A survey on topological games and their applications in analysis

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Dedicated to Petar S. Kenderov on the occasion of his 70th birthday.

Introduction

The purpose of this talk is twofold. This first is to celebrate Petar's contributions to analysis and topology over the past 40 plus years. Secondly, to take this opportunity to try to "sell" the use of topological games as a proof technique within analysis.

Although a combinatorial game was described back at the beginning of the 17^{th} century, the notion of a positional game (i.e., a two player game where the players alternate turns/moves in order to achieve a predefined winning condition) with perfect information (i.e., the players have available to them the same information concerning their next move, at the time of making that move, as they would have at the end of the game) was not formally introduced until the monograph of von Neumann and Morgenstern in 1944. In that monograph the authors considered finite positional games and proved that each such game can be reduced to a matrix game, and moreover, if the finite positional game is one with perfect information, then the corresponding matrix game has a saddle point.

However, infinite positional games with perfect information were discovered a little earlier. In 1935, Stanislaw Mazur proposed a game related to the Baire category theorem, which is described in Problem No. 43 of the Scottish book; its solution given by Stefan Banach is dated August 4, 1935. This game, now known as the Banach-Mazur game, is the first infinite positional game with perfect information.

In this talk we shall restrict ourselves to games that are essentially descendants of the Banach-Mazur game.

The Choquet Game

This game involves two players which we will call α and β. The "field/court" that the game is played on is a fixed topological space (X, τ). The name of the game is the Choquet game and is denoted by, $Ch(X)$.

After naming the game we need to describe how to "play" the $Ch(X)$-game. The player labeled β starts the game every time (life is not always fair). For his/her first move the player β must select nonempty open subset B_1 of X. Next, α gets a turn. For α's first move he/she must select a nonempty open subset A_1 of B_1. This ends the first round of the game.

In the second round, β goes first again and selects a nonempty open subset $B_2 \subseteq A_1$. Player α then gets to respond by choosing a nonempty open subset A_2 of B_2. This ends the second round of the game.

In general, after α and β have played the first n-rounds of the $Ch(X)$-game, β will have selected nonempty open subsets $B_1, B_2, \ldots, B_n$ and α will have selected nonempty open subsets $A_1, A_2, \ldots, A_n$ such that

$$A_n \subseteq B_n \subseteq A_{n-1} \subseteq B_{n-1} \subseteq \cdots \subseteq A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1.$$

At the start of the $(n+1)$-round of the game, β goes first (again!) and selects nonempty open subset B_{n+1} of A_n. As with the previous n-rounds, the player α gets to respond to this move by selecting a nonempty open subset A_{n+1} of B_{n+1}.

Continuing this procedure indefinitely (i.e., continuing on forever) the players α and β produce an infinite sequence

$$\big((A_k, B_k) : k \in \mathbb{N}\big)$$

called a play of the $Ch(X)$-game.

A partial play $\big((A_k, B_k) : 1 \leq k \leq n\big)$ of the $Ch(X)$-game consists of the first n-moves of a play of the $Ch(X)$-game.

As with any game, we need to specify a rule to determine who wins (otherwise, it is a very boring game). We shall declare that α wins a play $\big((A_k, B_k) : k \in \mathbb{N}\big)$ of the $Ch(X)$-game if: $\bigcap_{k\in\mathbb{N}} A_k = \bigcap_{k\in\mathbb{N}} B_k \neq \varnothing$.

If α does not win a play of the $Ch(X)$-game then we declare that β wins that play of the $Ch(X)$-game. So every play is won by either α or β and no play is won by both players.

Continuing further into game theory we need to introduce the notion of a strategy.

By a strategy t for the player β we mean a 'rule' that specifies each move of the player β in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for β is an inductively defined sequence of τ-valued functions. The domain of t_1 is the sequence of length zero, denoted by $\emptyset$. That is, $\mathrm{Dom}(t_1) = \{\emptyset\}$ and $t_1(\emptyset) \in (\tau \setminus \{\varnothing\})$. If $t_1, t_2, \ldots, t_k$ have been defined then the domain of t_{k+1} is:

$$\{(A_1, A_2, \ldots, A_k) \in \tau^k : (A_1, A_2, \ldots, A_{k-1}) \in \mathsf{Dom}(t_k) \text{ and } A_k \subseteq t_k(A_1, A_2, \ldots, A_{k-1}).\}$$

For each $(A_1, A_2, \ldots, A_k) \in \mathsf{Dom}(t_{k+1})$,

$$t_{k+1}(A_1, A_2, \ldots, A_k) := B_{k+1} \in \tau$$

is defined so that $\varnothing \neq B_{k+1} \subseteq A_k$.

