Fragmentability of groups and metric-valued function spaces

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Abstract. Let (X, τ) be a topological space and let ρ be a metric defined on X. We shall say that (X, τ) is *fragmented by* ρ if whenever $\varepsilon > 0$ and A is a nonempty subset of X there is a τ-open set U such that $U \cap A \neq \varnothing$ and $\rho-\text{diam}\,(U \cap A) < \varepsilon$. In this paper we consider the notion of fragmentability, and its generalisation σ-fragmentability, in the setting of topological groups and metric-valued function spaces. We show that in the presence of Baireness fragmentabilty of a topological group is very close to metrizability of that group. We also show that for a compact Hausdorff space X, σ-fragmentability of $(C(X), \|\cdot\|_\infty)$ implies that the space $C_p(X; M)$ of all continuous functions from X into a metric space M, endowed with the topology of pointwise convergence on X, is fragmented by a metric whose topology is at least as strong as the uniform topology on $C(X; M)$. The primary tool used is that of topological games.

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1 Introduction

Let (X, τ) be a topological space and let ρ be a metric defined on X. We shall say that (X, τ) is *fragmented by* ρ if whenever $\varepsilon > 0$ and A is a nonempty subset of X there is a τ-open set U such that $U \cap A \neq \varnothing$ and $\rho - \text{diam}\,(U \cap A) < \varepsilon$. The term "fragment" was coined by Jayne and Rogers in [13]. However, this notion had already been encountered before in the study of Banach spaces. In fact, the notion of fragmentability has, and continues to, appear in many guises in different areas of mathematics. For example in: (i) extensions of the Radon-Nikodým theorem from real-valued measures to vector-valued measures see, [4, 9, 21, 32]; (ii) the study of the differentiability properties of continuous convex functions defined on Banach spaces see, [20, 25, 26, 27, 34, 35]; (iii) topological dynamics see, [22, 23]; (iv) selection theorems see, [7, 13]; (v) variational principles see, [6, 36, 37] and (vi) fixed point theorems see, [8, 33], to name but a few.

Perhaps the appearance of the notion of fragmentability in these different areas can be explained by the fact that fragmentability enables one to use metric space techniques in places where the topology is far from being metrizable (e.g. the weak topology on an infinite dimensional Banach space).

Despite the utility of the notion of fragmentability there are still many situations in which a more general notion is appropriate. Specifically, if we are given a topological space (X, τ) that is also endowed with a metric ρ then we say that (X, τ) is *σ-fragmented by* ρ if for each $\varepsilon > 0$ there exists a cover $\{X_n^\varepsilon : n \in \mathbb{N}\}$ of X (i.e., $\bigcup_{n\in\mathbb{N}} X_n^\varepsilon = X$) such that for every $n \in \mathbb{N}$ and every nonempty subset A of X_n^ε there exists a τ-open set U such that $U \cap A \neq \varnothing$ and $\rho - \text{diam}\,(U \cap A) < \varepsilon$.

This notion was first introduced in [10] and many interesting properties of σ-fragmentability were investigated in [10, 11, 12], particularly in the case when X is a Banach space, τ is the weak topology on X and ρ is the natural metric on X induced by the norm on X. It turns out that in this situation σ-fragmentability is closely related to renorming theory. More precisely, it is related to Kadec and local uniform rotundity renorming, see [28, 30, 31]. Furthermore, in this setting it is also related to questions concerning separate and joint continuity of real-valued functions, [15, 16, 19, 29], and the study of the Namioka property in particular.

One approach to the study of σ-fragmentability was given in [17, 18], where the authors showed that fragmentability/σ-fragmentability can be characterised in terms of topological games. In this paper, we will follow this approach. However, before considering topological games in Section 3, we will first consider the impact, if any, of the notion of fragmentability/σ-fragmentability in the setting of groups.

In particular, we shall show that for topological groups that are also Baire spaces, fragmentability is equivalent to some well-known topological properties. In Section 3 we use the game approach to fragmentability to prove some results concerning metric-valued function spaces.

Throughout this paper we shall assume that all topological spaces are at least completely regular and that all Banach spaces are over the real numbers. Further, for a normed linear space $(X, \|\cdot\|)$ we shall denote by, B_X the closed unit ball in X, i.e., $B_X := \{x \in X : \|x\| \leq 1\}$. Finally, for a compact Hausdorff space X and a metric space (M, d) we shall denote by $C_p(X; M)$ [$C_p(X)$] the set of all continuous functions from X into M [the set of all real-valued continuous functions on X] endowed with the topology of pointwise convergence on X.

2 Fragmentability in topological groups

In this section we will examine the role of fragmentability in the setting of groups.

