Topological Groups and Topological Games

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Semitopological Groups

A triple $(G, \cdot, \tau)$ is called a semitopological group if:

(i) $(G, \cdot)$ is a group;

(ii) (G, τ) is a topological space;

(iii) multiplication, $(x, y) \mapsto x \cdot y$, from $G \times G$ into G is separately continuous.

A triple $(G, \cdot, \tau)$ is a topological group if:

(i) $(G, \cdot, \tau)$ is a semitopological group;

(ii) multiplication, $(x, y) \mapsto x \cdot y$, from $G \times G$ into G is jointly continuous;

(iii) inversion, $x \mapsto x^{-1}$, from G onto G is continuous.

Example 1. Let (X, τ) be a nonempty topological space and let G be a nonempty subset of X^X. If $(G, \circ)$ is a group (where "$\circ$" denotes the binary relation of function composition) and τ_p denotes the topology on X^X of pointwise convergence on X then $(G, \circ, \tau_p)$ is a semitopological group provided the members of G are continuous functions.

Example 2. Let $(G, \cdot)$ be a group and let (X, τ) be a topological space. Further, let $\pi : G \times X \to X$ be a mapping (group action) such that:

(i) $\pi(e, x) = x$ for all $x \in X$, where e denotes the identity element of G;

(ii) $\pi(g \cdot h, x) = \pi(g, \pi(h, x))$ for all $g, h \in G$ and $x \in X$;

(iii) for each $g \in G$, the mapping, $x \mapsto \pi(g, x)$, is a continuous function on X.

Then (G, X) is called a flow on X. Next, consider the mapping $\rho : G \to X^X$ defined by, $\rho(g)(x) = \pi(g, x)$ for all $x \in X$. Then $(\rho(G), \circ, \tau_p)$ is a semitopological group.

History

Research on the problem of which topological conditions on a semitopological group imply that it is a topological group possibly began in

[D. Montgomery, "Continuity in topological groups" *Bull. Amer. Math. Soc.* **42** (1936)]

when the author showed that each completely metrizable semitopological group has jointly continuous multiplication. Later, in

[R. Ellis, "A note on the continuity of the inverse" *Proc. Amer. Math. Soc.* **8** (1957) and "Locally compact transformation groups" *Duke Math. J.* **24** (1957)]

Ellis showed that each locally compact semitopological group is in fact a topological group. This answered a question raised by A. D. Wallace in

[A. D. Wallace, "The structure of topological semigroups" *Bull. Amer. Math.* **61** (1955)].

Next in

[W. Zelazko, "A theorem on B_0 division algebras" *Bull. Acad. Pol. Sci.* **8** (1960)]

Zelazko used Montgomery's result from 1936 to show that each completely metrizable semitopological group is a topological group. Much later, in

[A. Bouziad, "Every Čech-analytic Baire semitopological group is a topological group" *Proc. Amer. Math. Soc.* **124** (1996)]

Bouziad improved both of these results and answered a question raised by Pfister in

[H. Pfister, "Continuity of the inverse" *Proc. Amer. Math. Soc.* **95** (1985)]

by showing that each Čech-complete semitopological group is a topological group. (Recall that both locally compact and completely metrizable topological spaces are Čech-complete). To do this, it was sufficient for Bouziad to show that every Čech-complete semitopological group has jointly continuous multiplication since earlier, Brand

[N. Brand, "Another note on the continuity of the inverse" *Arch. Math.* **39** (1982)]

had proven that every Čech-complete semitopological group with jointly continuous multiplication is a topological group. Brand's proof of this was later improved and simplified in

[H. Pfister, "Continuity of the inverse" *Proc. Amer. Math. Soc.* **95** (1985)].

Apart from those named above there have been many other contributors to the question of when a semitopological group is a topological group. For example, Arhangel'skii, Cao, Choban, Kenderov, Korovin, Lawson, Piotrowski, Ravsky, Reznichenko and Tkachenko.

Topological Games

In [P. S. Kenderov, I. Kortezov and W.B. Moors, "Topological games and topological groups" *Topology Appl.* **109** (2001)] the authors used a two player topological game to determine some conditions on a semitopological group that imply it is a topological group. Using this game they were able to prove a theorem considerably more general than the following.

Theorem 1. Let $(G, \cdot, \tau)$ be a semitopological group such that (G, τ) is a regular Baire space. If any of the following conditions hold, then $(G, \cdot, \tau)$ is a topological group.

(i) (G, τ) is metrizable (or more generally, (G, τ) is a p-space);

(ii) (G, τ) is Čech-analytic (or more generally, has countable separation);

(iii) (G, τ) is locally countably compact.

