INVARIANT MEANS ON CHART GROUPS

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Abstract. The purpose of this paper is to give a stream-lined proof of the existence and uniqueness of a right-invariant mean on a CHART group. A CHART group is a slight generalisation of a compact topological group. The existence of an invariant mean on a CHART group can be used to prove Furstenberg's fixed point theorem.

1. Introduction and preliminaries

Given a nonempty set X and a linear subspace S of $\mathbb{R}^X$ that contains all the constant functions we say that a linear functional $m : S \to \mathbb{R}$ is a *mean on* S if:

(i) $m(f) \geq 0$ for all $f \in S$ that satisfy $f(x) \geq 0$ for all $x \in X$;
(ii) $m(\mathbf{1}) = 1$, where $\mathbf{1}$ is the function that is identically equal to 1.

If all the functions in S are bounded on X then this definition is equivalent to the following:

$$1 = m(\mathbf{1}) = \|m\|$$

where, $\|m\| := \sup\{m(f) : f \in S \text{ and } \|f\|_\infty \leq 1\}$.

If $(X, \cdot)$ is a semigroup then we can define, for each $g \in X$, $L_g : \mathbb{R}^X \to \mathbb{R}^X$ and $R_g : \mathbb{R}^X \to \mathbb{R}^X$ by,

$$L_g(f)(x) := f(gx) \text{ for all } x \in X \quad \text{ and } \quad R_g(f)(x) := f(xg) \text{ for all } x \in X.$$

Note that for all $g, h \in X$, $L_g \circ L_h = L_{hg}$, $R_g \circ R_h = R_{gh}$ and $L_g \circ R_h = R_h \circ L_g$.

If S is a subspace of $\mathbb{R}^X$ that contains all the constant functions and $L_g(S) \subseteq S$ [$R_g(S) \subseteq S$] for all $g \in X$ then we call a mean m on S *left-invariant* [*right-invariant*] if,

$$m(L_g(f)) = m(f) \quad [m(R_g(f)) = m(f)] \quad \text{for all } g \in X \text{ and all } f \in S.$$

We now need to consider some notions from topology. Suppose that X and Y are compact Hausdorff spaces and $\pi : X \to Y$ is a continuous surjection. Then $\pi^\# : C(Y) \to C(X)$ defined by, $\pi^\#(f) := f \circ \pi$ is an isometric algebra isomorphism into $C(X)$. Moreover, we know (from topology/functional analysis) that $f \in \pi^\#(C(Y))$ if, and only if, $f \in C(X)$ and f is constant on the fibers of π (i.e., f is constant on $\pi^{-1}(y)$ for each $y \in Y$).

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The final notion that we need for this section is that of a right topological group (left topological group). We shall call a triple $(G, \cdot, \tau)$ a *right topological group* (*left topological group*) if $(G, \cdot)$ is a group, (G, τ) is a topological space and, for each $g \in G$, the mapping $x \mapsto x \cdot g$ $(x \mapsto g \cdot x)$ is continuous on G. If $(G, \cdot, \tau)$ is both a right topological group and a left topological group then we call it a *semitopological group.*

If $(G, \cdot, \tau)$ and $(H, \cdot, \tau')$ are compact Hausdorff right topological groups and $\pi : G \to H$ is a continuous homomorphism then it easy to check that

$$R_g(\pi^{\#}(f)) = \pi^{\#}(R_{\pi(g)}(f)) \quad \text{for all } f \in C(H) \text{ and } g \in G.$$

If $\pi : X \to Y$ is surjective then $(\pi^{\#})^{-1} : \pi^{\#}(C(H)) \to C(H)$ exists. Therefore,

$$(\pi^{\#})^{-1}(R_g(h)) = R_{\pi(g)}((\pi^{\#})^{-1}(h)) \quad \text{for all } h \in \pi^{\#}(C(H)) \text{ and } g \in G.$$

From these equations we can easily establish our first result.

Proposition 1.1. *Let $(G, \cdot, \tau)$ and $(H, \cdot, \tau')$ be compact Hausdorff right topological groups and let $\pi : G \to H$ be a continuous epimorphism (i.e., a surjective homomorphism). If m is a right-invariant mean on $C(H)$ then $m^* : \pi^{\#}(C(H)) \to \mathbb{R}$ defined by, $m^*(f) := m((\pi^{\#})^{-1}(f))$ for all $f \in \pi^{\#}(C(H))$ is a right-invariant mean on $\pi^{\#}(C(H))$. If $C(H)$ has a unique right-invariant mean then $\pi^{\#}(C(H))$ has a unique right-invariant mean.*

We can now state and prove our main theorem for this section.