For our first result we need the notion of "countable separation". For a completely regular space X we shall say that X has *countable separation* if there exists a countable family $\{C_n : n \in \mathbb{N}\}$ of closed subsets of βX - the Stone-Cech compactification of X - such that for each $x \in X$ and $y \in \beta X \setminus X$ there exists an $n \in \mathbb{N}$ such that $|\{x, y\} \cap C_n| = 1$.

Proposition 2.1 *If $T : (X, \tau') \to (Y, \tau)$ is a continuous surjection from a second countable space (X, τ') onto a completely regular space (Y, τ), then Y is fragmented by a metric whose topology is at least as strong as τ.*

Proof: Let $\mathscr{B} := \{U_n : n \in \mathbb{N}\}$ be a base for τ'. Let

$$A := \{(U, V) \in \mathscr{B} \times \mathscr{B} \;:\; \text{there exists a continuous map } f : Y \to [0, 1] \text{ such that } T(U) \subseteq f^{-1}(0) \text{ and } T(V) \subseteq f^{-1}(1)\}.$$

Clearly, A is countable. Let $\{(U_n, V_n) : n \in \mathbb{N}\}$ be an enumeration of A. For each $n \in \mathbb{N}$ choose a continuous map $f_n : Y \to [0, 1]$ such that $T(U_n) \subseteq f_n^{-1}(0)$ and $T(V_n) \subseteq f_n^{-1}(1)$. Then define $d : Y \times Y \to [0, 1]$ by,

$$d(x, y) := \sum_{n=1}^{\infty} \frac{|f_n(x) - f_n(y)|}{2^n}.$$

It is routine to check that d is indeed a metric on Y; in fact the only non-trivial property to check is that d separates the points of Y. Moreover, by Weierstrass' M-test we get that for each $y \in Y$ and $0 < r$, $\{z \in Y : d(y, z) < r\} \in \tau$. Hence d fragments (Y, τ). Now, by [18, Proposition 4.1] we have that the continuous image of a second countable space has countable separation. The result then follows from [18, Proposition 4.2] which says that every fragmentable space with countable separation is fragmented by a metric whose topology is at least as strong as τ. ☺

Proposition 2.2 *Suppose that (X, τ) is a second Baire category topological space. If (X, τ) is fragmented by a metric ρ then X has a G_δ-point with respect to (X, τ). Moreover, if the ρ topology is at least as strong as τ then there exists a point $x \in X$ that has a countable local base.*

Proof: Fix $\varepsilon > 0$ and consider the following open subset of X:

$$O_\varepsilon := \bigcup\{U \in \tau : \rho - \operatorname{diam}(U) < \varepsilon\}.$$

We shall show that O_ε is dense in (X, τ). To this end, let W be a non-empty open subset of X. Since ρ fragments X there exists a nonempty relatively open (and hence open, since W is open) subset U of W such that $\rho - \operatorname{diam}(U) < \varepsilon$. Then

$$\varnothing \neq U \subseteq O_\varepsilon \cap W.$$

Therefore, O_ε is dense in (X, τ). Let $G := \bigcap_{n\in\mathbb{N}} O_{1/n}$. Since (X, τ) is a second Baire category space, $G \neq \varnothing$. It now only remains to observe that each point of G is a G_δ-point of (X, τ). Moreover, if the ρ topology is at least as strong as τ then every point of G has a countable local base. ☺

By combining the previous two results we immediately obtain the following.

Corollary 2.1 *If $f : X \to Y$ is a continuous map from a second countable topological space X onto a completely regular second Baire category space Y, then there exists a point $y \in Y$ with a countable local base.*

Proof: This follows directly from Proposition 2.1 and Proposition 2.2. ☺

A group $(G, \cdot)$ endowed with a topology τ is called a *semi-topological group* if for each $g \in G$, both mappings $x \mapsto x \cdot g$ and $x \mapsto g \cdot x$ are continuous on G. Since Ellis' result [5] that every locally compact semi-topological group is in fact a topology group, there has been continued interest in finding topological conditions on (G, τ) that are sufficient to ensure that a semi-topological group $(G, \cdot, \tau)$ is a topological group (i.e., multiplication and inversion are both continuous). In this regard, the most relevant result for us is in [14, Theorem 2] where it was shown that each semi-topological group $(G, \cdot, \tau)$ where (G, τ) is a Baire space with countable separation, is a topological group.

Corollary 2.2 *If $f : X \to G$ is a continuous map from a second countable topological space X onto a completely regular second category semi-topological group $(G, \cdot, \tau)$ then $(G, \cdot, \tau)$ is a metrizable topological group.*

Proof: From [18, Proposition 4.1] G has a countable separation and so by [14, Theorem 2], $(G, \cdot, \tau)$ a topological group. Furthermore, from Corollary 2.1 we get that G is first countable. Therefore, by the Birkhoff-Kakutani theorem, $(G, \cdot, \tau)$ is a metrizable topological group. ☺

The previous corollary says that any "topologically small" semi-topological group $(G, \cdot, \tau)$ is in fact a metrizable topological group, provided (G, τ) is a Baire space.

