The Pennsylvania State University
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The Convexity of Chebyshev Sets in Finite Dimensional Normed Linear Spaces

A term paper in

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by

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INTRODUCTION

It is well-known that every closed convex set in a Hilbert space is a Chebyshev set, i.e., contains a unique nearest point to each point in the space. However, a famous unsolved problem in approximation theory is whether or not every Chebyshev set in a Hilbert space is convex. In the finite dimensional case, the answer is known to be in the affirmative; in the infinite dimensional case, while some results have been obtained, the question is still unresolved (see Vlasov [23] and Narang [20]).

This paper is a historical and expository account of the results in finite dimensional normed linear spaces.

Historically, Bunt [8] in 1934, Motzkin [18] in 1935, and Kritikos [17] in 1938 all independently showed that in Euclidean spaces every Chebyshev set is convex. Bunt, working in $\mathbb{R}^n$, in his dissertation on convex sets, showed that more generally, a "basic set" is convex if it satisfies certain conditions. Motzkin proved his result in $\mathbb{R}^2$ and also in 1935 [19] gave a different proof for bounded Chebyshev sets in $\mathbb{R}^2$. This proof for bounded Chebyshev sets in $\mathbb{R}^2$ was reprinted, with credit given to Motzkin, in Bundgaard and Duerlune [5] in 1937 and in the survey paper of Beretta and Maxia [2] in 1940. Kritikos was led to his proof in $\mathbb{R}^n$ after an investigation of the relationship between "F-sets" and convex sets.

In 1940 Jessen [13], aware of Kritikos's proof, gave still another in $\mathbb{R}^n$. Busemann in 1947 [6] noted that Jessen's proof could be extended to "straight line spaces" and in 1955 [7] showed how this could be done. Since a finite dimensional normed linear space is a "straight line space" if and only if it is strictly convex (see section 5), Busemann's result is that in a smooth strictly convex finite dimensional normed linear space, every Chebyshev set is convex. Valentine [21] independently in 1964 gave essentially the same proof as Busemann.

In 1953, Klee [14] stated that in a finite dimensional normed linear space, every Chebyshev set is a "sun" and gave a characterization of Chebyshev sets in a smooth and strictly convex finite dimensional normed linear space. However, as he noted in 1961 [15] the argument in [14] was garbled, and he proceeded to prove a stronger result, which in the finite dimensional case, showed that the requirement of strict convexity could be dropped. Thus Klee was the first to show that in a smooth finite dimensional normed linear space, every Chebyshev set is convex. (It is easy to see that smoothness cannot be dropped - take $\mathbb{R}^2$ with norm $||(x,y)||=\max\{|x|, |y|\}$.

Then the union of the set of points above or on the lines y=x and y=2x is a nonconvex Chebyshev set.) In 1961 Vlasov [22] showed that a "boundedly compact" Chebyshev set in a smooth Banach space (of arbitrary dimension) is convex. In the finite dimensional case, this yields a short proof that in a smooth finite dimensional normed linear space, every Chebyshev set is convex.

The proofs fall into three classes. Bunt, Jessen and Busemann use proof by contradiction to find a ball in the complement of the Chebyshev set of maximal radius containing a fixed ball. Since the set is Chebyshev the maximal ball only touches at one point. They then proceed to move the maximal ball so that it still contains the fixed ball but is wholly contained in the complement of the Chebyshev set. This contradicts the maximality of the ball. Motzkin, Klee and Vlasov show that a Chebyshev set is a "sun". (The concept of "sun" did not exist in 1935 when Motzkin wrote his paper but was first used by Klee [14], although he did not name them; this was done later by Efimov and Steckin [11].) Brøndsted [4] noted that both Klee and Vlasov use a fixed point theorem and remarked that no elementary proof seems to be known.

Kritikos's proof seems to be different from any of the others.

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PRELIMINARIES

Definition Let X be a normed linear space and M a nonempty subset of X. The set-valued mapping $P_M: X \to 2^M$ defined by

$$P_M(x) = \{y \in M: ||x-y|| = d(x,M)\},$$

where $d(x,M) = \inf_{y \in M} ||x-y||$, is called the metric projection (or best approximation operator). The set M is proximinal (semi-Chebyshev) if $P_M(x)$ contains at least (at most) one element for every $x \in X$. M is a Chebyshev set if it is both proximinal and semi-Chebyshev; i.e., if $P_M(x)$ is a singleton for every $x \in X$.

Lemma 1 If M is a proximinal set in a normed linear space X, then M is closed.

Proof If M were not closed, choose $x \in \overline{M} \setminus M$. Then $d(x,M) = 0$ but $||x-m|| > 0$ for all $m \in M$. Hence $P_M(x) = \phi$, a contradiction.

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Definition For X a normed linear space, the closed ball with center x and radius r, denoted $B[x,r]$, is $B[x,r] \equiv \{y \in X: ||x-y|| \leq r\}$. The open ball with center x and radius r, denoted $B(x,r)$, is $B(x,r) \equiv \{y \in X: ||x-y|| < r\}$.

Proposition Let X be a normed linear space. For all $\lambda\in(o,r)$, if $p\in B[m,r]$ then there exists a $c\in B[m,r]$ such that $p\in B[c,\lambda]\subset B[m,r]$.

Proof By translating and scaling, if necessary, assume m=0 and r=1. If $\lambda\geq||p||$, take c=0. If $\lambda<||p||$, take $c = \frac{||p||-\lambda}{||p||}\, p$.

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Definition For any two points x and y in a linear space X, the interval (or segment) between x and y, denoted [x,y], is $[x,y]\equiv\{\lambda x+(1-\lambda)y:0\leq\lambda\leq 1\}$. The half-line from x through y, denoted [x,y> or $\overrightarrow{xy}$, is

$$\overrightarrow{xy} = [x,y\rangle\equiv\{x+\lambda(y-x):\lambda\geq 0\}.$$

Lemma 2 Let M be a Chebyshev set in a normed linear space X. Let $x\in X\setminus M$. Let $y(\lambda) = P_M(x) + \lambda(x-P_M(x))$. Then $K = \{\lambda\in R:\lambda\geq 0 \text{ and } P_M(y(\lambda)) = P_M(x)\}$ is a nonempty closed interval.

Proof By the triangle inequality, if $\eta\in K$, then $\alpha\in K$ for $0\leq\alpha\leq\eta$. Further, $1\in K$. Let $\{\alpha_i\}$ be a sequence in K converging to β.

$$\begin{aligned}||y(\beta)-P_M(x)|| &\leq ||y(\beta)-y(\alpha_i)||+||y(\alpha_i)-P_M(y(\alpha_i))|| \\ &\quad +||P_M(y(\alpha_i))-P_M(x)|| \\ &=||y(\beta)-y(\alpha_i)||+d(y(\alpha_i),M) \\ &\longrightarrow d(y(\beta),M)\end{aligned}$$

implies $||y(\beta)-P_M(x)||\leq d(y(\beta),M)$. So $P_M(y(\beta))=P_M(x)$, $\beta\in K$, and K is closed. ///.

Propositions 1) Let M be a nonempty set in a metric space X. Then $d(x,M)\equiv\inf\{d(x,m):m\in M\}$ is continuous.

2) In a finite dimensional normed linear space, every closed bounded set is compact.

3) Let M be a Chebyshev set in a finite dimensional normed linear space X. Then P_M is continuous.

Proofs 1) ([12], p. 77).

2) ([10], p. 246).

3) Follows by a compactness argument ([9]).

