USCO SELECTIONS OF DENSELY DEFINED SET-VALUED MAPPINGS

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A set-valued mapping \( \Phi : X \to 2^Y \) acting between topological spaces \( X \) and \( Y \) is said to be "lower demicontinuous" if the interior of the closure of the set \( \Phi^{-1}(V) := \{ x \in X : \Phi(x) \cap V \neq \emptyset \} \) is dense in the closure of \( \Phi^{-1}(V) \) for each open set \( V \) in \( Y \).

Čoban, Kenderov and Revalski (1994) showed that for every densely defined lower demicontinuous mapping \( \Phi \) acting from a Baire space \( X \) into subsets of a monotonely Čech-complete space \( Y \), there exist a dense and \( G_\delta \) subset \( X_1 \subseteq X \) and an usco mapping \( G : X_1 \to 2^Y \) such that \( G(x) \subseteq \Phi^*(x) \), for every \( x \in X_1 \), where the mapping \( \Phi^* : X \to 2^Y \) is the extension of \( \Phi \) defined by,

\[
\Phi^*(x) := \bigcap \{ \Phi(W) : W \text{ is a neighbourhood of } x \}.
\]

In this paper we present a proof of the above result with the notion of monotone Čech-completeness replaced by the weaker notion of partition completeness. In addition, we observe that if the range space also lies in Stegall's class then we may assume that the mapping \( G \) is single-valued on \( X_1 \).

1. INTRODUCTION

Selection theorems provide conditions under which there exists a continuous selection for a set-valued mapping. In a recent paper [4], on selection theorems the authors presented a selection theorem for quasi-lower semicontinuous mappings that map from Baire spaces into subsets of topological spaces that are fragmented by complete metrics. In this paper we improve this result by presenting a selection theorem for "lower demicontinuous" mappings that map from Baire spaces into partition complete spaces. Specifically, we show that for a lower demicontinuous mapping \( \Phi \) with closed graph acting from a Baire space \( X \) into a partition complete space \( Y \) there exist a dense and \( G_\delta \) subset \( X_1 \subseteq X \) and an usco mapping \( G : X_1 \to 2^Y \) such that \( G(x) \subseteq \Phi(x) \) for all \( x \in X_1 \). In addition we show that if the range space \( Y \) is partition complete and lies in Stegall's class then the mapping \( G \) may also be assumed to be single-valued on \( X_1 \). We also show that if the domain space \( X \) is \( \alpha \)-favourable and the range space is partition complete and

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belongs to the class of weakly Stegall spaces then the mapping \( G \) is single-valued on an everywhere second category subset of \( X \).

We end this section by giving some definitions, then in Section 2 we present the main result and finally in Section 3 we give some applications of our selection theorem.

Let \((Y, \tau)\) be a regular topological space, endowed with a pseudo-metric \( d \). A filter-base \( \mathcal{F} \) on \( Y \) is said to be \( d \)-Cauchy if for each \( \varepsilon > 0 \) there exists an \( F \in \mathcal{F} \) such that \( d - \text{diam}(F) < \varepsilon \) and the space itself is said to be partition complete if the pseudo-metric \( d \) satisfies the following properties:

\begin{enumerate}[(i)]
    \item every \( d \)-Cauchy filter-base \( \mathcal{F} \) on \( Y \) has a \( \tau \)-cluster point in \( Y \) (that is, \( \bigcap \{ F : F \in \mathcal{F} \} \neq \emptyset \));
    \item \( Y \) is "fragmented" by \( d \), that is, for every \( \varepsilon > 0 \) and every non-empty subset \( A \) of \( Y \) there exists a \( \tau \)-open subset \( B \) of \( Y \) such that \( A \cap B \neq \emptyset \) and \( d - \text{diam}(A \cap B) < \varepsilon \).
\end{enumerate}

(Note: It follows from (i) that in a partition complete space \( \bigcap \{ F : F \in \mathcal{F} \} \) is non-empty and compact for every \( d \)-Cauchy filter-base \( \mathcal{F} \).) The class of partition complete spaces is quite large including all the Čech complete spaces. More details on partition completeness can be found in [5].

