$. The first inequality is strict unless $x = 0$ and the second strict unless $r = 0$. Hence, ψ_θ and ψ_θ^{-1} are bounded, and so continuous. $\square$

DEFINITION 4.30. Let $1 < \theta \leq 2$ and define $M'' \subseteq E \times \mathbb{R}$ by $M'' := M' \cup \psi_\theta(M')$.

We would like to show that the properties of M' established in Proposition 4.26 are inherited by this extension. The next result is used to show that M'' has the $\frac{b}{4}$-boundary property.

LEMMA 4.31. *Let S be the ellipsoidal surface in $\mathbb{R}^n$ defined by the equation*

$$\frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\beta^2} + \cdots + \frac{x_n^2}{\beta^2} = 1,$$

where $\alpha > \beta > 0$. For any $y \in S$ there exists $w \in \mathrm{co}(S)$ such that $\|w - y\| = d(y,S) = \frac{\beta^2}{\alpha}$.

PROOF. See Appendix E. $\square$

PROPOSITION 4.32. *M'' has the following properties.*

(i) *$M \times \{0\} = M'' \cap (E \times \{0\})$, M'' is closed in $E \times \mathbb{R}$, $0 \notin M''$, and $(E \times \mathbb{R}) \setminus M''$ is nonempty, convex and bounded.*

(ii) *For every $x \in E$ there exists a unique $y \in M''$ such that*

$$\left\|x' - y\right\| = d(x', M'').$$

Furthermore, when $x \in E \setminus M$, $d(x', M'') = \rho(x)$.

(iii) *M'' has the $\frac{b}{4}$-boundary property.*

PROOF. (i) These statements follow from Proposition 4.26. Since $\psi_\theta(E \times \{0\}) = E \times \{0\}$, we have

$$M \times \{0\} = M' \cap (E \times \{0\}) = M'' \cap (E \times \{0\}).$$

Also, $\emptyset \neq (E \times [0,\infty)) \setminus M' \subseteq (E \times \mathbb{R}) \setminus M'$. It is straightforward to verify that

$$(E \times \mathbb{R}) \setminus M'' = (E \times [0,\infty)) \setminus M' \cup \psi_\theta\left((E \times [0,\infty)) \setminus M'\right).$$

Since ψ_θ^{-1} is continuous and M' is closed in $E \times \mathbb{R}$, the above set is the finite union of closed sets in $E \times \mathbb{R}$, and so is closed in $E \times \mathbb{R}$.

As $0 \notin M'$ and $\psi_\theta(0) = 0$, it follows that $0 \notin M''$.

Since $(E \times [0,\infty)) \setminus M'$ is convex and ψ_θ is linear, $\psi_\theta(E \times [0,\infty) \setminus M')$ is also convex. Thus, to prove that $(E \times \mathbb{R}) \setminus M''$ is convex we need only show that it contains $[x,y]$

where $x := (u, r) \in (E \times [0, \infty)) \setminus M'$ and $y := (v, s) \in \psi_\theta ((E \times [0, \infty)) \setminus M')$. Since $x \notin M'$,

$$\left\|u' - z\right\| \leq \|x - z\| < \rho(z) \leq d(z, M)$$

for some $z \in E \setminus M$. Therefore, $u' \in E \setminus M$. Similarly, $v' \in E \setminus M$. Since $r \geq 0$ and $s \leq 0$, whilst $E \setminus M$ is convex, there exists $\lambda \in [0, 1]$ such that $\lambda x + (1-\lambda) y \in E \setminus M \times \{0\}$. As

$$(E \setminus M) \times \{0\} \subseteq \left((E \times [0, \infty)) \setminus M'\right) \cap \psi_\theta \left((E \times [0, \infty)) \setminus M'\right)$$

and the two sets in this intersection are convex, it follows that $[x, y] \subseteq (E \times \mathbb{R}) \setminus M''$ as required.

Finally, since ψ_θ is bounded, it follows that $(E \times \mathbb{R}) \setminus M''$ is the finite union of bounded sets, so is bounded.

(ii) Let $x \in E$. If $x \in M$, then $x' \in M''$ and the claim holds trivially, so we may as well assume $x \in E \setminus M$. By part (ii) of Proposition 4.26, there exists a unique $y \in M'$ such that $\|x' - y\| = d(x', M') = \rho(x)$. Looking for a contradiction, suppose there exists $z \in M'' \setminus M'$, such that $\|x' - z\| \leq d(x', M')$. Since $x' - z \notin E \times \{0\}$, Lemma 4.29 gives

$$\begin{aligned}
\left\|x' - \psi_\theta^{-1}(z)\right\| &= \left\|\psi_\theta^{-1}(x') - \psi_\theta^{-1}(z)\right\| \quad \text{since } x' \in E \times \{0\} \\
&= \left\|\psi_\theta^{-1}(x' - z)\right\| \\
&< \left\|x' - z\right\| \\
&\leq d(x, M'),
\end{aligned}$$

which is impossible since $\psi_\theta^{-1}(z) \in M'$. Hence, y is the unique element of M'' such that

$$\left\|x' - y\right\| = d(x', M'') = \rho(x).$$

(iii) Let $x \in \mathrm{Bd}(M'')$. It is straightforward to verify that

$$\mathrm{Bd}(M'') = \mathrm{Bd}(M') \setminus (\mathrm{int}(M) \times \{0\}) \cup \psi_\theta \left(\mathrm{Bd}(M') \setminus (\mathrm{int}(M) \times \{0\}\right).$$