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Definition If X is a normed linear space, X^* will denote its dual space: the Banach space of all bounded linear functionals x^* on X with the norm

$$||x^*|| = \sup_{x\neq 0} \frac{|x^*(x)|}{||x||} .$$

Definition If X is a linear space, a flat (or linear variety) is a translate of a linear subspace of X. The dimension of a flat is the cardinality of a basis of the corresponding linear subspace. A hyperplane is a maximal proper flat of X.

Remark It can be shown that in a linear space X, H is a hyperplane if and only if there exists a nonzero linear functional f and $r\in\mathbb{R}$ such that $H=[f:r]\equiv\{x\in X:f(x)=r\}$ (See [16] or [21]).

Definition In a normed linear space X, a supporting hyperplane of a closed bounded set M at a point $y \in M$ is a hyperplane $[f:r]$ such that $f(y)=r$ and $f(z) \leq r$ for $z \in M$.

Notation 1) $\mathbb{R}^n$ will denote Euclidean n-space, i.e., the set of n-tuples $x = (x_1, x_2, \ldots, x_n)$, $x_i \in \mathbb{R}$, with $||x|| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$.

2) bd M denotes the boundary of the set M.

Definition The distance between two sets A and B, denoted $d(A,B)$, is $d(A,B) = \inf\limits_{\substack{a \in A \\ b \in B}} ||a-b||$.

Remark The following is used by Motzkin and Jessen. It was also noted by Bonnesen [3] (p.43) in 1929.

Lemma 3 If $M \subset \mathbb{R}^n$, $m \in \mathbb{R}^n \backslash M$, $d=d(m,M)$, $B[c,r] \subset B[m,d]$, $o \leq r \leq d$, $M \cap B[m,d] = \{p\}$, and $p \notin B[c,r]$, then there exists a ball with radius larger than d, containing $B[c,r]$, and disjoint from M.

Proof By translating and scaling, if necessary, assume $m=0$ and $d=1$. Let δ be the distance between the supporting hyperplane H of $B[0,1]$ at p and the ball $B[c,r]$. Let $H' = H - \frac{\delta}{2} p$. Thus $H' \cap [o,p]$ is the point λp for some λ, $0<\lambda<1$. So $H' = \{x: <x-\lambda p, p>=0\}$. Let $a,b \in \{x: ||x||=1 \text{ and } x \in H'\}$. Then for $\varepsilon>0$,

$||a+\varepsilon p||=||b+\varepsilon p||$(look at the square of the norms). Let $S=\{x:||x||\leq 1$ and $<x-\lambda p, p>\leq 0\}$. Then $S\subset B[-\varepsilon p, ||a+\varepsilon p||]$ since for $x\in S$, $||x+\varepsilon p||\leq||a+\varepsilon p||$. Thus $B[c,r]\subset B[-\varepsilon p,||a+\varepsilon p||]$. Let $T=\{x:x\in B[-\varepsilon p,||a+\varepsilon p||]$ and $<x-\lambda p, p>>0\}$. Then $T\subset B(0,1)$ since for $x\in T$, if $||x||\geq 1$, then $||x+\varepsilon p||>||a+\varepsilon p||$. Thus for ε sufficiently small, $B[-\varepsilon p, ||a+\varepsilon p||]$ has a radius larger than 1, contains $B[c,r]$, and is disjoint from M.

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<u>Definition</u> A normed linear space X is <u>strictly convex</u> if for every two distinct points x and y of unit norm, $||\lambda x+(1-\lambda)y||<1$ for $0<\lambda<1$.

<u>Definition</u> A normed linear space X is <u>smooth</u> if every point on the boundary of $B[0,1]$ has a unique supporting hyperplane.

<u>Lemma 4</u> In a smooth normed linear space X, the supporting hyperplane of $B[x,||x-y||]$ at y is $[x^* : x^* (x) + ||x-y||]$ where $x^*\in X^*$, $||x^*||=1$ and $x^*(y-x) = ||y-x||$.

<u>Proof</u> By a corollary to the Hahn-Banach Theorem ([12], p.214) $[x^*:1]$ is a supporting hyperplane of $B[0,1]$ at $\frac{y-x}{||y-x||}$ where $x^*\in X^*$, $||x^*||=1$ and $x^*(\frac{y-x}{||y-x||}) = 1$. By smoothness, it is unique. Thus the unique supporting hyperplane of $B[x,||x-y||]$ at y is (by translating) $[x^*:x^*(x)+||x-y||]$.

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Remark It is well-known that in a smooth normed linear space the union of all open balls with centers on a half-line and radius the distance between the center and the endpoint of the half-line is a half-space (see, for example, [1] where a somewhat more general result was established). This can be restated as follows.

Proposition In a smooth normed linear space X, let $[x^*:x^*(x)+||x-y||]$ be the supporting hyperplane of $B[x,||x-y||]$ at y. then

$$\{z\in X:x^*(z)<x^*(x)+||x-y||\}= \bigcup_{\lambda>0} B(y+\lambda(x-y),\lambda||x-y||).$$

Proof ([1]).

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Definition A Chebyshev set M in a normed linear space X is called a sun if for every $x\in X\setminus M$ and $w\in \overrightarrow{P_M(x)x}$, $P_M(x) = P_M(w)$.

Remark In [22] Vlasov proved the following (which had been stated previously by N. Efimov and S. Steckin). The following proof is from Amir and Deutsch [1], where a somewhat more general result was established.

Theorem 1 Let M be a Chebyshev set in a smooth normed linear space X. If M is a sun, then M is convex.

Proof If M were not convex, there would exist $y_1, y_2 \in M$, $\lambda \in (0,1)$ such that $x = \lambda y_1 + (1-\lambda)y_2 \notin M$.

Claim $\{z \in X : x^*(z) < x^*(x) + ||x-P_M(x)||\} \cap M = \phi$. Let $y \in \{z \in X : x^*(z) < x^*(x) + ||x-P_M(x)||\} \cap M$. Thus there exists a $\gamma > 0$ such that $y \in B(P_M(x) + \gamma(x-P_M(x)), \gamma||x-P_M(x)||)$; that is,

$$||P_M(x)+\gamma(x-P_M(x))-y|| < \gamma||x-P_M(x)||$$

$$= ||P_M(x)-\gamma(x-P_M(x))-P_M(x)||$$

which contradicts the fact that for

$$w = P_M(x) + \gamma(x-P_M(x)) \in \overrightarrow{P_M(x)x}, \quad P_M(x) = P_M(w),$$

since $y \in M$. Thus the claim is proved.

Thus, $x^*(y_i) \geq x^*(x) + ||x-P_M(x)||$ for $i = 1,2$. But

$$0 < ||x-P_M(x)|| = \lambda||x-P_M(x)|| + (1-\lambda)||x-P_M(x)||$$

$$\leq \lambda x^*(y_1-x) + (1-\lambda)x^*(y_2-x)$$

$$= x^*(\lambda y_1 + (1-\lambda)y_2 - x)$$

$$= x^*(0)$$

$$= 0,$$

a contradiction. Thus M is convex.

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Theorem 2 (Schauder-Tychonoff Fixed Point Theorem) Let B be a compact convex subset of a normed linear space X and $T:B\to B$ a continuous map. Then there exists an $x\in B$ such that $T(x) = x$.

Proof ([10], p. 456).

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Definition In a linear space X, the convex hull of a set M, denoted co(M), is the intersection of all convex sets which contain M.

Definition A convex combination of the elements $x_1, x_2, \ldots, x_n$ in a linear space X is any linear combination $\sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

Propositions 1) In a linear space X, a set M is convex if and only if M=co(M).