\section{Selection Theorem}

Let \( \Phi : X \to 2^Y \) be a set-valued mapping acting from a topological space \( X \) into subsets of a topological space \( Y \). We call the mapping \( \Phi \) lower demicontinuous on \( X \) if for every open set \( V \) in \( Y \), the interior of the closure of the set \( \Phi^{-1}(V) := \{ x \in X : \Phi(x) \cap V \neq \emptyset \} \) is dense in the closure of \( \Phi^{-1}(V) \), that is, \( \text{int}(\Phi^{-1}(V)) \) is dense in \( \Phi^{-1}(V) \). When \( \{ x \in X : \Phi(x) \neq \emptyset \} \) is dense in \( X \), we say \( \Phi \) is densely defined.

**Lemma 1.** Consider a lower demicontinuous mapping \( \Phi \) from a topological space \( X \) into subsets of a topological space \( Y \). For each pair of non-empty open sets \( U \) in \( X \) and \( V \) in \( Y \), the mapping \( \Phi_{(U,V)} \) from \( U \) into subsets of \( V \) defined by, \( \Phi_{(U,V)}(x) := \Phi(x) \cap V \) is a lower demicontinuous mapping on \( U \).

**Proof:** The proof of the lemma follows from the fact that for each open set \( W \subseteq V \), \( \Phi_{(U,V)}^{-1}(W) = \Phi^{-1}(W) \cap U \).

A set-valued mapping \( \Phi : X \to 2^Y \) acting between topological spaces \( X \) and \( Y \) is said to be an usco mapping if for each \( x \in X \), \( \Phi(x) \) is a non-empty compact subset of \( Y \) and for each open set \( W \) in \( Y \), \( \{ x \in X : \Phi(x) \subseteq W \} \) is open in \( X \).

**Theorem 1.** Let \( X \) be a Baire space and \( Y \) be a Hausdorff partition complete space and let \( \Phi \) be a densely defined lower demicontinuous set-valued mapping acting from \( X \) into subsets of \( Y \). Then there exist a dense and \( G_\delta \)-subset \( X_1 \subseteq X \) and an usco mapping \( G : X_1 \to 2^Y \) with \( G(x) \subseteq \Phi^*(x) \) for all \( x \in X_1 \), where the mapping...
\( \Phi^* : X \to 2^Y \) is defined by,

\[
\Phi^*(x) := \bigcap \{ \overline{\Phi(W)} : W \text{ is a neighbourhood of } x \}.
\]

In particular, \( \{ x \in X : \Phi^*(x) \neq \emptyset \} \) is residual in \( X \).

**Proof:** Let \( d \) be the fragmenting pseudo-metric on \( Y \) associated with the partition completeness of \( Y \). To prove our theorem we inductively construct a sequence of families of ordered pairs \( \mathcal{F}^n := \{(U^*_\alpha, \Phi^*_\alpha) : \alpha \in \Lambda^n \} \) consisting of non-empty open subsets \( \{U^*_\alpha : \alpha \in \Lambda^n \} \) of \( X \) and densely defined lower demicontinuous mappings \( \{\Phi^*_\alpha : \alpha \in \Lambda^n \} \) such that for each \( \alpha \in \Lambda^n \), \( \Phi^*_\alpha \) maps \( U^*_\alpha \) into subsets of \( Y \).

**Base Step.** Consider \( \Lambda^0 := \{0\} \), \( U^0 := X \) and \( \Phi^0 := \Phi \) and define,

\[
\mathcal{F}^0 := \{(U^0_\alpha, \Phi^0_\alpha) : \alpha \in \Lambda^0 \} \text{ and } W^0 := \bigcup \{U^0_\alpha : \alpha \in \Lambda^0 \} = X.
\]

For each \( n \in \mathbb{N} \), we require the family \( \mathcal{F}^n \) to have the following properties:

1. \( U^*_\alpha \cap U^*_\beta = \emptyset \) for each \( \alpha \neq \beta, \alpha, \beta \in \Lambda^n \);
2. \( W^n := \bigcup \{U^*_\alpha : \alpha \in \Lambda^n \} \) is dense in \( X \);
3. \( d - \text{diam}[\Phi^*_\alpha(U^*_\alpha)] < 1/n \) for each \( \alpha \in \Lambda^n \);
4. for each \( \alpha \in \Lambda^n \) there exists a \( \beta \in \Lambda^{n-1} \) such that \( U^*_\alpha \subseteq U^*_\beta \) and \( \Phi^*_\alpha(x) \subseteq \Phi^*_\beta^{-1}(x) \) for each \( x \in U^*_\alpha \).

