ININVARIANT MEANS ON CHART GROUPS

WARREN B. MOORS∗

Abstract. The purpose of this paper is to give a stream-lined proof of the existence and uniqueness of a right-invariant mean on a CHART group. A CHART group is a slight generalisation of a compact topological group. The existence of an invariant mean on a CHART group can be used to prove Furstenberg’s fixed point theorem.

1. Introduction and preliminaries

Given a nonempty set X and a linear subspace S of \( \mathbb{R}^X \) that contains all the constant functions we say that a linear functional \( m : S \rightarrow \mathbb{R} \) is a mean on \( S \) if:

(i) \( m(f) \geq 0 \) for all \( f \in S \) that satisfy \( f(x) \geq 0 \) for all \( x \in X \);
(ii) \( m(1) = 1 \), where \( 1 \) is the function that is identically equal to 1.

If all the functions in \( S \) are bounded on \( X \) then this definition is equivalent to the following:

\[
1 = m(1) = \|m\| \]

where, \( \|m\| := \sup\{m(f) : f \in S \text{ and } \|f\|_{\infty} \leq 1\} \).

If \( (X, \cdot) \) is a semigroup then we can define, for each \( g \in X \), \( L_g : \mathbb{R}^X \rightarrow \mathbb{R}^X \) and \( R_g : \mathbb{R}^X \rightarrow \mathbb{R}^X \) by,

\[
L_g(f)(x) := f(gx) \quad \text{for all } x \in X \quad \text{and} \quad R_g(f)(x) := f(xg) \quad \text{for all } x \in X.
\]

Note that for all \( g, h \in X \), \( L_g \circ L_h = L_{gh} \), \( R_g \circ R_h = R_{gh} \) and \( L_g \circ R_h = R_h \circ L_g \).

If \( S \) is a subspace of \( \mathbb{R}^X \) that contains all the constant functions and \( L_g(S) \subseteq S \) [\( R_g(S) \subseteq S \)] for all \( g \in X \) then we call a mean \( m \) on \( S \) left-invariant [right-invariant] if,

\[
m(L_g(f)) = m(f) \quad \text{[}m(R_g(f)) = m(f)\text{]} \quad \text{for all } g \in X \text{ and all } f \in S.
\]

We now need to consider some notions from topology. Suppose that \( X \) and \( Y \) are compact Hausdorff spaces and \( \pi : X \rightarrow Y \) is a continuous surjection. Then \( \pi^\# : C(Y) \rightarrow C(X) \) defined by, \( \pi^\#(f) := f \circ \pi \) is an isometric algebra isomorphism into \( C(X) \). Moreover, we know (from topology/functional analysis) that \( f \in \pi^\#(C(Y)) \) if, and only if, \( f \in C(X) \) and \( f \) is constant on the fibers of \( \pi \) (i.e., \( f \) is constant on \( \pi^{-1}(y) \) for each \( y \in Y \)).

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∗ Corresponding author.

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The final notion that we need for this section is that of a right topological group (left topological group). We shall call a triple \((G, \cdot, \tau)\) a right topological group (left topological group) if \((G, \cdot)\) is a group, \((G, \tau)\) is a topological space and, for each \(g \in G\), the mapping \(x \mapsto x \cdot g (x \mapsto g \cdot x)\) is continuous on \(G\). If \((G, \cdot, \tau)\) is both a right topological group and a left topological group then we call it a semitopological group.

If \((G, \cdot, \tau)\) and \((H, \cdot, \tau')\) are compact Hausdorff right topological groups and \(\pi : G \to H\) is a continuous homomorphism then it easy to check that
\[
R_g(\pi^#(f)) = \pi^#(R_{\pi(g)}(f)) \quad \text{for all } f \in C(H) \text{ and } g \in G.
\]
If \(\pi : X \to Y\) is surjective then \((\pi^#)^{-1} : \pi^#(C(H)) \to C(H)\) exists. Therefore,
\[
(\pi^#)^{-1}(R_g(h)) = R_{\pi(g)}((\pi^#)^{-1}(h)) \quad \text{for all } h \in \pi^#(C(H)) \text{ and } g \in G.
\]
From these equations we can easily establish our first result.

