An Elementary Proof of James' Characterisation of weak Compactness

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Background

The purpose of this talk is to give a self-contained proof of James' characterisation of weak compactness (in the case of separable Banach spaces). The proof is completely elementary and does not require recourse to integral representations nor Simons' inequality. It only requires results from linear topology (in particular the Krein-Milman Theorem) and Ekeland's variational principle (Bishop-Phelps Theorem).

The idea of the proof is due to V. Fonf, J. Lindenstrauss and R. Phelps.

Proposition 1 *Let $\{K_j : 1 \le j \le n\}$ be convex subsets of a vector space V. Then*

$$\mathit{co}\bigcup_{j=1}^{n} K_j = \left\{\sum_{j=1}^{n} \lambda_j k_j : (\lambda_j, k_j) \in [0,1] \times K_j \right.$$

$$\left. \text{for all } 1 \le j \le n \text{ and } \sum_{j=1}^{n} \lambda_j = 1 \right\}.$$

From this we may easily obtain the following result.

Theorem 1 *Let $\{K_j : 1 \leq j \leq n\}$ be weak* compact convex subsets of the dual of a Banach space X. Then co$\bigcup_{j=1}^{n} K_j$ is weak* compact.*

We say that a subset E of a set K in a vector space V is an extremal subset of K if $x, y \in E$ whenever

$$\lambda x + (1-\lambda) y \in E,\ \ x, y \in K \text{ and } 0 < \lambda < 1.$$

A point x is called an extreme point if the set $\{x\}$ is an extremal subset of K. For a set K in a vector space X we will denote the set of all extreme points of K by Ext(K).

Proposition 2 *Let K be a nonempty subset of a vector space V. Suppose that $E^* \subseteq E \subseteq K$. If E^* is an extremal subset of E and E is an extremal subset of K then E^* is an extremal subset of K. In particular, Ext$(E) \subseteq$ Ext(K).*

We may now present our first key result.

Theorem 2 *(Milman's Theorem) Let E be a nonempty subset of the dual of a Banach space X. If $K := \overline{\text{co}}^{\text{weak}^*}(E)$ is weak* compact then Ext$(K) \subseteq \overline{E}^{\text{weak}^*}$.*

Proof: Let e^* be any element of Ext(K) and let N be any weak* closed and convex weak* neighbourhood of $0 \in X^*$. Let $E^* := \overline{E}^{\text{weak}^*}$. Then $E^* \subseteq \bigcup_{x^* \in E^*}(x^* + N)$. So by compactness there exist a finite set $y_1^*, y_2^*, \ldots, y_n^*$ in E^* such that $E^* \subseteq \bigcup_{j=1}^n (y_j^* + N)$. For each $1 \leq j \leq n$, let $K_j := (y_j^* + N) \cap K$. Then each K_j is weak* compact and convex and $E \subseteq E^* \subseteq \bigcup_{j=1}^n K_j$. Therefore,

$$e^* \in K = \overline{\text{co}}^{\text{weak}^*}(E) \subseteq \overline{\text{co}}^{\text{weak}^*} \bigcup_{j=1}^{n} K_j = \text{co} \bigcup_{j=1}^{n} K_j.$$

Thus, $e^* = \sum_{j=1}^n \lambda_j k_j$ for some $(\lambda_j, k_j) \in [0,1] \times K_j$ with $\sum_{j=1}^n \lambda_j = 1$. Since $e^* \in$ Ext(K), there exists an

$$i \in \{1, 2, \ldots, n\}$$

such that $\lambda_i = 1$ (and $\lambda_j = 0$ for all $j \in \{1, 2, \dots n\} \setminus \{i\}$). Therefore, $e^* = k_i \in K_i \subseteq y_i^* + N \subseteq E^* + N$. Since N was an arbitrary weak* closed convex weak* neighbourhood of 0, $e^* \in E^*$. ☺

Theorem 3 *Let X be a Banach space. Then every nonempty weak* compact convex subset of X^* has an extreme point.*