If $x \in \mathrm{Bd}(M') \setminus (\mathrm{int}(M) \times \{0\})$, by Proposition 4.27, there exists $y \in E \times [0, \infty)$ such that

$$\|x - y\| = d(y, M') = \frac{b}{2}.$$

Using the same argument as in part (ii), we conclude that $\|x - y\| = d(y, M'') = \frac{b}{2}$. Therefore,

$$\left\|x - \frac{x + y}{2}\right\| = d\left(\frac{x + y}{2}, M''\right) = \frac{b}{4}.$$

Alternatively, $x \in \psi_\theta \left(\mathrm{Bd}(M') \setminus (\mathrm{int}(M) \times \{0\})\right)$. As before, we can find $y \in E \times [0, \infty)$ such that $\left\|\psi_\theta^{-1}(x) - y\right\| = d(y, M'') = \frac{b}{2}$. It follows that $\psi_\theta \left(B\left[y; \frac{b}{2}\right]\right) \subseteq \overline{(E \times \mathbb{R}) \setminus M''}$

is an ellipsoid centred at $\psi_\theta(y)$ with minor axes of length $\frac{b}{2}$ and a major axis of length $\frac{b\theta}{2}$ such that

$$x \in \mathrm{Bd}\left(\psi_\theta\left(B\left[y; \frac{b}{2}\right]\right)\right).$$

By Lemma 4.31, there exists $w \in \psi_\theta\left(B\left[y; \frac{b}{2}\right]\right)$ such that $B\left[w; \frac{b}{2\theta}\right] \subseteq \psi_\theta\left(B\left[y; \frac{b}{2}\right]\right)$ and $x \in B\left[w; \frac{b}{2\theta}\right]$. Hence,

$$\|w - x\| = d(w, M'') = \frac{b}{2\theta}.$$

Since $\theta \le 2$, we have that $\frac{b}{4} \le \frac{b}{2\theta}$. Therefore, there exists a $z \in [w, x] \subseteq E \times \mathbb{R}$ such that $\|z - x\| = d(z, M'') = \frac{b}{4}$.

□

The next result explains why we chose to define Q as in Definition 4.3.

PROPOSITION 4.33. *For any $x \in E^{(0)}$, $P_M(x) \times \{0\} = P_{M''}(x')$.*

PROOF. Let $x \in E^{(0)}$. If $x \in M$, then there is nothing to prove, so assume $x \notin M$. By assumption, $P_M(x)$ is a singleton. Proposition 4.32 tells us that $P_{M''}(x')$ is a singleton and $d(x', M'') = \rho(x)$.
We now show that $d(x, M) = \rho(x)$. Since $\rho(x) \le d(x, M)$, it is sufficient to check that $d(x, M) \le \rho(x)$, which is true provided $w := (x, d(x, M)) \in \overline{D}$. Clearly, $w \in C$, with $R(w) = 0$. Since $x \in E^{(0)}$, we have $w \in E^{(0)} \times \mathbb{R}$, and so $d(w, E^{(0)} \times \mathbb{R}) = 0 = \xi R(w)$. Thus, $w \in Q$, and so $w \in \overline{D}$ by Proposition 4.7. Finally, since $M \times \{0\} \subseteq M''$, it follows that $P_M(x) \times \{0\} = P_{M''}(x')$. □

The following two results will be used to prove that our non-convex Chebyshev set has bounded complement.

PROPOSITION 4.34.

$$\sup\{\|x\| : x \in (E \times \mathbb{R}) \setminus M''\} \le 3\theta \sup\{\|x\| : x \in E \setminus M\}.$$

PROOF. To begin with let

$$x \in (E \times [0, \infty)) \setminus M' = (E \times [0, \infty)) \cap \bigcup_{z \in E \setminus M} B\big(z'; \rho(z)\big).$$

Hence, there exists $y \in E \setminus M$ such that $\|y' - x\| < \rho(y)$. Since the distance function for M is non-expansive,

$$\begin{aligned} \|x\| &< \|y'\| + \rho(y) \\ &\le \|y\| + d(y, M) \\ &\le 2\|y\| + d(0, M) \\ &\le 3 \sup\{\|z\| : z \in E \setminus M\}. \end{aligned}$$

Thus,

$$\sup\{\|x\| : x \in (E \times [0, \infty)) \setminus M'\} \leq 3 \sup\{\|x\| : x \in E \setminus M\}.$$

Since

$$(E \times \mathbb{R}) \setminus M'' = \big((E \times [0, \infty)) \setminus M'\big) \cup \psi_\theta \big((E \times [0, \infty)) \setminus M'\big)$$

and $\theta > 1$, the result follows. □

THEOREM 4.35. *Let $(a_n)_{n=1}^{\infty}$ be a sequence with non-negative terms. Then the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ converges if, and only if, the infinite series $\sum_{n=1}^{\infty} a_n$ converges.*

PROOF. For a proof, see [69]. □

4.2.4. The non-convex Chebyshev set

DEFINITION 4.36. For any $K \subseteq E_n$, let $K \otimes 0 := \{(x_1, \ldots, x_n, 0, 0, \ldots) : (x_1, \ldots, x_n) \in K\}$. Similarly, for any $x := (x_1, \ldots, x_n) \in E_n$, define $x \otimes 0 := (x_1, \ldots, x_n, 0, 0, \ldots) \in E_n \otimes 0$.

DEFINITION 4.37. Let E be the set of all real sequences with only finitely many non-zero terms, that is

$$E := \bigcup_{n=1}^{\infty} (E_n \otimes 0).$$

Equip E with the inner product $\langle \cdot, \cdot \rangle$ defined by $\langle x, y \rangle := \sum_{k=1}^{\infty} x_k y_k$ for all $x := (x_1, x_2, \ldots)$ and $y := (y_1, y_2, \ldots) \in E$. The induced norm $\|\cdot\|$ on E is then given by $\|x\| := \sqrt{\langle x, x \rangle}$ for all $x \in E$.

