RICH FAMILIES, \textit{W}-SPACES AND THE PRODUCT OF BAIRE SPACES

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\textbf{Abstract.} In this paper we prove a theorem more general than the following. Suppose that \(X\) is a Baire space and \(Y\) is the product of hereditarily Baire metric spaces then \(X \times Y\) is a Baire space.

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1. \textbf{Introduction}

A topological space \(X\) is said to be a \textit{Baire} space if for each sequence \((O_n : n \in \mathbb{N})\) of dense open subsets of \(X\), \(\bigcap_{n \in \mathbb{N}} O_n\) is dense in \(X\) and a Baire space \(Y\) is called \textit{barely Baire} if there exists a Baire space \(Z\) such that \(Y \times Z\) is not Baire. It is well known that there exist metrizable barely Baire spaces, (see [5]). On the other hand it has recently been shown that the product of a Baire space \(X\) with a hereditarily Baire metric space \(Y\) is Baire, [7]. In that same paper the author claims in a “Remark” that the hypothesis on \(Y\) can be reduced to: \(Y\) is the product of hereditarily Baire metric spaces”. In this paper we substantiate this claim.

The main result of this paper relies upon two notions. The first, which is that of a \textit{W}-space [6], is recalled in Section 2. The second, which is that of a “rich family” is considered in Section 3. In Section 4, we shall prove our main theorem which states that the product of a Baire space with a \(W\)-space that possesses a rich family of Baire subspaces is Baire.

2. \textit{W}-spaces

In this paper all topological spaces are assumed to be Hausdorff and nonempty. Furthermore, if \(X\) is a topological space and \(a \in X\) then we shall always denote by \(\mathcal{N}(a)\) the set of all neighbourhoods of \(a\).

For any point \(a\) in a topological space \(X\) we can consider the following two player topological game, called the \(G(a)\)-\textit{game}. This game is played between the players \(\alpha\) and \(\beta\) and although it may seem unfair, \(\beta\) will always

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be granted the priviledge of the first move. To define this game we must first specify the rules and then also specify the definition of a win.

The moves of the player $\alpha$ are simple. He/she must always select a neighbourhood of the point $a$. However, the moves of the player $\beta$ depend upon the previous move of $\alpha$. Specifically, for his/her first move $\beta$ may select any point $x_1 \in X$. For $\alpha$'s first move, as mentioned earlier, $\alpha$ must select a neighbourhood $O_1$ of $a$. Now, for $\beta$'s second move he/she must select a point $x_2 \in O_1$. For $\alpha$'s second move he/she is entitled to select any neighbourhood $O_2$ of $a$. In general, if $\alpha$ has chosen $O_n \in \mathcal{N}(a)$ as his/her $n^{th}$ move of the $G(a)$-game then $\beta$ is obliged to choose a point $x_{n+1} \in O_n$. The response of $\alpha$ is then simply to choose any neighbourhood $O_{n+1}$ of $a$. Continuing in this fashion indefinitely, the players $\alpha$ and $\beta$ produce a sequence $((x_n,O_n) : n \in \mathbb{N})$ of ordered pairs with $x_{n+1} \in O_n \in \mathcal{N}(a)$ for all $n \in \mathbb{N}$, called a play of the $G(a)$-game. A partial play $((x_k,O_k) : 1 \leq k \leq n)$ of the $G(a)$-game consists of the first $n$ moves of a play of the $G(a)$-game. We shall declare $\alpha$ the winner of a play $((x_n,O_n) : n \in \mathbb{N})$ if, and only if, a \textit{strategy} for $\alpha$ is an inductively defined sequence of functions $t := (t_n : n \in \mathbb{N})$. The domain of $t_1$ is $X^1$ and for each $(x_1) \in X^1$, $t_1(x_1) \in \mathcal{N}(a)$, i.e., $((x_1,t_1(x_1)))$ is a partial play. Inductively, if $t_1,t_2,\ldots,t_n$ have been defined then the domain of $t_{n+1}$ is defined to be,

$$\{(x_1,x_2,\ldots,x_{n+1}) \in X^{n+1} : (x_1,x_2,\ldots,x_n) \in \text{Dom}(t_n) \quad \text{and} \quad x_{n+1} \in t_n(x_1,x_2,\ldots,x_n)\}.$$ 

For each $(x_1,x_2,\ldots,x_{n+1}) \in \text{Dom}(t_{n+1})$, $t_{n+1}(x_1,x_2,\ldots,x_{n+1}) \in \mathcal{N}(a)$. Equivalently, for each $(x_1,x_2,\ldots,x_{n+1}) \in \text{Dom}(t_{n+1})$, $((x_k,t_k(x_1,\ldots,x_k)) : 1 \leq k \leq n + 1)$ is a partial play.