Proof: Let K be a nonempty weak* compact convex subset of X^* and let $X \subseteq 2^K \setminus \{\varnothing\}$ be the set of all nonempty weak* compact convex extremal subsets of K. Then $X \neq \varnothing$ since $K \in X$. Now, $(X, \subseteq)$ is a nonempty partially ordered set. We will use Zorn's lemma to show that $(X, \subseteq)$ has a minimal element. To this end, let $T \subseteq X$ be a totally ordered subset of X (i.e., $(T, \subseteq)$ is a totally ordered set). Let $K_\infty := \bigcap_{C \in T} C$. Then $\varnothing \neq K_\infty$ is a weak* compact convex subset of K. Moreover, K_∞ is an extremal subset of K since if $x^*, y^* \in K$ and $0 < \lambda < 1$ and $\lambda x^* + (1 - \lambda) y^* \in K_\infty$ then for each

$C \in T$, $\lambda x^* + (1-\lambda)y^* \in C$; which implies that $x^*, y^* \in C$. That is, $x^*, y^* \in K_\infty$. Therefore, $K_\infty \in X$ and $K_\infty \subseteq C$ for every $C \in T$, i.e., T has a lower bound in X. Thus, by Zorn's Lemma, $(X \subseteq)$ has a minimal element K_M.

Claim: K_M is a singleton. Supppose, in order to obtain a contradiction, that K_M is not a singleton. Then there exist $x^*, y^* \in K_M$ such that $x^* \neq y^*$. Choose $x \in X$ such that $x^*(x) \neq y^*(x)$. Let

$$K^* := \{z^* \in K_M : \widehat{x}(z^*) = \max_{w^* \in K_M} \widehat{x}(w^*)\}.$$

Then $\varnothing \neq K^* \subseteq K_M$ and $K^* \in X$. Thus, $K^* = K_M$; which implies that $x^*(x) = y^*(x)$. Thus, we have obtained a contradiction and so K_M is indeed a singleton. It now follows from the definition of an extreme point that the only member of K_M is an extreme point of K. ☺

In order to prove the well-known consequence of this result we need a separation result (which we will not prove here).

Theorem 4 *Let K be a nonempty weak* compact convex subset of the dual of a Banach space X. If $x^* \in X^*$ is not a member of K then there exists an $x \in X$ such that*

$$\widehat{x}(x^*) > \max_{y^* \in K} \widehat{x}(y^*).$$

Theorem 5 (Krein-Milman Theorem) *Let K be a nonempty weak* compact convex subset of the dual of a Banach space X. Then $K = \overline{\mathsf{co}}^{\mathrm{weak}^*}\mathsf{Ext}(K)$.*

Proof: Suppose, in order to obtain a contradiction, that

$$\overline{\mathsf{co}}^{\mathrm{weak}^*}\mathsf{Ext}(K) \subsetneqq K.$$

Then there exists $x^* \in K \setminus \overline{\mathsf{co}}^{\mathrm{weak}^*}\mathsf{Ext}(K)$. Choose $x \in X$ such that $\widehat{x}(x^*) > \max\{\widehat{x}(y^*) : y^* \in \overline{\mathsf{co}}^{\mathrm{weak}^*}\mathsf{Ext}(K)\}$. Let

$$K^* := \{z^* \in K : \widehat{x}(z^*) = \max_{y^* \in K} \widehat{x}(y^*)\}.$$

Now, K^* is a nonempty weak* compact convex extremal subset of K. Therefore, by Theorem 3, there exists an $e^* \in \text{Ext}(K^*) \subseteq \text{Ext}(K)$. However, $e^* \notin \overline{\text{co}}^{\text{weak}^*}\text{Ext}(K)$. Thus, we have obtained a contradiction. Hence the statement of the Krein-Milman theorem holds ☺

This concludes the necessary linear topology required in order to prove James' Theorem.

Our next goal is to prove the Bishop-Phelps Theorem. To do this we start will some convex analysis.

Let $f : X \to \mathbb{R}$ be a continuous convex function defined on a Banach space X. Then for each $x_0 \in X$ we define the subdifferential of f at x_0 to be:

$$\partial f(x_0) := \{x^* \in X^* \;\; : \;\; x^*(x) + [f(x_0) - x^*(x_0)] \leq f(x) \text{ for all } x \in X\}.$$

Then for each $x \in X$, $\partial f(x)$, is a nonempty weak* compact convex subset of X^*. We will require two facts about the subdifferential:

(a) If $f(x_\infty) = \min_{x\in X} f(x)$ then $0 \in \partial f(x_\infty)$ (this follows directly from the definition);

(b) If $h : X \to \mathbb{R}$ is also a continuous convex function then

$$\partial(h + f)(x) = \partial h(x) + \partial f(x) \text{ for all } x \in X.$$

Next, we prove Ekeland's variational principle.