Theorem 1.2. *Let $(G, \cdot, \tau)$ and $(H, \cdot, \tau')$ be compact Hausdorff right topological groups and let $\pi : G \to H$ be a continuous epimorphism. If the mapping*

$$m : G \times ker(\pi) \to G \text{ defined by, } m(x, y) := x \cdot y \text{ for all } (x, y) \in G \times ker(\pi)$$

is continuous and $C(H)$ has a right-invariant mean then $C(G)$ has a right-invariant mean. Furthermore, if $C(H)$ has a unique right-invariant mean then so does $C(G)$.

Proof. Let $L := \ker(\pi)$. Then from the hypotheses and [1, Theorem 2] $(L, \cdot, \tau_L)$ (here τ_L is the relative τ-topology on L) is a compact topological group. Thus $(L, \cdot, \tau_L)$ admits a unique Borel probability measure λ (called the *Haar measure* on L) such that

$$\int_L L_g(f)(t)\, d\lambda(t) = \int_L R_g(f)(t)\, d\lambda(t) = \int_L f(t)\, d\lambda(t) \text{ for all } g \in L \text{ and } f \in C(L).$$

Let $P : C(G) \to \pi^{\#}(C(H))$ be defined by,

$$P(f)(g) := \int_L f(g \cdot t)\, d\lambda(t) \text{ i.e., } P(f)(g) \text{ is the “average” of } f \text{ over the coset } gL.$$

Firstly, since m is continuous on $G \times L$ (and L is compact) $P(f) \in C(G)$ for each $f \in C(G)$. Secondly, since λ is invariant on L it is routine to check that $P(f)$ is constant on the fibers of π. Hence, $P(f) \in \pi^{\#}(C(H))$. We now show that for each $g \in G$ and $f \in C(G)$,

$$\int_L L_g(f)(t)\, d\lambda(t) = \int_L f(g \cdot t)\, d\lambda(t) = \int_L f(t \cdot g)\, d\lambda(t) = \int_L R_g(f)(t)\, d\lambda(t). \quad (*)$$

To this end, fixed $g \in G$ and define $G : C(L) \to C(L)$ by, $G(f)(t) := f(g^{-1} \cdot t \cdot g)$. Since m is continuous, $t \mapsto (g^{-1} \cdot t) \cdot g$ is continuous and so G is well-defined, i.e., $G(f) \in C(L)$ for each $f \in C(L)$. We claim that

$$f \mapsto \int_L G(f)(t)\ \mathrm{d}\lambda(t)$$

is a right-invariant mean on $C(L)$. Clearly, this mapping is a mean so it remains to show that it is right-invariant. To see this, let $l \in L$. Then $g \cdot l \cdot g^{-1} \in L$ and

$$\begin{aligned}
\int_L G(R_l(f))(t)\ \mathrm{d}\lambda(t) &= \int_L R_l(f)(g^{-1} \cdot t \cdot g)\ \mathrm{d}\lambda(t) \\
&= \int_L f(g^{-1} \cdot t \cdot g \cdot l)\ \mathrm{d}\lambda(t) \\
&= \int_L f(g^{-1} \cdot [t \cdot (g \cdot l \cdot g^{-1})] \cdot g)\ \mathrm{d}\lambda(t) \\
&= \int_L G(f)(t \cdot (g \cdot l \cdot g^{-1}))\ \mathrm{d}\lambda(t) \\
&= \int_L R_{g \cdot l \cdot g^{-1}}(G(f))(t)\ \mathrm{d}\lambda(t) \\
&= \int_L G(f)(t)\ \mathrm{d}\lambda(t) \quad \text{since } \lambda \text{ is right-invariant.}
\end{aligned}$$

Now, since there is only one right-invariant mean on $C(L)$ we must have that

$$\int_L G(f)(t)\ \mathrm{d}\lambda(t) = \int_L f(g^{-1} \cdot t \cdot g)\ \mathrm{d}\lambda(t) = \int_L f(t)\ \mathrm{d}\lambda(t) \quad \text{for all } f \in C(L).$$