A partial $t$-play is a finite sequence $(x_1,x_2,\ldots,x_n) \in X^n$ such that $(x_1,x_2,\ldots,x_n) \in \text{Dom}(t_n)$ or, equivalently, if $x_{k+1} \in t_k(x_1,x_2,\ldots,x_k)$ for all $1 \leq k < n$. A $t$-play is an infinite sequence $(x_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(x_1,x_2,\ldots,x_n)$ is a partial $t$-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player $\alpha$ is said to be a winning strategy if each play of the form $((x_n,t_n(x_1,x_2,\ldots,x_n)) : n \in \mathbb{N})$ is won by $\alpha$, or equivalently, if $a \in \{x_n : n \in \mathbb{N}\}$ for each $t$-play $(x_n : n \in \mathbb{N})$.

A topological space $X$ is called a $W$-space if $\alpha$ has a winning strategy in the $G(a)$-game for each $a \in X$, [6].

In the remainder of this section we shall recall some relevant fact concerning $W$-spaces.

**Theorem 2.1.** [6, Theorem 3.3] Every first countable space is a $W$-space.
There are of course many W-spaces that are not first countable, see Example 2.7.

A topological space $X$ is said to have countable tightness if for each nonempty subset $A$ of $X$ and each $p \in \overline{A}$, there exists a countable subset $C \subseteq A$ such that $p \in \overline{C}$.

**Proposition 2.2.** [6, Corollary 3.4] Every W-space has countable tightness.

**Proposition 2.3.** [6, Theorem 3.1] If $X$ is a W-space and $\emptyset \neq A \subseteq X$ then $A$ is a W-space.

**Lemma 2.4.** [6, Theorem 3.9] Suppose that $X$ is a W-space and $a \in X$, then the player $\alpha$ possesses a strategy $s := (s_n : n \in \mathbb{N})$ in the $G(a)$-game such that every s-play converges to $a$.

For the remainder of this paper whenever we shall consider a W-space $X$ with $a \in X$ we shall assume that the player $\alpha$ is employing a strategy $t$, in the $G(a)$-game, in which every $t$-play converges to $a$.

Let $\{X_s : s \in S\}$ be a nonempty family of topological spaces and let $a \in \Pi_{s \in S}X_s$. The $\Sigma$-product of this family with base point $a$, denoted by $\Sigma_{s \in S}X_s(a)$, is the set of all $x \in \Pi_{s \in S}X_s$ such that $x(s) \neq a(s)$ for at most countably many $s \in S$. For each $x \in \Sigma_{s \in S}X_s(a)$, the support of $x$ is defined by $\text{supp}(x) := \{s \in S : x(s) \neq a(s)\}$.

**Theorem 2.5.** [6, Theorem 4.6] Suppose that $\{X_s : s \in S\}$ is a nonempty family of W-spaces. If $a \in \Pi_{s \in S}X_s$ then $\Sigma_{s \in S}X_s(a)$ is a W-space.

**Corollary 2.6.** [6, Theorem 4.1] If $\{X_n : n \in \mathbb{N}\}$ are W-spaces, then so is $\Pi_{n \in \mathbb{N}}X_n$.

**Example 2.7.** Suppose that $S$ is a nonempty set. For each $s \in S$, let $X_s := [0, 1]$ and define $a : S \rightarrow [0, 1]$ by, $a(s) := 0$ for all $s \in S$. Then by Theorem 2.5, $X := \Sigma_{s \in S}X_s(a)$ is a W-space. However, $X$ is not first countable whenever $S$ is uncountable.

### 3. Rich families

Let $X$ be a topological space, and let $\mathcal{F}$ be a family of nonempty, closed and separable subsets of $X$. Then $\mathcal{F}$ is rich if the following two conditions are satisfied:

(i) for every separable subspace $Y$ of $X$, there exists an $F \in \mathcal{F}$ such that $Y \subseteq F$;

(ii) for every increasing sequence $(F_n : n \in \mathbb{N})$ in $\mathcal{F}$, $\bigcup_{n \in \mathbb{N}}F_n \in \mathcal{F}$.