A partial t-play is a finite sequence $(A_1, A_2, \ldots, A_{n-1})$ such that $(A_1, A_2, \ldots, A_{n-1}) \in \mathsf{Dom}(t_n)$. A t-play is an infinite sequence $(A_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $(A_1, A_2, \ldots, A_{n-1})$ is a partial t-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player β is called a winning strategy if each play of the form:

$$((A_n, t_n(A_1, \ldots, A_{n-1})) : n \in \mathbb{N})$$

is won by β. Similarly we can define a strategy for α. By a strategy s for the player α we mean a 'rule' that specifies each move of the player α in every possible situation. More precisely, a strategy $s := (s_n : n \in \mathbb{N})$ for α is an inductively defined sequence of τ-valued functions. The domain of s_1 is $\{(B) : B \in \tau\setminus\{\varnothing\}\}$ and for each $B_1 \in \text{Dom}(s_1)$, $s_1(B_1) := A_1 \in \tau$ is defined so that $\varnothing \neq A_1 \subseteq B_1$.

If $s_1, s_2, \ldots, s_k$ have been defined then the domain of s_{k+1} is:

$$\{(B_1, B_2, \ldots, B_{k+1}) \in \tau^{k+1} : (B_1, B_2, \ldots, B_k) \in \text{Dom}(s_k) \\ \text{and } B_{k+1} \subseteq s_k(B_1, B_2, \ldots, B_k)\}.$$

For each $(B_1, B_2, \ldots, B_{k+1}) \in \text{Dom}(s_{k+1})$,

$$s_{k+1}(B_1, B_2, \ldots, B_{k+1}) := A_{k+1} \in \tau$$

is defined so that $\varnothing \neq A_{k+1} \subseteq B_{k+1}$.

A partial s-play is a finite sequence $(B_1, B_2, \ldots, B_n)$ such that $(B_1, B_2, \ldots, B_n) \in \mathrm{Dom}(s_n)$. An s-play is an infinite sequence $(B_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $(B_1, B_2, \ldots, B_n)$ is a partial s-play.

A strategy $s := (s_n : n \in \mathbb{N})$ for the player α is called a winning strategy if each play of the form:

$$\big((s_n(B_1, \ldots, B_n), B_n) : n \in \mathbb{N}\big)$$

is won by α.

Note that since it is not possible for any play of the $Ch(X)$-game to be won by both players, it is not possible for both players to possess a winning strategy in the $Ch(X)$-game. Hence, if for example, the player α has a winning strategy in the $Ch(X)$-game then it is not possible for the player β to have a winning strategy in the $Ch(X)$-game.

A space (X, τ) is called weakly α-favorable if α has a winning strategy in the $Ch(X)$-game.

Given the previous discussion it is natural to ask the question.

"What topological spaces (X, τ) are characterized by the fact that the player β does not have a winning strategy in the $Ch(X)$-game?"

THEOREM: A topological space (X, τ) is a Baire space, (i.e., the intersection of every countable family of dense open sets is dense), if, and only if, the player β does not have a winning strategy in the $Ch(X)$-game.

So clearly every weakly α-favourable space is a Baire space. However, the validity of the converse statement is not clear.

That is, do there exist topological spaces (X, τ) where neither β nor α possess a winning strategy ? The answer is YES.

One way to see this is to first show that if (X, τ) and (Y, τ') are both weakly α-favourable then so is $(X \times Y, \tau \times \tau')$.

Since weakly α-favourable spaces are Baire spaces, the product $X \times Y$ will be a Baire space. However, it is known that there exists Baire spaces (X, τ) and (Y, τ') such that $(X \times Y, \tau \times \tau')$ is not a Baire space. [These spaces are known as barely Baire spaces.]

Hence it follows that at least one of these spaces is a space where neither α nor β has a winning strategy.