**Proposition 1.1.** Let \((G, \cdot, \tau)\) and \((H, \cdot, \tau')\) be compact Hausdorff right topological groups and let \(\pi : G \to H\) be a continuous epimorphism (i.e., a surjective homomorphism). If \(m\) is a right-invariant mean on \(C(H)\) then \(m^* : \pi^#(C(H)) \to \mathbb{R}\) defined by, \(m^*(f) := m((\pi^#)^{-1}(f))\) for all \(f \in \pi^#(C(H))\) is a right-invariant mean on \(\pi^#(C(H))\). If \(C(H)\) has a unique right-invariant mean then \(\pi^#(C(H))\) has a unique right-invariant mean.

We can now state and prove our main theorem for this section.

**Theorem 1.2.** Let \((G, \cdot, \tau)\) and \((H, \cdot, \tau')\) be compact Hausdorff right topological groups and let \(\pi : G \to H\) be a continuous epimorphism. If the mapping
\[
m : G \times \ker(\pi) \to G \text{ defined by, } m(x, y) := x \cdot y \quad \text{for all } (x, y) \in G \times \ker(\pi)
\]
is continuous and \(C(H)\) has a right-invariant mean then \(C(G)\) has a right-invariant mean. Furthermore, if \(C(H)\) has a unique right-invariant mean then so does \(C(G)\).

**Proof.** Let \(L := \ker(\pi)\). Then from the hypotheses and [1, Theorem 2] \((L, \cdot, \tau_L)\) (here \(\tau_L\) is the relative \(\tau\)-topology on \(L\)) is a compact topological group. Thus \((L, \cdot, \tau_L)\) admits a unique Borel probability measure \(\lambda\) (called the Haar measure on \(L\)) such that
\[
\int_L L_g(f)(t) \, d\lambda(t) = \int_L R_g(f)(t) \, d\lambda(t) = \int_L f(t) \, d\lambda(t) \quad \text{for all } g \in L \text{ and } f \in C(L).
\]
Let \(P : C(G) \to \pi^#(C(H))\) be defined by,
\[
P(f)(g) := \int_L f(g \cdot t) \, d\lambda(t) \quad \text{i.e., } P(f)(g) \text{ is the “average” of } f \text{ over the coset } gL.
\]
Firstly, since \(m\) is continuous on \(G \times L\) (and \(L\) is compact) \(P(f) \in C(G)\) for each \(f \in C(G)\). Secondly, since \(\lambda\) is invariant on \(L\) it is routine to check that \(P(f)\) is constant on the fibers of \(\pi\). Hence, \(P(f) \in \pi^#(C(H))\). We now show that for each \(g \in G\) and \(f \in C(G)\),
\[
\int_L L_g(f)(t) \, d\lambda(t) = \int_L f(g \cdot t) \, d\lambda(t) = \int_L f(t \cdot g) \, d\lambda(t) = \int_L R_g(f)(t) \, d\lambda(t). \quad (*)
\]
To this end, fixed $g \in G$ and define $G : C(L) \to C(L)$ by, $G(f)(t) := f(g^{-1} \cdot t \cdot g)$. Since $m$ is continuous, $t \mapsto (g^{-1} \cdot t) \cdot g$ is continuous and so $G$ is well-defined, i.e., $G(f) \in C(L)$ for each $f \in C(L)$. We claim that

$$f \mapsto \int_G G(f)(t) \, d\lambda(t)$$

is a right-invariant mean on $C(L)$. Clearly, this mapping is a mean so it remains to show that it is right-invariant. To see this, let $l \in L$. Then $g \cdot l \cdot g^{-1} \in L$ and

$$\int_L G(R_l(f))(t) \, d\lambda(t) = \int_L R_l(f)(g^{-1} \cdot t \cdot g) \, d\lambda(t) = \int_L f(g^{-1} \cdot t \cdot g \cdot l) \, d\lambda(t) = \int_L f(g^{-1} \cdot [t \cdot (g \cdot l \cdot g^{-1})] \cdot g) \, d\lambda(t) = \int_L G(f) \cdot (t \cdot (g \cdot l \cdot g^{-1})) \, d\lambda(t) = \int_L R_{g \cdot l \cdot g^{-1}}(G(f))(t) \, d\lambda(t) = \int_L G(f)(t) \, d\lambda(t)$$

since $\lambda$ is right-invariant.