2) Let M be a subset of a linear space X. Then co(M) is the set of all convex combinations of elements of M.

3) (Caratheodory) Let $M\subset X$ where X is a linear space of dimension n. If $x\in co(M)$, then x is expressible as a convex combination of n+1 (or fewer) points of M.

4) Let M be a compact set in $\mathbb{R}^n$. Then co(M) is closed.

Proofs 1) Follows from the definition of convex hull.

2) Let B denote the set of all convex combinations of elements of M. $M\subset B$ and since B is convex $co(M)\subset B$. The

proof of the reverse inclusion, $B \subset co(M)$, follows by induction on the number of elements in a convex combination.

3) ([9], p. 17).

4) Follows by a compactness argument.

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1. Bunt (1934)

Lemma (Bunt) Let there be given in $\mathbb{R}^n$: a closed set M, a point $c \notin M$ but $c = \frac{1}{2}x + \frac{1}{2}y$, $x,y \in M$, and a ball $B[c,r]$ such that $B[c,r] \cap M = \phi$. Then the radii of the closed balls which contain $B[c,r]$ and which do not intersect M are bounded above by a positive number d.

Proof It will be shown that

$$d = \frac{||x-y||^2 + r^2}{2r}$$

works. If n=1, since

$$\frac{||x-y||^2 + r^2}{2r} \geq \frac{1}{2}||x-y||,$$

the result follows. If $n \geq 2$, let $B[c^*, r^*]$ be a ball which contains $B[c,r]$ and which does not intersect M. Let F be any two-dimensional flat containing x,y and c^*. If d works in F, it works in $\mathbb{R}^n$.

Let ℓ be the line in F passing through c^* and perpendicular to $[x,y]$. Let c' denote the intersection of ℓ and $[x,y]$. Let p be the point of intersection of the half-line $\overrightarrow{c^*c'}$ and bd $B[c^*, r^*]$ (in case $\overrightarrow{c^* = c'}$, choose one of the intersections of the line ℓ and bd $B[c^*, r^*]$). Let s and q be the points of intersection of $[x,y]$ and bd $B[c^*, r^*]$.

Applying the Pythagorean Theorem to the triangle $sc'c^*$ and noting that $||c^*-c'||=r^*-||c'-p||$, it is seen that

$$r^* = \frac{\frac{||s-q||^2}{4} + ||c'-p||^2}{2||c'-p||} .$$

Because $B[c,r] \subset B[c^*,r^*]$ and c is on the segment [x,y], $r \leq ||c'-p||$. Since

$$\frac{\pi}{2} \geq \sphericalangle psc^* \geq \sphericalangle psc' > 0$$

and

$$\sphericalangle psc^* = \sphericalangle spc^*,$$

$$\cos(\sphericalangle spc^*) \leq \cos(\sphericalangle psc')$$

then

$$\frac{||c'-p||}{||s-p||} \leq \frac{\frac{||s-q||}{2}}{||s-p||} .$$

Thus

$$0 \leq r \leq ||c'-p|| \leq \frac{||s-q||}{2} .$$

For some $\lambda \in (0,1)$, $r=\lambda||c'-p||$.

Also

$$||c'-p||^2 \leq \frac{||s-q||^2}{4}$$

implies

$$\lambda(1-\lambda)||c'-p||^2 \leq (1-\lambda)\frac{||s-q||^2}{4} ,$$

expanding this

$$\lambda[\frac{||s-q||^2}{4}+||c'-p||^2] \leq \frac{||s-q||^2}{4} + \lambda^2||c'-p||^2$$

and by dividing both sides by $2\lambda||c'-p||$,

$$r^* = \frac{\frac{||s-q||^2}{4}+||c'-p||^2}{2||c'-p||} \leq \frac{\frac{||s-q||^2}{4} + \lambda^2||c'-p||^2}{2\lambda||c'-p||}$$

$$= \frac{\frac{||s-q||^2}{4} + r^2}{2r} .$$

But $\frac{||s-q||}{2} < ||x-y||$ so

$$r^* < \frac{||x-y||^2+r^2}{2r} .$$

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Remark Bunt's original proof used induction on the dimension of $\mathbb{R}^n$. Bunt's Lemma can be extended to smooth finite dimensional normed linear spaces (see Busemann's Lemma below). However, the result is false without smoothness - take $\mathbb{R}^2$ with norm $||(x,y)||=|x|+|y|$. For $M=\{(-2,0),(2,0)\}$, $B[0,1]$ can be contained in balls of arbitrarily large radius not intersecting M.

Theorem In $\mathbb{R}^n$, every Chebyshev set is convex.

Proof (Bunt [8]) Let M be a Chebyshev set in $\mathbb{R}^n$ which is not convex. Then there exists points x and y on the boundary of M such that $\{\lambda x+(1-\lambda)y: 0<\lambda<1\} \cap M = \phi$. Since M is closed, there exists a ball B[c,r] with center at $c = \frac{1}{2}x + \frac{1}{2}y$ and with radius r>0 such that $M \cap B[c,r] = \phi$. Let d be the supremum of the radii of the closed balls which contain B[c,r] and which do not intersect M (d exists by Bunt's Lemma). Let $\{B[m_i,d_i]\}$ be a sequence of closed balls with $d_i \to d$ which contain B[c,r] and which do not intersect M. Since $||c-m_i|| \leq d_i \leq d$, the sequence of points $\{m_i\}$ is bounded and hence, by passing to a subsequence if necessary, $m_i \to m$ as $i \to \infty$.

Now B[m,d] contains B[c,r]. For if $q \in B[c,r]$ then $||q-m_i|| \leq d_i$ which implies $||q-m|| \leq d$.

Also $M \cap B(m,d) = \phi$. For if $q \in M$ then $||q-m_i|| > d_i$ which implies $||q-m|| \geq d$.

Since M is closed, the maximality of the radius implies $B[m,d] \cap M \neq \phi$. Since M is a Chebyshev set, the intersection of B[m,d] and M is a point; call it p.

Let B[z,r] be a ball such that $p \in B[z,r] \subset B[m,d]$. Since $B[c,r] \cap M = \phi$, $c \neq z$.

<u>Claim</u> There exists a $\delta>0$ such that :

1) $[B[m,d] + \lambda \frac{(c-z)}{||c-z||}] \cap M = \phi$, and

2) $B[c,r] \subset B[m,d] + \lambda \frac{(c-z)}{||c-z||}$

whenever $0<\lambda<\delta$.

Let $t \in [c+(p-z),p] \cap B(m,d)$ (always possible by strict convexity). Thus $||t-p|| \leq ||c-z||$. Choose $s>0$ such that $B[t,s] \subset B[m,d]$. $B[m,d] \backslash B(p,s)$ is bounded and closed and has no point in common with M and hence has positive distance γ from M. Let $\delta = \min(\gamma, ||t-p||)$. Note that

$$B[m,d] + \lambda \frac{(t-p)}{||t-p||} = B[m,d] + \lambda \frac{(c-z)}{||c-z||}.$$

Assume

$$q \in [B[m,d] + \lambda_o \frac{(t-p)}{||t-p||}] \cap M$$

for some $\lambda_o < \delta$. Now

$$d(q-\lambda_o \frac{(t-p)}{||t-p||}, M) \leq ||q- \lambda_o \frac{(t-p)}{||t-p||} - q||$$

$$= \lambda_o < \delta .$$

Thus

$$q-\lambda_o \frac{(t-p)}{||t-p||} \notin B[m,d]\backslash B(p,s)$$

so

$$q-\lambda_o \frac{(t-p)}{||t-p||} \in B(p,s).$$

Let

$$b = q - \lambda_o \frac{(t-p)}{||t-p||} + (t-p),$$

so

$$||b-t|| = \left|\left|q- \lambda_o \frac{(t-p)}{||t-p||} - p\right|\right| < s$$

which implies $b \in B(t,s) \subset B(m,d)$. Thus

$$[q-\lambda_o \frac{(t-p)}{||t-p||}, b] \subset B(m,d).$$

From the definition of b,

$$\left|\left|q-\lambda_o \frac{(t-p)}{||t-p||} - b\right|\right| = ||t-p||,$$

and

$$\left|\left|q-\lambda_o \frac{(t-p)}{||t-p||} - q\right|\right| = \lambda_o < \delta \leq ||t-p|| ;$$

thus

$$q \in [q-\lambda_o \frac{(t-p)}{||t-p||}, b] \subset B(m,d).$$

This is a contradiction as $M \cap B(m,d) = \phi$. Hence

$$[B(m,d] + \lambda \frac{(c-z)}{||c-z||}] \cap M = \phi$$

for $0<\lambda<\delta$, proving 1) of the claim.