For any topological space $X$, the collection of all rich families of subsets forms a partially ordered set, under the binary relation of set inclusion. This partially ordered set has a greatest element, $\mathcal{S}_X := \{S \in 2^X : S$ is a nonempty, closed and separable subset of $X\}$. On the other hand, if $X$ is a
separable space, then the partially ordered set has a least element, namely \{X\}.

Next we present an important property of rich families. For a proof of this see [2, Proposition 1.1].

**Proposition 3.1.** Suppose that \(X\) is a topological space. If \(\{F_n : n \in \mathbb{N}\}\) are rich families then so is \(\bigcap_{n \in \mathbb{N}} F_n\).

Suppose that \(X\) is a topological space and \(S\) is a separable subset, it can be easily verified that the family \(F_S := \{F \in \mathcal{S}_X : S \subseteq F\}\) is rich. Hence, whenever \(X\) is an infinite set and \(F\) is a rich family of subsets of \(X\), then we can always assume, by possibly passing to a sub-family, that all the members of \(F\) are infinite. Indeed, if \(X\) has a countably infinite subset \(A\), then by Proposition 3.1, \(F \cap F_A \subseteq F\) is a rich family whose members are all infinite.

**Proposition 3.2.** If \(X\) is a topological space with countable tightness (e.g. if \(X\) is a \(W\)-space) and \(E\) is a dense subset of \(X\) then

\[ F := \{F \in \mathcal{S}_X : E \cap F \text{ is dense in } F\} \]

is a rich family.

**Proof:** Let \(Y\) be a separable subspace of \(X\), then \(Y\) has a countable dense subset \(D := \{d_n : n \in \mathbb{N}\}\). Since \(X\) has countable tightness, for each \(n \in \mathbb{N}\), there is a countable subset \(C_n \subseteq E\) such that \(d_n \in C_n\). Let \(F := \bigcup_{n \in \mathbb{N}} C_n\), then \(Y = D \subseteq F \in \mathcal{S}_X\) and

\[ F = \bigcup_{n \in \mathbb{N}} C_n \subseteq E \cap F \subseteq F. \]

Therefore, \(F \in F\). Now suppose that \(\{F_n : n \in \mathbb{N}\}\) is an increasing sequence in \(F\). Then \(F' := \bigcup_{n \in \mathbb{N}} F_n \in \mathcal{S}_X\) and \(F' \cap E\) is dense in \(F'\). Therefore, \(F' \in F\). \(\square\)

**Theorem 3.3.** Suppose that \(X\) is a topological space with countable tightness (in particular if \(X\) is a \(W\)-space) that possesses a rich family \(F\) of Baire subspaces then \(X\) is also a Baire space.

**Proof:** Let \(\{O_n : n \in \mathbb{N}\}\) be dense open subsets of \(X\). For each \(n \in \mathbb{N}\), let \(F_n := \{F \in \mathcal{S}_X : O_n \cap F\text{ is dense in } F\}\), then \(F_n\) is a rich family by Proposition 3.2. Let \(F^* = \bigcap_{n \in \mathbb{N}} F_n \cap F\); then \(F^*\) is also a rich family by Proposition 3.1. For each \(F \in F^*\), \(\bigcap_{n \in \mathbb{N}} (O_n \cap F)\) is dense in \(F\) since \(F\) is a Baire space. Let \(x \in X\), then there is \(F \in F^*\) such that \(x \in F\). Then \(x \in \bigcap_{n \in \mathbb{N}} (O_n \cap F) \subseteq \bigcap_{n \in \mathbb{N}} (O_n \cap F)\). Therefore, \(\bigcap_{n \in \mathbb{N}} O_n = X\). \(\square\)

Suppose that \(\{X_s : s \in S\}\) is a nonempty family of topological spaces and \(a \in \prod_{s \in S} X_s\). A cube \(E\) in \(\Sigma_{s \in S} X_s(a)\) is any nonempty product \(\prod_{s \in S} E_s \subseteq \Sigma_{s \in S} X_s(a)\). The set \(C_E := \{s \in S : E_s \neq \{a(s)\}\}\) is at most countable and \(E\) is homeomorphic to \(\prod_{s \in C_E} E_s\). If for each \(s \in S\), \(F_s\) is a rich family of subsets of \(X_s\) then the \(\Sigma\)-product of the rich families, with the base point
a ∈ Πₘₜ∈S Xₙ, denoted by Σₘₜ∈S Fₙ(a), is the set of all cubes E := Πₘₜ∈S Eₙ in Σₘₜ∈S Xₙ(a) such that Eₙ ∈ Fₙ for each n ∈ Cₛ.

