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Fragmentable mappings and CHART groups

by

Warren B. Moors (Auckland)

Abstract. The purpose of this note is two-fold: firstly, to give a new and interesting result concerning separate and joint continuity, and secondly, to give a stream-lined (and self-contained) proof of the fact that "tame" CHART groups are topological groups.

1. Introduction. We shall call a triple $(G, \cdot, \tau)$ a *right topological group* (*left topological group*) if $(G, \cdot)$ is a group, (G, τ) is a topological space and, for each $g \in G$, the mapping $x \mapsto x \cdot g$ ($x \mapsto g \cdot x$) is τ-continuous on G. If $(G, \cdot, \tau)$ is both a right topological group and a left topological group then we call it a *semitopological group*. Let $(G, \cdot, \tau)$ be a right topological group and let $\Lambda(G, \tau)$ be the set of all $x \in G$ such that the map $y \mapsto x \cdot y$ is τ-continuous. If $\Lambda(G, \tau)$ is τ-dense in G then $(G, \cdot, \tau)$ is said to be *admissible*. A compact Hausdorff admissible right topological group $(G, \cdot, \tau)$ is called a *CHART group*.

The study of CHART groups has come from the study of topological dynamics; see [16–19]. Recently, it has been shown [6–10] that if the topology on a CHART group is sufficiently "nice" then this CHART group is actually a topological group. The proofs of these results have come as a by-product of the investigation of the representation theory of "tame" dynamical systems [9, 10]. The purpose of this note is two-fold: firstly, to give a new and interesting result concerning separate and joint continuity, and secondly, to give a stream-lined (and self-contained) proof of the fact that "tame" CHART groups are topological groups.

2. Fragmentable maps. Let (X, τ) be a topological space and (Y, μ) be a uniform space. Following [11,15] we shall say that a mapping $f : X \to Y$

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is (τ, μ)-*fragmented* if for every nonempty subset A of X and every $\varepsilon \in \mu$ there exists a τ-open subset U of X such that $U \cap A \neq \emptyset$ and $f(U \cap A)$ is ε-small in Y. When the topology τ and the uniformity μ are understood (from the context), then we simply say that the function f is *fragmented*. In the special case when the range space Y is a metric space, this definition simplifies to the following: a mapping $f : X \to Y$ is fragmented if for every nonempty subset A of X and every $\varepsilon > 0$ there exists a τ-open subset U of X such that $U \cap A \neq \emptyset$ and $\mathrm{diam}[f(U \cap A)] < \varepsilon$.

We shall write $\mathcal{F}(X) := \{f \in \mathbb{R}^X : f \text{ is fragmented}\}$. If $C(X)$ denotes the set of all real-valued continuous functions defined on (X, τ) then obviously $C(X) \subseteq \mathcal{F}(X)$. We shall denote by $C_p(X)$ and $\mathcal{F}_p(X)$ the sets $C(X)$ and $\mathcal{F}(X)$, respectively, equipped with the topology of pointwise convergence on X.

Prior to the study of fragmentability of functions, fragmentability was considered in regard to sets. In this setting the notion of fragmentability can be traced back to the early 1970's. See [14] for a brief review of the notion of fragmentability.

Our considerations will require the following well-known result that says that fragmentability is preserved by perfect mappings. Recall that a function $T : (X, \tau) \to (Y, \tau')$ between topological spaces is said to be *perfect* if it is: (i) surjective; (ii) continuous; (iii) maps closed sets to closed sets; and (iv) $T^{-1}(y)$ is a compact subset of X for each $y \in Y$.

Lemma 2.1. *Let $T : X \to Y$ be a perfect surjection acting between topological spaces (X, τ) and (Y, τ'). Suppose that $f \in \mathbb{R}^X$, $g \in \mathbb{R}^Y$ and $f = g \circ T$. Then $f \in \mathcal{F}(X)$ if, and only if, $g \in \mathcal{F}(Y)$.*