THEOREM 4.38 ([7, p. 1175]). *In the (incomplete) inner product space $(E, \langle \cdot, \cdot \rangle)$ there exists a non-convex Chebyshev set with bounded convex complement.*

PROOF. Firstly, we inductively define a sequence of sets $(M_n)_{n=1}^{\infty}$ such that, for all $n \in \mathbb{N}$, M_n is a nonempty closed subset of E_n, $E_n \setminus M_n$ is convex, $0 \notin M_n$, every point in $E_n^{(0)}$ has a unique nearest point in M_n, and M_n has the $\left(\frac{1}{4}\right)^{n-1}$-boundary property.
To begin let $M_1 =: E_1 \setminus (-2, 1)$. It is clear that M_1 is a closed, nonempty subset of E_1. Furthermore, it is straightforward to check that $0 \notin M_1$, $E_1 \setminus M_1$ is bounded and convex, M_1 has the 1-boundary property, and every point in $E_1^{(0)} = \{0\}$ has a unique nearest point in M_1.
Suppose then that M_k is defined for some $k \in \mathbb{N}$, such that M_k has the properties listed above. Construct M_k' as in Definition 4.25. Then let $M_{k+1} := M_k''$ where $M_k'' := M_k' \cup \psi_{\theta_k}(M_k')$ and $\theta_k := 1 + \dfrac{1}{k^2}$. By Proposition 4.32, it follows that M_{k+1} has the required properties, which completes the construction. Furthermore, by Proposition 4.32, Proposition 4.33, and Proposition 4.34, we have

$$M_n \times \{0\} = M_{n+1} \cap (E_n \times \{0\}), \quad P_{M_n}(x) \times \{0\} = P_{M_{n+1}}(x') \quad \text{for all } x \in E_n^{(0)}, \quad \text{and}$$

$$\sup\{\|x\| : x \in E_{n+1} \setminus M_{n+1}\} \leq 3\theta \sup\{\|x\| : x \in E_n \setminus M_n\} \quad \text{for all } n \in \mathbb{N}.$$

We now show that $M := \bigcup_{n=1}^{\infty} (M_n \otimes 0)$ is a Chebyshev set in $(E, \langle \cdot, \cdot \rangle)$. Firstly, we show that

$$M \cap (E_k \otimes 0) = M_k \otimes 0$$

for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. It is clear that $M_k \otimes 0 \subseteq M \cap (E_k \otimes 0)$, so suppose $x \in M \cap (E_k \otimes 0)$. Thus, for some $n \in \mathbb{N}$, $x \in (M_n \otimes 0) \cap (E_k \otimes 0)$. Repeatedly using the fact that $M_m \times \{0\} = M_{m+1} \cap (E_m \times \{0\})$ for all $m \in \mathbb{N}$, we conclude that $(M_n \otimes 0) \cap (E_k \times 0) \subseteq M_k \otimes 0$. Hence, $x \in M_k \otimes 0$, and we're done.

Now let $x \in E$. Since x can have only finitely many non-zero components, there exists $n \in \mathbb{N}$ such that $x \in E_n^{(0)} \otimes 0$. By construction, $P_{M_n \otimes 0}(x) = \{y\}$ for some $y \in M_n \otimes 0 \subseteq M$. Furthermore, since $P_{M_n \otimes 0}(x) = P_{M_{n+1} \otimes 0}(x)$ and $E_n^{(0)} \otimes 0 \subseteq E_{n+1}^{(0)} \otimes 0$, it follows by induction that $P_{M_m \otimes 0}(x) = P_{M_n \otimes 0}(x) = \{y\}$ for all $m \geq n$. To show that y is the unique nearest point to x in M, suppose there exists $z \in M$ such that $\|x - z\| \leq \|x - y\|$. Let $k \geq n$ be such that $z \in E_k \otimes 0$. Thus, $P_{M_k \otimes 0}(x) = \{y\}$. Since

$$z \in (E_k \otimes 0)) \cap M = M_k \otimes 0 \quad \text{and} \quad \|x - z\| \leq \|x - y\| = d(x, M_k \otimes 0),$$

it follows that $z = y$. Therefore, $P_M(x) = \{y\}$, and so M is a Chebyshev set.

We now show that M is non-convex. Clearly $P_{M_1 \otimes 0}(0) = \{(1) \otimes 0\}$. By the previous working, $P_M(0) = \{(1) \otimes 0\}$, and so $0 \notin M$. However, $(-2) \otimes 0, (1) \otimes 0 \in M_1 \otimes 0 \subseteq M$, so M is not convex.

Making use of the fact that $(M_n \otimes 0) \cap (E_k \times 0) \subseteq M_k \otimes 0$ for all $k, n \in \mathbb{N}$, we have that

$$\begin{aligned} E \setminus M &= \left(\bigcup_{n=1}^{\infty} (E_n \otimes 0) \right) \setminus \left(\bigcup_{n=1}^{\infty} (M_n \otimes 0) \right) \\ &= \bigcup_{n=1}^{\infty} (E_n \otimes 0) \setminus (M_n \otimes 0) \\ &= \bigcup_{n=1}^{\infty} ((E_n \setminus M_n) \otimes 0) . \end{aligned}$$

Hence, $X \setminus M$ is the union of expanding convex sets, so is convex. Furthermore, since

$$\sup\{\|z\| : z \in (E_{n+1} \setminus M_{n+1}) \otimes 0\} \leq 3\theta_k \sup\{\|z\| : z \in (E_n \setminus M_n) \otimes 0\}$$

for all $k \in \mathbb{N}$, we have by induction that

$$\begin{aligned} \sup\{\|z\| : z \in E \setminus M\} &\leq 3 \left(\prod_{n=1}^{\infty} \theta_n \right) \sup\{\|z\| : z \in (E_1 \setminus M_1) \otimes 0\} \\ &= 6 \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2} \right). \end{aligned}$$

By Theorem 4.35, the infinite product $\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)$ converges, since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Thus, the right hand side of the previous expression is finite, and so $E \setminus M$ is bounded. $\square$

Theorem 4.38 seems to be the closest known answer to the conjecture of V. Klee [53], that a nonconvex Chebyshev set exists in some infinite dimensional Hilbert space.