It now follows that equation $(*)$ holds. Next, we show that $R_g(P(f)) = P(R_g(f))$ for all $g \in G$ and $f \in C(G)$. To this end, let $g \in G$ and $f \in C(G)$. Then for any $x \in G$,

$$\begin{aligned}
R_g(P(f))(x) &= P(f)(x \cdot g) = \int_L f(x \cdot g \cdot t)\ \mathrm{d}\lambda(t) = \int_L f(x \cdot t \cdot g)\ \mathrm{d}\lambda(t) \quad \text{by } (*) \\
&= \int_L R_g(f)(x \cdot t)\ \mathrm{d}\lambda(t) = P(R_g(f))(x).
\end{aligned}$$

Let μ be the unique right-invariant mean on $\pi^{\#}(C(H))$, given to us by Proposition 1.1. Let $\mu^* : C(G) \to \mathbb{R}$ be defined by, $\mu^*(f) := \mu(P(f))$. It is now easy to verify that μ^* is a right-invariant mean on $C(G)$.

So it remains to prove uniqueness. Suppose that μ^* and ν^* are right-invariant means on $C(G)$. Since, by Proposition 1.1, we know that $\mu^*|_{\pi^{\#}(C(H))} = \nu^*|_{\pi^{\#}(C(H))}$ it will be sufficient to show that $\mu^*(f) = \mu^*(P(f))$ and $\nu^*(f) = \nu^*(P(f))$ for each $f \in C(G)$. We shall apply Riesz's representation theorem along with Fubini's theorem. Let μ be the probability measure on G that represents μ^* and let

$f \in C(G)$. Then

$$\begin{aligned}\mu^*(f) &= \int_G f(s)\ \mathrm{d}\mu(s) = \int_L \int_G f(s\cdot t)\ \mathrm{d}\mu(s)\ \mathrm{d}\lambda(t) \\ &= \int_G \int_L f(s\cdot t)\ \mathrm{d}\lambda(t)\ \mathrm{d}\mu(s) \\ &= \int_G P(f)(s)\ \mathrm{d}\mu(s) = \mu^*(P(f)).\end{aligned}$$

A similar argument show that $\nu^*(f) = \nu^*(P(f))$. This completes the proof. □

This paper is the culmination of work done many people, starting with the work of H. Furstenberg in [4] on the existence of invariant measures on distal flows. This work was later simplified and phrased in terms of CHART groups by I. Namioka in [8]. The results of Namioka were further generalised by R. Ellis, [3]. In 1992, P. Milnes and J. Pym, [5] showed that every CHART group (that satisfies some countability condition) admits a unique right-invariant mean (unique right-invariant measure) called the Haar mean (Haar measure). Later, in [6], Milne and Pym managed to remove the countability condition from the proof contained in [5] by appealing to a result from [3]. Finally, in [7], a direct proof of the existence and uniqueness of a right-invariant mean on a CHART group was given, however, this proof still relied upon the results from [5].

In the present paper we give a stream-lined proof (that does not require knowledge from topological dynamics) of the existence and uniqueness of a right-invariant mean on a CHART group.

2. Groups

Let $(G, \cdot, \tau)$ be a right topological group and let H be a subgroup of G. We shall denote by (H, τ_H) the set H equipped with the relative τ-topology. It is easy to see that $(H, \cdot, \tau_H)$ is also a right topological group.

Now let G/H be the set $\{xH : x \in G\}$ of all left cosets of H in G and give G/H the quotient topology $q(\tau)$ induced from (G, τ) by the map $\pi : G \to G/H$ defined by $\pi(x) := xH$.

Note that π is an open mapping because, if U is an open subset of G then

$$\pi^{-1}(\pi(U)) = UH = \bigcup\{Ux : x \in H\}$$

and this last set is open since right multiplication is a homeomorphism on G.

If H is a normal subgroup of a right(left)[semi] topological group $(G, \cdot, \tau)$ then one can check that $(G/H, \cdot, q(\tau))$ is also a right(left)[semi] topological group.

In order to continue our investigations further we need to introduce a new topology.

2.1. **The σ-topology.** Let $(G, \cdot, \tau)$ be a right topological group and let $\varphi : G \times G \to G$ be the map defined by

$$\varphi(x, y) := x^{-1} \cdot y.$$

Then the quotient topology on G induced from $(G \times G, \tau \times \tau)$ by the map φ is called the $\sigma(G, \tau)$-topology or σ-topology.