Assume $q \in B[c,r]$. Now $q \in B[m,d]$ since $B[c,r] \subset B[m,d]$. Also

$$0<\lambda<\delta \leq ||t-p|| \leq ||c-z||$$

and c and z are in $B[m,d]$. Thus

$$q-\lambda \frac{(c-z)}{||c-z||} \in B[m,d]$$

since

$$q+(z-c) \in B[z,r] \subset B[m,d].$$

so

$$q \in B[m,d] + \lambda \frac{(c-z)}{||c-z||}$$

and

$$B[c,r] \subset B[m,d] + \lambda \frac{(c-z)}{||c-z||}.$$

Thus the claim is proved.

Since

$$B[m+\lambda \frac{(c-z)}{||c-z||}, d] \cap M = \phi,$$

if follows that for $d_o>d$ and d_o sufficiently close to d that

$$B[m+\lambda \frac{(c-z)}{||c-z||}, d_o] \cap M = \phi.$$

However, this contradicts the maximality of the radius of B[m,d]. Thus, every Chebyshev set is convex.

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Remark Close inspection of the proof shows that only strict convexity, finite dimensionality and Bunt's Lemma were used. Thus in any strictly convex finite dimensional normed linear space X in which Bunt's Lemma holds (e.g., if X is also smooth - compare section 5 below) every Chebyshev set in X is convex.

2. Motzkin (1935)

Remark In [18] Motzkin gave the following theorem and proof and mentioned that it could be extended to any smooth two-dimensional normed linear spaces (although it is not entirely clear how this would be done). In addition, it is not clear whether Motzkin's proof can be extended from $\mathbb{R}^2$ to $\mathbb{R}^n$.

Theorem In $\mathbb{R}^2$, every Chebyshev set is convex.

Proof Let M be a Chebyshev in $\mathbb{R}^2$. It will be shown that M is a sun, hence by Theorem 1, it is convex. Let $x \in \mathbb{R}^2 \setminus M$ be given. It is necessary to show that every point on the half-line $\overrightarrow{P_M(x)x}$ has $P_M(x)$ as best approximation. Assume, by translation if necessary, that 0 is the last point on the half-line that has $P_M(x)$ as best approximation (it has $P_M(x)$ as best approximation by Lemma 2).

Define the deviation of a vector to be the angle between it and the vector $P_M(x)$. Since the metric projection is continuous, there exists a closed ball $B[o,r]$ such that for every point $y \in B[o,r]$, $P_M(y)-y$ has deviation less than ε $(<\frac{\pi}{6})$.

Let D be the diameter of $B[o,r]$ perpendicular to $P_M(x)$. Now D divides the circumference of $B[o,r]$ into two half-circumferences. Let C be the half-circumference which is on the opposite side of D as $P_M(x)$. Let c_1 and c_2 be points on C such that $-c_1$ and $-c_2$ have deviation equal to ε and let d_1 and d_2 be points on D, different from 0, such that d_1-c_1 and d_2-c_2 have deviation equal to ε (d_1 and d_2 exist since $\varepsilon<\frac{\pi}{6}$).

Let $\widehat{c_1c_2}$ be the curve on C with endpoints c_1 and c_2. Let $d:\widehat{c_1c_2}\to D$ be the function defined by $d(c) = \overrightarrow{cP_M(c)}\cap D$ (defined since every point of $\widehat{c_1c_2}$ has deviation less than ε). Now d is injective since if $d(c')=d(c'')$, $P_M(c'')$ would be a better approximation to c' by the triangle inequality. Note that $d(c_1)\in[d_1,0]$ and $d(c_2)\in[d_2,0]$ and that d is continuous since P_M is continuous. Since the continuous image of a connected set is connected, by restricting the range of d to $[d(c_1),d(c_2)]$, d is surjective. Thus there is a bijective correspondence between $\widehat{c_1c_2}$ and $[d(c_1),d(c_2)]$. So there exists a $\hat{c}\in\widehat{c_1c_2}$ such that $d(\hat{c})=0$. Thus $0\in[\hat{c},P_M(\hat{c})]$ and so by the triangle inequality $P_M(\hat{c}) = P_M(0) = P_M(x)$. Thus $\hat{c}$ is a point on the half-line $\overrightarrow{P_M(x)x}$ further from $P_M(x)$ than 0 that has $P_M(x)$ as best approximation. But this is a contradiction as 0 was assumed to be the last point on the half-line that has $P_M(x)$ as best approximation.

///.

Remark Motzkin's original proof showed that every sun in $\mathbb{R}^2$ is convex by noting that M is the intersection (over all $x\in\mathbb{R}^2\setminus M$) of the complement of the open half-space formed by taking the union of all open balls with centers on the half-line $\overrightarrow{P_M(x)x}$ and radius the distance between the center and $P_M(x)$.

Theorem In $\mathbb{R}^2$, every bounded Chebyshev set is convex.

Proof (Motzkin [19]) Let M be a bounded Chebyshev set in $\mathbb{R}^2$. Assume M is not convex. Hence $M \neq co(M)$.

Case 1 bd $[co(M)] \subset M$.

Let r = sup d(p,M) where p runs through co(M). Since M is closed, r>0. Since co(M) is compact there exists a point c such that r = d(c,M). Thus the interior of B[c,r] has no point in common with M. Since M is a Chebyshev set, B[c,r] intersects M at only one point, call it q. By Lemma 3, there exists a ball with larger radius than r that is disjoint from M. This contradicts the maximality of the radius of B[c,r]. Thus M is convex.

Case 2 bd $[co(M)] \not\subset M$.

Let c be a coundary point of co(M) which is not contained in M. Let H be a supporting hyperplane of co(M) passing through c (possible by finite dimensionality). If a supporting hyperplane only has one point in common with co(M), that point is contained in M. Hence H contains an interval [a,b] which contains c in its relative interior and has only a and b belonging to M (since M is closed).

Let $B[c,r]$ be a ball that does not have any point in common with M. Let x be the point on the boundary of $B[c,r]$ such that $[c,x]$ is perpendicular to $[a,b]$ and x is in the half-space supported by the hyperplane H. Let r_1 be the half-line from x passing through the closed triangle cxb, such that no point of M is contained in the open area bounded by r_1 and $\overrightarrow{xc}$, while r_1 has at least one point y in common with M. Let r_2 be the half-line from y, perpendicular to r_1, which eventually lies in the half-space not containing M. Let w be the point on r_2 such that $||w-y||=1$. Let $K=\{\alpha\in R: B(y+\alpha(w-y),\alpha)\cap M=\phi\}$. By taking α sufficiently small K is seen to be nonempty and if α is taken big enough, K is seen to be bounded above. Let $\beta=\sup K$.