Proof. Suppose that $g \in \mathcal{F}(Y)$. Let A be a nonempty subset of X and $\varepsilon > 0$. Since $g \in \mathcal{F}(Y)$, there exists a τ'-open subset U' of Y such that $U' \cap T(A) \neq \emptyset$ and $\mathrm{diam}[g(U' \cap T(A))] < \varepsilon$. Let $U := T^{-1}(U') \in \tau$. Then $U \cap A \neq \emptyset$ and

$$f(U \cap A) = (g \circ T)(U \cap A) \subseteq g(T(U) \cap T(A)) = g(U' \cap T(A)).$$

Therefore, $\mathrm{diam}[f(U \cap A)] < \varepsilon$. This shows that $f = g \circ T \in \mathcal{F}(X)$. Suppose that $f = g \circ T \in \mathcal{F}(X)$. Let A be a nonempty subset of Y and $\varepsilon > 0$. Since T is a perfect mapping there exists a minimal (with respect to set-inclusion) closed subset A' of X such that $A \subseteq T(A')$. Since $f \in \mathcal{F}(X)$, there exists a τ-open subset U' of X with $U' \cap A' \neq \emptyset$ and $\mathrm{diam}[f(U' \cap A')] < \varepsilon$. Now, $A \not\subseteq T(A' \setminus U')$ by the minimality of A'. Let $U := Y \setminus T(A' \setminus U') \in \tau'$. Then $\emptyset \neq U \cap A = A \setminus T(A' \setminus U') \subseteq T(A' \cap U')$ and so

$$g(U \cap A) \subseteq g(T(A' \cap U')) = (g \circ T)(A' \cap U') = f(A' \cap U').$$

Therefore, $\mathrm{diam}[g(U \cap A)] < \varepsilon$. This shows that $g \in \mathcal{F}(Y)$. ■

The following proposition, due to Jean Bourgain, lies at the very heart of our main result, Corollary 2.6.

PROPOSITION 2.2 ([2]). *Let (X,τ) be a second countable topological space and $K \subseteq \mathcal{F}_p(X)$ be compact. Then every nonempty G_δ-subset of K contains a G_δ-point.*

Proof. Let (X,τ) be a second countable topological space, $K \subseteq \mathcal{F}_p(X)$ be a compact subspace and G be a nonempty G_δ-subset of K. In order to show that G contains a G_δ-point it is sufficient to prove that for every nonempty closed G_δ-subset G' of G and every $\varepsilon > 0$ there exists a nonempty closed G_δ-subset G'' of G' such that $\|\cdot\|_\infty$-diam$(G'') \le \varepsilon$.

To this end, let $\varepsilon > 0$ and G' be a nonempty closed G_δ-subset of G. Let $\mathcal{X}$ denote the set of all pairs $(G^*, U^*) \in 2^G \times \tau$ such that G^* is a nonempty closed G_δ-subset of G' and

$$\sup\{|f(t) - g(t)| : f, g \in G^* \text{ and } t \in U^*\} < \varepsilon.$$

We can define a partial ordering on $\mathcal{X}$ as follows. Suppose that $(G_1,U_1) \in \mathcal{X}$ and $(G_2,U_2) \in \mathcal{X}$. Then we write $(G_1,U_1) \le (G_2,U_2)$ if either $(G_1,U_1) = (G_2,U_2)$, or $G_2 \subsetneq G_1$ and $U_1 \subsetneq U_2$. We will also write $(G_1,U_1) < (G_2,U_2)$ if $(G_1,U_1) \le (G_2,U_2)$ and $(G_1,U_1) \ne (G_2,U_2)$. It is not hard to show that $(\mathcal{X},\le)$ is a nonempty partially ordered set. We claim that $(\mathcal{X},\le)$ has a maximal element $(G_{\max},U_{\max})$. To see this, let $\{(G_\alpha,U_\alpha) : \alpha \in A\}$ be a totally ordered subset of $\mathcal{X}$. Let $U_\infty := \bigcup_{\alpha\in A} U_\alpha$. Since (X,τ) is hereditarily Lindelöf there exists a countable subset C of A such that $U_\infty = \bigcup_{\alpha\in C} U_\alpha$. Let $G_\infty := \bigcap_{\alpha\in C} G_\alpha$. Then $(G_\infty,U_\infty) \in \mathcal{X}$. Note that if $U_{\alpha^*} = U_\infty$ for some $\alpha^* \in A$, then $(G_{\alpha^*},U_{\alpha^*})$ is an upper bound for $\{(G_\alpha,U_\alpha) : \alpha \in A\}$. On the other hand, if U_α is a proper subset of U_∞ for each $\alpha \in A$ then (G_∞,U_∞) is an upper bound for $\{(G_\alpha,U_\alpha) : \alpha \in A\}$. So in either case we have an upper bound for $\{(G_\alpha,U_\alpha) : \alpha \in A\}$. Thus, by Zorn's lemma, $(\mathcal{X},\le)$ has a maximal element $(G_{\max},U_{\max})$.