**Step 1.** Consider \( \mathcal{F}^1 := \{(U^1_\alpha, \Phi^1_\alpha) : \alpha \in \Lambda^1 \} \) a family of ordered pairs satisfying the properties \((a_1), (c_1)\) and \((d_1)\) which is maximal with respect to set inclusion. By Zorn’s lemma such a maximal family exists. We shall show that \( \mathcal{F}^1 \) satisfies property \((b_1)\). If \( W^1 := \bigcup \{U^1_\alpha : \alpha \in \Lambda^1 \} \) is not dense in \( X \) then there exists a non-empty open subset \( U \) of \( X \) such that \( W^1 \cap U = \emptyset \). Since \( Y \) is fragmented by \( d \) and \( \Phi^0_\alpha \) is densely defined there exists an open set \( V \) in \( Y \) such that \( \Phi^0_\alpha(U) \cap V \neq \emptyset \) and \( d - \text{diam}[\Phi^0_\alpha(U) \cap V] < 1 \). By the lower demicontinuity of \( \Phi^0_\alpha \) on \( U \) there exists a non-empty open subset \( U' \) of \( U \) such that \( (\Phi^0_\alpha|_{U'}) \) is densely defined and lower demicontinuous on \( U' \) (by Lemma 1). Now \( (U', (\Phi^0_\alpha(U', V)) \notin \mathcal{F}^1 \) and \( \{(U', (\Phi^0_\alpha(U', V)) \} \cup \mathcal{F}^1 \) is a family satisfying the properties \((a_1), (c_1)\) and \((d_1)\). This contradicts the maximality of \( \mathcal{F}^1 \) and hence we may conclude that \( \mathcal{F}^1 \) satisfies property \((b_1)\).

Assuming that we have constructed the families \( \mathcal{F}^k \) in the sequence satisfying the properties \((a_k), (b_k), (c_k)\) and \((d_k)\) up to and including the \( n \)th step, we proceed to construct the next step.

**Step \((n + 1)\).** Consider \( \mathcal{F}^{n+1} := \{(U^{n+1}_\alpha, \Phi^{n+1}_\alpha) : \alpha \in \Lambda^{n+1} \} \) a family of ordered pairs satisfying the properties \((a_{n+1}), (c_{n+1})\) and \((d_{n+1})\) which is maximal with respect to set inclusion. We shall show that \( \mathcal{F}^{n+1} \) satisfies property \((b_{n+1})\). If \( W^{n+1} := \bigcup \{U^{n+1}_\alpha : \alpha \in \Lambda^{n+1} \} \) is not dense in \( X \) then there exists a non-empty open subset \( U \) of \( X \) such that \( W^{n+1} \cap U = \emptyset \). Since \( W^n \) is dense in \( X \), \( W^n \cap U \neq \emptyset \) and so we may assume that \( U \subseteq U^n \).
for some $\beta \in \Lambda^n$. Now since $Y$ is fragmented by $d$ and $\Phi^n_\beta$ is densely defined there exists an open set $V$ in $Y$ such that $\Phi^n_\beta(U) \cap V \neq \emptyset$ and $d - \text{diam}[\Phi^n_\beta(U) \cap V] < 1/(n + 1)
.

By the lower demicontinuity of $\Phi^n_\beta$ on $\mathcal{F}^n(U)$ there exists a non-empty open subset $U'$ of $U$ such that $(\Phi^n_\beta(U', V))$ is densely defined and lower demicontinuous on $U'$ (by Lemma 1).

Clearly, $(U', (\Phi^n_\beta(U', V))) \notin \mathcal{F}^{n+1}$ and $(U', (\Phi^n_\beta(U', V))) \cup \mathcal{F}^{n+1}$ is a family satisfying the properties $(a_{n+1}), (c_{n+1})$ and $(d_{n+1})$. This contradicts the maximality of $\mathcal{F}^{n+1}$ and hence we may conclude that $\mathcal{F}^{n+1}$ satisfies property $(b_{n+1})$. This completes the inductive step.

