SOME BAIRE SEMITOPOLOGICAL GROUPS THAT ARE TOPOLOGICAL GROUPS

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Dedicated to the memory of Rastislav Telgársky and his significant contribution to the use of games in topology

Abstract. A semitopological group (topological group) is a group endowed with a topology for which multiplication is separately continuous (multiplication is jointly continuous and inversion is continuous). In this paper we show that many Baire semitopological groups are in fact topological groups.

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1 Introduction

A triple $(G, \cdot, \tau)$ is called a *semitopological group* (*topological group*) if $(G, \cdot)$ is a group, (G, τ) is a topological space and the multiplication operation "$\cdot$" is separately continuous on $G \times G$ (jointly continuous on $G \times G$ and the inversion mapping, $g \mapsto g^{-1}$, is continuous on G).

Recall that a function $f : X \times Y \to Z$ that maps from a product of topological spaces X and Y into a topological space Z is said to be *jointly continuous* at a point $(x, y) \in X \times Y$ if for each neighbourhood W of $f(x, y)$ there exists a pair of neighbourhoods U of x and V of y such that $f(U \times V) \subseteq W$. If f is jointly continuous at each point of $X \times Y$ then we say that f is *jointly continuous on* $X \times Y$. A related, but weaker notion of continuity is the following: A function $g : X \times Y \to Z$ that maps from a product of topological spaces X and Y into a topological space Z is said to be *separately continuous on* $X \times Y$ if for each $x_0 \in X$ and $y_0 \in Y$ the functions $y \mapsto g(x_0, y)$ and $x \mapsto g(x, y_0)$ are both continuous on Y and X respectively.

Let us also take this opportunity, just to avoid any possible confusion later, to define what we mean by a *regular* topological space and by a *Baire* topological space. We will say that a topological space (X, τ) is a *regular* topological space if for every closed subset K of X and for every point $x \in X \setminus K$, there exist disjoint open sets U and V of X such that $x \in U$ and $K \subseteq V$ (Note: such spaces may not be T_1) and we shall say that a topological space (X, τ) is a *Baire* space if the intersection of any countable family of dense open sets is again dense in X.

The purpose of this paper is to give an up-to-date account of the following question:

Under what additional topological conditions on (G, τ) is a semitopological group $(G, \cdot, \tau)$ a topological group?

A very thorough and well-written account of this question was recently given in [42] and also in [4]. However, since these articles/books went to print some further advances have been made. In this note we will present some of these advances.

As this article is not targeted at experts in this area, we will start by giving some simple examples of semitopological groups, some of which will turn out to be topological groups.

Example 1. $(\mathbb{R}, +, \tau_S)$, *where τ_S (the Sorgenfrey topology) is the topology on $\mathbb{R}$ generated by the sets $\{[a, b) : a, b \in \mathbb{R}$ and $a < b\}$. Note that in this example $(x, y) \mapsto x + y$ is continuous, $(\mathbb{R}, \tau_S)$ is a Baire space but inversion is not continuous, i.e., $x \mapsto (-x)$ is not continuous.*

The canonical construction of a semitopological group is given next.

Example 2. *Let (X, τ) be a nonempty topological space and let G be a nonempty subset of X^X. If $(G, \circ)$ is a group (where "$\circ$" denotes the binary relation of function composition) and τ_p denotes the topology on X^X of pointwise convergence on X, then $(G, \circ, \tau_p)$ is a semitopological group, provided the members of G are continuous functions.*

Unsurprisingly, not all the semitopological groups described in Example 2 are topological groups.

Example 3. *Let G denote the set of all homeomorphisms on $(\mathbb{R}, \tau_S)$. From Example 2 we see that $(G, \circ, \tau_p)$ is a semitopological group. However, $(G, \circ, \tau_p)$ is not a topological group.*

To see this, define $g_n : \mathbb{R} \to \mathbb{R}$ by, $g_n(x) := [1 + 1/(n + 1)]x$, $a_n := 1 + 1/(2n)$ and

$$f_n(x) := \begin{cases} x & \text{if } x \notin [a_n, a_n + 1/(2n)) \cup [n, n + 1/(2n)) \\ n + (x - a_n) & \text{if } x \in [a_n, a_n + 1/(2n)) \\ a_n + (x - n) & \text{if } x \in [n, n + 1/(2n)). \end{cases}$$

Then both $(f_n : n \in \mathbb{N})$ and $(g_n : n \in \mathbb{N})$ converge pointwise to id - the identity map, however,

$$\lim_{n\to\infty} (f_n \circ g_n)(1) = \lim_{n\to\infty} f_n(g_n(1)) = \infty \neq (id \circ id)(1) = id(1) = 1.$$

This shows that the multiplication operation is not continuous.

On the other hand, sometimes Example 2 does give rise to topological groups.

Example 4. *Let (M, d) be a metric space and let G be the set of all isometries on (M, d). Then $(G, \circ, \tau_p)$ is a topological group.*

Proof. Firstly, let us recall that a local sub-base for the topology τ_p at an element $f \in G$ consists of all sets of the form: $W(f, x, \varepsilon) := \{g \in G : d(f(x), g(x)) < \varepsilon\}$ where $x \in M$ and $\varepsilon > 0$. Using this we shall show that "$\circ$" is continuous on $G \times G$. To this end, let $(f, g) \in G \times G$ and suppose that $x \in M$ and $\varepsilon > 0$ are given. We claim that: $W(f, g(x), \varepsilon/2) \circ W(g, x, \varepsilon/2) \subseteq W(f \circ g, x, \varepsilon)$; which is sufficient to show that "$\circ$" is continuous at (f, g). To prove the claim, suppose that $f' \in W(f, g(x), \varepsilon/2)$ and $g' \in W(g, x, \varepsilon/2)$. Then,