The proof of the next result can be found in [8, Theorem 1.1,Theorem 1.3] or [9, Lemma 4.3].

Lemma 2.1. *Let $(G, \cdot, \tau)$ be a right topological group. Then,*

(i) *(G, σ) is a semitopological group.*
(ii) *$\sigma \subseteq \tau$.*
(iii) *$(G/H, q(\tau))$ is Hausdorff provided the subgroup H is closed with respect to the σ-topology on G.*

2.2. **Admissibility and CHART groups.** Let $(G, \cdot, \tau)$ be a right topological group and let $\Lambda(G, \tau)$ be the set of all $x \in G$ such that the map $y \mapsto x \cdot y$ is τ continuous. If $\Lambda(G, \tau)$ is τ-dense in G then (G, τ) is said to be *admissible*.

The proof for the following proposition may be found in [8, Theorem 1.2, Corollary 1.1] or [9, Proposition 4.4, Proposition 4.5].

Proposition 2.2. *Let $(G, \cdot, \tau)$ be an admissible right topological group.*

(i) *If $\mathcal{U}$ is the family of all τ-open neighbuorhoods of e in G then $\{U^{-1}U : U \in \mathcal{U}\}$ is a base of open neighbuorhoods of e in (G, σ).*
(ii) *If $N(G, \tau) := \bigcap\{U^{-1}U : U \in \mathcal{U}\}$ then $N(G, \tau) = \overline{\{e\}}^{\sigma}$.*

A compact Hausdorff admissible right topological group $(G, \cdot, \tau)$ is called a *CHART group*.

The proof for the following result may be found [8, Proposition 2.1] or [9, Proposition 4.6].

Proposition 2.3. *Let $(G, \cdot, \tau)$ be a CHART group. Then the following hold:*

(i) *If L is a σ-closed normal subgroup of G, then so is $N(L, \sigma_L)$.*
(ii) *If $m : (G/N(L, \sigma_L), q(\tau)) \times (L/N(L, \sigma_L), q(\tau)) \to (G/N(L, \sigma_L), q(\tau))$ is defined by*

$$m(xN(L, \sigma_L), yN(L, \sigma_L)) := x \cdot yN(L, \sigma_L) \quad \text{for all } (x, y) \in G \times L$$

then m is well-defined and continuous.

Remark 2.4. By considering the mapping $\pi : G/N(L, \sigma_L) \to G/L$, Theorem 1.2 and Proposition 2.3 we see that if $(G/L, q(\tau))$ admits a unique right-invariant mean then so does $(G/N(L, \sigma_L), q(\tau))$. Hence if $N(L, \sigma_L)$ is a proper subset of L then we have made some progress towards showing that $G \cong G/\{e\}$ admits a unique right-invariant mean.

3. $N(L, \sigma_L) \neq L$

In this section we will show that if L is a nontrivial σ-closed normal subgroup of a CHART group $(G, \cdot, \tau)$ then $N(L, \sigma_L)$ is a proper subset of L.

Lemma 3.1. *Let $(H, \cdot)$ be a group and X be a nonempty set. Then for any $f : H \to X$, $S := \{s \in H : f(hs) = f(h) \text{ for all } h \in H\}$ is a subgroup of H.*

Proof. Clearly, $e \in S$. Now suppose that, $s_1, s_2 \in S$. Let h be any element of H then

$$f(h(s_1 s_2)) = f((hs_1)s_2) = f(hs_1) = f(h)$$

Therefore, $s_1 s_2 \in S$. Next, let s be any element of S and h be any element of H then

$$f(h) = f(h(s^{-1}s)) = f((hs^{-1})s) = f(hs^{-1}).$$

Therefore, $s^{-1} \in S$. □

Lemma 3.2. *Let $(G, \cdot, \tau)$ be a compact right topological group and let σ be a topology on G weaker than τ such that $(G, \cdot, \sigma)$ is also a right topological group. If U is a dense open subset of (G, σ) then U is also a dense subset of (G, τ).*

Proof. Let $C := G \backslash U$. Then C is a σ-closed (hence τ-closed) nowhere-dense subset of G. If U is not τ-dense in G then C contains a nonempty τ-open subset. By the compactness of (G, τ) there exists a finite subset F of G such that $G = \bigcup\{Cg : g \in F\}$. Now each Cg is nowhere dense in (G, σ) since each right multiplication is a homeomorphism. This forms a contradiction since a nonempty topological space can never be the union of a finite number of nowhere dense subsets. □