Now, since there is only one right-invariant mean on $C(L)$ we must have that

$$\int_L G(f)(t) \, d\lambda(t) = \int_L f(g^{-1} \cdot t \cdot g) \, d\lambda(t) = \int_L f(t) \, d\lambda(t) \quad \text{for all } f \in C(L).$$

It now follows that equation (\ast) holds. Next, we show that $R_g(P(f)) = P(R_g(f))$ for all $g \in G$ and $f \in C(G)$. To this end, let $g \in G$ and $f \in C(G)$. Then for any $x \in G$,

$$R_g(P(f))(x) = P(f)(x \cdot g) = \int_L f(x \cdot g \cdot t) \, d\lambda(t) = \int_L f(x \cdot t \cdot g) \, d\lambda(t) \quad \text{by (\ast)} = \int_L R_g(f)(x \cdot t) \, d\lambda(t) = P(R_g(f))(x).$$

Let $\mu$ be the unique right-invariant mean on $\pi^\#(C(H))$, given to us by Proposition 1.1. Let $\mu^* : C(G) \to \mathbb{R}$ be defined by, $\mu^*(f) := \mu(P(f))$. It is now easy to verify that $\mu^*$ is a right-invariant mean on $C(G)$.

So it remains to prove uniqueness. Suppose that $\mu^*$ and $\nu^*$ are right-invariant means on $C(G)$. Since, by Proposition 1.1, we know that $\mu^*|_{\pi^\#(C(H))} = \nu^*|_{\pi^\#(C(H))}$ it will be sufficient to show that $\mu^*(f) = \mu^*(P(f))$ and $\nu^*(f) = \nu^*(P(f))$ for each $f \in C(G)$. We shall apply Riesz’s representation theorem along with Fubini’s theorem. Let $\mu$ be the probability measure on $G$ that represents $\mu^*$ and let
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Then

$$\mu^*(f) = \int_G f(s) \, d\mu(s) = \int_L \int_G f(s \cdot t) \, d\mu(s) \, d\lambda(t)$$

$$= \int_G \int_L f(s \cdot t) \, d\lambda(t) \, d\mu(s)$$

$$= \int_G P(f)(s) \, d\mu(s) = \mu^*(P(f)).$$

A similar argument shows that $$\nu^*(f) = \nu^*(P(f))$$. This completes the proof. \qed

This paper is the culmination of work done by many people, starting with the work of H. Furstenberg in [4] on the existence of invariant measures on distal flows. This work was later simplified and phrased in terms of CHART groups by I. Namioka in [8]. The results of Namioka were further generalised by R. Ellis, [3]. In 1992, P. Milnes and J. Pym, [5] showed that every CHART group (that satisfies some countability condition) admits a unique right-invariant mean (unique right-invariant measure) called the Haar mean (Haar measure). Later, in [6], Milne and Pym managed to remove the countability condition from the proof contained in [5] by appealing to a result from [3]. Finally, in [7], a direct proof of the existence and uniqueness of a right-invariant mean on a CHART group was given, however, this proof still relied upon the results from [5].

In the present paper we give a streamlined proof (that does not require knowledge from topological dynamics) of the existence and uniqueness of a right-invariant mean on a CHART group.

2. Groups

Let $$(G, \cdot, \tau)$$ be a right topological group and let $$H$$ be a subgroup of $$G$$. We shall denote by $$(H, \tau_H)$$ the set $$H$$ equipped with the relative $$\tau$$-topology. It is easy to see that $$(H, \cdot, \tau_H)$$ is also a right topological group.

Now let $$G/H$$ be the set $$\{xH : x \in G\}$$ of all left cosets of $$H$$ in $$G$$ and give $$G/H$$ the quotient topology $$q(\tau)$$ induced from $$(G, \tau)$$ by the map $$\pi: G \to G/H$$ defined by $$\pi(x) := xH$$.