Lemma 3.4. Let \{Xₙ : n ∈ N\} be a nonempty family of topological spaces. For each s ∈ S, let (Eₙ : n ∈ N) be an increasing sequence of nonempty subsets of Xₙ. Then \(\bigcup_{n∈N}(Πₘₜ∈S Eₙ) = Πₘₜ∈S(\bigcup_{n∈N} Eₙ)\). 

Proof: It is easy to see that \(\bigcup_{n∈N}(Πₘₜ∈S Eₙ) \subseteq Πₘₜ∈S(\bigcup_{n∈N} Eₙ)\) since for all n ∈ N, \(Πₘₜ∈S Eₙ \subseteq Πₘₜ∈S(\bigcup_{n∈N} Eₙ)\).

Let x ∈ Πₘₜ∈S(\bigcup_{n∈N} Eₙ) and let U := Πₘₜ∈SUₙ be a basic neighbourhood of x. Then there exists y ∈ U ∩ Πₘₜ∈S(\bigcup_{n∈N} Eₙ). Let M be the finite set \(\{s ∈ S : Uₙ \neq Xₙ\}\), and let Nₙ := min\{n ∈ N : y(s) ∈ Eₙ\} for all s ∈ M. Let N := max\{Nₙ : s ∈ M\}, then y(s) ∈ Eₙ for all s ∈ M. Let a ∈ Πₘₜ∈S Eₙ and let y′/U be defined by y′(s) := y(s) for all s ∈ M and y′(s) := a(s) for all s ∈ S \ M. Since y′ ∈ Πₘₜ∈S Eₙ, \(U \cap \bigcup_{n∈N}(Πₘₜ∈S Eₙ) \neq ∅\). Therefore, \(x ∈ \bigcup_{n∈N}(Πₘₜ∈S Eₙ)\). □

Theorem 3.5. Suppose that \{Xₙ : n ∈ N\} is a nonempty family of topological spaces and a ∈ Πₘₜ∈S Xₙ. If for each s ∈ S, Fₙ is a rich family of subsets of Xₙ, then Σₘₜ∈S Fₙ(a) is a rich family of subsets of Σₘₜ∈S Xₙ(a).

Proof: Let Y be a separable subspace of Σₘₜ∈S Xₙ(a), then it has a countable dense subset D. Let C := \(\bigcup_{d∈D}\text{supp}(d)\), then C is a countable set. For each s ∈ C, let Pₙ be the projection of D onto Xₙ, then Pₙ is countable and hence there is some Eₙ ∈ Fₙ such that Pₙ ⊆ Eₙ. For each s ∈ S \ C, let Eₙ := \{a(s)\}. Let F := Πₘₜ∈S Eₙ, then F ∈ Σₘₜ∈S Fₙ(a) and Y ⊆ F.

Let (Eₙ : n ∈ N) be an increasing sequence in Σₘₜ∈S Fₙ(a). For each cube Eₙ ∈ Σₘₜ∈S Fₙ(a), let Eₙ := Πₘₜ∈S Eₙ. Then by Lemma 3.4

\[\bigcup_{n∈N} Eₙ = \bigcup_{n∈N}(Πₘₜ∈S Eₙ) = Πₘₜ∈S(\bigcup_{n∈N} Eₙ) = Πₘₜ∈S(\bigcup_{n∈N} Eₙ).\]

It now follows that \(\bigcup_{n∈N} Eₙ \in Σₘₜ∈S Fₙ(a)\). □

4. Baire spaces and \(Σ\)-products

A subset R of a topological space X is residual in X if there exist dense open subsets \{Oₙ : n ∈ N\} of X such that \(∩_{n∈N} Oₙ \subseteq R\).

For any subset R of a topological space X we can consider the following two player topological game, called the BM(R)-game. This game is played between two players \(α\) and \(β\) and, as with the G(a)-game, the player \(β\) is always granted the privilege of the first move. To define this game we must first specify the rules and then specify the definition of a win.