Theorem 6 *(E.V.P.) Suppose that $f : X \to \mathbb{R}$ is a bounded below lower semi-continuous function defined on a Banach space X. If $\varepsilon > 0$, $x_0 \in X$ and $f(x_0) \le \inf_{y\in X} f(y) + \varepsilon^2$ then there exists $x_\infty \in X$ such that $\|x_\infty - x_0\| \le \varepsilon$ and the function $f + \varepsilon\|\cdot\ - x_\infty\|$ attains its minimum value at x_∞. Moreover, if f is continuous and convex then*

$$0 \in \partial f(x_\infty) + \varepsilon B_{X^*}.$$

Proof: We shall inductively define a sequence $(x_n : n \in \mathbb{N})$ in X and a sequence $(D_n : n \in \mathbb{N})$ of closed subsets of X such that

(i) $D_n := \{x \in D_{n-1} : f(x) \leq f(x_{n-1}) - \varepsilon\|x - x_{n-1}\|\}$;

(ii) $x_n \in D_n$;

(iii) $f(x_n) \leq \inf_{x \in D_n} f(x) + \varepsilon^2/(n+1)$.

Set $D_0 := X$. In the base step we let

$$D_1 := \{x \in D_0 : f(x) \leq f(x_0) - \varepsilon\|x - x_0\|\}$$

and choose $x_1 \in D_1$ so that $f(x_1) \leq \inf_{x \in D_1} f(x) + \varepsilon^2/2$. Then at the $(n+1)^{\text{th}}$-step we let

$$D_{n+1} := \{x \in D_n : f(x) \leq f(x_n) - \varepsilon\|x - x_n\|\}$$

and we choose $x_{n+1} \in D_{n+1}$ such that

$$f(x_{n+1}) \leq \inf_{x \in D_{n+1}} f(x) + \varepsilon^2/(n+2).$$

This completes the induction.

Now, by construction, $\varnothing \neq D_{n+1} \subseteq D_n$ for all $n \in \mathbb{N}$. It is also easy to see that $\sup\{\|x - x_n\| : x \in D_{n+1}\} \leq \varepsilon/(n+1)$. Indeed, if $x \in D_{n+1}$ and $\|x - x_n\| > \varepsilon/(n+1)$ then

$$
\begin{aligned}
f(x) &< \big[f(x_n) - \varepsilon(\varepsilon/(n+1))\big] = f(x_n) - \varepsilon^2/(n+1) \\
&\leq \big[\inf_{y \in D_n} f(y) + \varepsilon^2/(n+1)\big] - \varepsilon^2/(n+1) = \inf_{y \in D_n} f(y);
\end{aligned}
$$

which contradicts the fact that $x \in D_{n+1} \subseteq D_n$.

Let $\{x_\infty\} := \bigcap_{n=1}^{\infty} D_n$. Fix $x \in X \setminus \{x_\infty\}$ and let n be the first natural number such that $x \notin D_n$, i.e., $x \in D_{n-1} \setminus D_n$. Then,

$$
f(x_\infty) - \varepsilon\|x - x_\infty\| \leq f(x_{n-1}) - \varepsilon\|x - x_{n-1}\| < f(x)
$$

since

$$
\begin{aligned}
f(x_\infty) &\leq f(x_{n-1}) - \varepsilon\|x_{n-1} - x_\infty\| \quad \text{since } x_\infty \in D_n \\
&\leq f(x_{n-1}) - \varepsilon\big[\|x - x_{n-1}\| - \|x - x_\infty\|\big].
\end{aligned}
$$

Hence, $f + \varepsilon\| \cdot - x_\infty\|$ attains its minimum at x_∞. Also note that $x_\infty \in D_1$ and so $\|x_\infty - x_0\| \leq \varepsilon$. ☺

We can now proceed to a proof of the Bishop-Phelps Theorem, but first we need a couple of definitions. Let K be a weak* compact convex body in the dual of a Banach space X. Define $p : X \to [0, \infty)$ by, $p(x) = \max_{x^* \in K} \widehat{x}(x^*)$. Then p is a continuous sublinear functional on X. Let

$$\begin{aligned} BP(K) &:= \{x^* \in K : x^*(x) = p(x) \text{ for some } x \neq 0\} \\ &= \bigcup_{x \neq 0} \partial p(x). \end{aligned}$$