Note that $$\pi$$ is an open mapping because, if $$U$$ is an open subset of $$G$$ then

$$\pi^{-1}(\pi(U)) = UH = \bigcup\{Ux : x \in H\}$$

and this last set is open since right multiplication is a homeomorphism on $$G$$.

If $$H$$ is a normal subgroup of a right(left)[semi] topological group $$(G, \cdot, \tau)$$ then one can check that $$(G/H, \cdot, q(\tau))$$ is also a right(left)[semi] topological group.

In order to continue our investigations further we need to introduce a new topology.
2.1. The $\sigma$-topology. Let $(G, \cdot, \tau)$ be a right topological group and let $\varphi : G \times G \to G$ be the map defined by

$$\varphi(x, y) := x^{-1} \cdot y.$$ 

Then the quotient topology on $G$ induced from $(G \times G, \tau \times \tau)$ by the map $\varphi$ is called the $\sigma(G, \tau)$-topology or $\sigma$-topology.

The proof of the next result can be found in [8, Theorem 1.1, Theorem 1.3] or [9, Lemma 4.3].

**Lemma 2.1.** Let $(G, \cdot, \tau)$ be a right topological group. Then,

(i) $(G, \sigma)$ is a semitopological group.

(ii) $\sigma \subseteq \tau$.

(iii) $(G/H, q(\tau))$ is Hausdorff provided the subgroup $H$ is closed with respect to the $\sigma$-topology on $G$.

2.2. Admissibility and CHART groups. Let $(G, \cdot, \tau)$ be a right topological group and let $\Lambda(G, \tau)$ be the set of all $x \in G$ such that the map $y \mapsto x \cdot y$ is $\tau$ continuous. If $\Lambda(G, \tau)$ is $\tau$-dense in $G$ then $(G, \tau)$ is said to be admissible.

The proof for the following proposition may be found in [8, Theorem 1.2, Corollary 1.1] or [9, Proposition 4.4, Proposition 4.5].

**Proposition 2.2.** Let $(G, \cdot, \tau)$ be an admissible right topological group.

(i) If $\mathcal{U}$ is the family of all $\tau$-open neighborhoods of $e$ in $G$ then

$$\{U^{-1}U : U \in \mathcal{U}\}$$

is a base of open neighborhoods of $e$ in $(G, \sigma)$.

(ii) If $N(G, \tau) := \bigcap\{U^{-1}U : U \in \mathcal{U}\}$ then $N(G, \tau) = \{e\}^\sigma$.

A compact Hausdorff admissible right topological group $(G, \cdot, \tau)$ is called a CHART group.

The proof for the following result may be found [8, Proposition 2.1] or [9, Proposition 4.6].

**Proposition 2.3.** Let $(G, \cdot, \tau)$ be a CHART group. Then the following hold:

(i) If $L$ is a $\sigma$-closed normal subgroup of $G$, then so is $N(L, \sigma_L)$.

(ii) If $m : (G/N(L, \sigma_L), q(\tau)) \times (L/N(L, \sigma_L), q(\tau)) \to (G/N(L, \sigma_L), q(\tau))$ is defined by

$$m(xN(L, \sigma_L), yN(L, \sigma_L)) := x \cdot yN(L, \sigma_L)$$

for all $(x, y) \in G \times L$

then $m$ is well-defined and continuous.

**Remark 2.4.** By considering the mapping $\pi : G/N(L, \sigma_L) \to G/L$, Theorem 1.2 and Proposition 2.3 we see that if $(G/L, q(\tau))$ admits a unique right-invariant mean then so does $(G/N(L, \sigma_L), q(\tau))$. Hence if $N(L, \sigma_L)$ is a proper subset of $L$ then we have made some progress towards showing that $G \cong G/\{e\}$ admits a unique right-invariant mean.
3. \( N(L, \sigma_L) \neq L \)

In this section we will show that if \( L \) is a nontrivial \( \sigma \)-closed normal subgroup of a CHART group \((G, \cdot, \tau)\) then \( N(L, \sigma_L) \) is a proper subset of \( L \).