The advantage to the "game" approach is that it covers many different cases at once. However, there is a disadvantage to using games too. Namely, people find the use of games unappealing, artificial, hard to read and hard to understand. Hence some of the consequences of the paper [KKM] have gone unnoticed.

So now, on to the dreaded game.

The game that we shall consider involves two players which we will call player α and player β. The "field/court" that the game is played on is a fixed topological space (X, τ) with a fixed dense subset D.

The name of the game is the "$G_S(D)$-game".

After naming the game we need to describe how to "play" the $G_S(D)$-game.

The player labeled β starts the game every time (life is not always fair). For their first move the player β must select a nonempty open subset B_1 of X. Next, α gets a turn. For α's first move he/she must select a nonempty open subset A_1 of B_1. This ends the first round of the game. In the second round, β goes first (again) and selects a nonempty open subset B_2 of A_1. α then gets to respond by choosing a nonempty open subset A_2 of B_2. This ends the second round of the game. At this stage we have

$$A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1.$$

In general, after α and β have played the first n-rounds of the $G_S(D)$-game, β will have selected nonempty open sets $B_1, B_2, \ldots, B_n$ and α will have selected nonempty open sets $A_1, A_2, \ldots, A_n$ such that:

$$A_n \subseteq B_n \subseteq A_{n-1} \subseteq B_{n-1} \subseteq \cdots \subseteq A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1.$$

At the start of the $(n+1)$-round of the game, β goes first (again!) and selects a nonempty open subset B_{n+1} of A_n. As with the previous n-rounds, player α gets to respond to this move by selecting a nonempty open subset A_{n+1} of B_{n+1}. Continuing this process indefinitely (i.e., continuing-on forever) the players produce an infinite sequence, (called a play of the $G_S(D)$-game)

$$\{(A_n, B_n) : n \in \mathbb{N}\}$$

of pairs of nonempty open subsets of X such that

$$A_{n+1} \subseteq B_{n+1} \subseteq A_n \subseteq B_n \quad \text{for all } n \in \mathbb{N}.$$

As with any game, we need a rule to determine who wins (otherwise it is a very boring game). We shall declare that α wins a play $\{(A_n, B_n) : n \in \mathbb{N}\}$ of the $G_S(D)$-game if:

(i) $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and

(ii) each sequence $(x_n : n \in \mathbb{N})$ in D with $x_n \in A_n$ for all $n \in \mathbb{N}$, has a cluster-point in X.

If α does not win a play of the $G_S(D)$-game then we declare that β wins that play of the $G_S(D)$-game. So every play is won by either α or β and no play is won by both players. Continuing further into game theory we need to introduce the notion of a strategy. A strategy for the player β (player α) is a "rule" that specifies how the player β (player α) must respond/move in every possible situation that may occur during the course of the game. [A more precise mathematical description of a strategy is possible, but we shall not give it here.]

We may now finally define a "strongly Baire" space. We shall say that a topological space (X, τ) is strongly Baire if it is regular and there exists a dense subset D of X such that the player β (i.e., the player with the privilege of going first) does NOT have a winning strategy in the $G_S(D)$-game played on X (that is to say, that no matter what strategy player β adopts there is always at least one play of the $G_S(D)$-game where α wins.). Clearly, if α actually possesses a winning strategy himself/herself then β cannot possibly possess a winning strategy as well and so all spaces (X, τ) in which α has a winning strategy in the $G_S(D)$-game are strongly Baire.

Note: there are some strongly Baire spaces in which the player α does not possess a winning strategy.

Known results

Theorem 2. [KKM, 2001] Let $(G, \cdot, \tau)$ be a semitopological space. If (G, τ) is a strongly Baire space then $(G, \cdot, \tau)$ is a topological group.

Corollary 1. [KM, 2012] Let $(G, \cdot, \tau)$ be a semitopological space. If (G, τ) is regular, T_1, Baire and has a countable network then $(G, \cdot, \tau)$ is a metrizable topological group.

Given Theorem 1. part (iii) which says that every semitopological group $(G, \cdot, \tau)$ such that (G, τ) is regular and locally countably compact is a topological group it is natural ask:

Question 1. If $(G, \cdot, \tau)$ is a semitopological group and (G, τ) is completely regular and pseudocompact is $(G, \cdot, \tau)$ a topological group?

Recall that a completely regular topological space (X, τ) is called pseudocompact if each real-valued continuous function defined on it is bounded and that every countably compact space is pseudocompact.