$$\begin{aligned} 0 &\leq d((f' \circ g')(x), (f \circ g)(x)) \\ &\leq d((f' \circ g')(x), (f' \circ g)(x)) + d((f' \circ g)(x), (f \circ g)(x)) \\ &= d(g'(x), g(x)) + d(f'(g(x)), f(g(x))) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Next we need to show that inversion is continuous. To this end, let $f \in G$ and suppose that $x \in M$ and $\varepsilon > 0$ are given. We claim that: $[W(f, f^{-1}(x), \varepsilon)]^{-1} \subseteq W(f^{-1}, x, \varepsilon)$; which is sufficient to show that inversion is continuous at f. To prove the claim, suppose that $f' \in W(f, f^{-1}(x), \varepsilon)$. Then,

$$0 \leq d((f')^{-1}(x), f^{-1}(x)) = d(x, f'(f^{-1}(x)) = d(f(f^{-1}(x)), f'(f^{-1}(x))) < \varepsilon.$$

Thus, $(G, \circ, \tau_p)$ is a topological group. $\square$

Remarks 1. *It is not hard to show that if (M, d) is a compact metric space, then $(G, \circ, \tau_p)$ is in fact a compact topological group (and hence an amenable group).*

Semitopological groups also naturally arise in the study of group actions (topological dynamics).

Example 5. *Let $(G, \cdot)$ be a group and let (X, τ) be a topological space. Further, let $\pi : G \times X \to X$ be a mapping (i.e., a group action) such that:*

(i) *$\pi(e, x) = x$ for all $x \in X$, where e denotes the identity element of G;*

(ii) *$\pi(g \cdot h, x) = \pi(g, \pi(h, x))$ for all $g, h \in G$ and $x \in X$;*

(iii) *for each $g \in G$, the mapping, $x \mapsto \pi(g, x)$, is a continuous function on X.*

Then (G, X) is called a flow on X. If we consider the mapping $\rho : G \to X^X$ defined by,

$$\rho(g)(x) := \pi(g, x) \quad \text{for all } x \in X,$$

then ρ is a group homomorphism and $(\rho(G), \circ, \tau_p)$ is a semitopological group, see Example 2.

For further information on the mapping ρ, the semitopological group $(\rho(G), \circ, \tau_p)$ and the amenability of $(\rho(G), \circ, \tau_p)$ relative to the amenability of the group G, see [10, 31, 39].

Examples 1 and 3 show that, in general, semitopological groups may fail to be topological groups. In fact there are even examples of completely regular pseudo-compact semitopological groups that are not topological groups, see [21, 42]. Hence, it is perhaps natural to ask what additional properties of a semitopological group $(G, \cdot, \tau)$ are required in order that it is a topological group. In this direction one could ask "what additional topological conditions on (G, τ) are sufficient to ensure that a semitopological group $(G, \cdot, \tau)$ is a topological group." Some answers to this question are well-known. For example, in [15, 16] it is shown that if $(G, \cdot, \tau)$ is a semitopological group and (G, τ) is a locally compact Hausdorff space, then $(G, \cdot, \tau)$ is a topological group. However, results of this type go back much earlier than this, to at least 1936, (see [28]), and possibly even earlier. Perhaps the best result of this type, to-date, is the result of A. Bouziad in [8], that every semitopological group $(G, \cdot, \tau)$ for which (G, τ) is Čech-complete, is a topological group. Having said that, E. Reznichenko, has just recently (see [38] and also [42]) made another significant advancement.

In his unpublished manuscript [38] Reznichenko showed that many *paratopological* groups (i.e., a semitopological group whose multiplication is jointly continuous) are in fact topological groups. It is the aim of this paper to extend Reznichenko's approach, using Δ-Baire spaces, to situations where the multiplication operation is not assumed to be jointly continuous (perhaps only quasicontinuous at a single point). The benefit of this is that there are many results in the literature of the type: "Suppose that $f : X \times Y \to Z$ is a separately continuous function and suppose also that X, Y and Z satisfy some topological conditions, then f has a point of quasicontinuity." By using these results we are able to obtain results for semitopological groups rather than just paratopological groups. As this has been a much studied area of research it would be remiss of us not to mention some of the many contributions to this area of research, see [1–9, 11, 13, 15, 16, 19–21, 25–29, 32–35, 37, 40, 42–44], just to name a small selection of them.

We shall proceed from here by first recalling the results of Reznichenko from [38]. We will then show how to extend these results to semitopological groups where the multiplication operation is only "feebly" continuous. After that we will mention some of the existing results concerning when the multiplication operation on a semitopological group is quasicontinuous (and hence feebly continuous) and then finally, we give some examples of Δ-Baire spaces.

Let (X, τ) be a topological space. Following E. Reznichenko, [38] we shall say that a subset $W \subseteq X \times X$ is *separately open, in the second variable,* if for each $x \in X$, $\{z \in X : (x, z) \in W\} \in \tau$ and we shall say that a topological space (X, τ) is a Δ-*Baire space* if for each separately open, in the second variable set W, containing $\Delta_X := \{(x, y) \in X \times X : x = y\}$, there exists a nonempty open subset U of X such that $U \times U \subseteq \overline{W}^{\tau\times\tau}$.

Our first task is to show that there are many Δ-Baire spaces.