**Lemma 3.1.** Let \((H, \cdot)\) be a group and \( X \) be a nonempty set. Then for any \( f : H \to X \), \( S := \{s \in H : f(hs) = f(h) \text{ for all } h \in H\} \) is a subgroup of \( H \).

**Proof.** Clearly, \( e \in S \). Now suppose that, \( s_1, s_2 \in S \). Let \( h \in H \) then
\[
f(h(s_1s_2)) = f((hs_1)s_2) = f(hs_1) = f(h)
\]
Therefore, \( s_1s_2 \in S \). Next, let \( s \) be any element of \( S \) and \( h \) be any element of \( H \) then
\[
f(h) = f(h(s^{-1}s)) = f((hs^{-1})s) = f(hs^{-1}).
\]
Therefore, \( s^{-1} \in S \). \( \square \)

**Lemma 3.2.** Let \((G, \cdot, \tau)\) be a compact right topological group and let \( \sigma \) be a topology on \( G \) weaker than \( \tau \) such that \((G, \cdot, \sigma)\) is also a right topological group. If \( U \) is a dense open subset of \((G, \sigma)\) then \( U \) is also a dense subset of \((G, \tau)\).

**Proof.** Let \( C := G \setminus U \). Then \( C \) is a \( \sigma \)-closed (hence \( \tau \)-closed) nowhere-dense subset of \( G \). If \( U \) is not \( \tau \)-dense in \( G \) then \( C \) contains a nonempty \( \tau \)-open subset. By the compactness of \((G, \tau)\) there exists a finite subset \( F \) of \( G \) such that \( G = \bigcup \{Cg : g \in F\} \). Now each \( Cg \) is nowhere dense in \((G, \sigma)\) since each right multiplication is a homeomorphism. This forms a contradiction since a nonempty topological space can never be the union of a finite number of nowhere dense subsets. \( \square \)

**Lemma 3.3.** Let \((G, \cdot, \tau)\) be a CHART group and let \( \Lambda = \Lambda(G, \tau) \). If \( A \) and \( B \) are nonempty open subsets of \((G, \tau)\), then \( A^{-1}B = (A \cap \Lambda)^{-1}B \).

**Proof.** Let \( x \in A^{-1}B \). Then for some \( a \in A, ax \in B \). Since \( B \) is open and \( A \cap \Lambda \) is dense in \( A \) there is a \( c \in A \cap \Lambda \) such that \( cx \in B \). Hence \( x \in c^{-1}B \subseteq (A \cap \Lambda)^{-1}B \). Thus, \( A^{-1}B \subseteq (A \cap \Lambda)^{-1}B \). The reverse inclusion is obvious. \( \square \)

**Lemma 3.4.** Let \((G, \cdot, \tau)\) be a compact Hausdorff right topological group. If \( S \) is a nonempty subsemigroup of \( \Lambda(G, \tau) \) then \( \overline{S} \) is a subgroup of \( G \).

**Proof.** In this proof we shall repeatedly use the following fact, [2, Lemma 1] “Every nonempty compact right topological semigroup admits an idempotent element (i.e., an element \( u \) such that \( u \cdot u = u \)). Firstly, it is easy to see that \( \overline{S} \) is a subsemigroup of \( G \). Hence, \((\overline{S}, \cdot)\) is a nonempty compact right topological semigroup and so has an idempotent element \( u \). However, since \( G \) is a group it has only one idempotent element, namely \( e \). Therefore, \( e = u \in \overline{S} \). Next, let \( s \) be any element of \( \overline{S} \). Then \( \overline{S} \cdot s \) is a nonempty compact right topological semigroup of \( \overline{S} \). Therefore, there exists an element \( s' \in \overline{S} \) such that \( (s' \cdot s) \cdot (s' \cdot s) = (s' \cdot s) \). Again, since \( G \) is a group, \( s' \cdot s = e \). By multiplying both sides of this equation by \( s^{-1} \) we see that \( s^{-1} = s' \in \overline{S} \). \( \square \)

The following lemma is a simplified form of the structure theorem found in [7].
Lemma 3.5. Let \((G, \cdot, \tau)\) be a CHART group and let \(\sigma\) denote its \(\sigma\)-topology. Suppose \(L\) is a nontrivial \(\sigma\)-closed subgroup of \(G\). Then \(N(L, \sigma_L)\) is a proper subset of \(L\).