Appendix A

THEOREM (Primitive Ekeland Theorem [36]). Let (X, d) be a complete metric space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a bounded below, lower semi-continuous function on X. If $\varepsilon > 0$ and $x_0 \in X$, then there exists $x_\infty \in X$ such that:

(i) $f(x_\infty) \leq f(x_0) - \varepsilon d(x_\infty, x_0)$ and

(ii) $f(x_\infty) - \varepsilon d(x_\infty, x) < f(x)$ for all $x \in X \setminus \{x_\infty\}$.

PROOF. We shall inductively define a sequence $(x_n)_{n=1}^\infty$ in X and a sequence $(D_n)_{n=1}^\infty$ of closed subsets of X such that:

(i) $D_n := \{x \in D_{n-1} : f(x) \leq f(x_{n-1}) - \varepsilon d(x, x_{n-1})\}$;

(ii) $x_n \in D_n$;

(iii) $f(x_n) \leq \inf_{x \in D_n} f(x) + \varepsilon^2/(n+1)$.

Set $D_0 := X$. In the base step we let

$$D_1 := \{x \in D_0 : f(x) \leq f(x_0) - \varepsilon d(x, x_0)\}$$

and choose $x_1 \in D_1$ so that $f(x_1) \leq \inf_{x \in D_1} f(x) + \varepsilon^2/2$. Then at the $(n+1)^{\text{th}}$-step we let

$$D_{n+1} := \{x \in D_n : f(x) \leq f(x_n) - \varepsilon d(x, x_n)\}$$

and we choose $x_{n+1} \in D_{n+1}$ such that

$$f(x_{n+1}) \leq \inf_{x \in D_{n+1}} f(x) + \varepsilon^2/(n+2).$$

This completes the induction.

Now, by construction, the sets $(D_n)_{n=1}^\infty$ are closed and $\emptyset \neq D_{n+1} \subseteq D_n$ for all $n \in \mathbb{N}$. It is also easy to see that $\sup\{d(x, x_n) : x \in D_{n+1}\} \leq \varepsilon/(n+1)$. Indeed, if $x \in D_{n+1}$ and $\frac{\varepsilon}{n+1} < d(x, x_n)$, then

$$\begin{aligned} f(x) &< \left[f(x_n) - \varepsilon\left(\frac{\varepsilon}{n+1}\right)\right] = f(x_n) - \frac{\varepsilon^2}{n+1} \\ &\leq \left[\inf_{y \in D_n} f(y) + \frac{\varepsilon^2}{n+1}\right] - \frac{\varepsilon^2}{n+1} = \inf_{y \in D_n} f(y); \end{aligned}$$

which contradicts the fact that $x \in D_{n+1} \subseteq D_n$.

Let $\{x_\infty\} := \bigcap_{n=1}^\infty D_n$. Fix $x \in X \setminus \{x_\infty\}$ and let n be the first natural number such that $x \notin D_n$, i.e., $x \in D_{n-1} \setminus D_n$. Then

$$f(x_{n-1}) - \varepsilon d(x, x_{n-1}) < f(x). \qquad (*)$$

On the other hand, since $x_\infty \in D_n$,

$$f(x_\infty) \leq f(x_{n-1}) - \varepsilon d(x_\infty, x_{n-1}),$$

and so

$$\begin{aligned} f(x_\infty) - \varepsilon d(x, x_\infty) &\leq f(x_{n-1}) - \varepsilon[d(x, x_\infty) + d(x_\infty, x_{n-1})] \\ &\leq f(x_{n-1}) - \varepsilon d(x, x_{n-1}) \quad \text{by the triangle inequality} \\ &< f(x) \qquad \text{by } (*). \end{aligned}$$

Finally, note that $f(x_\infty) \leq f(x_0) - \varepsilon d(x_\infty, x_0)$, since $x_\infty \in D_1$. □

Appendix B

PROPOSITION. For any $x \in E$ and $r > 0$,

$$K[x; r] = E \times [0, \infty) \cap \bigcup_{\lambda \in [0,1]} B\Big[(x, (1 - 2\lambda)r); \lambda\sqrt{2}r\Big].$$

PROOF. Let $(y, t) \in K[x; r]$. Hence, $t \geq 0$ and $\|y - x\| + t \leq r$. Define

$$\lambda := \frac{r + \|y - x\| - t}{2r}.$$

Since

$$\|y - x\| - t \leq \|y - x\| + t \leq r$$

and

$$0 \leq 2\|y - x\| = \|y - x\| + t + \|y - x\| - t \leq r + \|y - x\| - t,$$

we see that $\lambda \in [0, 1]$. It is straightforward to show that

$$\|(y, t) - (x, (1 - 2\lambda)r)\|^2 = \|y - x\|^2 + (t - (1 - 2\lambda)r)^2 = 2\|y - x\|^2$$

whilst

$$\left(\lambda\sqrt{2}r\right)^2 = \frac{1}{2}(r + \|y - x\| - t)^2 \geq 2\|y - x\|^2.$$

Therefore, $(y, t) \in B\big[(x, (1 - 2\lambda)r); \lambda\sqrt{2}r\big]$ and we have set inclusion in one direction.