More on Strategies

Since strategies play an important role in game theory, they deserve further consideration. Let (X, τ) be a topological space and let us assume that the player α adopts a strategy $s := (s_n : n \in \mathbb{N})$ in the $Ch(X)$-game, then one can consider the space of all s-plays, $P(s)$, endowed with the Baire metric d. That is, if

$$p := (B_n)_{n \in \mathbb{N}} \quad \text{and} \quad p' := (B'_n)_{n \in \mathbb{N}}$$

are two s-plays, then

$$d(p, p') = 0 \quad \text{if} \quad p = p'$$

and otherwise

$$d(p, p') = 1/n$$

where $n := \min\{k \in \mathbb{N} : B_k \neq B'_k\}$.

It is can be shown that $(P(s), d)$ is a complete metric space. The study of this space can lead to a deeper understanding of the strategy s.

It is also possible to compare two strategies.

Given two strategies s and σ for the player α in $Ch(X)$-game, we say that σ refines s, denoted by, $\sigma \preceq s$, if each σ play is an s-play. If $s^1, s^2, \cdots, s^n$ are strategies for the player α in the $Ch(X)$-game then there exists a strategy s for the player α in the $Ch(X)$-game played on X such that s refines each s^j, $1 \leq j \leq n$.

Moreover, we have the following theorem.

THEOREM: Suppose that (X, τ) is a topological space. If $(s^n)_{n\in\mathbb{N}}$ is a countable family of strategies for the player α in the $Ch(X)$-game then there exists a strategy s for the player α in the $Ch(X)$-game such that for each s-play $(B_n)_{n\in\mathbb{N}}$, and each $k \in \mathbb{N}$, $(B_n)_{n\geq k}$ is a s^k-play.

This theorem enables the exposition of several known results concerning the Choquet game to be simplified.

The Structure of Game Theoretic Proofs

The use of Banach-Mazur type games can often simplify the presentation of certain inductive arguments. One can design a game that exactly suits/fits the particular inductive argument under consideration. That is, the game can be tailor made to fit the situation. The proof then divides into two parts. In one part we use the tailor made game to expedite the proof of the inductive argument. Strategies offering an effect way of recording the inductive hypotheses. The other part of the proof is then to determine those space/situations where the game conditions are satisfied. This dividing the proof into two parts is an important feature of the game approach. A good example of this is in the paper by R. Deville and E. Matheron on a solution to the Eikonal equation, where the game considered in that paper exactly isolated the geometric property of the underlying space that was required for the inductive construction of the desired function.

Examples

Games are used in many places within analysis. Some of these are listed below.

- study of the Namioka Property;
- study of weak Asplund spaces and Gâteaux differentiability spaces;
- in the theory of selections (of set-valued mappings);
- optimization of continuous and lower semi-continuous functions;
- active boundaries of set-valued mappings (involves a game defined on filter bases);
- closed graph theorems;
- fragmentability and σ-fragmentability;

- Baire category arguments;
- differentiability theory;
- semi-topological groups/topological groups.

Plus many other places.

Semitopological Groups

As an illustration of how games can be exploited in order to simplify, and at the same time generalize, known results we shall consider the problem of determining when a semitopological group is a topological group.

This example also highlights the way in which the game approach breaks the problem into two parts (as described earlier). In the first part we use a tailor made game to expedite the inductive arguments. Then in the second part we show that certain known topological properties/conditions imply our game theoretic hypotheses.

A semitopological group (topological group) is a group endowed with a topology for which multiplication is separately continuous (multiplication is jointly continuous and inversion is continuous). Ever since the paper of Montgomery in 1936 there has been continued interest in determining topological properties of a semitopological group that are sufficient to ensure that it is a topological group. There have been many significant contributions to this field. One such contribution is due to Ahmed Bouziad (1996) who introduced the use of games to this area and showed that every Čech-complete semitopological group is a topological group. This answered, in the positive, a question raised by Pfister in 1985.

Later in 2001, the authors [KKM] further exploited the game approach to extend the results of Bouziad. In particular, the authors in [KKM] considered spaces where neither player (in a Choquet-type game) possessed a winning strategy.

The way in which one usually exploits the hypothesis/condition that β does not possess a winning strategy is the following.

One uses a proof by contradiction. This is, assume that the conclusion of the statement (that one wants to prove) is false. Then use this additional information to construct a strategy

t for the player β. The fact that t is not a winning strategy for the player β then yields the existence of a play $(A_n)_{n\in\mathbb{N}}$ where α wins. This play $(A_n)_{n\in\mathbb{N}}$ is then used to obtain the requirred contradiction.

To prove our result we need to introduce two new games that are tailor made for the situation.