Proof. Let \(U\) denote the family of all open neighbourhoods of \(e\) in \((G, \tau)\). Then it follows from Proposition 2.2 that \(\mathcal{V} = \{U^{-1}U : U \in \mathcal{U}\}\) is a base for the system of open neighbourhoods of \(e\) in \((G, \sigma)\). Then \(\{V \cap L : V \in \mathcal{V}\}\) is a basis for the system of neighbourhoods of \(e\) in \((L, \sigma_L)\). From the definition of \(N(L, \sigma_L)\) (see Proposition 2.2 part (ii)) it follows that
\[
N(L, \sigma_L) = \bigcap\{(V \cap L)^{-1}(V \cap L) : V \in \mathcal{V}\}.
\]
The proof is by contradiction. So assume that \(N(L, \sigma_L) = L\). Then
\[
L = \bigcap\{(V \cap L)^{-1}(V \cap L) : V \in \mathcal{V}\}.
\]
Hence, for each \(V \in \mathcal{V}\), \((V \cap L)^{-1}(V \cap L) = L\), or equivalently, for each \(V \in \mathcal{V}\), \((V \cap L)\) is dense in \((L, \sigma_L)\). That is, for each \(U \in \mathcal{U}\), \((U^{-1}U \cap L)\) is open and dense in \((L, \sigma_L)\) and hence, by Lemma 3.2, dense in \((L, \tau_L)\).

Since \(L \neq \{e\}\), there exists a point \(a \in L\) such that \(a \neq e\). Note that since \((G, \tau)\) is compact and Hausdorff there is a continuous function \(f\) on \((G, \tau)\) such that \(f(e) = 0\) and \(f \equiv 1\) on a \(\tau\)-neighbourhood of \(a\).

For the rest of the proof, the topology always refers to \(\tau\) and we shall denote \(\Lambda(G, \tau)\) by \(\Lambda\). By induction on \(n\), we construct a sequence \(\{U_n : n \in \mathbb{N}\}\) in \(\mathcal{U}\), a sequence \(\{V_n : n \in \mathbb{N}\}\) of nonempty open subsets of \(G\), each of which intersects \(L\) and sequences \(\{u_n : n \in \mathbb{N}\}\) and \(\{v_n : n \in \mathbb{N}\}\) in \(G\) which satisfy the following conditions:

(i) \(v_n \in U_{n-1}^{-1}U_{n-1} \cap (V_{n-1} \cap \Lambda) = (U_{n-1} \cap \Lambda)^{-1}U_{n-1} \cap (V_{n-1} \cap \Lambda)\); by Lemma 3.3.

(ii) \(u_n \in U_{n-1} \cap \Lambda\);

(iii) \(V_n \subset \bigcap_{n \in \mathbb{N}} V_{n-1} \subset f^{-1}(1)\) and \(V_n \cap L \neq \emptyset\);

(iv) \(u_nV_n \subset U_{n-1}\);

(v) if \(H_n\) denotes the semigroup generated by \(\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n\}\); which we enumerate as: \(H_n := \{h^n_j : j \in \mathbb{N}\}\) and
\[
U_n := \{t \in G : |f(h^n_i t) - f(h^n_j)| < 1/n \quad \text{for} \quad 1 \leq i, j \leq n\}\]
then \(H_n \subset \Lambda\) and \(e \in U_n \subset \overline{U_n} \subset U_{n-1}\).