For the opposite direction, suppose

$$(y, t) \in \mathbb{R}^n \times [0, \infty) \cap \bigcup_{\lambda \in [0,1]} B\Big[(x, (1 - 2\lambda)r); \lambda\sqrt{2}r\Big].$$

Therefore, $t \geq 0$ and $\|(y, t) - (x, (1 - 2\lambda)r)\|^2 = \|y - x\|^2 + (t - (1 - 2\lambda)r)^2 \leq (\sqrt{2}\lambda r)^2$ for some $\lambda \in [0, 1]$. Since

$$(r - t)^2 - \left((\sqrt{2}\lambda r)^2 - (t - (1 - 2\lambda)r)^2\right) = \left(\sqrt{2}(\lambda - 1)r + \sqrt{2}t\right)^2 \geq 0,$$

it follows that $\|y-x\|^2 \leq (r-t)^2$. Finally, for $\lambda \in [0,1]$,

$$t \leq (1-2\lambda)r + \lambda\sqrt{2}r = r + \lambda(\sqrt{2}-2)r \leq r,$$

and so we conclude that $\|y-x\| + t \leq r$. Therefore, $(y,t) \in K[x;r]$ and we have set inclusion in the opposite direction. $\square$

COROLLARY. If $x \in E$ and $r > 0$, then $K[x;r]$ is smooth at any $(y,t) \in E \times (0,r)$, such that $\|y-x\| + t = r$.

PROOF. Let $(y,t) \in E \times (0,r)$ and suppose $\|y-x\| + t = r$. Clearly, $(y,t) \in \mathrm{Bd}(K[x;r])$. By Proposition 4.14, there exists $\lambda \in [0,1]$ such that

$$(y,t) \in E \times [0,\infty) \cap B\Big[(x,(1-2\lambda)r); \lambda\sqrt{2}r\Big] \subseteq K[x;r]\,.$$

Since $t \neq r$, we have $\lambda > 0$, and so $B\big[(x,(1-2\lambda)r); \lambda\sqrt{2}r\big]$ is smooth. Furthermore, since $t \neq 0$, (y,t) is a smooth point of $E \times [0,\infty) \cap B\big[(x,(1-2\lambda)r); \lambda\sqrt{2}r\big]$. Therefore, $K[x;r]$ is smooth at (y,t). $\square$

PROPOSITION. Let $x \in E \setminus M$ and suppose $(x, d(x,M)) \in \overline{D}$. Then $K[x; d(x,M)] \subseteq \overline{D}$.

PROOF. Since $\overline{D}$ is convex and $K[x; d(x,M)]$ is simply the convex hull of the set

$$\{(x, d(x,M))\} \cup B[x; d(x,M)] \times \{0\},$$

we need only show that $B[x; d(x,M)] \times \{0\} \subseteq \overline{D}$. This follows from Proposition 4.9 and the fact that $B[x; d(x,M)] \subseteq \overline{E \setminus M}$. $\square$

Appendix C

One of the fundamental tools of elementary analysis is the “Sandwich Theorem” (sometimes called the “Squeeze Theorem”). The theory of differentiation is no different. In particular, we have the following result.

If $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ are functions defined on a normed linear space $(X, \|\cdot\|)$ and

(i) $f(y) \leq g(y) \leq h(y)$ for all $y \in X$,

(ii) $f(x) = g(x) = h(x)$ for some $x \in X$ and

(iii) both $\nabla f(x)$ and $\nabla h(x)$ exist then $\nabla g(x)$ exists and equals $\nabla f(x) = \nabla h(x)$. This result holds for both Gâteaux and Fréchet derivatives. Furthermore, this result has direct implications for the differentiability of distance functions.

Suppose that K is a Chebyshev subset of a Gâteaux (Fréchet) smooth normed linear space $(X, \|\cdot\|)$. Then for each $x \in X \setminus K$ the distance function, $y \mapsto d(y, K)$, is Gâteaux (Fréchet) differentiable at each point of $(x, p_K(x))$.

To see this, let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ be defined by

$$f(y) := d(x, K) - \|y - x\|, \quad g(y) := d(y, K), \quad \text{and} \quad h(y) := \|y - p_K(x)\|.$$

Since g is nonexpansive and $f(x) = g(x)$, we have that $f(y) \leq g(y)$ for all $y \in X$. Similarly, since $h(p_K(x)) = g(p_K(x)) = 0$, we have that $g(y) \leq h(y)$ for all $y \in X$. Therefore, $f(y) \leq g(y) \leq h(y)$ for all $y \in X$. On the other hand, $h - f$ is convex and $(h - f)(x) = (h - f)(p_K(x)) = 0$. Therefore, $(h - f)(y) \leq 0$ for all $y \in [x, p_K(x)]$, and so $h(y) \leq f(y)$ for all $y \in [x, p_K(x)]$. Thus, we have that $f(y) = g(y) = h(y)$ for all $y \in [x, p_K(x)]$. Finally, both f and h are Gâteaux (Fréchet) differentiable on $(x, p_K(x))$. In fact, f is Gâteaux (Fréchet) differentiable on $X \setminus \{x\}$ and h is Gâteaux (Fréchet) differentiable on $X \setminus \{p_K(x)\}$. The result now follows from the "Sandwich Theorem" given above.

The following results, which are in essence due to Simon Fitzpatrick [38], are an attempt to extend the aforementioned result on the differentiability of the distance function on $(x, p_K(x))$ to the differentiability of the distance function on $[x, p_K(x))$.