Proposition 1 ([38])**.** *Every Baire metric space (X, d) is a Δ-Baire space.*

Proof. Let (X, d) be a Baire metric space and let W be a separately open, in the second variable set, that contains Δ_X. For each $n \in \mathbb{N}$, let

$$F_n := \{x \in X : \{x\} \times B(x; 1/n) \subseteq W\}.$$

Then, since W is a separately open, in the second variable set, containing Δ_X, $X = \bigcup_{n\in\mathbb{N}} F_n$. Now, because (X, d) is a Baire space there exists a $k \in \mathbb{N}$ such that $\mathrm{int}(\overline{F_k}) \neq \varnothing$. Next, let us choose $x_0 \in \mathrm{int}(\overline{F_k})$ and $0 < r < 1/(2k)$ such that $B(x_0; r) \subseteq \mathrm{int}(\overline{F_k})$. Let $U := B(x_0; r)$ and let $x \in U \cap F_k$. Then $U = B(x_0; r) \subseteq B(x; 1/k)$ and so

$$\{x\} \times U = \{x\} \times B(x_0; r) \subseteq \{x\} \times B(x; 1/k) \subseteq W.$$

Since $U \cap F_k$ is dense in U we have that $U \times U \subseteq \overline{W}^{\tau\times\tau}$; which shows that (X, d) is a Δ-Baire space. $\square$

E. Reznichenko's interest, and indeed our interest in Δ-Baire spaces, follows from the next lemma.

Lemma 1 ([38])**.** *Suppose that $(G, \cdot, \tau)$ is a semitopological group, W is an open neighbourhood of e and $\varphi : G \times G \to G$ is defined by $\varphi(h, g) := h^{-1} \cdot g$. If (G, τ) is a Δ-Baire space then there exists a nonempty open subset U of G such that $\varphi(U \times U) = U^{-1} \cdot U \subseteq \overline{W \cdot W}$.*

Proof. Let W be an open neighbourhood of e. Since (G, τ) is a Δ-Baire space and $\varphi^{-1}(W)$ is a separately open, in the second variable set, that contains Δ_G, there exists a nonempty open subset U of G such that $U \times U \subseteq \overline{\varphi^{-1}(W)}^{\tau\times\tau}$. We claim that $\varphi(U \times U) = U^{-1} \cdot U \subseteq \overline{W \cdot W}$. So let us suppose, in order to obtain a contradiction, that there exists an $(x, y) \in U \times U$ such that $\varphi(x, y) = x^{-1} \cdot y \notin \overline{W \cdot W}$ (i.e., $e \notin x \cdot \overline{W \cdot W} \cdot y^{-1}$). From this it follows that there exists an open neighbourhood N of e such that $N \cap (x \cdot W \cdot W \cdot y^{-1}) = \varnothing$, or equivalently, $(x^{-1} \cdot N \cdot y) \cap W \cdot W = \varnothing$. This in turn, implies that $(W^{-1} \cdot x^{-1} \cdot N \cdot y) \cap W = \varnothing$. Thus, we get that:

$$\varphi(x \cdot W \times N \cdot y) \cap W = \left[(x \cdot W)^{-1} \cdot (N \cdot y)\right] \cap W = (W^{-1} \cdot x^{-1} \cdot N \cdot y) \cap W = \varnothing;$$

which is impossible since, $(x \cdot W) \times (N \cdot y) \cap (U \times U) \cap \varphi^{-1}(W) \neq \varnothing$. Hence, $\varphi(U \times U) \subseteq \overline{W \cdot W}$. $\square$

Remarks 2 ([38,42]). *It follows from the previous lemma that if $(G, \cdot, \tau)$ is a paratopological group and (G, τ) is a regular Δ-Baire space, then $(G, \cdot, \tau)$ is a topological group.*

In order to extend this result we need to weaken the hypothesis on the multiplication operation to something weaker than joint continuity.

If $f : (X, \tau) \to (Y, \tau')$ is a mapping acting between topological spaces (X, τ) and (Y, τ') then we say that f is *feebly continuous on* X if for each open subset V of Y such that $V \cap f(X) \neq \varnothing$, $\mathrm{int}[f^{-1}(V)] \neq \varnothing$, [11, 17] and we say that f is *quasicontinuous* at a point $x_0 \in X$ if for every neighbourhood U of x_0 and every neighbourhood W of $f(x_0)$ there exists a nonempty open subset V of U such that $f(V) \subseteq W$, [18].

Proposition 2 ([25, 26]). *If $(G, \cdot, \tau)$ is a semitopological group and the multiplication operation is feebly continuous on $G \times G$ then for each neighbourhood N of e, there exists an open neighbourhood V of e, and an element $n \in N$, such that $V \cdot V \cdot n \subseteq N$.*

Proof. To see this, note that by feeble continuity there exist nonempty open subsets U' and V' such that $V' \cdot U' \subseteq N$. Choose $v \in V'$ and let $V := V' \cdot (v^{-1})$ and $U := v \cdot U'$. Note that V is an open neighbourhood of e and $V \cdot U = V' \cdot U' \subseteq N$. Choose $n \in U$ and note that since $e \in V$, $n \in N$. By possibly making V smaller we can assume that $V \cdot n \subseteq U$. Then $V \cdot V \cdot n \subseteq N$. $\square$

Theorem 1. *Suppose that $(G, \cdot, \tau)$ is a semitopological group and (G, τ) is a regular Δ-Baire space. If the multiplication operation on G is feebly continuous then $(G, \cdot, \tau)$ is a topological group.*

Proof. Given that $(G, \cdot, \tau)$ is a semitopological group and (G, τ) is regular, to prove that $(G, \cdot, \tau)$ is a topological group it is sufficient to show that for each open neighbourhood W of e there exists a nonempty open subset U of G such that $U^{-1} \cdot U \subseteq \overline{W}$.

Let $\varphi : G \times G \to G$ be defined by $\varphi(h, g) := h^{-1} \cdot g$ and let W be an arbitrary open neighbourhood of e. Since (G, τ) is a Δ-Baire space and $\varphi^{-1}(W)$ is a separately open, in the second variable set, that contains Δ_G, there exists a nonempty open set U such that $U \times U \subseteq \overline{\varphi^{-1}(W)}^{\tau \times \tau}$. We claim that $\varphi(U \times U) = U^{-1} \cdot U \subseteq \overline{W}$. So let us suppose, in order to obtain a contradiction, that there exists an $(x, y) \in U \times U$ such that $\varphi(x, y) = x^{-1} \cdot y \notin \overline{W}$ (i.e., $e \notin x \cdot \overline{W} \cdot y^{-1}$). Then we may choose an open neighbourhood N of e such that

(i) $N \cdot y \subseteq U$ and

(ii) $\overline{N} \cap (x \cdot \overline{W} \cdot y^{-1}) = \varnothing$, or equivalently, $(x^{-1} \cdot \overline{N} \cdot y) \cap \overline{W} = \varnothing$.