If $G_{\max}$ is a singleton then we are done. So suppose that $G_{\max}$ is not a singleton. In this case we claim that $U_{\max} = X$. Suppose not; then $X \setminus U_{\max}$ is a nonempty subset of X. Let $\{U_n : n \in \mathbb{N}\}$ be a countable base (of nonempty subsets) for the relative topology on $X \setminus U_{\max}$. For each $n \in \mathbb{N}$, let

$$G^n_{\max} := \{f \in G_{\max} : \sup\{|f(t) - f(t')| : t,t' \in U_n\} \le \varepsilon/3\}.$$

Then each $G^n_{\max}$ is closed, and since $G_{\max} \subseteq \mathcal{F}(X)$, $G_{\max} = \bigcup_{n\in\mathbb{N}} G^n_{\max}$. By the Baire category theorem there exists an $l \in \mathbb{N}$, an $f \in G^l_{\max}$ and a neighbourhood W of f in $\mathcal{F}_p(X)$ that is both closed and a G_δ-set such that

$$f \in W \cap G_{\max} \subseteq G^l_{\max} \quad \text{and} \quad W \cap G_{\max} \text{ is a proper subset of } G_{\max}.$$

Let $t \in U_l$, and set $U^* := U_{\max} \cup U_l$ and

$$G^* := \{g \in G_{\max} : g(t) = f(t)\} \cap W \cap G_{\max}.$$

Then $(G^*, U^*) \in \mathcal{X}$ but $(G_{\max}, U_{\max}) < (G^*, U^*)$, which contradicts the maximality of $(G_{\max}, U_{\max})$. Hence, $U_{\max} = X$. ∎

For our purposes the following reformulation of Proposition 2.2 is useful.

COROLLARY 2.3. *Let (X, τ) be a second countable topological space and $K \subseteq \mathcal{F}_p(X)$ be compact. If $f \in K$ and C is a countable subset of X then there exists an element $g \in K$ such that:*

(i) *$g|_C = f|_C$;*
(ii) *g is a G_δ-point of K.*

To simplify the statement of the subsequent lemma we shall recall the following definition. If $f : (X, \tau) \to (Y, \tau')$ is a function between topological spaces, and $x \in X$, then we say that f is *quasi-continuous at x* if for each neighbourhood W of $f(x)$ and each neighbourhood U of x there exists a nonempty open subset $V \subseteq U$ such that $f(V) \subseteq W$ [13]. If f is quasi-continuous at each point of X then we simply say that f is *quasi-continuous on X*.

LEMMA 2.4. *Let (X, τ) be a second countable Baire space and L be a subset of $C(X)$ such that $K := \overline{L}^{\tau_p} \subseteq \mathcal{F}(X)$ is compact. Then the evaluation function $e : K \times X \to \mathbb{R}$, defined by $e(f, t) := f(t)$, is quasi-continuous on $K \times X$.*

Proof. Let $(f, t) \in K \times X$. Let U_1 be a τ_p-open neighbourhood of f in K, U_2 be a τ-open neighbourhood of t in X, and W be an open neighbourhood of $f(t)$ in $\mathbb{R}$. Since $e : K \times X \to \mathbb{R}$ is continuous in the first variable (i.e., for every $x \in X$, the mapping $g \mapsto e(g, x) = g(x)$ is continuous) we may assume, without loss of generality, that $f \in L$. Furthermore, since $\mathbb{R}$ is regular, to prove that e is quasi-continuous at (f, t) it is sufficient to show that there exists a nonempty τ_p-open subset V_1 of U_1 and a nonempty τ-open subset V_2 of U_2 such that $e(V_1 \times V_2) \subseteq \overline{W}$. Now, since f is continuous there exists an open neighbourhood N of t, contained in U_2, and an $\varepsilon > 0$ such that