PROPOSITION. Let $(X, \|\cdot\|)$ be a Banach space and suppose that $g : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ are functions. Let $\delta, \varepsilon > 0$ and $x_0 \in X$. Suppose also that (a) $g(x) \leq h(x)$ for all $x \in X$ and $g(x_0) = h(x_0)$ and
(b) h is Gâteaux differentiable at x_0 with Gâteaux derivative $\nabla h(x_0)$. If there exists an $x^* \in X^*$ and functions $f_\lambda : X \to \mathbb{R}$ (for $0 < \lambda < \delta$) such that

(i) $f_\lambda(x) \leq g(x)$ for all $x \in X$,

(ii) $\displaystyle\lim_{\lambda \to 0^+} \frac{f_\lambda(x_0) - g(x_0)}{\lambda} = 0$ and

(iii) $\displaystyle\liminf_{\lambda \to 0^+} \frac{f_\lambda(x_0 + \lambda y) - f_\lambda(x_0)}{\lambda} \geq x^*(y) - \varepsilon$ for all $y \in S_X$

then

$$x^*(y) - \varepsilon \leq \liminf_{\lambda \to 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \limsup_{\lambda \to 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \nabla h(x_0)(y)$$

for all $y \in S_X$.

PROOF. The straight-forward proof is left as an exercise for the reader. $\square$

THEOREM ([7, Theorem 1.4]). Let K be a nonempty closed subset of a Fréchet smooth normed linear space $(X, \|\cdot\|)$ and let $x_0 \in X \setminus K$. If $P_K(x_0)$ is nonempty and

$$\lim_{t\to 0^+} \frac{d(x_0+tv, K) - d(x_0, K)}{t} = 1$$

for some $v \in S_X$, then the distance function for K is Gâteaux differentiable at x_0.

PROOF. We shall apply the previous Proposition. Let ε be an arbitrary positive real number and let $\delta := 1$. Let $g : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ be defined by $g(x) := d(x, K)$ and $h(x) := \|x - p_K(x_0)\|$. Then $g(x) \leq h(x)$ for all $x \in X$, $g(x_0) = h(x_0)$, and $\nabla h(x_0) = \nabla \|x_0 - p_K(x_0)\|$. Let $x^* = \nabla\|v\|$ and choose $t > 0$ so that

$$\left| \frac{\|v+ty\| - \|v\|}{t} - \nabla \|v\| (y) \right| < \varepsilon \quad \text{for all } y \in S_X.$$

Note that this is possible since the norm $\|\cdot\|$ is Fréchet differentiable at $v \in S_X$. For each $0 < \lambda < 1$, let $f_\lambda : X \to \mathbb{R}$ be defined by $f_\lambda(x) := d(x_0 + t^{-1}\lambda v, K) - \left\|x - (x_0 + t^{-1}\lambda v)\right\|$. Then

(i) $f_\lambda(x) \leq g(x) \quad$ for all $x \in X$,

(ii)

$$\begin{aligned}
\lim_{\lambda\to 0^+} \frac{f_\lambda(x_0) - g(x_0)}{\lambda} &= \lim_{\lambda\to 0^+} \frac{d(x_0 + t^{-1}\lambda v, K) - d(x_0, K) - t^{-1}\lambda}{\lambda} \\
&= \lim_{\lambda\to 0^+} t^{-1} \left(\frac{d(x_0 + t^{-1}\lambda v, K) - d(x_0, K)}{t^{-1}\lambda} - 1 \right) \\
&= t^{-1} \lim_{s\to 0^+} \left(\frac{d(x_0 + sv, K) - d(x_0, K)}{s} - 1 \right) = 0.
\end{aligned}$$

(iii) Let $0 < \lambda < 1$ and $y \in S_X$. Then

$$\begin{aligned}
\frac{f_\lambda(x_0+\lambda y) - f_\lambda(x_0)}{\lambda} &= -\left(\frac{\left\|\lambda y - t^{-1}\lambda v\right\| - \left\|t^{-1}\lambda v\right\|}{\lambda} \right) \\
&= -t^{-1}\lambda \left(\frac{\|v - ty\| - \|v\|}{\lambda} \right) \\
&= -\left(\frac{\|v + t(-y)\| - \|v\|}{t} \right) \\
&\geq -\nabla \|v\| (-y) - \varepsilon = \nabla \|v\| (y) - \varepsilon = x^*(y) - \varepsilon.
\end{aligned}$$

Therefore, $\displaystyle\liminf_{\lambda\to 0^+} \frac{f_\lambda(x_0+\lambda y) - f_\lambda(x_0)}{\lambda} \geq x^*(y) - \varepsilon \quad$ for all $y \in S_X$.

Hence, by the previous propostion,

$$x^*(y) - \varepsilon \leq \liminf_{\lambda \to 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \limsup_{\lambda \to 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \nabla h(x_0)(y)$$

for all $y \in S_X$. Since $\varepsilon > 0$ was arbitrary, we have that

$$x^*(y) \leq \liminf_{\lambda \to 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \limsup_{\lambda \to 0^+} \frac{g(x_0 + \lambda y) - g(x_0)}{\lambda} \leq \nabla h(x_0)(y)$$

for all $y \in S_X$. Thus, $x^* = \nabla h(x_0)$, and so $\nabla h(x_0)(y) = \lim_{\lambda \to 0^+} \dfrac{g(x_0 + \lambda y) - g(x_0)}{\lambda}$. It is now routine to show that $\nabla h(x_0)(y) = \lim_{\lambda \to 0} \dfrac{g(x_0 + \lambda y) - g(x_0)}{\lambda} = \nabla g(x_0)(y)$ for all $y \in S_X$. $\square$

Note that by being a little more careful in the proof of the above theorem, one can show that the assumption that the norm is Fréchet smooth can be relaxed to the norm being Gâteaux smooth.