The player \(β\)'s first move is to select a nonempty open subset \(B₁\) of X. For \(α\)'s first move he/she must also select a nonempty open subset \(A₁\) of \(B₁\). Now, for \(β\)'s second move he/she must select a nonempty open subset \(B₂\) of \(A₁\). For \(α\)'s second move he/she must select a nonempty open subset \(A₂\)
of $B_2$. In general, if $\alpha$ has chosen $A_n$ as his/her $n^{th}$ move of the BM($R$)-

game then $\beta$ is obliged to select a nonempty open subset $B_{n+1}$ of $A_n$. The

response of $\alpha$ is then simply to select any nonempty open subset $A_{n+1}$ of $B_{n+1}$. Continuing in this fashion indefinitely the players $\alpha$ and $\beta$ produce a

sequence $((B_n, A_n) : n \in \mathbb{N})$ of ordered pairs of nonempty open subsets of $X$
such that $B_{n+1} \subseteq A_n \subseteq B_n$ for all $n \in \mathbb{N}$, called a play of the BM($R$)-game.

A partial play $((B_k, A_k) : 1 \leq k \leq n)$ of the BM($R$)-game consists of the first $n$
moves of a play of the BM($R$)-game. We shall declare $\alpha$ the winner of a play $((B_n, A_n) : n \in \mathbb{N})$ of the BM($R$)-game if $\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} B_n \subseteq R$, otherwise, $\beta$ is declared the winner. That is, $\beta$ is the winner if, and only if, $\bigcap_{n \in \mathbb{N}} B_n \not\subseteq R$.

A strategy for the player $\alpha$ is an inductively defined sequence of functions $t := (t_n : n \in \mathbb{N})$. The domain of $t_1$ is the family of all nonempty open subsets of $X$ and for each $B_1 \in \text{Dom}(t_1)$, $t_1(B_1)$ must be a nonempty open subset of $B_1$ or, equivalently, for each $B_1 \in \text{Dom}(t_1)$, $t_1(B_1)$ is defined so that $((B_1, t_1(B_1)))$ is a partial play of the BM($R$)-game. Inductively, if $t_1, t_2, \ldots, t_n$ have been defined then the domain of $t_{n+1}$ is defined to be:

$$\{ (B_1, B_2, \ldots, B_{n+1}) : (B_1, B_2, \ldots, B_n) \in \text{Dom}(t_n) \text{ and } B_{n+1} \text{ is a nonempty open subset of } t_n(B_1, B_2, \ldots, B_n) \}.$$

For each $(B_1, B_2, \ldots, B_{n+1}) \in \text{Dom}(t_{n+1})$, $t_{n+1}(B_1, B_2, \ldots, B_{n+1})$ must be a nonempty open subset of $B_{n+1}$. Alternatively, but equivalently, for each $(B_1, B_2, \ldots, B_{n+1}) \in \text{Dom}(t_{n+1})$, $t_{n+1}(B_1, B_2, \ldots, B_{n+1})$ is defined so that $((B_k, t_k(B_1, B_2, \ldots, B_k)) : 1 \leq k \leq n + 1)$ is a partial play. A partial $t$-play is a finite sequence $(B_1, B_2, \ldots, B_n)$ such that $(B_1, B_2, \ldots, B_n) \in \text{Dom}(t_n)$ or, equivalently, $B_{k+1}$ is a nonempty open subset of $t_k(B_1, B_2, \ldots, B_k)$ for all $1 \leq k < n$. A $t$-play is an infinite sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(B_1, B_2, \ldots, B_n)$ is a partial $t$-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player $\alpha$ is said to be a winning strategy if each play of the form $((B_n, t_n(B_1, B_2, \ldots, B_n)) : n \in \mathbb{N})$ is won by $\alpha$, or equivalently, if $\bigcap_{n \in \mathbb{N}} B_n \subseteq R$ for each $t$-play $(B_n : n \in \mathbb{N})$. For more information on the BM($R$)-game see [3].

Our interest in the BM($R$)-game is revealed in the next lemma.

**Lemma 4.1** ([9]). Let $R$ be a subset of a topological space $X$. Then $R$ is residual in $X$ if, and only if, the player $\alpha$ has a winning strategy in the BM($R$)-game played on $X$.

The next simple result plays a key role in the proof of our main theorem (Theorem 4.3).

**Lemma 4.2.** Let $X$ and $Y$ be topological spaces and let $O$ be a dense open subset of $X \times Y$. Given nonempty open subsets $V_1, V_2, \ldots, V_m$ of $Y$ and a nonempty open subset $U$ of $X$, there exists a nonempty open subset $W \subseteq U$ and elements $y_i \in V_i, 1 \leq i \leq m$, such that $W \times \{y_1, \ldots, y_m\} \subseteq O$. 
Such that for all \( (\text{BM}, \text{winning strategy for the player } \alpha) \{ \)

Moreover, without loss of generality, we can also assume that all the sets are dense and open in \( X \times Y \) is finite (and hence has the discrete topology) follows from Lemma 4.2. Thus we can assume that all the members of \( F \) are infinite. Moreover, without loss of generality, we can also assume that all the sets \( \{O_n : n \in \mathbb{N} \} \) are decreasing. For each \( a \in Y \), let \( t^n_a := (t^n_a : n \in \mathbb{N}) \) be a winning strategy for the player \( \alpha \) in the \( G(a) \)-game.