$$\overline{f(N)} + [-\varepsilon, \varepsilon] \subseteq W.$$

Let D be a countable dense subset of X that includes the point t. By Corollary 2.3 there exists a G_δ-point $g \in U_1$ such that $f|_D = g|_D$. In particular, $g(N \cap D) + [-\varepsilon, \varepsilon] \subseteq W$. Since $g \in \mathcal{F}(X)$ there exists a nonempty open subset O of N such that $\operatorname{diam}[g(O)] = \operatorname{diam}[g(O \cap N)] < \varepsilon$. Therefore, $g(O) \subseteq W$. Let $\{G_n : n \in \mathbb{N}\}$ be a local base for the topology on K at g. Next, for each $n \in \mathbb{N}$, let $O_n := \{u \in O : h(u) \in W \text{ for all } h \in G_n\}$. Then $O = \bigcup_{n \in \mathbb{N}} O_n$, and so by the Baire category theorem there exists an $l \in \mathbb{N}$ such that $\operatorname{int}(\overline{O_l}) \neq \emptyset$. In fact $\operatorname{int}(\overline{O_l}) \cap O \neq \emptyset$. Finally, one can check that if $V_2 := \operatorname{int}(\overline{O_l}) \cap O$ and $V_1 := G_l \cap U_1$ then $e(V_1 \times V_2) \subseteq \overline{W}$. ∎

If $f : (X, \tau) \to (Y, \tau')$ is a function between topological spaces then we say that f is *feebly continuous on* X if for each open subset W of Y such that $f(X) \cap W \neq \emptyset$, $\mathrm{int}[f^{-1}(W)] \neq \emptyset$ (see [3, 5]).

THEOREM 2.5. *Let* (X, τ) *be a compact Hausdorff space and* L *be a subset of* $C(X)$ *such that* $K := \overline{L}^{\tau_p} \subseteq \mathcal{F}(X)$ *is compact. Then the evaluation function* $e : K \times X \to \mathbb{R}$*, defined by* $e(f, t) := f(t)$*, is feebly continuous on* $K \times X$*.*

Proof. The proof, which is indirect, comprises two parts. In Part (I) we reduce the problem to the case when L is countable. Then, in Part (II), we apply Lemma 2.4, via a factorisation argument, to obtain the desired contradiction.

Part (I). Suppose, in order to obtain a contradiction, that the evaluation mapping e is not feebly continuous on $K \times X$. Then, due to the regularity of $\mathbb{R}$, there exists an open subset W of $\mathbb{R}$ and $(f, t) \in K \times X$, with $e(f, t) = f(t) \in W$, such that for each nonempty open subset U'' of K and each nonempty open subset V'' of X there exists a $g \in U'' \cap L$ such that $g(V'') \not\subseteq \overline{W}$. For each set $F \subseteq C(X)$ we shall denote by $\tau(F)$ the weak topology on X generated by F. It is not hard to see that if F is (at most) countable then $\tau(F)$ is second countable. Since $C_p(X)$ has countable tightness (see [1,20]) it follows that for each countable subset $C \subseteq L$ there exists a countable subset $C^* \subseteq L$, containing C, such that for each $g \in C$, each neighbourhood U'' of g and each nonempty $\tau(C)$-open subset V'' of X there exists an element $g^* \in C^*$ such that $g^*(V'') \not\subseteq \overline{W}$.

Let $L_1 := \{f\}$ and, for each $n \in \mathbb{N}$, let $L_{n+1} := (L_n)^*$. Furthermore, let $L_\infty := \bigcup_{n\in\mathbb{N}} L_n \subseteq L$ and $K_\infty := \overline{L_\infty}^{\tau_p} \subseteq K$. Note: (i) $f \in L_\infty$, (ii) L_∞ is countable, (iii) $\bigcup_{n\in\mathbb{N}} \tau(L_n)$ is a topological base for $\tau(L_\infty)$, and (iv) for each nonempty open subset U'' of K_∞ and each nonempty $\tau(L_\infty)$-open subset V'' of X there exists a $g \in U'' \cap L_\infty$ such that $g(V'') \not\subseteq \overline{W}$.