Now, since multiplication is feebly continuous on $G \times G$ there exists, by Proposition 2, an open neighbourhood V of e and an element $n \in N$ such that $V \cdot V \cdot n \subseteq N$. Therefore, $\overline{V \cdot V} \cdot n \subseteq \overline{N}$. By Lemma 1, there exists an open neighbourhood A of e such that $A^{-1} \cdot A \subseteq \overline{V \cdot V}$. Therefore, $A^{-1} \cdot A \cdot n \subseteq \overline{N}$ and so by (ii) we have that

$$\varnothing = x^{-1} \cdot (A^{-1} \cdot A \cdot n) \cdot y \cap \overline{W} = (A \cdot x)^{-1} \cdot A \cdot (n \cdot y) \cap \overline{W}.$$

Let $y' := n \cdot y$. Then by (i), $y' \in U$ and $\varphi(A \cdot x \times A \cdot y') \cap \overline{W} = \varnothing$. However, this is impossible since

$$(A \cdot x \times A \cdot y') \cap (U \times U) \cap \varphi^{-1}(W) \neq \varnothing.$$

Hence, $\varphi(U \times U) = U^{-1} \cdot U \subseteq \overline{W}$. $\square$

Remarks 3. *The above theorem generalises [25, Theorem 1]. However, the main benefit of the previous theorem is that its proof is much simpler than the corresponding proofs in [25, 26].*

In order to fully exploit Theorem 1 we need to be able to find some natural conditions on (G, τ) that imply that the separately continuous multiplication operation on a semitopological group $(G, \cdot, \tau)$ is feebly continuous.

Suppose that (X, τ), (Y, τ') and (Z, τ'') are topological spaces and $f : X \times Y \to Z$ is a separately continuous function. In the literature there are many results of the type where X is a Baire space (with possibly some additional completeness property), Y contains a q-point, or some generalisation of a q-point, and Z is a regular space, that conclude that there is at least one point where f is quasicontinuous, see [6, 7, 13, 19, 23–25, 27, 33–35]. Versions of this type of result go right back to H. Hahn. However, it is not the purpose of this current paper to delve deeply into the question of when such separately continuous functions are feebly continuous. The interested reader should consult the papers mentioned above.

On the other hand, it would be nice to give at least one concrete, self contained, example where Theorem 1 can be applied to a semitopological group. So we will give a theorem, due to Y. Mibu, which shows that under suitable circumstances a separately continuous function is quasicontinuous. Note that the proof given below is not the same as the one given in [23] where the space Z was assumed to be metrisable.

Lemma 2. *Suppose that (X, τ), (Y, τ'') and (Z, τ''') are topological spaces and $f : X \times Y \to Z$ is a separately continuous mapping. If (X, τ) is a Baire space, (Z, τ''') is regular and $y_0 \in Y$ has a countable local base, then f is quasicontinuous at each point of $X \times \{y_0\}$.*

Proof. Let $x_0 \in X$, W be an open neighbourhood of $f(x_0, y_0)$ and U be an open neighbourhood of x_0. Let $(V_n : n \in \mathbb{N})$ be a countable local base for the topology at y_0. Since, $x \mapsto f(x, y_0)$ is continuous and $f(x_0, y_0) \in W$, we may assume, by possibly making U smaller that $f(U \times \{y_0\}) \subseteq W$. For each $n \in \mathbb{N}$, let $F_n := \{x \in U : f(\{x\} \times V_n) \subseteq W\}$. Then $U = \bigcup_{n \in \mathbb{N}} F_n$. Now, because X is a Baire space there exists a $k \in \mathbb{N}$ such that $\text{int}(\overline{F_k}) \cap U \neq \varnothing$. We claim that if $U' := \text{int}(\overline{F_k}) \cap U$ and $V' := V_k$ then $f(U' \times V') \subseteq \overline{W}$. To see this, consider $(x, y) \in U' \times V'$ such that $f(x, y) \notin \overline{W}$. Since f is separately continuous there exists an open neighbourhood N of x, contained in U' such that $f(N \times \{y\}) \subseteq Z \setminus \overline{W}$. However, this is impossible since $N \cap F_k \neq \varnothing$ and

$$f(N \cap F_k \times \{y\}) \subseteq f(N \cap F_k \times V_k) \subseteq f(F_k \times V_k) \subseteq W.$$

Therefore, it must be the case that $f(U' \times V') \subseteq \overline{W}$. Finally, since Z is regular, this is sufficient to show that f is quasicontinuous at (x_0, y_0). ☐

Corollary 1. *Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is: (i) a first countable Baire space and (ii) a regular Δ-Baire space, then $(G, \cdot, \tau)$ is a topological group. In particular, if (G, τ) is a metrisable Baire space, then $(G, \cdot, \tau)$ is a topological group.*

In the last part of this paper we will give some examples of Δ-Baire spaces. The most economical way of doing this is to use topological games, [12, 36, 41].