Assume $\beta\notin K$ (a contradiction will be shown). Then $B(w+\beta(w-y),\beta)\cap M\neq\phi$. Let $z=P_M(w+\beta(w-y))$. Then $z\in B(w+\beta(w-y),\beta)$ and so

$$||w+\beta(w-y)-z||=d(w+\beta(w-y),M)<\beta.$$

Choose $\varepsilon>0$ such that

$$d(w+\beta(w-y),M)<\beta-\frac{2\varepsilon}{3}<\beta-\frac{\varepsilon}{3}<\beta.$$

Then

$$\begin{aligned}||w+(\beta-\tfrac{\varepsilon}{3})(w-y)-z|| &\leq ||w+\beta(w-y)-z||+||\tfrac{\varepsilon}{3}(w-y))|| \\ &=d(w+\beta(w-y),M)+\frac{\varepsilon}{3} \\ &<\beta-\frac{2\varepsilon}{3}+\frac{\varepsilon}{3} \\ &=\beta-\frac{\varepsilon}{3}.\end{aligned}$$

so $z \in B(w+(\beta-\frac{\varepsilon}{3})(w-y),\beta-\frac{\varepsilon}{3})$, a contradiction. Thus $\beta \in K$.

Since M is a Chebyshev set, $B[w+\beta(w-y),\beta] \cap M=\{y\}$. Let $S = B[w+\beta(w-y),\beta] \setminus B(y,\varepsilon)$ where $\varepsilon<\beta$ and ε is sufficiently small so that $M \cap B(y,\varepsilon)$ is on one side of r_1. Now S is closed and bounded and $S \cap M=\phi$. Thus $d(S,M)= \delta>0$. Thus $B[w+(\beta+\frac{\delta}{2})(w-y),\beta+\frac{\delta}{2}]$ intersects M only within $B(y,\varepsilon)$. But r_1 is the supporting hyperplane of $M \cap B(y,\varepsilon)$ at y, and the balls with centers on r_2 have r_1 as a tangent line. Thus the intersection of $B[w+(\beta+\frac{\delta}{2})(w-y),\beta+\frac{\delta}{2}]$ and $M \cap B(y,\varepsilon)$ is y. This is a contradiction as $\beta=\sup K$. Thus M is convex.

///.

3. Kritikos (1938)

Definition A closed set M in $\mathbb{R}^n$ is an F-set, if for any $r>0$ and any point $p \in \mathbb{R}^n \setminus M$ there exists a closed ball of radius r containing p and contained in $\mathbb{R}^n \setminus M$.

Definition A closed set M in $\mathbb{R}^n$ is an H-set if for every point $p \in \mathbb{R}^n \setminus M$ there exists a hyperplane passing through p such that M is contained in one of the closed half-spaces formed by the hyperplane. (Kritikos called such sets G-sets).

Lemma In $\mathbb{R}^n$, every F-set is an H-set.

Proof (Kritikos [17]) Let M be an F-set in $\mathbb{R}^n$ and let $p \in \mathbb{R}^n \setminus M$. Thus there exists a sequence of balls $\{B[m_i, d_i]\}$ with $d_i \to \infty$ containing p and contained in $\mathbb{R}^n \setminus M$. Let $q \in M$. Then $||q-m_i|| > d_i$ and $||q-m_i|| \to \infty$ as $i \to \infty$. Since $\Big| ||m_i - q|| - ||q-p|| \Big| \leq ||m_i - p||$, $||m_i - p|| \to \infty$ as $i \to \infty$. Thus, for i large, $m_i - p \neq 0$. Hence by passing to a subsequence if necessary, $p + \dfrac{m_i - p}{||m_i - p||} \to c$. So the direction of $\overrightarrow{pm_i}$ converges to the direction of $\overrightarrow{pc}$. Let H be the hyperplane perpendicular to $\overrightarrow{pc}$ and passing through p.

Let t be any point in the open half-space of H containing c. So the angle $\measuredangle(\overrightarrow{pc}, \overrightarrow{pt}) < \frac{\pi}{2}$. Let s_i be the point of intersection of $\overrightarrow{pt}$ and bd $B[m_i, d_i]$. Let b be the point of intersection of $\overrightarrow{pm_i}$ and the hyperplane T perpendicular to $\overrightarrow{ps_i}$ passing through s_i.

Claim. For i large, $||p-b||\geq d_i$.

For i sufficiently large, $\overrightarrow{pm_i}$ is in the direction of $\overrightarrow{pc}$. Assume for purposes of the claim, by scaling and translating if necessary, that $m_i=0$ and $d_i=1$. Let $u=\beta p\cap \text{bd } B[0,1]$, $\beta\in\mathbb{R}$. Thus $p=\lambda u$, $0\leq\lambda\leq 1$. Now $H=\{x:<u,x-p>=0\}$. So $<u,s_i-p><0$ and $<u,s_i><\lambda$. Also $T=\{x:<p-s_i,x-s_i>=0\}$. Since $b=T\cap\alpha u$, $\alpha\in\mathbb{R}$, $<p-s_i,\alpha u-s_i>=0$ which implies

$$\alpha = \frac{\lambda<u,\ s_i>-1}{\lambda-<u,\ s_i>} .$$

Thus

$$||p-b|| = \frac{|\lambda^2 - 2\lambda<u,\ s_i> + 1|}{\lambda -<u,\ s_i>} .$$

So it sufficies to show that

$$|\lambda^2 - 2\lambda<u,\ s_i>+1|-\lambda+<u,\ s_i>\geq 0.$$

If $\lambda=0$, $1+<u,\ s_i>\geq 0$ since $|<u,\ s_i>|\leq||u||\ ||s_i||$. If $\lambda=1$, $2|1-<u,\ s_i>|-(1-<u,\ s_i>)\geq 0$.

Case 1 $\lambda^2 - 2\lambda<u,\ s_i>+1\geq 0$.

So

$$f(\lambda) = \lambda^2 - 2\lambda<u,\ s_i>+1-\lambda+<u,\ s_i>,$$

$$f'(\lambda) = 2\lambda - 2<u,\ s_i>-1,$$

and f has a minimum when $f'(\lambda) = 0$; that is, $\lambda = \langle u, s_i\rangle + \frac{1}{2}$. But $0 \leq \lambda \leq 1$, hence $-\frac{1}{2} \leq \langle u, s_i\rangle \leq \frac{1}{2}$. Thus $f(\langle u,s_i\rangle+\frac{1}{2}) = -\langle u,s_i\rangle^2+\frac{3}{4}>0$.

<u>Case 2</u> $\lambda^2 - 2\lambda\langle u, s_i\rangle+1<0$.

So

$$g(\lambda) = -\lambda^2 + 2\lambda\langle u, s_i\rangle-1-\lambda+\langle u, s_i\rangle,$$

$$g'(\lambda) = -2\lambda+2\langle u, s_i\rangle-1,$$

and

$$g''(\lambda) = -2<0.$$

Thus g is concave down.

So

$$|\lambda^2 - 2\lambda\langle u, s_i\rangle+1|-\lambda+\langle u, s_i\rangle \geq 0$$

for $0 \leq \lambda \leq 1$ and the claim is proved.