Lemma 3.3. *Let $(G, \cdot, \tau)$ be a CHART group and let $\Lambda = \Lambda(G, \tau)$. If A and B are nonempty open subsets of (G, τ), then $A^{-1}B = (A \cap \Lambda)^{-1}B$.*

Proof. Let $x \in A^{-1}B$. Then for some $a \in A$, $ax \in B$. Since B is open and $A \cap \Lambda$ is dense in A there is a $c \in A \cap \Lambda$ such that $cx \in B$. Hence $x \in c^{-1}B \subseteq (A \cap \Lambda)^{-1}B$. Thus, $A^{-1}B \subseteq (A \cap \Lambda)^{-1}B$. The reverse inclusion is obvious. □

Lemma 3.4. *Let $(G, \cdot, \tau)$ be a compact Hausdorff right topological group. If S is a nonempty subsemigroup of $\Lambda(G, \tau)$ then $\overline{S}$ is a subgroup of G.*

Proof. In this proof we shall repeatedly use the following fact, [2, Lemma 1] "Every nonempty compact right topological semigroup admits an idempotent element (i.e., an element u such that $u \cdot u = u$). Firstly, it is easy to see that $\overline{S}$ is a subsemigroup of G. Hence, $(\overline{S}, \cdot)$ is a nonempty compact right topological semigroup and so has an idempotent element u. However, since G is a group it has only one idempotent element, namely e. Therefore, $e = u \in \overline{S}$. Next, let s be any element of $\overline{S}$. Then $\overline{S} \cdot s$ is a nonempty compact right topological semigroup of $\overline{S}$. Therefore, there exists an element $s' \in \overline{S}$ such that $(s' \cdot s) \cdot (s' \cdot s) = (s' \cdot s)$. Again, since G is a group, $s' \cdot s = e$. By multiplying both sides of this equation by s^{-1} we see that $s^{-1} = s' \in \overline{S}$. □

The following lemma is a simplified form of the structure theorem found in [7].

Lemma 3.5. *Let $(G,\cdot,\tau)$ be a CHART group and let σ denote its σ-topology. Suppose L is a nontrivial σ-closed subgroup of G. Then $N(L,\sigma_L)$ is a proper subset of L.*

Proof. Let $\mathcal{U}$ denote the family of all open neighbuorhoods of e in (G,τ). Then it follows from Proposition 2.2 that $\mathcal{V}=\{U^{-1}U : U\in\mathcal{U}\}$ is a base for the system of open neighbourhoods of e in (G,σ). Then $\{V\cap L : V\in\mathcal{V}\}$ is a basis for the system of neighbourhoods of e in (L,σ_L). From the definition of $N(L,\sigma_L)$ (see Proposition 2.2 part (ii)) it follows that

$$N(L,\sigma_L)=\bigcap\{(V\cap L)^{-1}(V\cap L) : V\in\mathcal{V}\}.$$

The proof is by contradiction. So assume that $N(L,\sigma_L)=L$. Then

$$L=\bigcap\{(V\cap L)^{-1}(V\cap L) : V\in\mathcal{V}\}.$$

Hence, for each $V\in\mathcal{V}$, $(V\cap L)^{-1}(V\cap L)=L$, or equivalently, for each $V\in\mathcal{V}$, $(V\cap L)$ is dense in (L,σ_L). That is, for each $U\in\mathcal{U}$, $(U^{-1}U\cap L)$ is open and dense in (L,σ_L) and hence, by Lemma 3.2, dense in (L,τ_L).

Since $L\neq\{e\}$, there exists a point $a\in L$ such that $a\neq e$. Note that since (G,τ) is compact and Hausdorff there is a continuous function f on (G,τ) such that $f(e)=0$ and $f\equiv 1$ on a τ-neighbuorhood of a.

