Fixed Point Theorems and Applications

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What is a fixed point theorem?

The first question you might ask yourself is:

"What is a fixed point theorem?"

Answer: Given a nonempty set K and a function $f : K \to K$ (i.e., a self-map) a fixed point theorem gives conditions on K, or f, or both K and f, such that we are guaranteed at least one point $x \in K$ such that $f(x) = x$.

The point x, above, is called a fixed point of f.

Next, we consider some simple examples of self-mappings that do not possess any fixed points.

Examples

(a) Suppose that c is a nonzero number. Let $f_c : \mathbb{R} \to \mathbb{R}$ be defined by,

$$f_c(x) := x + c.$$

Then f_c has no fixed points.

(b) Let $f : [-1, 1] \setminus \{0\} \to [-1, 1] \setminus \{0\}$ be defined by,

$$f(x) := -x.$$

Then f has no fixed points.

(c) Let $K := \{(x, y) \in \mathbb{R}^2 : 0 < \|(x, y)\| \leq 1\}$ and let $f_\vartheta : K \to K$ be the rotation through ϑ radians ($0 < \vartheta < 2\pi$). Then f_ϑ has no fixed points.

These examples show that some extra conditions are required, other than the continuity of f, in order to obtain fixed points.

Some famous fixed point theorems

Let us first recall a basic definition.

A metric space (M, d) in which every Cauchy sequence converges is called complete.

Examples: $(\mathbb{R}^n, \| \cdot \|_2)$ - Euclidean space; $(C[a, b], \| \cdot \|_\infty)$ and $(\ell_2(\mathbb{N}), \| \cdot \|_2)$.

Banach's Fixed Point Theorem

Let (M, d) be a complete metric space and let $f : M \to M$. If there exists an $0 < r < 1$ such that $d(f(x), f(y)) \leq rd(x, y)$ for all $x, y \in M$ then f has a unique fixed point.

Brouwer's Fixed Point Theorem

Let K be a nonempty compact convex subset of $\mathbb{R}^n$ and suppose that $f : K \to K$ is continuous then f has a fixed point.

This theorem was generalized to Banach spaces by Schauder in 1930 and to locally convex spaces by Tychonoff in 1935.

Tychonoff's Fixed Point Theorem

Let K be a nonempty compact convex subset of a locally convex space and let $f : K \to K$ be continuous. Then f has a fixed point.

Recall that a space is called locally convex if it is a vector space with a topology such that vector addition is continuous, scalar multiplication is continuous, and 0 has a local base of convex open subsets.

There are also some set-valued versions of these theorems.

Why do people care about fixed point theorems?

Answer: Because they have many applications.

Examples:

(a) Newton-Raphson method for finding roots of functions on $\mathbb{R}$;

(b) solutions to ordinary differential equations;

(c) implicit and inverse function theorems;

(d) existence of steady states for Markov chains;

(e) invariant measures in dynamical systems;

(f) existence of equilibrium points in abstract economies.

DOWN SIDE

Fixed point theorems are usually hard to prove!!

SOME EASY CASES

Proof of Banach's fixed point theorem

First we consider the case when:

$$\operatorname{diam}(M) := \sup\{d(x,y) : x, y \in M\} < \infty.$$

For each $n \in \mathbb{N}$, let $C_n := f^n(M)$. Then

$$C_{n+1} = f^{n+1}(M) = f^n(f(M)) \subseteq f^n(M) = C_n$$

for all $n \in \mathbb{N}$. Therefore, $\{C_n : n \in \mathbb{N}\}$ is a decreasing sequence of nonempty subsets of M.

Next, notice that

$$0 \le \operatorname{diam}(C_{n+1}) \le r \cdot \operatorname{diam}(C_n) \quad \text{for all } n \in \mathbb{N}$$

and so, by induction,

$$0 \leq \operatorname{diam}(C_{n+1}) \leq r^n \cdot \operatorname{diam}(M) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $\lim_{n\to\infty} \operatorname{diam}(\overline{C_n}) = \lim_{n\to\infty} \operatorname{diam}(C_n) = 0$. It then follows from Cantor's intersection property that

$$\bigcap_{n\in\mathbb{N}} \overline{C_n} = \{x\} \quad \text{for some } x \in M.$$

Moreover, since $x \in \overline{C_n}$,

$$f(x) \in f(\overline{C_n}) \subseteq \overline{f(C_n)} = \overline{C_{n+1}} \subseteq \overline{C_n},$$

$f(x) \in \bigcap_{n\in\mathbb{N}} \overline{C_n} = \{x\}$. That is, $f(x) = x$.