Theorem 7 *(Bishop-Phelps Theorem) Let K be a weak* compact convex body with $0 \in$ int(K) in the dual of a Banach space X. Then $BP(K)$ is dense in the boundary of K.*

Proof: Let x_0^* be an arbitrary element of the boundary of K and let $0 < \varepsilon < 1$. Without loss of generality we may assume that $\varepsilon < M := (\sup_{x^*\in K}\|x^*\|)^{-1}$. Now, $x_0^* \notin (1-\varepsilon^2)K$. Hence we may choose $x \in X$ such that

$$(1-\varepsilon^2)p(x) = \max_{x^*\in(1-\varepsilon^2)K} \widehat{x}(x^*) < x_0^*(x) \leq p(x).$$

Without loss of generality we may assume that $p(x) = 1$ and so $(1-\varepsilon^2) < x_0^*(x) \leq 1$. It also follows that $M \leq \|x\|$. Let $h : X \to [0,\infty)$ be defined by, $h := p - x_0^*$. Then

$$0 \leq h(x) = p(x) - x_0^*(x) = 1 - x_0^*(x) < \varepsilon^2.$$

By Ekeland's variation principle there exists $x_\infty \in X$ such that $\|x_\infty - x\| \leq \varepsilon < M$ (and so $\|x_\infty\| \neq 0$) and

$$\begin{aligned} 0 &\in \partial h(x_\infty) + \varepsilon B_{X^*} \\ &= \partial p(x_\infty) - x_0^* + \varepsilon B_{X^*}. \end{aligned}$$

Hence there exists $x^* \in \partial p(x_\infty) \in BP(K)$ and $y^* \in B_{X^*}$ such that $\|x^* - x_0^*\| = \varepsilon\|-y^*\| \leq \varepsilon$. ☺

The Main Theorem

Ever since R. C. James first proved that, in any Banach space X, a closed bounded convex subset C of X is weakly compact if, and only if, every continuous linear functional attains its supremum over C, there has been continued interest in trying to simplify his proof. Some success was made when G. Godefroy used Simons' inequality to deduce James' theorem in the case of a separable Banach space. However, although the proof of Simons' inequality is elementary, it is certainly not easy and so the search for a simple proof continued. Later Fonf, Lindenstrauss and Phelps used the notion of (I)-generation to provide an alternative proof of James' theorem (in the separable Banach space case) without recourse to Simons' inequality. Their proof was short and reasonably elementary. However, it still relied upon integral representation theorems, as well as, the Bishop-Phelps theorem. In this part of the talk we will show how to modify the proof of FLP in order to remove the integral representations.

Let K be a weak* compact convex subset of the dual of a Banach space X. A subset B of K is called a boundary of K if for every $x \in X$ there exists an $x^* \in B$ such that

$$x^*(x) = \sup\{y^*(x) : y^* \in K\}.$$

We shall say that B, (I)-generates K, if for every countable cover $\{C_n : n \in \mathbb{N}\}$ of B by weak* compact convex subsets of K, the convex hull of $\bigcup_{n\in\mathbb{N}} C_n$ is norm dense in K.

The main theorem relies upon the following prerequisite result.

Lemma 1 *Suppose that K, S and $\{K_n : n \in \mathbb{N}\}$ are weak* compact subsets of the dual of a Banach space X. Suppose also that $S \cap K = \varnothing$ and $S \subseteq \overline{\bigcup_{n\in\mathbb{N}} K_n}^{w^*}$. If for each weak* open neighbourhood W of 0 there exists an $N \in \mathbb{N}$ such that*

$K_n \subseteq K + W$ for all $n > N$ then $S \subseteq \bigcup_{1 \le n \le M} K_n$ for some $M \in \mathbb{N}$.