Part (II). Let $L_\infty = \{f_n : n \in \mathbb{N}\}$ and define $T : X \to \mathbb{R}^{\mathbb{N}}$ by $T(x)(n) := f_n(x)$ for all $x \in X$ and $n \in \mathbb{N}$. We shall consider $\mathbb{R}^{\mathbb{N}}$ endowed with the topology of pointwise convergence on $\mathbb{N}$. With this topology T is continuous and so $T(X)$ is compact. Since the topology on $\mathbb{R}^{\mathbb{N}}$ is metrisable we see that $X' := T(X)$ is second countable (and Baire). Let $T^* : \mathbb{R}^{X'} \to \mathbb{R}^X$ be defined by $T^*(h) := h \circ T$ for all $h \in \mathbb{R}^{X'}$. We shall consider $\mathbb{R}^{X'}$ endowed with the topology of pointwise convergence on X'. With this topology, T^* is a homeomorphic embedding of $\mathbb{R}^{X'}$ onto a closed subspace $\mathbb{R}^X$. It is easy to see that $L_\infty \subseteq T^*(C(X'))$ since $f_n = T^*(\pi_n) = \pi_n \circ T$ for each $n \in \mathbb{N}$, where $\pi_n : X' \to \mathbb{R}$ is defined by $\pi_n(x') := x'(n)$ for all $x' \in X' \subseteq \mathbb{R}^{\mathbb{N}}$. Thus,

$$K_\infty = \overline{L_\infty}^{\tau_p} \subseteq T^*(\mathbb{R}^{X'}).$$

Let $K'_\infty := (T^*)^{-1}(K_\infty)$ and $L'_\infty := (T^*)^{-1}(L_\infty)$. Then $L'_\infty \subseteq C(X')$ and $K'_\infty = \overline{L'_\infty}^{\tau_p}$. Furthermore, by Lemma 2.1, $K'_\infty \subseteq \mathcal{F}(X')$. Let $f' \in L'_\infty \subseteq C(X')$ be chosen so that $T^*(f') = f$ (i.e., $f' \circ T = f$). Then

$$f'(t') = f'(T(t)) = (f' \circ T)(t) = f(t) \in W, \qquad \text{where } t' = T(t).$$

By Lemma 2.4 there exists a nonempty open subset U' of K'_∞ and a nonempty open subset V' of X' such that $g'(x') \in W$ for all $(g', x') \in U' \times V'$. Let $U := T^*(U')$ and $V := T^{-1}(V')$. Then U is open on K_∞ and V is $\tau(L_\infty)$-open in X.

Now, if $g \in U$ and $x \in V$ then $g = T^*(g')$ for some $g' \in U'$. Therefore,

$$g(x) = T^*(g')(x) = g'(T(x)) \subseteq g'(V') \subseteq W.$$

However, this contradicts (iv) above [at the end of Part (I)]. ∎

Suppose that X, Y and Z are sets and $f : X \times Y \to Z$ is a function. Then for each $x \in X$ we define $f_{[x]} : Y \to Z$ by $f_{[x]}(y) := f(x, y)$, and for each $y \in Y$ we define $f_{[y]} : X \to Z$ by $f_{[y]}(x) := f(x, y)$.

Corollary 2.6. *Let (X, τ) and (Y, τ') be compact Hausdorff spaces and (Z, τ'') be a completely regular topological space. Suppose that $f : X \times Y \to Z$ is a function such that:*

(i) *for each $y \in Y$, $f_{[y]}$ is continuous;*
(ii) *there exists a dense subset D of X such that $f_{[x]}$ is continuous for each $x \in D$;*
(iii) *for every $g \in C(Z)$ and every $x \in X$, $(g \circ f)_{[x]} \in \mathcal{F}(Y)$.*

Then f is quasi-continuous on $X \times Y$.