In the case when $\operatorname{diam}(M) = \infty$ some extra work is required.

In this case we choose any $x_0 \in M$ and let

$$M' := \overline{\{f^n(x_0) : n \in \mathbb{N}\}}.$$

Then $f(M') \subseteq M'$ and

$$\operatorname{diam}(M') \leq \frac{d(f(x_0), x_0)}{(1 - r)} < \infty.$$

Hence from the previous argument there exists a point $x \in M' \subseteq M$ such that $f(x) = x$. ☺

Proof of Brouwer's fixed point theorem in $\mathbb{R}$

Let $K := [a, b]$ and consider the function $g : [a, b] \to \mathbb{R}$ defined by,

$$g(x) := f(x) - x.$$

If $f(a) = a$ or $f(b) = b$ then we are done. So suppose that $a < f(a)$ and $f(b) < b$. Then $0 < g(a)$ and $g(b) < 0$. Therefore, by the intermediate value theorem, there exists $c \in [a, b]$ such that $g(c) = 0$, or equivalently, $f(c) = c$. ☺

Proof of Tychonoff's fixed point theorem in the affine case

Let us recall that an affine mapping $f : K \to K$ is any mapping such that:

$$T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y)$$

for all $x, y \in K$ and $0 \leq \lambda \leq 1$

Let us just prove this in the case of a normed linear space $(X, \|\cdot\|)$. Let x be any element of K and for each $n \in \mathbb{N}$ define,

$$x_n := \frac{T(x) + T^2(x) + \cdots + T^n(x)}{n}.$$

Then $x_n \in K$ for each $n \in \mathbb{N}$ and

$$T(x_n) = \frac{T^2(x) + T^3(x) + \cdots + T^{n+1}(x)}{n}.$$

Thus,

$$T(x_n) - x_n = \frac{1}{n}\sum_{j=1}^{n}[T^{j+1}(x) - T^j(x)] = \frac{1}{n}[T^{n+1}(x) - T(x)].$$

Therefore,

$$\begin{aligned} \|T(x_n) - x_n\| &= \frac{\|T^{n+1}(x) - T(x)\|}{n} \\ &\leq \frac{\|T^{n+1}(x)\| + \|T(x)\|}{n} \\ &\leq \frac{2M}{n} \end{aligned}$$

where $M := \sup\{\|y\| : y \in K\} < \infty$.

Now since K is compact $(x_n : n \in \mathbb{N})$ has a convergent subsequence $(x_{n_k} : k \in \mathbb{N})$. Let $x_\infty = \lim_{n\to\infty} x_{n_k} \in K$. Then,

$$\begin{aligned} \|T(x_\infty) - x_\infty\| &= \|T(\lim_{n\to\infty} x_{n_k}) - \lim_{k\to\infty} x_{n_k}\| \\ &= \lim_{k\to\infty} \|T(x_{n_k}) - x_{n_k}\| \\ &\leq \lim_{k\to\infty} \frac{2M}{n_k} = 0. \end{aligned}$$

That is, $T(x_\infty) = x_\infty$. ☺

Application

Let $P := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}$ is a probability vector$\}$, that is, if $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ then $0 \leq x_i \leq 1$ for all $1 \leq i \leq n$ and $\sum_{i=1}^n x_i = 1$.

Then P is a compact convex subset of $\mathbb{R}^n$. Let A be any $n \times n$ matrix such that $AP \subseteq P$ (such matrices are called stochastic matrices and are characterized by the fact that their column vectors are probability vectors). Then A has a fixed point, i.e., there exists a probability vector $\boldsymbol{x}$ such that

$$A\boldsymbol{x} = \boldsymbol{x}.$$

This shows that every Markov chain has a steady state.

Common fixed point theorems

Suppose that K is a nonempty compact convex subset of a locally convex space. Suppose also that $\mathcal{S}$ is a family of mappings from K into K. Then we say that a point $x \in K$ is a common fixed point of $\mathcal{S}$ if $s(x) = x$ for all $s \in \mathcal{S}$.

Markov-Kakutani Fixed Point Theorem

Suppose that K is a nonempty compact convex subset of a locally convex space. Suppose also that $\mathcal{S}$ is a commuting family (i.e., $s \circ s' = s' \circ s$ for all $s, s' \in \mathcal{S}$) of continuous affine self-mappings on K. Then there is a common fixed point of $\mathcal{S}$ in K.