Let $X_1 := \bigcap_{n=1}^{\infty} W^n$. Clearly $X_1$ is a dense-$G_\delta$ subset of $X$ and for each $x \in X_1$ and $n \in \mathbb{N}$ there exists a unique $a_n(x) \in \Lambda^n$ such that $x \in U^n_{a_n(x)}$. Therefore we can define a set-valued mapping $\Psi : X_1 \to 2^Y$ by,

$$\Psi(x) := \bigcap_{n=1}^{\infty} \Phi^n_{a_n(x)}(U^n_{a_n(x)}).$$

Clearly, $\Psi$ is non-empty and compact-valued since for each $x \in X_1$,

$$\mathcal{F}(x) := \{\Phi^n_{a_n(x)}(U^n_{a_n(x)}) : n \in \mathbb{N}\}$$

is a $d$-Cauchy filter-base on $Y$. So to show that $\Psi$ is an usco, it remains to show that $\Psi$ is upper semicontinuous. To this end, consider $x \in X_1$ and $O$ an open set containing $\Psi(x)$. Since $\Psi(x)$ is compact it will suffice to show that there exists an open neighbourhood $U$ of $x$ such that $\Psi(U) \subseteq \overline{O}$. We claim that for some $n_0 \in \mathbb{N}$, $\Phi^n_{a_{n_0}(x)}(U^n_{a_{n_0}(x)}) \subseteq O$, for otherwise, $\mathcal{F}^*(x) := \{\Phi^n_{a_n(x)}(U^n_{a_n(x)}) \setminus O : n \in \mathbb{N}\}$ would be a $d$-Cauchy filter-base on $Y$ which would have a cluster point in $Y \setminus O$. But this is impossible since,

$$\emptyset \neq \bigcap_{F \in \mathcal{F}^*} \overline{F} \subseteq \bigcap_{F \in \mathcal{F}} \overline{F} = \Psi(x) \subseteq O.$$

Therefore there is some $n_0 \in \mathbb{N}$ such that $\Phi^n_{a_{n_0}(x)}(U^n_{a_{n_0}(x)}) \subseteq O$ and so,

$$\Psi(y) = \bigcap_{n=1}^{\infty} \Phi^n_{a_{n_0}(y)}(U^n_{a_{n_0}(y)}) \subseteq \Phi^n_{a_{n_0}(y)}(U^n_{a_{n_0}(y)}) = \Phi^n_{a_{n_0}(y)}(U^n_{a_{n_0}(y)}) \subseteq \overline{O}$$

for all $y \in U^n_{a_{n_0}(x)} \cap X_1$.

We now define the mapping $G : X_1 \to 2^Y$ by, $G(x) := \Psi(x) \cap \Phi^*(x)$ for all $x \in X_1$. We claim that the mapping $G$ is an usco. Obviously $G$ has a closed graph as both $\Psi$ and $\Phi^*$ have closed graphs. Moreover, as $\text{Gr}(G) \subseteq \text{Gr}(\Psi)$ and $\Psi$ is an usco, we have that $G$ is also an usco (see, [1, page 309]), provided we can show that $G$ has non-empty images. So in order to obtain a contradiction, let us suppose that for some $x_0 \in X_1$, $G(x_0) = \emptyset$.

This means that the non-empty compact set $\{x_0\} \times \Psi(x_0)$ does not intersect the graph of
We consider the Banach-Mazur game played between two players \( \alpha \) and \( \beta \). A play of the game is a decreasing sequence of, alternately chosen, non-empty open subsets \( A_n \subseteq B_n \subseteq \ldots B_2 \subseteq A_1 \subseteq B_1 \), where the sets \( A_n \) are chosen by player \( \alpha \) and the sets \( B_n \) by player \( \beta \). Player \( \alpha \) is said to have won a play of the game if \( \bigcap_{n \in \mathbb{N}} A_n \neq \emptyset \). Otherwise player \( \beta \) is said to have won the play. A strategy \( s \) for player \( \alpha \) is a rule that tells him or her how to play (possibly depending on all the previous moves of player \( \beta \)). Since the moves of player \( \alpha \) may depend on the moves of player \( \beta \), we denote the \( n \)th move of player \( \alpha \) by \( s(B_1, B_2, \ldots, B_n) \). We say that \( s \) is a winning strategy, if using it, player \( \alpha \) wins every play, independently of the moves of player \( \beta \). More information on Banach-Mazur game can be found in [6].