Appendix D

Let us start by giving a quick proof that the norm on a Hilbert space is rotund. In any Hilbert space $(H, \langle \cdot, \cdot \rangle)$ the norm satisfies the identities $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle$ and $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle$. Therefore, $\|x - y\|^2 + \|x + y\|^2 = 2(\|x\|^2 + \|y\|^2)$ - **the parallelogram law**. Hence,

$$\left\| \frac{x+y}{2} \right\|^2 = \frac{\|x\|^2 + \|y\|^2}{2} - \left\| \frac{x-y}{2} \right\|^2 < \frac{\|x\|^2 + \|y\|^2}{2} = 1 \quad \text{for all } x \neq y \in S_H.$$

This shows that $(H, \langle \cdot, \cdot \rangle)$ is rotund.

THEOREM ([23, Theorem 9]). If $(X, \|\cdot\|)$ is a separable normed linear space, then X admits an equivalent rotund norm.

PROOF. If $(Y, \|\cdot\|_Y)$ is a rotund normed linear space and $T : X \to Y$ is a 1-to-1 bounded linear mapping, then $\|x\|' := \|x\| + \|T(x)\|_Y$ is an equivalent rotund norm on X (apply Lemma 2.23 for the proof of rotundity). Since X is separable, there exists a dense subset $\{x_n\}_{n=1}^\infty$ of S_X. Now, by applying the Hahn-Banach Extension Theorem, there exists a subset $\{x_n^*\}_{n=1}^\infty$ of S_{X^*} such that $x_n^*(x_n) = 1$ for all $n \in \mathbb{N}$. Define $T : (X, \|\cdot\|) \to (\ell_2(\mathbb{N}), \|\cdot\|_2)$ by

$$T(x) := \left(\frac{x_1^*(x)}{2}, \frac{x_2^*(x)}{4}, \ldots, \frac{x_n^*(x)}{2^n}, \ldots \right).$$

Then T is 1-to-1, bounded and linear. Thus, X has an equivalent rotund norm. $\square$

Note that the argument above works whenever the dual space X^* is separable with respect to the weak* topology on X^*. That is, the following statement is true. If $(X, \|\cdot\|)$ is a normed linear space and (X^*, weak^*) is separable, then X admits an equivalent rotund norm. In particular, $\ell_\infty(\mathbb{N})$ admits an equivalent rotund norm. On the the hand, one should also note that not every Banach space admits an equivalent strictly convex norm. For example, ℓ^∞/c_0 does not possess an equivalent strictly convex norm, see [16, 43, 59].

Appendix E

LEMMA. Let S be the ellipsoidal surface in $\mathbb{R}^n$ defined by the equation

$$\frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\beta^2} + \cdots + \frac{x_n^2}{\beta^2} = 1,$$

where $\alpha > \beta > 0$. For any $y \in S$ there exists $w \in \text{co}(S)$ such that $\|w - y\| = d(y, S) = \frac{\beta^2}{\alpha}$.

PROOF. Let $y := (y_1, \ldots, y_n) \in S$, and consider $\hat{y} := \left(y_1\left(1 - \frac{\beta^2}{\alpha^2}\right), 0, \ldots, 0\right) \in \text{co}(S)$. We will show that y is a nearest point to $\hat{y}$ in S. To do this we will minimise the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x_1, x_2, \ldots, x_n) := \left\|(x_1, x_2, \ldots, x_n) - \left(y_1\left(1 - \frac{\beta^2}{\alpha^2}\right), 0, \ldots, 0\right)\right\|^2,$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, subject to the constraint that

$$\frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\beta^2} + \cdots + \frac{x_n^2}{\beta^2} = 1.$$

Since $f(x_1, x_2, \ldots, x_n) = \left(x_1 - y_1\left(1 - \frac{\beta^2}{\alpha^2}\right)\right)^2 + x_2^2 + \cdots + x_n^2$, substituting in the constraint shows that we need only minimise (without constraint) the function $g : \mathbb{R} \to \mathbb{R}$, defined by

$$g(x_1) := \left(x_1 - y_1\left(1 - \frac{\beta^2}{\alpha^2}\right)\right)^2 + \beta^2 - \frac{x_1^2\beta^2}{\alpha^2}.$$

As $1 - \frac{\beta^2}{\alpha^2} > 0$,

$$g'(x_1) = 2\left(x_1 - y_1\left(1 - \frac{\beta^2}{\alpha^2}\right)\right) - \frac{2x_1\beta^2}{\alpha^2}$$

is zero only when $x_1 = y_1$. Furthermore, since $g''(x_1) = 2\left(1 - \frac{\beta^2}{\alpha^2}\right) > 0$, we see that g is minimised at y_1. Therefore, f, subject to the given constraint, is minimised at $(y_1, x_2, \ldots, x_n)$, where $\frac{y_1^2}{\alpha^2} + \frac{x_2^2}{\beta^2} + \cdots + \frac{x_n^2}{\beta^2} = 1$. Hence, f is minimised (amongst other points) at $(y_1, y_2, \ldots y_n)$, and so y is a nearest point to $\hat{y}$ in S. Finally, since

$$\begin{aligned}
\left\|(y_1, y_2, \ldots, y_n) - \left(y_1\left(1 - \frac{\beta^2}{\alpha^2}\right), 0, \ldots, 0\right)\right\|^2 &= \left(\frac{y_1\beta^2}{\alpha^2}\right)^2 + y_2^2 + \cdots + y_n^2 \\
&\geq \left(\frac{y_1\beta^2}{\alpha^2}\right)^2 + \frac{y_2^2\beta^2}{\alpha^2} + \cdots + \frac{y_n^2\beta^2}{\alpha^2} \\
&= \left(\frac{\beta^2}{\alpha}\right)^2 \left(\frac{y_1^2}{\alpha^2} + \frac{y_2^2}{\beta^2} + \cdots + \frac{y_n^2}{\beta^2}\right) \\
&= \left(\frac{\beta^2}{\alpha}\right)^2,
\end{aligned}$$

there exists $w \in [\hat{y}, y] \subseteq \mathrm{co}(S)$ such that $\|w - y\| = d(y, S) = \frac{\beta^2}{\alpha}$. $\square$