Proof. Suppose that $(x, y) \in X \times Y$ and W is an open neighbourhood of $f(x, y)$. Suppose also that $U \times V$ is an open neighbourhood of (x, y). We will show that there exist nonempty open subsets U'' of U and V'' of V such that $f(U'' \times V'') \subseteq W$. Choose $g \in C(Z)$ such that $g(f(x, y)) = 1$ and $g(Z \setminus W) = \{0\}$. Define $\varphi : (\overline{U}^\tau, \tau) \to \mathcal{F}_p(\overline{V}^{\tau'})$ by $\varphi(u)(v) := g(f(u, v))$ for all $v \in \overline{V}^{\tau'}$. By the hypotheses φ is well-defined and continuous and $L := \varphi(D \cap U) \subseteq C(\overline{V}^{\tau'})$. Let

$$K := \overline{L}^{\tau_p} = \overline{\varphi(D \cap U)}^{\tau_p} = \varphi(\overline{D \cap U}^\tau) = \varphi(\overline{U}^\tau) \subseteq \mathcal{F}(\overline{V}^{\tau'})$$

and note that $\varphi(x)(y) = 1$. Let $W' := (1/2, \infty)$. Then by Theorem 2.5 there exists a nonempty τ_p-open subset U' of K and a nonempty open subset V' of $\overline{V}^{\tau'}$ such that $g(v) \in W'$ for all $(g, v) \in U' \times V'$. Let $U'' := \varphi^{-1}(U') \cap U$ and $V'' := V' \cap V$. Then U'' and V'' are nonempty open subsets and $g(f(u, v)) = (g \circ f)(u, v) \in (1/2, \infty)$ for all $(u, v) \in U'' \times V''$. Thus, $f(U'' \times V'') \subseteq W$. ∎

COROLLARY 2.7. *Let (X,τ) and (Y,τ') be compact Hausdorff spaces and (Z,τ'') be a completely regular topological space. Suppose that $f : X \times Y \to Z$ is a function such that:*

(i) *for each $y \in Y$, $f_{[y]}$ is continuous;*
(ii) *there exists a dense subset D of X such that $f_{[x]}$ is continuous for each $x \in D$;*
(iii) *for every $x \in X$, there exists a sequence $(d_n : n \in \mathbb{N})$ in D such that $x = \lim_{n\to\infty} d_n$.*

Then f is quasi-continuous on $X \times Y$.

Proof. To prove this result it is sufficient to show that condition (iii) from Corollary 2.6 is satisfied. To this end, let $g \in C(Y)$, $x \in X$, $\varepsilon > 0$ and let A be a nonempty subset of Y. From the hypotheses there exists a sequence $(d_n : n \in \mathbb{N})$ in D such that $x = \lim_{n\to\infty} d_n$.

We claim that $(g \circ f)_{[x]}$ is the pointwise limit of the continuous functions $((g \circ f)_{[d_n]} : n \in \mathbb{N})$. To see this, let $y \in Y$. Then

$$\begin{aligned}(g \circ f)_{[x]}(y) &= (g \circ f)(x,y) = g(f(x,y)) = g(f_{[y]}(x)) \\ &= g\Big(\lim_{n\to\infty} f_{[y]}(d_n)\Big) \quad \text{since } f_{[y]} \in C(Y) \text{ and } x = \lim_{n\to\infty} d_n \\ &= \lim_{n\to\infty} g(f_{[y]}(d_n)) \quad \text{since } g \in C(Z) \\ &= \lim_{n\to\infty} g(f(d_n,y)) = \lim_{n\to\infty} (g \circ f)(d_n,y) = \lim_{n\to\infty} (g \circ f)_{[d_n]}(y).\end{aligned}$$

By Osgood's theorem [12, p. 86], $(g \circ f)_{[x]}|_{\overline{A}}$ has a point of continuity at some point $y_0 \in \overline{A}$. Therefore, there exists an open neighbourhood U of y_0 such that $\mathrm{diam}[(g \circ f)_{[x]}(U \cap A)] < \varepsilon$. Note also that $U \cap A \neq \emptyset$. Hence $(g \circ f)_{[x]} \in \mathcal{F}(Y)$. ∎

We end this section with the following question.