**Corollary 1.** Let \( X \) be a Baire (an \( \alpha \)-favourable) space and \( Y \) be a partition complete space that lies in Stegall’s class (the class of weakly Stegall spaces). Suppose that \( \Phi : X \to 2^Y \) is a densely defined lower demicontinuous mapping with closed graph. Then there exist a residual (everywhere second category) set \( X_1 \subseteq X \) and a continuous selection \( \sigma : X_1 \to Y \) of \( \Phi \) on \( X_1 \).

**Proof:** First we shall consider the case when \( X \) is a Baire space, \( Y \) is partition complete and in Stegall’s class. From Theorem 1 there exists an usco mapping \( G : R \to 2^Y \) acting from a residual subset \( R \) of \( X \) into \( Y \) such that \( G(x) \subseteq \Phi(x) \) for all \( x \in R \). As every usco mapping contains a minimal usco mapping (see, [2, page 649]), the mapping \( G \) contains a minimal usco mapping \( S : R \to 2^Y \). Now since the range space \( Y \) belongs
to Stegall's class the mapping \( S \) is single-valued on a residual subset \( X_1 \subseteq \mathbb{R} \). The restriction of the mapping \( S \) to the set \( X_1 \) gives rise to the desired selection of \( \Phi \) on \( X_1 \). In the case when the space \( Y \) belongs to the class of weakly Stegall spaces and \( X \) is \( \alpha \)-favourable the proof follows in a similar fashion except that one requires the additional fact that a residual subset of an \( \alpha \)-favourable space is again \( \alpha \)-favourable.

3. Applications

We say that a mapping \( f : X \rightarrow Y \) from a topological space \( X \) into a topological space \( Y \) is **demi-open** if for every open set \( U \) in \( X \) the set \( \text{int} f(U) \) is dense in \( f(U) \). It is easy to verify that \( f^{-1} : Y \rightarrow 2^X \) is lower demicontinuous on \( Y \) if the mapping \( f : X \rightarrow Y \) is demi-open on \( X \).

**Corollary 2.** Let \( f : X \rightarrow Y \) be a demi-open mapping with closed graph acting from a partition complete space \( X \) which lies in Stegall’s class (the class of weakly Stegall spaces) into a dense subset of a Baire space (an \( \alpha \)-favourable space) \( Y \). Then there exists a continuous mapping \( \sigma \) from a residual (everywhere second category) subset \( Y_1 \subseteq Y \) into \( X \) such that \( (f \circ \sigma)(x) = x \) for all \( x \) in \( Y_1 \).

**Proof:** Let us consider the inverse mapping \( f^{-1} : Y \rightarrow 2^X \). This is a densely defined lower demicontinuous mapping with closed graph. Hence from Corollary 1, there exist a residual (everywhere second category) subset \( Y_1 \subseteq Y \) and a continuous selection \( \sigma : Y_1 \rightarrow X \) of \( f^{-1} \) on \( Y_1 \). It follows then that \( (f \circ \sigma)(x) = x \) for all \( x \) in \( Y_1 \).

**Corollary 3.** Let \( h : G \rightarrow K \) be a homomorphism acting from a partition complete group \( G \) into a Baire topological group \( K \). If \( h \) is demi-open, has a closed graph and dense range then the mapping is open and onto \( K \).

**Proof:** The inverse mapping \( h^{-1} : K \rightarrow 2^G \) is densely defined and lower demicontinuous with closed graph. Hence by Theorem 1 the domain of \( h^{-1} \) is residual in \( K \), that is, the range of \( h \) is residual in \( K \). However, as \( h(G) \) is a subgroup of \( K \) it must be the case that \( h(G) = K \). To show that \( h \) is open it suffices to show that for each non-empty open set \( U \) in \( G \), \( h(U) \) is somewhere residual in \( K \) and this follows by applying Theorem 1 to the inverse of the restriction of \( h \) to \( U \).

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