The game that we shall consider involves two players which we will call α and β. The "field/court" that the game is played on is a fixed topological space (X, τ). The name of the game is the $\mathcal{G}_R$*-game*. After naming the game we need to describe how to "play" the $\mathcal{G}_R$-game. The player labelled β

starts the game every time (life is not always fair). For his/her first move the player β must select a pair (B_1, B_1^*) consisting of nonempty open subsets B_1 and B_1^* of X. Next, α gets a turn. For α's first move he/she must select a nonempty open subset A_1 of B_1. This ends the first round of the game. In the second round, β goes first again and selects a pair (B_2, B_2^*) consisting of nonempty open subsets B_2 and B_2^* of A_1. Player α then gets to respond by choosing a nonempty open subset A_2 of B_2. This ends the second round of the game. In general, after α and β have played the first n-rounds of the $\mathcal{G}_R$-game, β will have selected pairs $(B_1, B_1^*), (B_2, B_2^*), \ldots, (B_n, B_n^*)$ consisting of nonempty open sets $B_1, B_2, \ldots, B_n$ and $B_1^*, B_2^*, \ldots, B_n^*$ of X and α will have selected nonempty open subsets $A_1, A_2, \ldots, A_n$ such that

$$A_n \subseteq B_n \subseteq A_{n-1} \subseteq B_{n-1} \subseteq \cdots \subseteq A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1.$$

and $B_{k+1}^* \subseteq A_k$ for all $1 \leq k < n$.

At the start of the $(n+1)$-round of the game, β goes first (again!) and selects a pair (B_{n+1}, B_{n+1}^*) consisting of nonempty open subsets B_{n+1} and B_{n+1}^* of A_n. As with the previous n-rounds, the player α gets to respond to this move by selecting a nonempty open subset A_{n+1} of B_{n+1}. Continuing this procedure indefinitely (i.e., continuing on forever) the players α and β produce an infinite sequence $(A_n, (B_n, B_n^*))_{n\in\mathbb{N}}$ called a *play* of the $\mathcal{G}_R$-game. A *partial play* $\big((A_k, (B_k, B_k^*)) : 1 \leq k \leq n\big)$ of the $\mathcal{G}_R$-game consists of the first n-moves of the $\mathcal{G}_R$-game.

As with any game, we need to specify a rule to determine who wins (otherwise, it is a very boring game). We shall declare that α *wins* a play $(A_n, (B_n, B_n^*))_{n\in\mathbb{N}}$ of the $\mathcal{G}_R$-game if:

$$\overline{\left(\bigcup_{n\in\mathbb{N}} B_n^*\right)} \cap \bigcap_{n\in\mathbb{N}} B_n \neq \varnothing.$$

If α does not win a play of the $\mathcal{G}_R$-game then we declare that β *wins* that play of the $\mathcal{G}_R$-game. So every play is won by either α or β and no play is won by both players.

Note that if α wins a play $(A_n, (B_n, B_n^*))_{n\in\mathbb{N}}$ of the $\mathcal{G}_R$-game then $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$.

Continuing further into game theory we need to introduce the notion of a strategy. By a *strategy* t for the player β we mean a *'rule'* that specifies each move of the player β in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for β is an inductively defined sequence of $\tau \times \tau$-valued functions. The domain of t_1 is the sequence of length zero, denoted by $\emptyset$. That is, $\mathrm{Dom}(t_1) := \{\emptyset\}$ and $t_1(\emptyset) \in (\tau \setminus \{\varnothing\}) \times (\tau \setminus \{\varnothing\})$. If $t_1, t_2, \ldots, t_k$ have been defined then the domain of t_{k+1} is:

$$\{(A_1, A_2, \ldots, A_k) \in (\tau \setminus \{\varnothing\})^k : (A_1, A_2, \ldots, A_{k-1}) \in \mathrm{Dom}(t_k)$$
$$\text{and } A_k \subseteq B_k, \text{ where } (B_k, B_k^*) := t_k(A_1, A_2, \ldots, A_{k-1})\}.$$

For each $(A_1, A_2, \ldots, A_k) \in \mathrm{Dom}(t_{k+1})$, $t_{k+1}(A_1, A_2, \ldots, A_k) := (B_{k+1}, B_{k+1}^*) \in (\tau\setminus\{\varnothing\})\times(\tau\setminus\{\varnothing\})$ is defined so that B_{k+1} and B_{k+1}^* are subsets of A_k.

A *partial t-play* is a finite sequence $(A_1, A_2, \ldots, A_{n-1})$ such that $(A_1, A_2, \ldots, A_{n-1}) \in \mathrm{Dom}(t_n)$. A *$t$-play* is an infinite sequence $(A_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $(A_1, A_2, \ldots, A_{n-1})$ is a partial t-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player β is called a *winning strategy* if each play of the form: $(A_n, t_n(A_1, \ldots, A_{n-1}))_{n\in\mathbb{N}}$ is won by β. We will call a topological space (X, τ) a *Reznichenko* space if the player β does **not** have a winning strategy in the $\mathcal{G}_R$-game played on X.

In addition to the $\mathcal{G}_R$-game we will also need to consider the following topological game.

The *Choquet game* $\mathcal{G}$ played on a topological space (X,τ) is similar to the $\mathcal{G}_R$-game played on (X,τ). In the Choquet game the two players α and β alternately choose nonempty open subsets: the A_n's by α and the B_n's by β, in such a way that $B_{n+1} \subseteq A_n \subseteq B_n$ for all $n \in \mathbb{N}$. The player α is declared the *winner* of a play $((A_n, B_n))_{n\in\mathbb{N}}$ if $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$. Otherwise, the player β is declared the winner. Strategies for the players α and β in this game are defined analogously to those in the $\mathcal{G}_R$-game, see [12].

The significance of the Choquet game to this paper is revealed in the next theorem.

Theorem 2 ([14, 22, 30, 40, 41])**.** *A topological space (X,τ) is a Baire space if, and only if, the player β does not have a winning strategy in the Choquet game played on (X,τ).*

It follows from Theorem 2 that every Reznichenko space is a Baire space since, if $t := (t_n : n \in \mathbb{N})$ is a winning strategy for the player β in the Choquet game played on (X,τ) and we write $B_1 := t_1(\emptyset)$ and $B_n := t_n(A_1, \ldots, A_{n-1})$ for all $n \geq 2$ then we can define a winning strategy $t' := (t'_n : n \in \mathbb{N})$ for the player β in the $\mathcal{G}_R$-game played on (X,τ) by, $t'_1(\emptyset) := (B_1, B_1)$ and $t'_n(A_1, \ldots, A_{n-1}) := (B_n, B_n)$ for all $n \geq 2$. However, what is more important, for our current considerations, is the following theorem.