QUESTION 2.8. *Note that the conclusions of Corollaries* 2.6 *and* 2.7 *remain valid if one weakens the hypotheses on both spaces (X,τ) and (Y,τ') from being compact Hausdorff to being locally compact Hausdorff. Does the conclusion of Corollary* 2.6 *still hold if we further weaken the hypothesis on (X,τ) to being a completely regular Čech-complete space?*

One possible approach to this is to use topological games [4].

3. CHART groups. In this section we apply Corollary 2.6 to CHART groups.

LEMMA 3.1. *If $(G,\cdot,\tau)$ is a CHART group and N is an open neighbourhood of e then for any $g \in G$,*

$$g \cdot N^{-1} \cdot N = [g \cdot N^{-1} \cap \Lambda(G)] \cdot N \in \tau.$$

Proof. We will first show that for any dense subset D of G and any nonempty open subsets A and B of (G,τ), $A^{-1}\cdot B = (A\cap D)^{-1}\cdot B$. Let $x\in A^{-1}\cdot B$. Then for some $a\in A$, $a\cdot x\in B$. Since B is open and $A\cap D$ is dense in A there is a $c\in A\cap D$ such that $c\cdot x\in B$. Hence $x\in c^{-1}\cdot B\subseteq (A\cap D)^{-1}\cdot B$. Thus, $A^{-1}\cdot B\subseteq (A\cap D)^{-1}\cdot B$. The reverse inclusion is obvious. We now prove the statement given in the lemma. Let N be an open neighbourhood of e and let $g\in G$. Let $D := \Lambda(G)\cdot g$, which is dense in (G,τ) since $x\mapsto x\cdot g$ is a homeomorphism on G. Note also that $[\Lambda(G)]^{-1} = \Lambda(G)$. Therefore,

$$\begin{aligned} g\cdot N^{-1}\cdot N &= g\cdot [N\cap \Lambda(G)\cdot g]^{-1}\cdot N \qquad \text{by the above}\\ &= g\cdot [N^{-1}\cap g^{-1}\cdot \Lambda(G)]\cdot N = [g\cdot N^{-1}\cap \Lambda(G)]\cdot N. \ \blacksquare \end{aligned}$$

PROPOSITION 3.2. *If $(G,\cdot,\tau)$ is a CHART group and multiplication is feebly continuous then $(G,\cdot,\tau)$ is a topological group.*

Proof. Let $\pi : G\times G\to G$ be defined by $\pi(g,h) := g\cdot h$ for all $(g,h)\in G\times G$. We will first show that π is continuous at (e,e). To this end, let W be an open neighbourhood of e and let

$$W^* := \{g\in G : (g,e)\in \operatorname{int}(\pi^{-1}(W))\}.$$

We claim that $e\in \overline{W^*}$. To justify this claim let us consider an arbitrary open neighbourhood N of e. Since multiplication is feebly continuous there exist nonempty open subsets U and V of G such that $\pi(U\times V) = U\cdot V\subseteq N\cap W$. Let $\lambda\in V\cap \Lambda(G)$. Then

$$(U\cdot\lambda)\cdot(\lambda^{-1}\cdot V) = U\cdot V\subseteq W\cap N\subseteq W.$$

Now, $e\in \lambda^{-1}\cdot V$ and so $U\cdot\lambda\subseteq W^*$. On there other hand,

$$U\cdot\lambda = (U\cdot\lambda)\cdot e\subseteq (U\cdot\lambda)\cdot(\lambda^{-1}\cdot V)\subseteq W\cap N\subseteq N.$$

Therefore $\emptyset\neq U\cdot\lambda\subseteq W^*\cap N$. This completes the proof of the claim.