We shall inductively define a strategy \( s := (s_n : n \in \mathbb{N}) \) for the player \( \alpha \) in the BM(R)-game played on \( X \), but first let us choose \( y \in Y \), set \( z_{(i,j,0)} := y \) for all \((i,j) \in \mathbb{N}^2\), set \( Z_0 := \{z_{(1,1,0)}\} \) and let \( F_0 \) be any countable subset of \( Y \) such that \( Z \subseteq F_0 \in F \).

Base Step: Suppose that \((B_1)\) is a partial \( s \)-play. We shall define the following:

\( i \) a countable set \( F_1 := \{f_{(1,n)} : n \in \mathbb{N}\} \) such that \( Z_0 \cup F_0 \subseteq F_1 \in F \);

\( ii \) \( s_1(B_1) \) and \( z_{(1,1,1)} \) so that:

\( a \) \( s_1(B_1) \) is a nonempty open subset of \( B_1 \);

\( b \) \( z_{(1,1,1)} \in t^{(1,1)}_1(z_{(1,1,0)}) \), i.e., \( (z_{(1,1,0)}, z_{(1,1,1)}) \in \text{Dom}(t^{(1,1)}_1) \);

\( c \) \( s_1(B_1) \times \{z_{(1,1,1)}\} \subseteq O_1 \).

Note that this is possible by Lemma 4.2.

Finally, define \( Z_1 := \{z_{(1,1,1)}\} \).

Inductive Hypothesis: Suppose that \((B_1, \ldots, B_k)\) is a partial \( s \)-play, and for each \( 1 \leq n \leq k \), the following terms have been defined, \( F_n = \{f_{(n,j)} : j \in \mathbb{N}\} \), \( Z_n = \{z_{(i,j,l)} : (i,j,l) \in \mathbb{N}^3 \) and \( i + j + l \leq n + 2 \) \) and \( s_n \) so that:
Lemma 4.1, Suppose that $\forall$ open subset $U \times V$ then

$$s_n(B_1, ..., B_n) \times \{z_{(i,j,l)} \in \alpha : i + j + l = n + 2\} \subseteq O_n.$$

**Inductive Step:** Suppose that $(B_1, ..., B_{k+1})$ is a partial s-play, that is, $(B_1, ..., B_k) \in \text{Dom}(s_k)$ and $B_{k+1}$ is a nonempty open subset of $s_k(B_1, ..., B_k)$. Then:

1. $Z_k \cup \mathcal{F}_k$ is countable, hence it is contained in some $F \in \mathcal{F}$. Define $\mathcal{F}_{k+1} := \{f_{(k+1,n)} : n \in \mathbb{N}\}$ to be a countable dense subset of $F$;
2. by the inductive hypothesis, $z_{(i,j,0)} \in \text{Dom}(t_{i,j}^{(i,j)})$ for all $i + j + l = k + 2$. By re-indexing and noting $z_{(i,j,0)} \in \text{Dom}(t_{i,j}^{(i,j)})$ for all $i + j = (k + 1) + 2$, we get that $z_{(i,j,0)} \in \text{Dom}(t_{i,j}^{(i,j)})$ for all $i + j + l = (k + 1) + 2$.

Next, we define $s_{k+1}(B_1, ..., B_{k+1})$ and $z_{(i,j,l)}$ for all $i + j + l = (k + 1) + 2$ so that:

(a) $s_{k+1}(B_1, ..., B_{k+1})$ is a nonempty open subset of $B_{k+1}$;

(b) $z_{(i,j,l)} \in t_{i,j}^{(i,j)}(z_{(i,j,0)}, ..., z_{(i,j,l-1)})$ for all $i + j + l = (k + 1) + 2$,

i.e., $(z_{(i,j,0)}, ..., z_{(i,j,l)}) \in \text{Dom}(t_{i,j}^{(i,j)})$ for all $i + j + l = (k + 1) + 2$;

(c) $s_{k+1}(B_1, ..., B_{k+1}) \times \{z_{(i,j,l)} : i + j + l = (k + 1) + 2\} \subseteq O_{k+1}$.