Hence

$$||s_i-p|| \geq d_i \frac{||s_i-p||}{||p-b||}$$

$$= d_i \cos \measuredangle (\overrightarrow{pm_i}, \overrightarrow{pt}).$$

Since

$$\lim_{i\to\infty} \cos\measuredangle(\overrightarrow{pm_i}, \overrightarrow{pt}) = \cos\measuredangle(\overrightarrow{pc},\overrightarrow{pt})>0, ||s_i-p||\to\infty$$

as $i\to\infty$. Thus, eventually, $t\in[p,s_i]$ and hence $t\in B[m_i,d_i]$ and so $t \in \mathbb{R}^n \setminus M$. Since t was an arbitrary point in the open half-space of H containing c, M is contained in the closed half-space not containing c. Hence M is an H-set.

///.

Lemma In $\mathbb{R}^n$, every H-set is either convex or contained in a hyperplane.

Proof (Kritikos [17]) Let M be a H-set in $\mathbb{R}^n$ and assume M is not contained in a hyperplane. If n=1 and if M contains two points q_1 and q_2, it must contain all the points of the segment $[q_1, q_2]$ since M is an H-set. Hence assume $n \geq 2$. If M is not convex, there exists two points q_1 and q_2 of M such that the segment $[q_1, q_2]$ contains a point p_2 in $\mathbb{R}^n \setminus M$. Choose a point q_3 in M not on the line connecting q_1 and q_2; such a point exists since $n \geq 2$ and M is not contained in a hyperplane. On the segment $[q_3, p_2]$ choose a point p_3 in the relative interior of $[q_3, p_2]$ such that $p_3 \in \mathbb{R}^n \setminus M$; this is possible since $p_2 \in \mathbb{R}^n \setminus M$. If $n > 3$, one can continue this construction until one has n+1 points $q_1, q_2, \ldots, q_{n+1}$ in M which determine an n-dimensional simplex whose interior contains a point p_{n+1} of $\mathbb{R}^n \setminus M$. But this implies a contradiction, as it is impossible to pass a hyperplane through p_{n+1} that does not separate some of the points $q_1, q_2, \ldots, q_{n+1}$. Thus M is convex.

///.

<u>Theorem</u> In $\mathbb{R}^n$, every Chebyshev set is convex.

<u>Proof</u> Let M be a Chebyshev set in $\mathbb{R}^n$. It may be assumed that $0 \in M$ and that span M is $\mathbb{R}^n$ (otherwise, restrict attention to span M).

<u>Claim</u> M is an F-set.

If $p \in \mathbb{R}^n \setminus M$, $B[p,r] \subset \mathbb{R}^n \setminus M$ for $r < d(p,M)$. Assume the radii of the closed balls containing p and contained in $\mathbb{R}^n \setminus M$ are bounded and let d be the supremum. Hence all closed balls containing p and contained in $\mathbb{R}^n \setminus M$ with radius d have at least one point in common with M.

Let $\{B[m_i,d_i]\}$ be a sequence of closed balls containing p and contained in $\mathbb{R}^n \setminus M$ with $d_i \to d$. Since $||p-m_i|| \leq d_i \leq d$, the sequence of points $\{m_i\}$ is bounded. Hence, by passing to a subsequence if necessary, $\{m_i\}$ converges to a point m.

Every interior point of B[m,d] is an interior point of $B[m_i,d_i]$ for i sufficiently large; hence every interior point of B[m,d] is contained in $\mathbb{R}^n \setminus M$. Also $||m_i - p|| \leq d$ and $\lim_{i\to\infty} ||m_i - p|| = ||m-p||$ so $p \in B[m,d]$. Since the radius is d, $B[m,d] \cap M \neq \phi$; since M is a Chebyshev set, the intersection of B[m,d] and M is a point, call it q.

Let H be the supporting hyperplane of B[m,d] at q. Let $\delta = d(p,M)$ and $\alpha = \min\{\frac{d}{2}, \frac{\delta}{2}\}$. Let $H' = H + \alpha\frac{(m-q)}{||m-q||}$.

The translation defined by the vector $\lambda\frac{(m-q)}{||m-q||}$, where $0<\lambda\leq\alpha$, translates the part of the ball B[m,d] on the same side of H' as q into the interior of the ball B[m,d]. Thus, this part of the ball is translated into $\mathbb{R}^n\setminus M$. The remaining part of the ball is a closed set in $\mathbb{R}^n\setminus M$, thus there is a minimum distance ε from it to M. Hence as long as $\lambda<\varepsilon$, its translation is contained in $\mathbb{R}^n\setminus M$. Also $p\in B[m,d] + \lambda\frac{(m-q)}{||m-q||}$ if $\lambda\leq\alpha$. Thus for $\lambda<\min\{\alpha,\varepsilon\}$, $B[m,d] + \lambda\frac{(m-q)}{||m-q||}$ is a ball with radius d which contains p, is contained in $\mathbb{R}^n\setminus M$, and has no point in common with M. Thus for $d_o>d$ and d_o sufficiently close to d, $B[m + d_o\frac{(m-q)}{||m-q||}, d_o]\cap M = \phi$. However, this contradicts the maximality of the radius of B[m,d]. Thus, M is an F-set.

Consequently, M is an H-set. Since span $M = \mathbb{R}^n$, M is convex. ///.

Remark Kritikos's original proof used an induction argument on the dimension n to show that M is convex. That one can assume that M is not contained in a hyperplane was suggested by Professor Deutsch.

4. Jessen (1940)

Remark The first part of Jessen's proof is similar to Bunt's proof.

Theorem In $\mathbb{R}^n$, every Chebyshev set is convex.

Proof (Jessen [13]) Let M be a Chebyshev set in $\mathbb{R}^n$ which is not convex. Then there exists points x and y on the boundary of M such that $\{\lambda x+(1-\lambda)y:0<\lambda<1\}\cap M = \phi$. Since M is closed, there exists a ball $B[c,r]$ with center at $c = \frac{1}{2}x + \frac{1}{2}y$ and with radius $r>0$ such that $M\cap B[c,r] = \phi$. Let $S =\{s\in\mathbb{R}^n\setminus M:B[c,r]\subset B[s,||s-P_M(s)||]\}$. Since P_M is continuous, S is closed. Since M is in $\mathbb{R}^n$, at least one of x and y is outside of $B[s,||s-P_M(s)||]$ for all $s\in S$. From this it follows that S is bounded (a rigorous proof of this is given in section 5 (Busemann's Lemma)). Thus S is compact. Hence the continuous function $d(.,M)$ attains a maximum on the set S; say at a point $\hat{s}$. Then the ball $B[\hat{s},||\hat{s}-P_M(s)||]$ contains $B[c,r]$ and only intersects M at $P_M(\hat{s})$. Thus by Lemma 3, there exists a ball B with radius larger than d, containing $B[c,r]$ and disjoint from M. Thus the center of the ball B belongs to the set S. But this contradicts the maximality of the radius of $B[\hat{s},||\hat{s}-P_M(\hat{s})||]$. Hence every Chebyshev set in R^n is convex.

///.

Remark Jessen did not give a detailed proof that S is bounded, but only stated this fact.

5. Busemann (1947)

Definition In a metric space X with metric d let (xyz) denote the statement that the three points x,y and z are different and the equation $d(x,y) + d(y,z) = d(x,z)$ holds. X is a straight line space if the following conditions are satisfied:

1) Each bounded infinite set has an accumulation point.
2) For any two different points x and z, there is a point y with (xyz).
3) For any two different points x and y, there is a point z with (xyz).
4) If (xyz_1) and (xyz_2), and $d(y,z_1) = d(y,z_2)$, then $z_1 = z_2$.

Theorem Let X be a normed linear space. Then X is a straight line space if and only if X if finite dimensional and strictly convex.