Construction. We let \(U_0 := G\) and let \(V_0\) be the interior of \(f^{-1}(1)\) and \(u_0, v_0\) are not defined. Assume that \(n \in \mathbb{N}\) and that \(U_k, V_k\) are defined for \(0 \leq k \leq n\) and \(v_k, u_k\) are defined for \(0 < k < n\). By our assumption there exists an \(x \in (U_{n-1} \cap \Lambda)^{-1}U_{n-1} \cap (V_{n-1} \cap L)\). So there is a \(u_n \in U_{n-1} \cap \Lambda\) such that \(u_nx \in U_{n-1}\). Since \(u_n \in \Lambda\), \(x \in V_{n-1} \cap \Lambda\) and \(U_{n-1}\) is open, there is an open neighbourhood \(V_n\) of \(x\) such that \(x \in V_n \subset \bigcap_{n \in \mathbb{N}} V_{n-1} \subset U_{n-1}\) and \(u_nV_n \subset U_{n-1}\). Then \(V_n \cap L \neq \emptyset\) since \(x \in V_n \cap L\). Thus (ii)-(iv) are satisfied. Let \(v_n\) be any element of \(V_n \cap \Lambda\), then by (iv) and (ii), (i) is satisfied and \(H_n \subset \Lambda\) is defined. Finally, since the map \(t \mapsto |f(gt) - f(g)|\) is continuous for \(g \in \Lambda\), the set \(U_n\) is an open neighbourhood of \(e\) and so condition (v) is satisfied. This completes the construction.

We let
\[
U_\infty = \bigcap \{\overline{U_n} : n \in \mathbb{N}\} \quad \text{and} \quad H = \bigcup \{H_n : n \in \mathbb{N}\}
\]
and let \( u_\infty, v_\infty \) be cluster points of the sequences \( \{ u_n : n \in \mathbb{N} \} \), \( \{ v_n : n \in \mathbb{N} \} \) respectively. Clearly \( u_\infty \in U_\infty, v_\infty \in V_0 \) and \( \overline{H} \) is a subgroup of \( G \), by Lemma 3.4. Moreover, by the construction, \( f(ht) = f(h) \) for each \( h \in H \) and each \( t \in \bigcap \{ U_n : n \in \mathbb{N} \} \). Therefore, if we let \[
 S = \{ s \in \overline{H} : f(hs) = f(h) \text{ for each } h \in H \} = \{ s \in \overline{H} : f(hs) = f(h) \text{ for each } h \in \overline{H} \}
\]
then \( \bigcap \{ U_n : n \in \mathbb{N} \} \cap \overline{H} \subset S \) and \( S \) is a subgroup of \( G \) by Lemma 3.1. Furthermore, by (ii), \( u_\infty \in U_\infty \cap \overline{H} \subset S \) and by (iv) \( u_\infty v_\infty \in \overline{U_{n-1}} \cap \overline{H} \) for each \( n \in \mathbb{N} \). Hence \( u_\infty v_\infty \in \bigcap_{n \in \mathbb{N}} U_{n-1} \cap \overline{H} \subset S \).

Therefore, \( v_\infty = u_\infty^{-1}(u_\infty v_\infty) \in S^{-1}S \subset S \). Now, \( f(s) = 0 \) for all \( s \in S \) since \( f(es) = f(e) = 0 \) for all \( s \in S \). Therefore, \( f(v_\infty) = 0 \). On the other hand, since \( v_\infty \in V_0 \subset f^{-1}(1) \), \( f(v_\infty) = 1 \). This contradiction completes the proof. \( \square \)

4. INARIANT MEANS ON CHART GROUPS

In this section we will show that every CHART group admits a unique right-invariant mean.

**Theorem 4.1.** Every CHART group \((G, \cdot, \tau)\) possesses a unique right-invariant mean \( m \) on \( C(G) \).

**Proof.** Let \( \mathcal{L} \) be the family of all \( \sigma \)-closed normal subgroups \( L \) of \( G \) for which \( C(G/L) \) has a unique right-invariant mean. Clearly, \( \mathcal{L} \neq \emptyset \) as \( G \in \mathcal{L} \). Now, \((\mathcal{L}, \subseteq)\) is a partially ordered set. We claim that \((\mathcal{L}, \subseteq)\) possesses a minimal element. To prove this, it is sufficient to show that every totally ordered subfamily \( \mathcal{M} \) of \( \mathcal{L} \) has a lower bound (in \( \mathcal{L} \)). To this end, let \( \mathcal{M} := \{ M_\alpha : \alpha \in A \} \) be a nonempty totally ordered subfamily of \( \mathcal{L} \). Let \( M_0 := \bigcap \{ M_\alpha : \alpha \in A \} \).