Theorem 3 ([38])**.** *Every Reznichenko space (X,τ) is a Δ-Baire space.*

Proof. We shall start by introducing some notation. Let $\pi_1 : \tau \times \tau \to \tau$ and $\pi_2 : \tau \times \tau \to \tau$ be defined by $\pi_1(A, B) := A$ and $\pi_2(A, B) := B$. We will show that if (X,τ) is not a Δ-Baire space then (X,τ) is not a Reznichenko space, (i.e., the player β has a winning strategy in the $\mathcal{G}_R$-game played on X). Since (X,τ) is not a Δ-Baire space there exists a separately open, in the second variable set W, containing Δ_X, such that for each nonempty open subset U of X, $U \times U \not\subseteq \overline{W}^{\tau\times\tau}$. We will use the set W to inductively define a winning strategy $t := (t_n : n \in \mathbb{N})$ for the player β in the $\mathcal{G}_R$-game played on (X,τ).

Base Step: Define $t_1(\emptyset) := (B_1, B_1^*)$, where B_1 and B_1^* are any nonempty open subsets of X such that $(B_1 \times B_1^*) \cap W = \varnothing$.

Now suppose that $t_1, t_2, \ldots, t_n$ have been defined.

Inductive Step: Suppose that $(A_1, A_2, \ldots, A_n)$ is a partial t-play, i.e., $(A_1, A_2, \ldots, A_{n-1}) \in \mathrm{Dom}(t_n)$ and A_n is a nonempty open subset of $\pi_1(t_n(A_1, A_2, \ldots, A_{n-1}))$. Since $A_n \times A_n \not\subseteq \overline{W}^{\tau\times\tau}$ there exist points $x, y \in X$ such that $(x, y) \in (A_n \times A_n) \setminus \overline{W}^{\tau\times\tau}$. Then since $\overline{W}^{\tau\times\tau}$ is closed there exist open neighbourhoods B_{n+1} of x and B_{n+1}^* of y (that are both contained in A_n) such that $(B_{n+1} \times B_{n+1}^*) \cap W = \varnothing$. Define $t_{n+1}(A_1, A_2, \ldots, A_n) := (B_{n+1}, B_{n+1}^*)$.

This completes the definition to $t := (t_n : n \in \mathbb{N})$.

We claim that $t := (t_n : n \in \mathbb{N})$ is a winning strategy for the player β in the $\mathcal{G}_R$-games played on (X,τ). To this end, let $(A_n : n \in \mathbb{N})$ be a t-play and let $(B_n, B_n^*) = t_n(A, A_2, \ldots, A_{n-1})$ for all $n \in \mathbb{N}$. Suppose, in order to obtain a contradiction, that $\overline{\left(\bigcup_{n\in\mathbb{N}} B_n^*\right)} \cap \bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$. Let $x \in \overline{\left(\bigcup_{n\in\mathbb{N}} B_n^*\right)} \cap \bigcap_{n\in\mathbb{N}} B_n$ and let $W_x := \{y \in X : (x, y) \in W\}$. Then, W_x is an open neighbourhood of x and so $W_x \cap \bigcup_{n\in\mathbb{N}} B_n^* \neq \varnothing$. In particular, for some $k \in \mathbb{N}$, $W_x \cap B_k^* \neq \varnothing$. Let $y \in W_x \cap B_k^*$. Then $(x, y) \in (B_k \times B_k^*) \cap W$; which contradicts the fact that, by the construction of the strategy t, $(B_k \times B_k^*) \cap W = \varnothing$.

This shows that t is indeed a winning strategy for the player β, which in turn, shows that (X,τ) is not a Reznichenko space. $\square$

It is not hard to show that the Bouziad spaces considered in [25], the nearly strongly Baire spaces considered in [26] and the (β, G_{Π})-unfavourable spaces considered in [2] are all Reznichenko spaces. In this way we see that Theorem 1 generalises many of the results contained in [2,25,26].

In order to more easily establish whether a space (X,τ) is a Reznichenko space we need to consider spaces where the player α has a strategy.

By a *strategy* s for the player α we mean a *'rule'* that specifies each move of the player α in every possible situation. More precisely, a strategy $s := (s_n : n \in \mathbb{N})$ for α is an inductively defined sequence of $(\tau \setminus \{\varnothing\})$-valued functions. The domain of s_1 is $(\tau \setminus \{\varnothing\}) \times (\tau \setminus \{\varnothing\})$ and for each $(B_1, B_1^*) \in \mathrm{Dom}(s_1)$, $s_1((B_1, B_1^*))$ is a nonempty open subset of B_1. If $s_1, s_2, \ldots, s_k$ have been defined then the domain of s_{k+1} is:

$$\{((B_1,B_1^*),\ldots,(B_{k+1},B_{k+1}^*)) \in ((\tau\setminus\{\varnothing\})\times(\tau\setminus\{\varnothing\}))^{k+1} : ((B_1,B_1^*),\ldots,(B_k,B_k^*)) \in \mathrm{Dom}(s_k) \\ \text{and } B_{k+1} \cup B_{k+1}^* \subseteq s_k((B_1,B_1^*),\ldots,(B_k,B_k^*))\}.$$

For each $((B_1, B_1^*), \ldots, (B_{k+1}, B_{k+1}^*)) \in \mathrm{Dom}(s_{k+1})$, $s_{k+1}((B_1, B_1^*), \ldots, (B_{k+1}, B_{k+1}^*))$ is a nonempty open subset of B_{k+1}.