Proof: Since $K \cap S = \varnothing$ there exists a weak* open neighbourhood W of 0 such that $K + W \subseteq X^* \setminus S$. By making W smaller, we may assume that $K + \overline{W}^{\text{weak}^*} \subseteq X^* \setminus S$. From the hypotheses there exists a $M \in \mathbb{N}$ such that

$$\bigcup_{n>M} K_n \subseteq K + W$$

and so

$$\overline{\bigcup_{n>M} K_n}^{\text{weak}^*} \subseteq K + \overline{W}^{\text{weak}^*} \subseteq X^* \setminus S,$$

since $K + \overline{W}^{\text{weak}^*}$ is weak* closed. On the other hand,

$$S \subseteq \overline{\bigcup_{n \in \mathbb{N}} K_n}^{\text{weak}^*} = \overline{\bigcup_{n>M} K_n}^{\text{weak}^*} \cup \bigcup_{1 \le n \le M} K_n.$$

Therefore, $S \subseteq \bigcup_{1 \le n \le M} K_n$. ☺

We may now state and prove the main theorem.

Theorem 8 *Let K be a weak* compact convex subset of the dual of a Banach space X and let B be a boundary of K. Then B, (I)-generates K.*

Proof: After possibly translating K we may assume that $0 \in B$. Suppose that $B \subseteq \bigcup_{n\in\mathbb{N}} C_n$ where $\{C_n : n \in \mathbb{N}\}$ are weak* compact convex subsets of K. Fix $\varepsilon > 0$. We will show that $K \subseteq \text{co}[\bigcup_{n\in\mathbb{N}} C_n] + 2\varepsilon B_{X^*}$. For each $n \in \mathbb{N}$, let $K_n := C_n + (\varepsilon/n)B_{X^*}$ and let $V^* := \overline{\text{co}}^{\text{weak}^*} \bigcup_{n\in\mathbb{N}} K_n$. Clearly, $B \subseteq \bigcup_{n\in\mathbb{N}} K_n$ and so $K = \overline{\text{co}}^{\text{weak}^*}(B) \subseteq V^*$. It is also clear that V^* is a weak* compact convex body in X^* with $0 \in \text{int}(V^*)$. Let x^* be any element of $BP(V^*)$ and let $x \in X$ be chosen so that $x^*(x) = \max_{y^*\in V^*} \widehat{x}(y^*) = 1$. It is easy to see that if

$$F := \{y^* \in V^* : y^*(x) = 1\}$$

then $F \cap K = \varnothing$. Indeed, if $F \cap K \neq \varnothing$ then

$$\max\{y^*(x) : y^* \in K\} = 1$$

and because B is a boundary for K it follows that for some $j \in \mathbb{N}$ there is a $b^* \in C_j \cap B$ such that $b^*(x) = 1$. However, as $b^* \in b^* + (\varepsilon/j)B_{X^*} \subseteq K_j \subseteq V^*$, this is impossible. Now,

$$\begin{aligned} \text{Ext}(F) &\subseteq \text{Ext}(V^*) \quad \text{since } F \text{ is an extremal subset of } V^* \\ &\subseteq \overline{\bigcup_{n\in\mathbb{N}} K_n}^{\text{weak}^*} \quad \text{by Milman's theorem.} \end{aligned}$$

Thus, $\text{Ext}(F) \subseteq F \cap \overline{\bigcup_{n\in\mathbb{N}} K_n}^{\text{weak}^*} \subseteq \overline{\bigcup_{n\in\mathbb{N}} K_n}^{\text{weak}^*}$ and so by Lemma 1, applied to the weak* compact set

$$S := F \cap \overline{\bigcup_{n\in\mathbb{N}} K_n}^{\text{weak}^*},$$

there exists an $M \in \mathbb{N}$ that that $\text{Ext}(F) \subseteq S \subseteq \bigcup_{1\leq n\leq M} K_n$.

Hence,

$$\begin{aligned}
x^* \in F &= \overline{\text{co}}^{\text{weak}^*}\text{Ext}(F) \quad \text{by the Krein-Milman theorem} \\
&\subseteq \text{co} \bigcup_{1 \le n \le M} K_n \\
&\subseteq \text{co} \bigcup_{1 \le n \le M} C_n + \varepsilon B_{X^*} \subseteq \text{co} \bigcup_{n \in \mathbb{N}} C_n + \varepsilon B_{X^*}.
\end{aligned}$$

Since $x^* \in BP(V^*)$ was arbitrary, we have by the Bishop-Phelps theorem, which says that $BP(V^*)$ is dense in ∂V^*, that

$$\partial V^* \subseteq \text{co} \bigcup_{n \in \mathbb{N}} C_n + 2\varepsilon B_{X^*}.$$

However, since $0 \in B$ (and hence in some C_n) it follows that $K \subseteq V^* \subseteq \text{co}[\bigcup_{n \in \mathbb{N}} C_n] + 2\varepsilon B_{X^*}$. Since $\varepsilon > 0$ was arbitrary we are done. ☺

There are many applications of this theorem. In particular, we have the following.