However, the answer to this question is “no”.

[A. V. Korovin, “Continuous actions on pseudocompact groups and topological axioms” *Comment Math. Univ. Carolin.* **33** (1992)].

Proposition 1. Let (X, τ) be a completely regular topological space. Then (X, τ) is pseudocompact if, and only if, for each decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of X, $\bigcap_{n\in\mathbb{N}} \overline{U_n} \neq \varnothing$.

Despite the previous negative result. It is possible to obtain some positive results for “nice” pseudocompact semitopological groups.

Theorem 3. [E. Reznichenko, 1994] Let $(G, \cdot, \tau)$ be a completely regular pseudocompact semitopological group. If any of the following conditions hold, then $(G, \cdot, \tau)$ is a topological group.

(i) (G, τ) has countable tightness;

(ii) (G, τ) is separable;

(iii) (G, τ) is a k-space.

The question now is, can we extend the game approach to include pseudocompactness?

Strongly bounded sets

We will say that a subset A of a topological space (X, τ) is bounded in X if for any sequence $(W_n : n \in \mathbb{N})$ of open sets in X such that $W_{n+1} \subseteq W_n$ and $A \cap W_n \neq \varnothing$ for all $n \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} \overline{W_n} \neq \varnothing$. When the space X is bounded in itself and completely regular it is pseudocompact. In this talk we need a stronger notion. A subset A of a topological space (X, τ) is said to be strongly bounded in X if for every infinite subset C of A there exists a separable subspace S of X such that the set $C \cap S$ is infinite and bounded in S.

Every countably compact space, as well as, every separable pseudocompact space is strongly bounded in itself and it is easy to show that every strongly bounded set in X is bounded in X.

We may now introduce the $G_S^*(D)$-game played on a topological space (X, τ). The $G_S^*(D)$-game is identical to the

$G_S(D)$-game in all regards except one. Namely, in the definition of a win for α. In the $G_S^*(D)$-game we declare that α **wins** a play $\{(A_n, B_n) : n \in \mathbb{N}\}$ of the $G_S(D)$-game if:

(i) $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and

(ii) each sequence $(x_n : n \in \mathbb{N})$ in D with $x_n \in A_n$ for all $n \in \mathbb{N}$, $\{x_n : n \in \mathbb{N}\}$ is strongly bounded in X.

We shall say that a topological space (X, τ) is **strongly boundedly Baire** if it is completely regular and there exists a dense subset D of X such that the player β does NOT have a winning strategy in the $G_S^*(D)$-game played on X.

The main result of this talk is the following.

Theorem 4. [CKM, 2012] *Let $(G, \cdot, \tau)$ be a semitopological space. If (G, τ) is a strongly boundedly Baire space then $(G, \cdot, \tau)$ is a topological group.*

As every talk should contain at least one proof let me give an idea of part of the above proof. Specifically, let me prove the following fact:

Every regular pseudocompact semitopological group $(G, \cdot, \tau)$ with continuous multiplication (i.e., a paratopological group) is a topological group.

To prove this we need to introduce the following definition.

Suppose that $f : (X, \tau) \to (Y, \tau')$ is a mapping acting between topological spaces (X, τ) and (Y, τ') and $x_0 \in X$. Then we say that f is quasicontinuous at x_0 if for each pair of open neighbourhoods U of x_0 and W of $f(x_0)$ there exists a nonempty open subset V of U such that $f(V) \subseteq W$. Hence, if f is NOT quasicontinuous at a point $x_0 \in X$ then there exists a pair of open neighbourhoods U of x_0 and W of $f(x_0)$ such that for each nonempty open subset V of U, $f(V) \not\subseteq W$.

Lemma 1. Let $(G, \cdot, \tau)$ be a paratopological group. If inversion is quasicontinuous at e then $(G, \cdot, \tau)$ is a topological group.

Proof: Since $(G, \cdot, \tau)$ is a semitopological group it will suffice to show that inversion is continuous at $e \in G$. To this end, let W be any neighbourhood of e. Since G is a paratopological group there exists a neighbourhood U of e such that $U \cdot U$ is a subset of W. Now since inversion is quasicontinuous at e there is a nonempty open subset V of U such that $V^{-1} \subseteq U$. Hence $V \cdot V^{-1}$ is an open neighbourhood of e and

$$(V \cdot V^{-1})^{-1} = V \cdot V^{-1} \subseteq U \cdot U \subseteq W.$$

This completes the proof. ☺

Lemma 2. Let $(G, \cdot, \tau)$ be a paratopological group. If (G, τ) is pseudocompact then inversion is quasicontinuous at e.