Proof of the Markov-Kakutani fixed point theorem

For each $s \in \mathcal{S}$ let $F_s := \{x \in K : s(x) = x\}$. From the previous theorem we know that $F_s \neq \varnothing$ (and it is easy to see that F_s is compact and convex). So to restate the conclusion of the Markov-Kakutani theorem we must show that $\bigcap_{s \in \mathcal{S}} F_s \neq \varnothing$. Since K is compact, we have, by the finite intersection property (for compact sets) that we need only show that $\bigcap_{s \in S'} F_s \neq \varnothing$ for each nonempty finite subset S' of $\mathcal{S}$. To this end, let $S' := \{s_1, s_2, \ldots, s_n\}$ be a nonempty finite subset of $\mathcal{S}$. We shall proceed by induction.

Let x be any element of F_{s_1} then

$$s_1(s_2(x)) = s_2(s_1(x)) = s_2(x).$$

That is, $s_2(x)$ is a fixed point of s_1 and so $s_2(x) \in F_{s_1}$. Thus, $s_2(F_{s_1}) \subseteq F_{s_1}$. Hence, by the previous theorem, s_2 has a fixed point in F_{s_1}. Therefore, $F_{s_1} \cap F_{s_2} \neq \varnothing$.

Now, suppose that

$$F_{s_1} \cap F_{s_2} \cap \cdots \cap F_{s_j} \neq \varnothing \quad \text{where, } 1 \leq j < n$$

Let $K' := F_{s_1} \cap F_{s_2} \cap \cdots \cap F_{s_j}$. Then K' is nonempty, compact and convex. Let x be any element of K' and let $1 \leq i \leq j$ then,

$$s_i(s_{j+1}(x)) = s_{j+1}(s_i(x)) = s_{j+1}(x)$$

That is, $s_{j+1}(x)$ is a fixed point of s_i and so $s_{j+1}(x) \in F_{s_i}$. Since $1 \leq i \leq j$ was arbitrary,

$$s_{j+1}(x) \in F_{s_1} \cap F_{s_2} \cap \cdots \cap F_{s_j} = K'.$$

Thus, $s_{j+1}(K') \subseteq K'$. Hence, by the previous theorem, s_{j+1} has a fixed point in K'. Therefore,

$$F_{s_1} \cap F_{s_2} \cap \cdots \cap F_{s_j} \cap F_{s_{j+1}} \neq \varnothing.$$

By induction, we see that $\bigcap_{s \in S'} F_s \neq \varnothing$. This completes the proof. ☺

Application

Every locally compact Hausdorff Abelian topological group has a non-zero translation invariant positive measure.

Counter-Example

In 1969 Boyce and Hunke independently gave examples of two continuous functions $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ that commute but have no common fixed points. Hence we cannot simply remove the affine hypothesis.

Removing Commutativity

To remove commutativity we need to work harder. We need to use more geometry.

Geometry

Let A be a nonempty subset of a vector space X. Then we say that a point $x \in A$ is an extreme point of A if whenever $y, z \in A$ and $x = (1/2)(x + y)$ then $x = y = z$. We shall denote by $\mathrm{Ext}(A)$ the set of all extreme points of A.

We need the following famous theorems.

Krein-Milman Theorem

Every nonempty compact convex subset of a Hausdorff locally convex space has an extreme point.

Milman's Theorem

Let K be a compact convex subset of a locally convex space. If $B \subseteq K$ is a closed subset of K and $\overline{\mathrm{co}}(B) = K$ then $\mathrm{Ext}(K) \subseteq B$.

We shall also need the technical notion of distality. Let K be a nonempty compact convex subset of a locally convex space X and let $\mathcal{S}$ be a family of continuous affine mapping on K (into K). Then $\mathcal{S}$ is said to be distal if for each distinct $x, y \in K$, $0 \notin \overline{\{s(x) - s(y) : s \in \mathcal{S}\}}$.

$\mathcal{S}$ is said to be a semi-group if $s' \circ s \in \mathcal{S}$ for each $s', s \in \mathcal{S}$. That is, $\mathcal{S}$ is closed under composition.

Hahn's Fixed Point Theorem (1967)

Let K be a nonempty compact convex subset of a locally convex space. Suppose that $\mathcal{S}$ is a semi-group of continuous affine mappings on K. If $\mathcal{S}$ is distal then $\mathcal{S}$ has a common fixed point.