Proof If X is a straight line space, then 1) implies B(X) is compact and hence X is finite dimensional. Let $u, v \in X$, $||u||=||v||=1$, and $||u+v||=2$. Set $x=u$, $z_1=-u$, $z_2=-v$, and $y=0$. Then for i=1,2

$$||x-y||+||y-z_i||=2=||x-z_i||$$

so by 4) $z_1=z_2$, i.e. u=v. Thus X is strictly convex.

Conversely, if X is a finite dimensional (strictly convex) space, conditions 1), 2), and 3) are obviously satisfied. For i=1,2, let x,y, and z_i be three distinct points with

$$||x-y||+||y-z_i||=||x-z_i||$$

and $||y-z_1||=||y-z_2||$. Then $||x-z_1||=||x-z_2||$. Let $\lambda = \frac{||x-y||}{||x-z_i||}$, $u = \frac{x-y}{||x-y||}$, and $v_i = \frac{y-z_i}{||y-z_i||}$ $(i=1,2)$. Then $||u||=||v||=1$ and $||\lambda u+(1-\lambda)v||=1$. By strict convexity, $u=v_i$ $(i=1,2)$, i.e. $z_1=z_2$. Thus condition 4) is satisfied and X is a straight line space. ///.

Remark The following extends Bunt's Lemma.

Lemma (Busemann) Let there be given in a smooth finite dimensional normed linear space X: a closed set M, a point $c \notin M$ but $c = \frac{1}{2}x + \frac{1}{2}y$, $x,y \in M$, and a ball $B[c,r]$ such that $B[c,r] \cap M = \phi$. Then the radii of the closed balls which do not intersect M are bounded above by a positive number d.

Proof (Busemann [7]) Assume $\{B[m_i,d_i]\}$ is a sequence of balls with $d_i \to \infty$ which contain $B[c,r]$ and which do not intersect M. Define

$$(1) \quad w_i \equiv m_i + (||c-m_i||+r) \frac{(c-m_i)}{||c-m_i||} .$$

Since $||w_i-c||=r$, $w_i \in \text{bd } B[c,r]$. Thus $B[m_i,||m_i-w_i||] \subset B[m_i,d_i]$ since $w_i \in B[c,r] \subset B[m_i,d_i]$. Also $M \cap B[m_i,d_i] = \phi$ implies $M \cap B[m_i,||m_i-w_i||] = \phi$, so $x,y \notin B[m_i,||m_i-w_i||]$ for all i.

From (1)

$$m_i = w_i + \left[\frac{||c-m_i||+r}{r}\right] (c-w_i)$$

and

$$||m_i - w_i|| = \left(\frac{||c-m_i||+r}{r}\right) ||c-w_i||.$$

so

$$B[m_i, ||m_i - w_i||] = B[w_i + \lambda_i(c-w_i), \lambda_i ||c-w_i||]$$

where

$$\lambda_i = \left[\frac{||c-m_i||+r}{r}\right].$$

Since bd B[c,r] is closed and bounded, it is compact. Thus, by passing to a subsequence if necessary, $w_i \to w$. Consider the unique supporting hyperplane $[x^*: x^*(c)+||w-c||]$ of B[c,r] at w. Either $x^*(x-c)<||w-c||$ or $x^*(y-c)<||w-c||$. For if not

$$x^*(x-c) \geq ||w-c|| \text{ and } x^*(y-c) \geq ||w-c||.$$

Thus

$$\frac{1}{2} x^*(x-c) + \frac{1}{2} x^*(y-c) \geq ||w-c||$$

so

$$||w-c|| \leq x^*(\tfrac{1}{2}x + \tfrac{1}{2}y - c) = x^*(0) = 0.$$

This is a contradiction, as $||w-c||=r>0$. Assume without loss of generality that

$$x^*(y-c)<||w-c||.$$

Thus there exists a $\lambda>0$ such that $y\in B(w+\lambda(c-w),\lambda||c-w||)$. Assume $y\notin B(w_i+(\lambda+1)(c-w_i),(\lambda+1)||c-w_i||)$ frequently (a contradiction will be shown). Thus, for infinitely many i,

$$||y-w_i-(\lambda+1)(c-w_i)||\geq(\lambda+1)||c-w_i||.$$

So

$$\lambda||c-w_i||\leq||y-w_i-(\lambda+1)(c-w_i)||-||c-w_i||$$

$$\leq||y-w_i-\lambda(c-w_i)||.$$

By passing to the limit on both sides of the inequality

$$\lambda||c-w||\leq||y-w-\lambda(c-w)||.$$

But $y\in B(w+\lambda(c-w),\lambda||c-w||)$ implies

$$||y-w-\lambda(c-w)||<\lambda||c-w||$$

which is a contradiction. Thus

$$y\in B(w_i+(\lambda+1)(c-w_i),(\lambda+1)||c-w_i||)$$

eventually.

To show $\lambda_i\to\infty$ it suffices to show that $||c-m_i||\to\infty$. Assume $||c-m_i||<M$ for all i. Now

$$||x-m_i||\leq||x-c||+||c-m_i||$$

$$<||x-c||+M.$$

Thus for $d_i \geq ||x-c||+M$, $x \in B[m_i,d_i]$. This is a contradiction of the definition of $B[m_i,d_i]$.

Choose N such that for $i \geq N$, $\lambda_i > \lambda+1$ and

$$y \in B(w_i+(\lambda+1)(c-w_i),\ (\lambda+1)||c-w_i||).$$

That is, for $i \geq N$,

$$y \in B(w_i+\lambda_i(c-w_i),\lambda_i||c-w_i||)=B(m_i,||m_i-w_i||).$$

But this is a contradiction of the construction of $B[m_i,||m_i-w_i||]$.

///.

Theorem In a strictly convex smooth finite dimensional normed linear space X, every Chebyshev set is convex.

Proof Busemann's Lemma can be used in place of Bunt's Lemma in the proof of Section 1.

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Remark Busemann claimed that Lemma 3 could be extended to straight line spaces; however, this is not clear to me.

6. Klee (1961)

Remark In [15] Klee showed that a Chebyshev set M in a smooth reflexive Banach space X, with every point of $X\setminus M$ having a neighborhood on which the restricted metric projection is both continuous and weakly continuous, is convex. In the finite dimensional case, the theorem and proof reduce to the following.

Theorem In a smooth finite dimensional normed linear space X, every Chebyshev set is convex.

Proof (Klee [15]) Let M be a Chebyshev set in X. It will be shown that M is a sun, hence by Theorem 1 it is convex. Let $x \in X\setminus M$ be given. It is necessary to show that every point on the half-line $\overrightarrow{P_M(x)x}$ has $P_M(x)$ as best approximation. Assume, by translation if necessary, that 0 is the last point on the half-line that has $P_M(x)$ as best approximation (it has $P_M(x)$ as best approximation by Lemma 2).

Let x^* be a norm one functional such that $x^*(P_M(x)) = ||P_M(x)||$. Let $H = x^{*-1}(0)$. Hence every point $y \in X$ can be written uniquely in the form $y = y' + x^*(y)p$, $y' \in H$, $p = \frac{P_M(x)}{||P_M(x)||}$.

Choose ε such that $0 < \varepsilon < \frac{1}{6}||P_M(x)||$. Thus $B[P_M(x), \varepsilon] \cap H = \phi$. For if $y \in B[P_M(x), \varepsilon]$,

(1) $$x^*(y) = x^*(y-P_M(x)) + x^*(P_M(x))$$

$$\geq -||y-P_M(x)||+||P_M(x)||$$

$$> -\frac{1}{6}||P_M(x)||+||P_M(x)||$$

$$> 0.$$

Choose $\delta>0$ such that

1. $0<\delta<\frac{1}{6}||P_M(x)||$

2. $B[0,\delta]\cap M = \phi$

and 3. $y\in B[0,\delta]\Rightarrow P_M(y)\in B[P_M(x),\varepsilon]$

(possible since P_M is continuous).