Note that this is possible by Lemma 4.2.

Finally, define $Z_{k+1} := \{z_{(i,j,l)} : i + j + l \leq (k + 1) + 2\}$. This completes the inductive definition of $s$.

Consider an s-play $(B_n : n \in \mathbb{N})$ of the BM($R$)-game played on $X$. For any $x \in \bigcap_{n \in \mathbb{N}} B_n$, let $F_x := \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \in \mathcal{F}$. Clearly, $Z \subseteq F_x$. Let $N \in \mathbb{N}$, we will show that the set $\{y \in F_x : (x, y) \in O_N\}$ is dense in $F_x$. For any open subset $U$ of $Y$ that intersects $F_x$, there is $f_{(i,j)} \in U \cap (\bigcup_{n \in \mathbb{N}} \mathcal{F}_n)$. Since $t_{i,j}^{(i,j)}$ is a winning strategy for the player $\alpha$ in the $G(f_{(i,j)})$-game, there is $m > N$ such that $z_{(i,j,m)} \in U \cap F_x$. Moreover, according to the definition of the strategy $s$, $(x, z_{(i,j,m)}) \in O_{i+j+m-2} \subseteq O_m \subseteq O_N$. Therefore, $\{y \in F_x : (x, y) \in O_N\}$ is dense in $F_x$. Hence $\bigcap_{n \in \mathbb{N}} B_n \subseteq R$, which means $s$ is a winning strategy for the player $\alpha$ is the BM($R$)-game. Hence, by Lemma 4.1, $R$ is residual in $X$. □

**Theorem 4.4.** Suppose that $Y$ is a $W$-space and $X$ is a Baire space. If $Y$ possesses a rich family $\mathcal{F}$ of Baire subspaces then $X \times Y$ is a Baire space. In fact, if $Z$ is any topological space that contains $Y$ as a dense subspace then $X \times Z$ is also a Baire space.

**Proof:** Suppose that $\{O_n : n \in \mathbb{N}\}$ are dense open subsets of $X \times Y$ and $U \times V$ is the product of a nonempty open subset $U$ of $X$ with a nonempty open subset $V$ of $Y$; we will show that $(U \times V) \cap \bigcap_{n \in \mathbb{N}} O_n \neq \emptyset$. To this end, choose $y \in V$ and set $Z := \{y\}$. By the previous theorem there
exists a residual subset $R$ of $X$ such that for each $x \in R$ there exists an $F_x \in \mathcal{F}$ such that (i) $y \in F_x$ and (ii) \{$y' \in F_x : (x, y') \in \bigcap_{n \in \mathbb{N}} O_n$\} is dense in $F_x$. Choose $x_0 \in U \cap R \neq \emptyset$ and $F_{x_0} \in \mathcal{F}$ such that $y \in F_{x_0}$ and \{$y' \in F_{x_0} : (x_0, y') \in \bigcap_{n \in \mathbb{N}} O_n$\} is dense in $F_{x_0}$. In particular, \{$y' \in F_{x_0} : (x_0, y') \in \bigcap_{n \in \mathbb{N}} O_n \cap V \neq \emptyset$\}. Hence, if we choose $y_0 \in \{y' \in F_{x_0} : (x_0, y') \in \bigcap_{n \in \mathbb{N}} O_n \cap V$ then $(x_0, y_0) \in (U \times V) \cap \bigcap_{n \in \mathbb{N}} O_n$. This completes the first part of the proof. To see that $X \times Z$ is a Baire space it is sufficient to realise that $X \times Y$ is a dense Baire subspace of $X \times Z$. \hfill \Box

There are many examples of spaces that admit a rich family of Baire spaces that are not hereditarily Baire. For example, if (i) $X$ is a separable Hausdorff space that is not hereditarily Baire; in which case $\mathcal{F} := \{X\}$ is a rich family of Baire spaces, [1] or (ii) $Y$ is a hereditarily Baire $W$-space such that $Y \times Y$ is not hereditarily Baire, [1], then the family of all nonempty closed separable rectangles gives a rich family of Baire subspaces of $Y \times Y$.

**Corollary 4.5.** Suppose that $\{X_s : s \in S\}$ is a nonempty family of $W$-spaces. If each $X_s$, $s \in S$, possesses a rich family of Baire subspaces $\mathcal{F}_s$, then for each $a \in \Pi_{s \in S} X_s$, $\Sigma_{s \in S} X_s(a)$ is a $W$-space with a rich family of Baire subspaces. In particular, $\Sigma_{s \in S} X_s(a)$ is a Baire space.