Corollary 1 *Let K be a weak* compact convex subset of the dual of a Banach space X, let B be a boundary for K and let $f_n : K \to \mathbb{R}$ be weak* lower semi-continuous convex functions. If $\{f_n : n \in \mathbb{N}\}$ are equicontinuous with respect to the norm and $\limsup_{n\to\infty} f_n(b^*) \leq 0$ for each $b^* \in B$ then $\limsup_{n\to\infty} f_n(x^*) \leq 0$ for each $x^* \in K$.*

Proof: Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, let

$$C_n := \{y^* \in K : f_k(y^*) \leq (\varepsilon/2) \text{ for all } k \geq n\}.$$

Then $\{C_n : n \in \mathbb{N}\}$ is a countable cover of B by weak* compact convex subsets of K. Therefore, co$[\bigcup_{n\in\mathbb{N}} C_n] = \bigcup_{n\in\mathbb{N}} C_n$ is norm dense in K. Since $\{f_n : n \in \mathbb{N}\}$ are equicontinuous (with respect to the norm) it follows that $\limsup_{n\to\infty} f_n(x^*) < \varepsilon$ for all $x^* \in K$. ☺

The classical Rainwater's theorem follows from this by setting: $K := B_{X^*}$; $B := \text{Ext}(K)$ and for any bounded set

$\{x_n : n \in \mathbb{N}\}$ in X that converges to $x \in X$ with respect to the topology of pointwise convergence on $\text{Ext}(B_{X^*})$, let $f_n : K \to [0, \infty)$ be defined by, $f_n(x^*) := |x^*(x_n) - x^*(x)|$.

We may also obtain the following well known result.

Corollary 2 *(Simons' Equality) Let K be a weak* compact convex subset of the dual of a Banach space X, let B be a boundary for K and let $\{x_n : n \in \mathbb{N}\}$ be a bounded subset of X. Then*

$$\sup_{b^* \in B} \left\{ \limsup_{n \to \infty} \widehat{x}_n(b^*) \right\} = \sup_{x^* \in K} \left\{ \limsup_{n \to \infty} \widehat{x}_n(x^*) \right\}.$$

Proof: Since clearly,

$$\sup_{b^* \in B} \left\{ \limsup_{n \to \infty} \widehat{x}_n(b^*) \right\} \leq \sup_{x^* \in K} \left\{ \limsup_{n \to \infty} \widehat{x}_n(x^*) \right\}$$

we need only show that

$$\sup_{x^* \in K} \left\{ \limsup_{n \to \infty} \widehat{x}_n(x^*) \right\} \leq \sup_{b^* \in B} \left\{ \limsup_{n \to \infty} \widehat{x}_n(b^*) \right\}.$$

To this end let

$$r := \sup_{b^* \in B} \left\{ \limsup_{n\to\infty} \widehat{x}_n(b^*) \right\}$$

and for each $n \in \mathbb{N}$, let $f_n : K \to \mathbb{R}$ be defined by,

$$f_n(x^*) := \sup\{\widehat{x}_k(x^*) : k \geq n\} - r.$$

Then $\{f_n : n \in \mathbb{N}\}$ are weak* lower semicontinuous, convex and equicontinuous with respect to the norm. Moreover, $\lim_{n\to\infty} f_n(b^*) \leq 0$ for all $b^* \in B$. Therefore, by Corollary 1, $\lim_{n\to\infty} f_n(x^*) \leq 0$ for all $x^* \in K$. The result now easily follows. ☺

As promised, we give a simple proof of James' theorem valid for separable, closed and bounded convex sets. In the proof of this theorem we shall denote the natural embedding of a Banach space X into its second dual X^{**} by, $\widehat{X}$ and similarly, we shall denote the natural embedding of an element $x \in X$ by, $\widehat{x}$.