For the rest of the proof, the topology always refers to τ and we shall denote $\Lambda(G,\tau)$ by Λ. By induction on n, we construct a sequence $\{U_n : n\in\mathbb{N}\}$ in $\mathcal{U}$, a sequence $\{V_n : n\in\mathbb{N}\}$ of nonempty open subsets of G, each of which intersects L and sequences $\{u_n : n\in\mathbb{N}\}$ and $\{v_n : n\in\mathbb{N}\}$ in G which satisfy the following conditions:

(i) $v_n\in U_{n-1}^{-1}U_{n-1}\cap(V_{n-1}\cap\Lambda)=(U_{n-1}\cap\Lambda)^{-1}U_{n-1}\cap(V_{n-1}\cap\Lambda)$; by Lemma 3.3.
(ii) $u_n\in U_{n-1}\cap\Lambda$;
(iii) $V_n\subset\overline{V_n}\subset V_{n-1}\subset f^{-1}(1)$ and $V_n\cap L\neq\emptyset$;
(iv) $u_nV_n\subset U_{n-1}$;
(v) if H_n denotes the semigroup generated by $\{u_1,v_1,u_2,v_2,\ldots,u_n,v_n\}$; which we enumerate as: $H_n:=\{h_j^n : j\in\mathbb{N}\}$ and

$$U_n:=\{t\in G : |f(h_j^i t)-f(h_j^i)|<1/n \quad\text{for}\quad 1\le i,j\le n\}$$

then $H_n\subset\Lambda$ and $e\in U_n\subset\overline{U_n}\subset U_{n-1}$.

Construction. We let $U_0:=G$ and let V_0 be the interior of $f^{-1}(1)$ and u_0, v_0 are not defined. Assume that $n\in\mathbb{N}$ and that U_k, V_k are defined for $0\le k<n$ and v_k, u_k are defined for $0<k<n$. By our assumption there exists an $x\in(U_{n-1}\cap\Lambda)^{-1}U_{n-1}\cap(V_{n-1}\cap L)$. So there is a $u_n\in U_{n-1}\cap\Lambda$ such that $u_nx\in U_{n-1}$. Since $u_n\in\Lambda$, $x\in V_{n-1}$ and U_{n-1} is open, there is an open neighbourhood V_n of x such that $x\in V_n\subset\overline{V_n}\subset V_{n-1}$ and $u_nV_n\subset U_{n-1}$. Then $V_n\cap L\neq\emptyset$ since $x\in V_n\cap L$. Thus (ii)-(iv) are satisfied. Let v_n be any element of $V_n\cap\Lambda$, then by (iv) and (ii), (i) is satisfied and $H_n\subset\Lambda$ is defined. Finally, since the map $t\mapsto|f(gt)-f(g)|$ is continuous for $g\in\Lambda$, the set U_n is an open neighbourhood of e and so condition (v) is satisfied. This completes the construction.

We let

$$U_\infty=\bigcap\{\overline{U_n} : n\in\mathbb{N}\}\quad\text{ and }\quad H=\bigcup\{H_n : n\in\mathbb{N}\}$$

and let u_∞, v_∞ be cluster points of the sequences $\{u_n : n \in \mathbb{N}\}$, $\{v_n : n \in \mathbb{N}\}$ respectively. Clearly $u_\infty \in U_\infty$, $v_\infty \in V_0$ and $\overline{H}$ is a subgroup of G, by Lemma 3.4. Moreover, by the construction, $f(ht) = f(h)$ for each $h \in H$ and each $t \in \bigcap\{\overline{U_n} : n \in \mathbb{N}\}$. Therefore, if we let

$$\begin{aligned} S &= \{s \in \overline{H} : f(hs) = f(h) \text{ for each } h \in H\} \\ &= \{s \in \overline{H} : f(hs) = f(h) \text{ for each } h \in \overline{H}\} \end{aligned}$$

then $\bigcap\{\overline{U_n} : n \in \mathbb{N}\} \cap \overline{H} \subset S$ and S is a subgroup of G by Lemma 3.1. Furthermore, by (ii), $u_\infty \in U_\infty \cap \overline{H} \subset S$ and by (iv) $u_n v_\infty \in \overline{U_{n-1}} \cap \overline{H}$ for each $n \in \mathbb{N}$. Hence

$$u_\infty v_\infty \in \bigcap_{n \in \mathbb{N}} \overline{U_{n-1}} \cap \overline{H} \subset S.$$

Therefore, $v_\infty = u_\infty^{-1}(u_\infty v_\infty) \in S^{-1}S \subset S$. Now, $f(s) = 0$ for all $s \in S$ since

$$f(es) = f(e) = 0 \quad \text{for all } s \in S.$$

Therefore, $f(v_\infty) = 0$. On the other hand, since $v_\infty \in V_0 \subset f^{-1}(1)$, $f(v_\infty) = 1$. This contradiction completes the proof. □

4. Invariant means on CHART groups

In this section we will show that every CHART group admits a unique right-invariant mean.