Then for $y\in B[0,\delta]$

(2) $$x^*(y)<x^*(P_M(y))$$

since

$$x^*(P_M(y)-y) = x^*(P_M(y) - P_M(x) + P_M(x) - y)$$

$$= ||P_M(x)||+x^*(P_M(y) - P_M(x)) - x^*(y)$$

$$\geq ||P_M(x)||-||P_M(y) - P_M(x)||-||y||$$

$$\geq ||P_M(x)|| - \frac{1}{6}||P_M(x)|| - \frac{1}{6}||P_M(x)||$$

$$= \frac{2}{3}||P_M(x)||$$

$$> 0 .$$

Also, for $y \in B[o,\delta]$,

(3) $||P_M(y)'-y'|| \leq x^*(P_M(y))-x^*(y)$

since

$$||P_M(y)'-y'|| \leq ||P_M(y)-x^*(P_M(y))p||+||y-x^*(y)p||$$

$$\leq ||P_M(y)-x^*(P_M(y)-P_M(x))\frac{P_M(x)}{||P_M(x)||} - P_M(x)||+2||y||$$

$$\leq \; 2||P_M(y)-P_M(x)||+2||y||$$

$$\leq \; 2 \cdot \frac{1}{6}||P_M(x)||+2 \cdot \frac{1}{6}||P_M(x)||$$

$$= \frac{2}{3}||P_M(x)||$$

$$\leq x^*(P_M(y)-y).$$

Assume without loss of generality, by scaling if necessary, that $\delta=2$ and hence $||P_M(x)||>12>2$. Let $W=\{x \in H: ||x|| \leq 1\}$ and $q = \frac{p}{||p||}$. Thus $W-q \subset B[0,2]$. By (1) and (2), for each $w \in W$, $x^*(w-q)<0<x^*(P_M(w-q))$, so the

segment from w-q to $P_M(w-q)$ must intersect H in a unique point, call it f(w). Thus

$$f(w) = (1-\lambda)(w-q) + \lambda[P_M(w-q)]$$

where

$$\lambda = \frac{x^*(q)}{x^*(q) + x^*(P_M(w-q))}$$

since $x^*(f(w)) = 0$.

Since P_M is continuous, f is continuous. For each $w \in W$,

$$f(w)-w = [f(w)-w]'$$

$$= \lambda[P_M(w-q)' - (w-q)']$$

and by (3)

$$||P_M(w-q)' - (w-q)'|| \leq x^*(P_M(w-q) - x^*(w-q)$$

$$= \frac{x^*(q)}{\lambda}$$

so $||f(w) - w|| \leq ||x^*(q)|| \leq ||q|| = 1$.

For each $z \in 2W$, let $g(z) = z - f(\frac{1}{2} z)$. Then g is continuous and $||g(z)|| \leq ||\frac{1}{2} z|| + ||\frac{1}{2} z - f(\frac{1}{2} z)|| \leq 2$, so $g(2W) \subset 2W$. Now 2W is compact and convex. Hence by the

Schauder-Tychonoff Fixed Point Theorem there exists a $\hat{w} \in 2W$ such that $g(\hat{w}) = \hat{w}$. Thus $f(\frac{1}{2}\hat{w}) = 0$, so $0 \in [\frac{1}{2}\hat{w}-q, P_M(\frac{1}{2}\hat{w}-q)]$. Thus by the triangle inequality $P_M(\frac{1}{2}\hat{w}-q) = P_M(0) = P_M(x)$. Thus $\frac{1}{2}\hat{w}-q$ is a point on the half-line $\overrightarrow{P_M(x)x}$ further from $P_M(x)$ than 0 that has $P_M(x)$ as best approximation. But this is a contradiction as 0 was assumed to be the last point on the half-line that has $P_M(x)$ as best approximation.

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Remark Klee showed that every sun in a smooth space is convex by noting that M is the intersection (over all $x \in X \setminus M$) of the complement of the open half-space formed by taking the union of all open balls with centers on the half-line $\overrightarrow{P_M(x)x}$ and radius the distance between the center and $P_M(x)$.

7. Vlasov (1961)

Remark In [22] Vlasov showed that in any smooth Banach space every boundedly compact Chebyshev set is convex (a set M is boundedly compact if every bounded sequence in M has a subsequence which converges to a point in M). In the finite dimensional case, the theorem and proof reduce to the following.

Theorem In a smooth finite dimensional normed linear space X, every Chebyshev set is convex.

Proof (Vlasov [22]) Let M be a Chebyshev set in X. It will be shown that M is a sun, hence by Theorem 1 it is convex. Let $x \in X \setminus M$ be given. It is necessary to show that every point on the half-line $\overrightarrow{P_M(x)x}$ has $P_M(x)$ as best approximation. Assume, by translation if necessary, that 0 is the last point on the half-line that has $P_M(x)$ as best approximation (it has $P_M(x)$ as best approximation by Lemma 2).

Let $T:B[0,||P_M(x)||] \to B[0,||P_M(x)||]$ be defined by

$$T(z) = - \left|\left|\frac{P_M(x)}{P_M(z)}\right|\right| \quad P_M(z).$$

Now $B[0,||P_M(x)||]$ is compact and convex, and T is continuous since P_M is continuous. Thus by the Schauder-Tychonoff Fixed Point Theorem there exists a $\hat{z} \in B[0,||P_M(x)||]$ such that

$T(\hat{z}) = \hat{z}$; that is

$$\hat{z} = -\frac{||P_M(x)||}{||P_M(\hat{z})||} P_M(\hat{z}).$$

Thus $\hat{z} \neq 0$ and

$$0 = \frac{||P_M(\hat{z})||}{||P_M(\hat{z})||+||P_M(x)||} \hat{z} + \frac{||P_M(x)||}{||P_M(\hat{z})||+||P_M(x)||} P_M(\hat{z}).$$

so $0 \in [\hat{z}, P_M(\hat{z})]$. Thus by the triangle inequality $P_M(\hat{z}) = P_M(0) = P_M(x)$. Thus $\hat{z}$ is a point on the half-line $\overrightarrow{P_M(x)x}$ further from $P_M(x)$ than 0 that has $P_M(x)$ as best approximation. But this is a contradiction as 0 was assumed to be the last point on the half-line that has $P_M(x)$ as best approximation.

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Summary

Recall that the proofs fall into three classes. Bunt, Jessen and Busemann use proof by contradiction to find a maximal ball and then move it. Motzkin, Klee and Vlasov show that a Chebyshev set is a sun, hence convex. Kritikos's proof uses the concept of F-set and H-set.

Two approaches to finding a proof not using a fixed point theorem for the finite dimensional case readily present themselves.

(1) Try to extend Motzkin's proof. Recall that Motzkin did not use a fixed point theorem but instead used the idea of deviation. The idea of deviation can be extended to any normed linear space; Motzkin's proof is probably capable of generalization.

(2) Try to extend Lemma 3 (concerning the existence of a ball larger than the maximal ball) from $\mathbb{R}^n$ to any finite dimensional normed linear space. In conjunction with Busemann's Lemma this would give a new proof (which would be fixed point free) that in a smooth finite dimensional normed linear space, every Chebyshev set is convex.

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