Proof: In order to obtain a contradiction let us assume that inversion is not quasicontinuous at $e \in G$. Then there exist neighbourhoods U and W of e such that for each nonempty open subset V of U, $V^{-1} \not\subseteq W$. Note that by possibly making U smaller (and using the fact that $(G, \cdot, \tau)$ is a paratopological group) we may assume that $\overline{U \cdot U} \subseteq W$. We inductively define sequences $(x_n : n \in \mathbb{N})$, $(U_n : n \in \mathbb{N})$ and $(A_n : n \in \mathbb{N})$ but first we set (for notational reasons): $A_0 := U$ and $x_0 := e$.

Step 1. Choose $x_1 \in A_0$ so that

$$(x_0^{-1} \cdot x_1)^{-1} = x_1^{-1} \notin W.$$

Then choose U_1 to be any open neighbourhood of e, contained in U, such that $x_1 \cdot \overline{U_1} \subseteq A_0$. Then define $A_1 := x_1 \cdot U_1$. Now, suppose that x_j, U_j and A_j have been defined for each

$1 \leq j \leq n$ so that:

(i) $x_j \in A_{j-1}$ and $(x_{j-1}^{-1} \cdot x_j)^{-1} \notin W$;

(ii) U_j is an open neighbourhood of e, contained in U, and $x_j \cdot \overline{U_j} \subseteq A_{j-1}$;

(iii) $A_j := x_j \cdot U_j$.

Step $n+1$. Choose $x_{n+1} \in A_n$ so that

$$(x_n^{-1} \cdot x_{n+1})^{-1} \notin W.$$

Note this is possible since $x_n^{-1} \cdot A_n$ is a nonempty open set and

$$x_n^{-1} \cdot A_n \subseteq x_n^{-1} \cdot (x_n \cdot U_n) = U_n \subseteq U.$$

Then choose U_{n+1} to be any open neighbourhood of e, contained in U, such that $x_{n+1} \cdot \overline{U_{n+1}} \subseteq A_n$. Finally, define $A_{n+1} := x_{n+1} \cdot U_{n+1}$.

This completes the induction.

We claim that there exists a $2 \leq k \in \mathbb{N}$ such that

$$A_{k-1} \subseteq \overline{(\bigcap_{n\in\mathbb{N}} A_n) \cdot U}.$$

Since otherwise, $\{A_k \setminus \overline{(\bigcap_{n\in\mathbb{N}} A_n) \cdot U} : k \in \mathbb{N}\}$ would be a decreasing sequence of nonempty open subsets of the pseudocompact space G and so

$$\begin{aligned} \varnothing &\neq \bigcap_{k\in\mathbb{N}} \overline{[A_k \setminus \overline{(\bigcap_{n\in\mathbb{N}} A_n) \cdot U}]} \\ &\subseteq \bigcap_{k\in\mathbb{N}} [\overline{A_k} \setminus (\bigcap_{n\in\mathbb{N}} A_n) \cdot U] \\ &\subseteq \bigcap_{k\in\mathbb{N}} [\overline{A_k} \setminus \bigcap_{n\in\mathbb{N}} A_n] \\ &= \bigcap_{k\in\mathbb{N}} [\overline{A_k} \setminus \bigcap_{n\in\mathbb{N}} \overline{A_n}] = \varnothing \end{aligned}$$

since $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} \overline{A_n}$ (as $\overline{A_{n+1}} \subseteq A_n$ for all $n \in \mathbb{N}$).

Thus,

$$\begin{aligned} x_k \in A_{k-1} &\subseteq \overline{(\bigcap_{n\in\mathbb{N}} A_n) \cdot U} \\ &\subseteq \overline{A_{k+1} \cdot U} \\ &\subseteq \overline{x_{k+1} \cdot U_{k+1} \cdot U} \\ &\subseteq \overline{x_{k+1} \cdot U \cdot U} = x_{k+1} \cdot \overline{U \cdot U} \subseteq x_{k+1} \cdot W. \end{aligned}$$

Therefore, $(x_k^{-1} \cdot x_{k+1})^{-1} = x_{k+1}^{-1} \cdot x_k \in W$. However, this contradicts the way x_{k+1} was chosen. This shows that inversion is quasicontinuous at e. ☺

By putting Lemma 1 and Lemma 2 together we obtain the desired result that every regular pseudocompact paratopological group is a topological group.

A PDF version of this talk is available at:

www.math.auckland.ac.nz/∼moors/

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