Note also that since W^* is an open set, $e\in \overline{W^*\cap \Lambda(G)}$. Since (G,τ) is compact and Hausdorff, to show that π is continuous at $(e,e)\in G\times G$ it is sufficient to prove that $\bigcap_{N\in\mathcal{N}(e)} \overline{\pi(N\times N)}\subseteq \{e\}$, where $\mathcal{N}(e)$ denotes the set of all open neighbourhoods of e in G. So let $z\in \bigcap_{N\in\mathcal{N}(e)} \overline{\pi(N\times N)}$. Since (G,τ) is regular and Hausdorff, to show that $z=e$ it will be sufficient to prove that $z\in \overline{V}$ for each $V\in\mathcal{N}(e)$. Let $V\in\mathcal{N}(e)$ and $\lambda\in V^*\cap\Lambda(G)$. Then there exists an $N\in\mathcal{N}(e)$ such that $(\lambda\cdot N)\cdot N = \lambda\cdot\pi(N\times N)\subseteq V$ and so $\lambda\cdot z\in \lambda\cdot\overline{\pi(N\times N)}\subseteq \overline{V}$. Now, as $e\in \overline{V^*\cap\Lambda(G)}$, we have $z = e\cdot z\in\overline{V}$. Thus, π is continuous at (e,e). In fact, it is not hard to see that π is continuous at each point of $\Lambda(G)\times G$.

Next, we shall show that inversion is continuous at e. As with multiplication, it is sufficient to show that $\bigcap_{N\in\mathcal{N}(e)} \overline{N^{-1}}\subseteq\{e\}$. Let $y\in \bigcap_{N\in\mathcal{N}(e)} \overline{N^{-1}}$. Since (G,τ) is T_1, to prove that $y=e$ it will be sufficient to show that $e\in V$

for every open neighbourhood V of y. Thus, let V be an open neighbourhood of $y = e \cdot y$. By the continuity of π at $(e, y) \in \Lambda(G) \times G$ there exists a neighbourhood N of e and a neighbourhood U of y such that $\pi(N \times U) \subseteq V$. Since $y \in \overline{N^{-1}}$, we have $N^{-1} \cap U \neq \emptyset$. Thus, $e \in N \cdot U \subseteq V$. This shows that inversion is continuous at e. Furthermore, since $N^{-1} \cdot N = (N \cap \Lambda(G))^{-1} \cdot N$ for each $N \in \mathcal{N}(e)$ (see Lemma 3.1), $\{N^{-1} \cdot N : N \in \mathcal{N}(e)\}$ is a local base for τ, at e, consisting of τ-open sets.

To prove that inversion is continuous on G we simply need to note that for each $g \in G$:

(i) $\{N^{-1} \cdot N \cdot g : N \in \mathcal{N}(e)\}$ is a local base for τ at g;
(ii) $(N^{-1} \cdot N \cdot g)^{-1} = g^{-1} \cdot N^{-1} \cdot N = (g^{-1} \cdot N^{-1} \cap \Lambda(G)) \cdot N \in \tau$ (see Lemma 3.1).

From this and the fact that $(G, \cdot, \tau)$ is a right topological group it follows that $(G, \cdot, \tau)$ is also a left topological group, i.e. $(G, \cdot, \tau)$ is a semitopological group. However, since π is continuous at (e, e) it now follows that π is continuous on $G \times G$. Therefore, $(G, \cdot, \tau)$ is a topological group. ∎

Following [6, 7] we shall say that a CHART group $(G, \cdot, \tau)$ is *tame* if for every $g \in G$ the mapping $L_g : (G, \tau) \to (G, \mu)$, defined by $L_g(x) := g \cdot x$, is (τ, μ)-fragmented, where μ is the unique uniformity on G that is compatible with the topology τ on G. It is shown in [7, Lemma 2.3(3)] that this definition is equivalent to the following. A CHART group $(G, \cdot, \tau)$ is *tame* if for every $f \in C(G)$ and $g \in G$, the mapping $x \mapsto f(g \cdot x)$ is fragmented.

Theorem 3.3 ([9,10]). *If $(G, \cdot, \tau)$ is a tame CHART group then $(G, \cdot, \tau)$ is a topological group.*

Proof. By Proposition 3.2 it is sufficient to show that the multiplication operation on $(G, \cdot, \tau)$ is feebly continuous. Let $X := Y := Z := G$ and let $f : X \times Y \to Z$ be defined by $f(x, y) := x \cdot y$. Since $(G, \cdot, \tau)$ is a tame CHART group it follows that f satisfies the hypotheses of Corollary 2.6. Therefore, f is quasi-continuous (and hence feebly continuous) on $G \times G$. ∎

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Warren B. Moors
Department of Mathematics
The University of Auckland
Private Bag 92019
Auckland 1142, New Zealand
E-mail: moors@math.auckland.ac.nz