Then \( M_0 \) is a \( \sigma \)-closed normal subgroup of \( G \) and \( M_0 \subseteq M_\alpha \) for every \( \alpha \in A \). Thus, to complete the proof of the claim we must show that \( M_0 \in \mathcal{L} \), i.e., show that \( C(G/M_0) \) admits a unique right-invariant mean. For each \( \alpha \in A \), let \( \pi_\alpha : G/M_0 \to G/M_\alpha \) be defined by \( \pi_\alpha(gM_0) := gM_\alpha \). Then \( \pi_\alpha \) is a continuous, open and onto map and its dual map \( \pi_\alpha^* : C(G/M_\alpha) \to C(G/M_0) \) is an isometric algebra isomorphism of \( C(G/M_\alpha) \) into \( C(G/M_0) \). By Proposition 1.1, for each \( \alpha \in A \), there exists a unique right-invariant mean \( m_\alpha \) on \( \pi_\alpha^*(C(G/M_\alpha)) \). From the Hahn-Banach extension theorem it follows that each mean \( m_\alpha \) has an extension to a mean \( m_\alpha^* \) on \( C(G/M_0) \). Let \( \mathcal{A} := \bigcup \{ \pi_\alpha^*(C(G/M_\alpha)) : \alpha \in A \} \). Then \( \mathcal{A} \) is a subalgebra of \( C(G/M_0) \), that contains all the constant functions and separates the point of \( G/M_0 \) since \( M_0 := \bigcap \{ M_\alpha : \alpha \in A \} \). Therefore, by the Stone-Weierstrass theorem, \( \mathcal{A} \) is dense in \( C(G/M_\alpha) \). Let \( m \) be a weak* cluster-point of the net \( (m_\alpha^* : \alpha \in A) \) in \( B_{C(G/M_\alpha)^*} \). Clearly, \( m \) is a mean on \( C(G/M_0) \). Furthermore, it is routine to show that (i) \( m|_{\mathcal{A}} \) is a right-invariant mean on \( \mathcal{A} \) and (ii) \( m|_{\mathcal{A}} \) is the
only (unique) right-invariant mean on \( \mathcal{A} \). It now follows from continuity that \( m \) is the one and only right-invariant mean on \( C(G/M_0) \), i.e., \( M_0 \in \mathcal{L} \).

Let \( L_0 \) be a minimal element of \( \mathcal{L} \). Then by Remark 2.4, \( N(L_0, \sigma_{L_0}) \subseteq L_0 \) and \( L_0 \) is a minimal element of \( \mathcal{L} \) we must have that \( N(L, \sigma_L) = L_0 \). Thus, by Lemma 3.5, it must be the case that \( L_0 = \{ e \} \). This completes the proof.

Let us now note that the unique right-invariant mean given above is also partially left invariant in the sense that for each \( g \in \Lambda(G, \tau) \), \( m(L_g(f)) = m(f) \) for all \( f \in C(G) \). To see why this is true, consider the mean \( m^* \) on \( C(G) \) defined by, \( m^* (f) := m(L_g(f)) \) for each \( f \in C(G) \) and some \( g \in \Lambda(G, \tau) \). Then for any \( h \in G \),

\[
m^* (R_h (f)) = m(L_g(R_h(f))) = m(R_h (L_g(f))) = m(L_g (f)) = m^* (f).
\]

Therefore, \( m^* \) is a right-invariant mean on \( C(G) \). Thus, \( m^* = m \) and so

\[
m(L_g(f)) = m^* (f) = m(f) \text{ for all } f \in C(G) \text{ and all } g \in \Lambda(G, \tau).
\]

References

Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland, New Zealand.

E-mail address: moors@math.auckland.ac.nz