**Proof:** The fact that $\Sigma_{s \in S} X_s(a)$ is a $W$-space follows directly from Theorem 2.5. Moreover, from Theorem 3.5 we know that $\Sigma_{s \in S} \mathcal{F}_s(a)$ is a rich family, so it remains to show that all the members of $\Sigma_{s \in S} \mathcal{F}_s(a)$ are Baire spaces. To this end, suppose that $E := \Pi_{s \in S} E_s \in \Sigma_{s \in S} \mathcal{F}_s(a)$. Then $E$ is homeomorphic to $\Pi_{s \in C_E} E_s$. However, by [6, Theorem 3.6] $E$ is a separable first countable space. Therefore, by [8, Theorem 3], $\Pi_{s \in C_E} E_s$ is a Baire space. Finally, the fact that $\Sigma_{s \in S} X_s(a)$ is a Baire space now follows from Theorem 3.3. \hfill \Box

**Corollary 4.6.** Suppose that $\{X_s : s \in S\}$ is a nonempty family of $W$-spaces. If each $X_s$, $s \in S$, possesses a rich family of Baire subspaces $\mathcal{F}_s$ then $\Pi_{s \in S} X_s$ is a Baire space.

**Proof:** This follows directly from Corollary 4.5 since for any $a \in \Pi_{s \in S} X_s$, $\Sigma_{s \in S} X_s(a)$ is a dense Baire subspace. \hfill \Box

As a tribute to Professor I. Namioka, let us end this paper with what is essentially a folklore result, apart from the phrasing in terms of rich families, concerning the Namioka property.

Recall that a Baire space $X$ has the Namioka property if for each compact Hausdorff space $K$ and continuous mapping $f : X \to C_p(K)$ there exists a dense subset $D$ of $X$ such that $f$ is continuous with respect to the $\| \cdot \|_{\infty}$-topology on $C(K)$ at each point of $D$.

**Theorem 4.7.** Suppose that $X$ is a topological space with countable tightness (in particular if $X$ is a $W$-space) that possesses a rich family $\mathcal{F}$ of Baire subspaces then $X$ has the Namioka property.
Proof: In order to obtain a contradiction let us suppose that $X$ does not have the Namioka property. Then there exists a compact Hausdorff space $K$ and a continuous mapping $f : X \to C_p(K)$ that does not have a dense set of points of continuity with respect to the $\| \cdot \|_{\infty}$-topology. In particular, since $X$ is a Baire space (by Theorem 3.3), this implies that for some $\varepsilon > 0$ the open set:

$$O_\varepsilon := \bigcup \{ U \in 2^X : U \text{ is open and } \| \cdot \|_{\infty}\text{-diam}[f(U)] \leq 2\varepsilon \}$$

is not dense in $X$. That is, there exists a nonempty open subset $W$ of $X$ such that $W \cap O_\varepsilon = \emptyset$. For each $x \in X$, let $F_x := \{ y \in X : \| f(y) - f(x) \|_{\infty} > \varepsilon \}$. Then $x \in F_x$ for each $x \in W$. Moreover, since $X$ has countable tightness, for each $x \in W$, there exists a countable subset $C_x$ of $F_x$ such that $x \in C_x$.

Next, we inductively define an increasing sequence of separable subspaces $(F_n : n \in \mathbb{N})$ of $X$ such that:

(i) $W \cap F_1 \neq \emptyset$;

(ii) $\bigcup \{ C_x : x \in D_n \cap W \} \cup F_n \subseteq F_{n+1} \in \mathcal{F}$ for all $n \in \mathbb{N}$, where $D_n$ is any countable dense subset of $F_n$.

Note that since the family $\mathcal{F}$ is rich this construction is possible.

Let $F := \bigcup_{n \in \mathbb{N}} F_n$ and $D := \bigcup_{n \in \mathbb{N}} D_n$. Then $\overline{D} = F \in \mathcal{F}$ and $\| \cdot \|_{\infty}$-diam$[f(U)] \geq \varepsilon$ every nonempty open subset $U$ of $F \cap W$. Therefore, $f|_F$ has no points of continuity in $F \cap W$ with respect to the $\| \cdot \|_{\infty}$-topology. This however, contradicts [10, Theorem 6] which states the every separable Baire space has the Naimoka property. Therefore, the space $X$ must have the Naimoka property. □

This theorem improves upon some results from [4].

References


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