Index of notation and assumed knowledge

- The **natural numbers**, $\mathbb{N} := \{1, 2, 3, \ldots\}$.
- The **integers**, $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2 \ldots\}$.
- The **rational numbers**, $\mathbb{Q} := \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$.
- For any set X, $\mathcal{P}(X)$ is the set of all subsets of X.
- For any subset A of a topological space (X, τ), we define
  - $\mathrm{int}(A)$, called the **interior** of A, is the union of all open sets contained in A;
  - $\overline{A}$, called the **closure** of A, is the intersection of all closed sets containing A;
  - $\mathrm{Bd}(A)$, called the **boundary** of A, is $\overline{A} \setminus \mathrm{int}(A)$,
- For any points x and y in a vector space X, we define the following intervals:
  - $[x, y] := \{x + \lambda(y - x) : 0 \leq \lambda \leq 1\}$;
  - $(x, y) := \{x + \lambda(y - x) : 0 < \lambda < 1\}$;
  - $[x, y) := \{x + \lambda(y - x) : 0 \leq \lambda < 1\}$;
  - $(x, y] := \{x + \lambda(y - x) : 0 < \lambda \leq 1\}$.
- For any normed linear space $(X, \|\cdot\|)$, we define
  - $B[x; r] := \{y \in X : \|x - y\| \leq r\}$, for any $x \in X$ and $r \geq 0$ (note: this implies that $B[x; 0] = \{x\}$, for any $x \in X$);
  - $B(x; r) := \{y \in X : \|x - y\| < r\}$, for any $x \in X$ and $r > 0$;
  - $B_X := B[0; 1]$;

– $S_X := \{x \in X : \|x\| = 1\}$.

- For any inner product space $(X, \langle\cdot,\cdot\rangle)$, $\angle xyz$ will denote the angle between the vectors $(y-x)$ and $(y-z)$, where x, y, z are distinct points in X. That is,

$$\angle xyz := \cos^{-1}\left(\frac{\langle y-x, y-z\rangle}{\|y-x\|\,\|y-z\|}\right),$$

where $\|\cdot\|$ is the norm induced by the inner product.

- Given a compact Hausdorff space K, we write $C(K)$ for the set of all real-valued continuous functions on K. This is a vector space under the operations of pointwise addition and pointwise scalar multiplication. $C(K)$ becomes a Banach space when equipped with the uniform norm $\|\cdot\|_\infty$, defined by

$$\|f\|_\infty := \sup_{x\in K} |f(x)|, \quad \text{for all } f \in C(K).$$

- Let A and B be sets. Given a function $f : A \to B$, we define $f(A) := \bigcup_{a\in A}\{f(x)\}$. Similarly, given a set valued mapping $\Phi : A \to \mathcal{P}(B)$, we define $\Phi(A) := \bigcup_{a\in A} \Phi(x)$.

- For a normed linear space $(X, \|\cdot\|)$, X^*, the set of bounded linear maps from X to $\mathbb{R}$, is called the **dual space** of X. X^* is a Banach space when equipped with the operator norm, given by

$$\|f\| := \sup_{x\in B_X} \|f(x)\| \text{ for all } f \in X^*.$$

- Let X be a set and Y a totally ordered set. For any function $f : X \to Y$ we define

$$\begin{aligned}\text{argmax}(f) &:= \{x \in X : f(y) \le f(x) \text{ for all } y \in X\},\\ \text{argmin}(f) &:= \{x \in X : f(x) \le f(y) \text{ for all } y \in X\}.\end{aligned}$$

- Let A be a subset of a vector space X. Then the **convex hull** of A, denoted by $\mathbf{co}(\boldsymbol{A})$, is defined to be the intersection of all convex subsets of X that contain A.

- Let X be a set and let $f : X \to \mathbb{R} \cup \{\infty\}$ a function. Then

$$\text{Dom}(f) := \{x \in X : f(x) < \infty\}.$$

- If X is a normed linear space, $f : X \to \mathbb{R}$ is a function and f is Gâteaux (Fréchet) differentiable at $x_0 \in X$, then we write $\nabla f(x_0)$ for the Gâteaux (Fréchet) derivative of f at x_0.

- If f is a convex function defined on a nonempty convex subset K of a normed linear space $(X, \|\cdot\|)$ and $x \in K$, then we define the **subdifferential of $\boldsymbol{f}$ at $\boldsymbol{x}$** to be the set $\boldsymbol{\partial f(x)}$ of all $x^* \in X^*$ satisfying

$$x^*(y-x) \le f(y) - f(x) \quad \text{for all } y \in K.$$

- It is assumed that the reader has a basic working knowledge of metric spaces and normed linear spaces. In particular, it is assumed that the reader is familiar with the basics of linear topology (including the Hahn-Banach Theorem). In particular, the weak topology on a normed linear space. In this regard, knowledge of the following theorems is assumed.

  THEOREM ([8, 15]). Let $(X, \|\cdot\|)$ be a normed linear space. Then X is reflexive if, and only if, B_X with the relative weak topology is compact.

  THEOREM (Eberlein-Šmulian Theorem, [27, 31]). Every nonempty weakly compact subset K of a Banach space $(X, \|\cdot\|)$ is weakly sequentially compact, i.e., every sequence in K possesses a weakly convergent subsequence (with the limit in K).

  For a good introduction to the theory of Banach spaces see any of: [27, 29, 30, 37, 78]

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