A *partial s-play* is a sequence $((B_1, B_1^*), \ldots, (B_n, B_n^*))$ such that $((B_1, B_1^*), \ldots, (B_n, B_n^*)) \in \mathrm{Dom}(s_n)$. An *$s$-play* is an infinite sequence $((B_n, B_n^*))_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $((B_1, B_1^*), \ldots, (B_n, B_n^*))$ is a partial s-play.

A strategy $s := (s_n : n \in \mathbb{N})$ for the player α is called a *winning strategy* if each play of the form: $(s_n((B_1, B_1^*), \ldots, (B_n, B_n^*)), (B_n, B_n^*))_{n\in\mathbb{N}}$ is won by α.

We will say that a space (X,τ) is *conditionally α-favourable* if the player α in the $\mathcal{G}_R$-game played on (X,τ) has a strategy s such that for every s-play $\big((B_n, B_n^*) : n \in \mathbb{N}\big)$ either, $\bigcap_{n\in\mathbb{N}} B_n = \varnothing$ or $\overline{\big(\bigcup_{n\in\mathbb{N}} B_n^*\big)} \cap \bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$.

In order to simply the proof of our final theorem we will give two preliminary results concerning strategies in the $\mathcal{G}_R$-game and Choquet game.

Proposition 3. *Let $t := (t_n : n \in \mathbb{N})$ be a strategy for the player β in the $\mathcal{G}_R$-game played on (X,τ) and let $s := (s_n : n \in \mathbb{N})$ be a strategy for the player α in the $\mathcal{G}_R$-game played on (X,τ). Then there exists a strategy $t' := (t'_n : n \in \mathbb{N})$ for the player β in the $\mathcal{G}_R$-game played on (X,τ) such that, for every t'-play, $(A_n : n \in \mathbb{N})$:*

(i) *$(A_n : n \in \mathbb{N})$ is a t-play and*

(ii) *$\big(t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_n(A_1, A_2, \ldots, A_{n-1}), \ldots\big)$ is an s-play.*

Proof. We shall start by introducing some notation. Let $\pi_1 : \tau \times \tau \to \tau$ and $\pi_2 : \tau \times \tau \to \tau$ be defined by $\pi_1(A, B) := A$ and $\pi_2(A, B) := B$. Suppose that $t := (t_n : n \in \mathbb{N})$ is a strategy for the player β in the $\mathcal{G}_R$-game played on (X,τ) and that $s := (s_n : n \in \mathbb{N})$ is a strategy for the player α in the $\mathcal{G}_R$-game played on (X,τ). We shall define the strategy $t' := (t'_n : n \in \mathbb{N})$ inductively.

Base Step: Define $t'_1(\emptyset)$ by $\pi_1(t'_1(\emptyset)) := s_1(t_1(\emptyset))$ and $\pi_2(t'_1(\emptyset)) := \pi_2(t_1(\emptyset))$. This makes sense since $\big(t_1(\emptyset)\big)$ is a partial s-play.

Now suppose that $t'_1, t'_2, \ldots, t'_n$ have been defined such that:

(i) every partial t'-play of length $(n-1)$ is a partial t-play;

(ii) for every partial t'-play $(A_1, A_2, \ldots, A_{n-1})$, $\big(t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_n(A_1, A_2, \ldots, A_{n-1})\big)$ is a partial s-play.

(iii) $\pi_1(t'_n(A_1, A_2, \ldots, A_{n-1})) := s_n((t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_n(A_1, A_2, \ldots, A_{n-1})))$ and $\pi_2(t'_n(A_1, A_2, \ldots, A_{n-1})) := \pi_2(t_n(A_1, A_2, \ldots, A_{n-1}))$.

Inductive Step: Suppose that $(A_1, A_2, \ldots, A_n)$ is a partial t'-play. That is, $A_n \in \tau$, $(A_1, A_2, \ldots, A_{n-1})$ is a partial t'-play and $\varnothing \neq A_n \subseteq \pi_1(t'_n(A_1, A_2, \ldots, A_{n-1}))$. By assumption $(A_1, A_2, \ldots, A_{n-1})$ is a partial t-play and $\big(t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_n(A_1, A_2, \ldots, A_{n-1})\big)$ is a partial s-play. Therefore,

$$\begin{aligned} A_n \subseteq \pi_1(t'_n(A_1, A_2, \ldots, A_{n-1})) &= s_n(t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_n(A_1, A_2, \ldots, A_{n-1})) \\ &\subseteq \pi_1(t_n(A_1, A_2, \ldots, A_{n-1})). \end{aligned}$$

Hence, $(A_1, A_2, \ldots, A_n)$ is a partial t-play. Furthermore, if $j \in \{1, 2\}$ then

$$\begin{aligned} \pi_j(t_{n+1}(A_1, A_2, \ldots, A_n)) &\subseteq A_n \\ &\subseteq \pi_1(t'_n(A_1, A_2, \ldots, A_{n-1})) \\ &= s_n\big(t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_n(A_1, A_2, \ldots, A_{n-1})\big). \end{aligned}$$

Therefore, $\big(t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_{n+1}(A_1, A_2, \ldots, A_n)\big)$ is a partial s-play. We may now define $t'_{n+1}(A_1, A_2, \ldots, A_n)$ by,

$\pi_1(t'_{n+1}(A_1, A_2, \ldots, A_n)) := s_{n+1}((t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_{n+1}(A_1, A_2, \ldots, A_n))) \subseteq A_n$ and $\pi_2(t'_{n+1}(A_1, A_2, \ldots, A_n)) := \pi_2(t_{n+1}(A_1, A_2, \ldots, A_n)) \subseteq A_n$.