Let (X, τ) be a topological space and let D be a dense subset of X. The $\mathscr{G}(D)$-game is a two player game. A play of the $\mathscr{G}(D)$-game is a sequence $(A_n, B_n, b_n)_{n\in\mathbb{N}}$ defined inductively in the following way: player β begins by choosing a pair (B_1, b_1) consisting of a nonempty open subset B_1 of X and a point $b_1 \in D$; player α then chooses a nonempty open subset A_1 of B_1. When (A_i, B_i, b_i), $i = 1, 2, \ldots, (n-1)$, have been defined, player β chooses a pair (B_n, b_n) consisting of a nonempty open subset B_n of A_{n-1} and a point $b_n \in A_{n-1} \cap D$. Player α then chooses a nonempty open subset A_n of B_n.

Player α is declared the winner if:

$$\bigcap_{n\in\mathbb{N}}\overline{\{b_k : k \geq n\}} \cap \bigcap_{n\in\mathbb{N}} B_n \neq \varnothing.$$

We shall call a topological space (X, τ) nearly strongly Baire if it is a regular topological space and there exists a dense subset D of X such that the player β does NOT have a winning strategy in the $\mathscr{G}(D)$-game played on X.

In this talk we also consider another game. Let (X, τ) be a topological space, $a \in X$, and let D be a dense subset of X. The $\mathscr{G}_p(a, D)$-game is a two player game. A play of the $\mathscr{G}_p(a, D)$-game is a sequence $(A_n, b_n)_{n\in\mathbb{N}}$ defined inductively in the following way: player β begins by choosing a point $b_1 \in D$; player α then chooses an open neigbourhood A_1 of a. When (A_i, b_i), $i = 1, 2, \ldots, (n-1)$, have been defined, player β chooses a point $b_n \in A_{n-1} \cap D$. Player α then chooses an open neighbourhood A_n of a. Player α is declared the winner if the sequence $(b_n)_{n\in\mathbb{N}}$ has a cluster-point in X.

We shall call a point a a nearly q_D-point if the player α has a winning strategy in the $\mathscr{G}_p(a, D)$-game played on X.

We can now state (and prove) our first result.

LEMMA 1. Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is nearly strongly Baire then for each pair of open neighbourboods U and W of identity element $e \in G$ there exists a nonempty open subset V of U such that $V^{-1} \subseteq W \cdot W \cdot W$.

Proof: Suppose, in order to obtain a contradiction, that there exists a pair of open neighbourhoods U and W of $e \in G$ such that for each nonempty open subset V of U,

$$V^{-1} \not\subseteq W \cdot W \cdot W.$$

From this it follows that for each nonempty open subset V of U and each dense subset D' of V there exists a point $x \in V \cap D'$ such that $x^{-1} \notin W \cdot W$, because otherwise,

$$V^{-1} \subseteq (\overline{V \cap D'})^{-1} \subseteq W \cdot (V \cap D')^{-1} \subseteq W \cdot W \cdot W.$$

Recall that for any nonempty subset A of a semitopological group $(H, \cdot, \tau)$ and any open neighbourhood W of the identity element $e \in H$, $(\overline{A})^{-1} \subseteq W \cdot A^{-1}$.

Now, let D be any dense subset of G such that β does not have a winning strategy in the $\mathscr{G}(D)$-game played on G. We will define a (necessarily non-winning) strategy t for β in the $\mathscr{G}(D)$-game played on G, but first we set, for notational reasons, $A_0 := U$ and $b_0 := e$.

Step 1. Choose $b_1 \in A_0 \cap D$ so that

$$(b_0^{-1} \cdot b_1)^{-1} = b_1^{-1} \notin W \cdot W.$$

Then choose U_1 to be any open neighbourhood of e, contained in $U \cap W$, such that $b_1 \cdot \overline{U_1} \subseteq A_0$. Then define $t_1(\emptyset) := (b_1 \cdot U_1, b_1)$.

Now, suppose that b_j, U_j and $t_j(A_1, \ldots, A_{j-1})$ have been defined for each $1 \leq j \leq n$ so that:

(i) $b_j \in A_{j-1} \cap D$ and $(b_{j-1}^{-1} \cdot b_j)^{-1} \notin W \cdot W$;

(ii) U_j is an open neighbourhood of e, contained in $U \cap W$, such that $b_j \cdot \overline{U_j} \subseteq A_{j-1}$;

(iii) $t_j(A_1, \ldots, A_{j-1}) := (b_j \cdot U_j, b_j)$.