Proof of Hahn's fixed point theorem

By Zorn's lemma there is a minimal compact convex subset M of K such that $s(M) \subseteq M$ for all $s \in \mathcal{S}$. If M is a singleton then we are done. If not, we can choose distinct $x, y \in M$. Let $z := (1/2)(x+y) \in M$ and let $Q := \overline{\{s(z) : s \in \mathcal{S}\}}$. Then since $\mathcal{S}$ is a semi-group, $s(Q) \subseteq Q$ for all $s \in \mathcal{S}$. By the minimality of M, $M = \overline{\text{co}}(Q)$, since $s(\overline{\text{co}}(Q)) \subseteq \overline{\text{co}}(Q)$ for each $s \in \mathcal{S}$. Let u be an extreme point of M (which exists by the Krein-Milman theorem). Then $u \in Q$ by Milman's theorem. Hence there is a net (s_α) in $\mathcal{S}$ such that $\lim_\alpha s_\alpha(z) = u$. By passing to a subnet we may assume that $\lim_\alpha s_\alpha(x) =: a \in M$ and $\lim_\alpha s_\alpha(y) =: b \in M$ exist. Since each s_α is affine

$$u = \lim_\alpha s_\alpha(z) = \lim_\alpha \frac{1}{2}(s_\alpha(x) + s_\alpha(y)) = \frac{1}{2}(a+b).$$

Since u is an extreme point of M, $a = u = b$. It then follows that $0 = \lim_\alpha(s_\alpha(x) - s_\alpha(y)) \in \overline{\{s(x) - s(y) : s \in \mathcal{S}\}}$. Since $x \neq y$ this contradicts the assumption that $\mathcal{S}$ is distal. Therefore, M must be a singleton. ☺

Application

Every compact Hausdorff topological group $(G,\cdot,\tau)$ admits a left translation-invariant probability measure. That is, a probability measure μ defined on the Borel subsets of G such that $\mu(gA)=\mu(A)$ for every $g\in G$ and Borel subset A of G.

Proof: Let $X := C(G)$ - the space or all real-valued continuous functions on G and consider the Banach space $(X, \|\cdot\|_\infty)$. Let $K := \{x^* \in X^* : \|x^*\| = x^*(\mathbf{1}) = 1\}$, where $\mathbf{1} : G \to \mathbb{R}$ is defined by, $\mathbf{1}(g) := 1$ for all $g \in G$. Then K is a nonempty compact convex subset of (X^*, weak^*). For each $g \in G$ define $S_g : X \to X$ by, $S_g(f)(x) := f(gx)$ for all $x \in G$. Then S_g is linear and $\|S_g(f)\|_\infty = \|f\|_\infty$ for each $f \in X$. Furthermore, for each fixed $f \in X$, the mapping $g \mapsto S_g(f)$ from (G,τ) into $(X, \|\cdot\|_\infty)$ is continuous.

Next, for each $g \in G$ define $T_g : K \to K$ by,

$$T_g(x^*) := x^* \circ S_g.$$

Then T_g is weak*-to-weak* continuous, affine and

$$\{T_g : g \in G\}$$

forms a semi-group (under composition). We claim that $\{T_g : g \in G\}$ is distal. To see this, consider $x^*, y^* \in K$ with $x^* \neq y^*$. Now, for each $z^* \in K$, the mapping, $g \mapsto T_g(z^*)$, from (G, τ) into (K, weak^*) continuous. [Note: $T_g(z^*)(f) = z^*(S_g(f))$, $g \mapsto S_g(f)$ is continuous and so is $z^* : X \to \mathbb{R}$] Therefore the mapping $g \mapsto T_g(x^*) - T_g(y^*)$ is continuous. Since G is compact, $\{T_g(x^*) - T_g(y^*) : g \in G\}$ is compact and so

$$\{T_g(x^*) - T_g(y^*) : g \in G\} = \overline{\{T_g(x^*) - T_g(y^*) : g \in G\}}.$$

Further,

$$\begin{aligned} \|T_g(x^*) - T_g(y^*)\| &= \|x^* \circ S_g - y^* \circ S_g\| \\ &= \|(x^* - y^*) \circ S_g\| = \|x^* - y^*\| > 0 \end{aligned}$$

since $x^* \neq y^*$. Therefore,

$$0 \notin \{T_g(x^*) - T_g(y^*) : g \in G\} = \overline{\{T_g(x^*) - T_g(y^*) : g \in G\}}$$

Hence by Hahn's fixed point theorem, $\{T_g : g \in G\}$ has a common fixed point. It then follows from Riesz's representation theorem that this corresponds to a left translation invariant probability measure. ☺

—————————— The End ——————————