Theorem 9 *Let C be a closed and bounded convex subset of a Banach space X. If C is separable and every continuous linear functional on X attains its supremum over C then C is weakly compact.*

Proof: Let $K := \overline{\widehat{C}}^{\text{weak}^*}$. To show that C is weakly compact it is sufficient to show that for every $\varepsilon > 0$,

$$K \subseteq \widehat{C} + 2\varepsilon B_{X^{**}}.$$

To this end, fix $\varepsilon > 0$ and let $\{x_n : n \in \mathbb{N}\}$ be any dense subset of C. For each $n \in \mathbb{N}$, let $C_n := K \cap [\widehat{x_n} + \varepsilon B_{X^{**}}]$. Then $\{C_n : n \in \mathbb{N}\}$ is a cover of $\widehat{C}$ by weak* closed convex subsets of K. Since $\widehat{C}$ is a boundary of K,

$$K \subseteq \overline{\text{co}} \bigcup_{n \in \mathbb{N}} C_n \subseteq \widehat{C} + 2\varepsilon B_{X^{**}} \qquad ☺$$

If we are willing to invest a little more effort we can extend Theorem 9 to the setting where B_{X^*} is weak* sequentially compact. To see this we need the following lemma.

Lemma 2 *Let C be a closed and bounded convex subset of a Banach space X. If (B_{X^*}, weak^*) is sequentially compact and every continuous linear functional on X attains its supremum over C then for each $\mathscr{F} \in B_{X^{***}}$ there exists an $x^* \in B_{X^*}$ such that $\mathscr{F}|_{\overline{\widehat{C}}^{w^*}} = \widehat{x^*}|_{\overline{\widehat{C}}^{w^*}}$.*

Proof: Let $K := \overline{\widehat{C}}^{w^*}$ and note that $\widehat{C}$ is a boundary of K. Let $B_p(K)$ $[C_p(K)]$ denote the bounded real-valued [weak* continuous real-valued] functions defined on K, endowed with the topology of pointwise convergence on K. For an arbitrary subset Y of K let $\tau_p(Y)$ denote the topology on $B(K)$ of pointwise convergence on Y. Consider, $S : (B_{X^*}, \text{weak}^*) \to (C(K), \tau_p(\widehat{C}))$ defined by, $S(x^*) := \widehat{x^*}|_K$. Since S is continuous, $S(B_{X^*})$ is sequentially $\tau_p(\widehat{C})$-compact. Hence, from Corollary 1, $S(B_{X^*})$ is sequentially $\tau_p(K)$-compact. It then follows from Grothendieck's Theorem that $S(B_{X^*})$ is a compact subset of $C_p(K)$ and so a compact subset of $B_p(K)$. In particular, $S(B_{X^*})$ is a closed subset of $B_p(K)$. Next, consider $T : (B_{X^{***}}, \text{weak}^*) \to B_p(K)$ defined by, $T(\mathscr{F}) := \mathscr{F}|_K$. Then T is continuous and so $T(B_{\widehat{X^*}})$ is dense in $T(B_{X^{***}})$, since $B_{\widehat{X^*}}$ is weak* dense in $B_{X^{***}}$ by Goldstine's Theorem. However, $T(B_{\widehat{X^*}}) = S(B_{X^*})$; which is closed in $B_p(K)$. Therefore, $T(B_{X^{***}}) = S(B_{X^*}) = T(B_{\widehat{X^*}})$. This completes the proof. ☺

Theorem 10 *Let C be a closed and bounded convex subset of a Banach space X. If (B_{X^*}, weak^*) is sequentially compact and every continuous linear functional on X attains its supremum over C then C is weakly compact.*

Proof: Let $K := \overline{\widehat{C}}^{w^*}$. In order to obtain a contradiction, suppose that $\widehat{C} \subsetneq K$. Let $F \in K \setminus \widehat{C}$. Then there exists a $\mathscr{F} \in B_{X^{***}}$ such that $\mathscr{F}(F) > \sup_{\widehat{c} \in \widehat{C}} \mathscr{F}(\widehat{c})$. However, by Lemma 2 there exists an $x^* \in B_{X^*}$ such that $\widehat{x^*}|_K = \mathscr{F}|_K$.

Therefore,

$$\widehat{x^*}(F) = \mathscr{F}(F) > \sup_{\widehat{c}\in\widehat{C}} \mathscr{F}(\widehat{c}) = \sup_{\widehat{c}\in\widehat{C}} \widehat{x^*}(\widehat{c}) = \max_{G\in K} \widehat{x^*}(G);$$

which contradicts the fact that $F \in K$. Therefore, $K = \widehat{C}$ and so C is weakly compact. ☺

The End