Step $n+1$. Choose $b_{n+1} \in A_n \cap D$ so that

$$(b_n^{-1} \cdot b_{n+1})^{-1} \notin W \cdot W.$$

Note that this is possible since $b_n^{-1} \cdot (A_n \cap D)$ is a dense subset of $b_n^{-1} \cdot A_n$ and

$$b_n^{-1} \cdot A_n \subseteq b_n^{-1} \cdot (b_n \cdot U_n) = U_n \subseteq U.$$

Then choose U_{n+1} to be any neighbourhood of e, contained in $U \cap W$, such that $b_{n+1} \cdot \overline{U_{n+1}} \subseteq A_n$. Finally, define $t_{n+1}(A_1, \ldots, A_n) := (b_{n+1} \cdot U_{n+1}, b_{n+1})$.

Note that:

(i) $b_{n+1} \in A_n \cap D$ and $(b_n^{-1} \cdot b_{n+1})^{-1} \notin W \cdot W$;

(ii) U_{n+1} is an open neighbourhood of e, contained in $U \cap W$, such that $b_{n+1} \cdot \overline{U_{n+1}} \subseteq A_n$;

(iii) $t_{n+1}(A_1, \ldots, A_n) := (b_{n+1} \cdot U_{n+1}, b_{n+1})$.

This completes the definition of t. Since t is not a winning strategy for β there exists a play $(A_n, t_n(A_1, \ldots, A_{n-1}))_{n\in\mathbb{N}}$ where α wins. Let $b_\infty \in \bigcap_{n\in\mathbb{N}} \overline{\{b_k : k \geq n\}} \cap \bigcap_{n\in\mathbb{N}} B_n$. Choose $k \in \mathbb{N}$ so that

$$b_k \in b_\infty \cdot W \subseteq A_{k+1} \cdot W \subseteq b_{k+1} \cdot U_{k+1} \cdot W \subseteq b_{k+1} \cdot W \cdot W.$$

Therefore, $(b_k^{-1} \cdot b_{k+1})^{-1} = b_{k+1}^{-1} \cdot b_k \in W \cdot W$. However, this contradicts the way b_{k+1} was chosen. This completes the proof. $\square$

We now (only) state the next two results, which are proved using games.

LEMMA 2. Let $(G, \cdot, \tau)$ be a semitopological group and let D be a dense subset of G. If (G, τ) is nearly strongly Baire and the identity element $e \in G$ is a nearly q_D-point then the multiplication operation, $(h, g) \mapsto h \cdot g$, is continuous on $G \times G$.

By putting these two results together we obtain the following result.

THEOREM Let $(G, \cdot, \tau)$ be a semitopological group and let D be a dense subset of G. If (G, τ) is nearly strongly Baire and the identity element $e \in G$ is a nearly q_D-point then $(G, \cdot, \tau)$ is a topological group.

We now come to the second part of the game approach, which is to show that our game theoretic hypotheses are satisfied by a large class of spaces.

EXAMPLE Suppose that $\{X_s : s \in S\}$ is a family of nonempty Čech-complete spaces. Then $X := \prod_{s\in S} X_s$ is nearly strongly Baire and each point of X is a nearly q_D-point with respect to some dense subset D of X.

Proof: For each $a \in X = \prod_{s\in S} X_s$ the

Σ-product of $\{X_s : s \in S\}$ with base point a,

denoted $\Sigma_{s\in S} X_s(a)$, is the set of all $x \in X$ such that

$$\{s \in S : x(s) \neq a(s)\}$$

is at most countable. Obviously, for each $a \in X$, $\Sigma_{s\in S} X_s(a)$ is dense in X.

For each $x \in \Sigma_{s\in S} X_s(a)$, let

$$\mathsf{supp}(x) = \{s \in S : x(s) \neq a(s)\}.$$

By considering only those factors X_s such that

$$s \in \bigcup_{1\leq k\leq n} \mathsf{supp}(b_k)$$

we can show that for an arbitrary $a \in X$, the player α has a winning strategy in the $\mathscr{G}(\Sigma_{s\in S} X_s(a))$-game played on X. Furthermore, it follows in a similar way that for each $a \in X$, the player α has a winning strategy in the $\mathscr{G}_p(a, \Sigma_{s\in S} X_s(a))$-game played on X. $\square$

A PDF version of this talk is available at:

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