This completes the definition of t'. It is now easy to see, with this construction, that for every t'-play $(A_n : n \in \mathbb{N})$:

(i) $(A_n : n \in \mathbb{N})$ is a t-play and

(ii) $\big(t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_n(A_1, A_2, \ldots, A_{n-1}), \ldots\big)$ is an s-play.

This completes the proof. $\square$

Proposition 4. *Let $t' := (t'_n : n \in \mathbb{N})$ be a strategy for the player β in the $\mathcal{G}_R$-game played on (X, τ). Then there exists a strategy $t'' := (t''_n : n \in \mathbb{N})$ for the player β in the Choquet game $\mathcal{G}$ played on (X, τ) such that every t''-play is a t'-play.*

Proof. We shall start by introducing some notation. Let $\pi_1 : \tau \times \tau \to \tau$ and $\pi_2 : \tau \times \tau \to \tau$ be defined by $\pi_1(A, B) := A$ and $\pi_2(A, B) := B$. Suppose that $t' := (t'_n : n \in \mathbb{N})$ is a strategy for the player β in the $\mathcal{G}_R$-game played on (X, τ). We shall define the strategy $t'' := (t''_n : n \in \mathbb{N})$ inductively.

Base Step: Define $t''_1(\emptyset) := \pi_1(t'_1(\emptyset))$; which is a nonempty open subset of X.

Now suppose that $t''_1, t''_2, \ldots, t''_n$ have been defined such that (i) every partial t''-play of length $n-1$ is a partial t'-play and (ii) $t''_n(A_1, A_2, \ldots, A_{n-1}) := \pi_1(t'_n(A_1, A_2, \ldots, A_{n-1}))$ for every partial t'-play $(A_1, A_2, \ldots, A_{n-1})$.

Inductive Step: Suppose that $(A_1, A_2, \ldots, A_n)$ is a partial t''-play. That is, $A_n \in \tau$, $(A_1, A_2, \ldots, A_{n-1})$ is a partial t''-play and $\varnothing \neq A_n \subseteq t''_n(A_1, A_2, \ldots, A_{n-1})$. By assumption $(A_1, A_2, \ldots, A_{n-1})$ is a partial t'-play and because

$$\varnothing \neq A_n \subseteq t''_n(A_1, A_2, \ldots, A_{n-1}) = \pi_1(t'_n(A_1, A_2, \ldots, A_{n-1})),$$

$(A_1, A_2, \ldots, A_n)$ is a partial t'-play. We may now define

$$t''_{n+1}(A_1, A_2, \ldots, A_n) := \pi_1(t'_{n+1}(A_1, A_2, \ldots, A_n)) \subseteq A_n.$$

This completes the definition of t''. Furthermore, it is easy to see from this construction that every t''-play is a t'-play. $\square$

Theorem 4. *Every conditionally α-favourable Baire space (X, τ) is a Reznichenko space (and hence a Δ-Baire space).*

Proof. Let $t := (t_n : n \in \mathbb{N})$ be a strategy for the player β in the $\mathcal{G}_R$-game played on (X, τ) and let $s := (s_n : n \in \mathbb{N})$ be a strategy for the player α in the $\mathcal{G}_R$-game played on (X, τ) such that for every s-play $\big((B_n, B_n^*) : n \in \mathbb{N}\big)$ either, $\bigcap_{n\in\mathbb{N}} B_n = \varnothing$ or $\overline{\big(\bigcup_{n\in\mathbb{N}} B_n^*\big)} \cap \bigcap_{n\in\mathbb{N}} B_n \neq \varnothing$. Since (X, τ) is conditionally α-favourable such a strategy for α exists. We need to construct a t-play $(A_n : n \in \mathbb{N})$ in the $\mathcal{G}_R$-game, in which α wins.

By Proposition 3 there exists a strategy $t' := (t'_n : n \in \mathbb{N})$ for the player β in the $\mathcal{G}_R$-game played on (X, τ) such that for every t'-play $(A_n : n \in \mathbb{N})$: (i) $(A_n : n \in \mathbb{N})$ is a t-play and (ii) $\big(t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_n(A_1, A_2, \ldots, A_{n-1}), \ldots\big)$ is an s-play. By Proposition 4 there exists a strategy $t'' := (t''_n : n \in \mathbb{N})$ for the player β in the Choquet game played on (X, τ) such that every t''-play is a t'-play. Since (X, τ) is a Baire space, we have, by Theorem 2, the existence of a t''-play $(A_n : n \in \mathbb{N})$ where α wins in the Choquet game played on (X, τ), i.e., where $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. Since every t''-play is a t'-play we have that $(A_n : n \in \mathbb{N})$ is a t'-play. Furthermore, by the properties of the strategy t', we have that $(A_n : n \in \mathbb{N})$ is also a t-play and $\big(t_1(\emptyset), t_2(A_1), t_3(A_1, A_2), \ldots, t_n(A_1, A_2, \ldots, A_{n-1}), \ldots\big)$ is an s-play. By the properties of the strategy s, and the fact that $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$, we must have that α wins the play $((A_n, t_n(A_1, A_2, \ldots, A_{n-1})) : n \in \mathbb{N})$. Hence $(A_n : n \in \mathbb{N})$ is a t-play where α-wins (in the $\mathcal{G}_R$-game played on (X, τ)). $\square$

Remarks 4. *It follows from Corollary 1 that the semitopological group $(\mathbb{R}, +, \tau_S)$ considered in Example 1 is not a Δ-Baire space (although it is a Baire space). Furthermore, it follows from Theorem 4 that $(\mathbb{R}, +, \tau_S)$ is not a conditionally α-favourable space either. On the other hand, the semitopological group $(G, \cdot, \tau)$ given in [21] is pseudocompact and hence α-favourable in the $\mathcal{G}_R$-game. Thus, this space is a Reznichenko space and hence a Δ-Baire space. Therefore, by Theorem 1, the multiplication operation on $(G, \cdot, \tau)$ is not feebly continuous.*

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