Theorem 4.1. *Every CHART group $(G, \cdot, \tau)$ possesses a unique right-invariant mean m on $C(G)$.*

Proof. Let $\mathcal{L}$ be the family of all σ-closed normal subgroups L of G for which $C(G/L)$ has a unique right-invariant mean. Clearly, $\mathcal{L} \neq \emptyset$ as $G \in \mathcal{L}$. Now, $(\mathcal{L}, \subseteq)$ is a partially ordered set. We claim that $(\mathcal{L}, \subseteq)$ possesses a minimal element. To prove this, it is sufficient to show that every totally ordered subfamily $\mathcal{M}$ of $\mathcal{L}$ has a lower bound (in $\mathcal{L}$). To this end, let $\mathcal{M} := \{M_\alpha : \alpha \in A\}$ be a nonempty totally ordered subfamily of $\mathcal{L}$. Let

$$M_0 := \bigcap\{M_\alpha : \alpha \in A\}.$$

Then M is a σ-closed normal subgroup of G and $M_0 \subseteq M_\alpha$ for every $\alpha \in A$. Thus, to complete the proof of the claim we must show that $M_0 \in \mathcal{L}$, i.e., show that $C(G/M_0)$ admits a unique right-invariant mean. For each $\alpha \in A$, let $\pi_\alpha : G/M_0 \to G/M_\alpha$ be defined by, $\pi_\alpha(gM_0) := gM_\alpha$. Then π_α is a continuous, open and onto map and its dual map $\pi_\alpha^\# : C(G/M_\alpha) \to C(G/M_0)$ is an isometric algebra isomorphism of $C(G/M_\alpha)$ into $C(G/M_0)$. By Proposition 1.1, for each $\alpha \in A$, there exists a unique right-invariant mean m_α on $\pi_\alpha^\#(C(G/M_\alpha))$. From the Hahn-Banach extension theorem it follows that each mean m_α has an extension to a mean m_α^* on $C(G/M_0)$. Let $\mathscr{A} := \bigcup\{\pi_\alpha^\#(C(G/M_\alpha)) : \alpha \in A\}$. Then $\mathscr{A}$ is a subalgebra of $C(G/M_0)$, that contains all the constant functions and separates the point of G/M_0 since $M_0 := \bigcap\{M_\alpha : \alpha \in A\}$. Therefore, by the Stone-Weierstrass theorem, $\mathscr{A}$ is dense in $C(G/M_\alpha)$. Let m be a weak* cluster-point of the net $(m_\alpha^* : \alpha \in A)$ in $B_{C(G/M_0)^*}$. Clearly, m is a mean on $C(G/M_0)$. Furthermore, it is routine to show that (i) $m|_{\mathscr{A}}$ is a right-invariant mean on $\mathscr{A}$ and (ii) $m|_{\mathscr{A}}$ is the only (unique) right-invariant mean on $\mathscr{A}$. It now follows from continuity that m is the one and only right-invariant mean on $C(G/M_0)$, i.e., $M_0 \in \mathcal{L}$.

Let L_0 be a minimal element of $\mathcal{L}$. Then by Remark 2.4, $N(L_0, \sigma_{L_0}) \in \mathcal{L}$. However, since $N(L, \sigma_{L_0}) \subseteq L_0$ and L_0 is a minimal element of $\mathcal{L}$ we must have that $N(L, \sigma_L) = L_0$. Thus, by Lemma 3.5, it must be the case that $L_0 = \{e\}$. This completes the proof. □

Let us now note that the unique right-invariant mean given above is also partially left invariant in the sense that for each $g \in \Lambda(G, \tau)$, $m(L_g(f)) = m(f)$ for all $f \in C(G)$. To see why this is true, consider the mean m^* on $C(G)$ defined by, $m^*(f) := m(L_g(f))$ for each $f \in C(G)$ and some $g \in \Lambda(G, \tau)$. Then for any $h \in G$,

$$m^*(R_h(f)) = m(L_g(R_h(f))) = m(R_h(L_g(f))) = m(L_g(f)) = m^*(f).$$

Therefore, m^* is a right-invariant mean on $C(G)$. Thus, $m^* = m$ and so

$$m(L_g(f)) = m^*(f) = m(f) \text{ for all } f \in C(G) \text{ and all } g \in \Lambda(G, \tau).$$

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