The previous corollary also suggests that there might be a relationship between fragmentability and metrizability of topological groups.

Theorem 2.1 *Let $(G, \cdot, \tau)$ be a topological group that possesses a nonempty G_δ subset H of G. If H with the relative topology is second category then the following are equivalent:*

(i) *(G, τ) is fragmentable;*

(ii) *(H, τ) is fragmentable;*

(iii) *(H, τ) has a G_δ-point;*

(iv) (G,τ) *has a* G_δ*-point.*

Proof: The implication (i) $\Rightarrow$ (ii) is obvious and the implication (ii) $\Rightarrow$ (iii) follows from Proposition 2.2. The implication (iii) $\Rightarrow$ (iv) follows from the easily proven fact that a G_δ subset of a G_δ subset is, itself, a G_δ subset of the whole space. So the only remaining implication is (iv) $\Rightarrow$ (i); which is what we do now. Suppose that $(G,\cdot,\tau)$ has a G_δ-point. Then without loss of generality we can assume that e - the identity element of G - is a G_δ-point. That is, there exist neighbourhoods $(U_n : n \in \mathbb{N})$ of e such that $\bigcap_{n\in\mathbb{N}} U_n = \{e\}$. By induction we can construct neighbourhoods $(W_n : n \in \mathbb{N})$ of e such that: (i) $W_n^{-1} = W_n$ and (ii) $W_{n+1}\cdot W_{n+1} \subseteq W_n \subseteq U_n$ for all $n \in \mathbb{N}$. If we let $\mathscr{W}_n := \{(x,y) \in G\times G : xy^{-1} \in W_n\}$ then $(\mathscr{W}_n : n \in \mathbb{N})$ is a base for a metrizable uniformity on G. Moreover, if d denotes the metric generating this uniformity then the topology τ is at least as strong as the topology generated by d. Hence, (G,τ) is fragmented by d. ☺

From this theorem, we see that in the presence of Baireness, fragmentability of a topological group reduces to the existence of a G_δ-point. Likewise, in the presence of Baireness, fragmentability of a topological group by a metric whose topology is at least as strong as τ is equivalent to metrizability of (G,τ). Hence, in the presence of Baireness, it does not make sense to consider fragmentability of groups.

However, in the absence of Baireness, fragmentability may be a strictly weaker property than the existence of a G_δ-point.

Example 2.1 *Let* Γ *be an uncountable set. Let* $X := \ell_2(\Gamma)$ *then* $(X,+,\mathit{weak})$ *is an Abelian topological group. Since* $(X,\|\cdot\|_2)$ *is reflexive, it is* σ*-fragmentable by the norm and hence fragmentable by a metric whose topology is at least as strong as the weak topology on* X*. However,* (X,weak) *is not first countable, in fact,* (X,weak) *does not even posses a* G_δ*-point.*

Proof: By Corollary 6.3.1 in [12] it follows that (X,weak) is σ-fragmented by the norm. It then follows from Proposition 3.3 that (X,weak) is fragmented by a metric whose topology is at least as strong as the weak topology on X. On the other hand, if (X,weak) possessed a G_δ-point then it would follow that every point of X is a G_δ-point with respect to the weak topology on X. In particular, 0 would be a G_δ-point with respect to (X,weak). This in turn would imply that there exists a countable set $\{x_n^* : n \in \mathbb{N}\} \subseteq B_{X^*}$ such that $\bigcap_{n\in\mathbb{N}} \mathrm{Ker}(x_n^*) = \{0\}$. Thus, if we defined $d : B_X \times B_X \to [0,\infty)$ by,

$$d(x,y) := \sum_{n=1}^{\infty} \frac{|x_n^*(x-y)|}{2^n},$$

then d would be a metric on B_X. Moreover the d-topology on B_X would coincide with the weak topology on B_X. Thus, (B_X,d) and so (B_X,weak) would be separable. Hence there would exist a countable set $\{x_n : n \in \mathbb{N}\} \subseteq B_X$ such that

$$B_X = \overline{\{x_n : n \in \mathbb{N}\}}^{\text{weak}} \subseteq \overline{\text{co}}\{x_n : n \in \mathbb{N}\} \subseteq B_X$$

(i.e., $B_X = \overline{\text{co}}\{x_n : n \in \mathbb{N}\}$). Therefore, B_X would be norm separable, which would then imply that Γ is countable. ☺

Fragmentability has been extensively studied in the setting of continuous function spaces. However, we shall briefly show here that fragmentability also has implications for spaces of uniformly continuous functions as well.

Our first result in this direction involves the notion of a space being "countably determined". Suppose that X is a completely regular topological space. Then we says that X is *countably* *determined* if there exists a countable family $\{K_n : n \in \mathbb{N})$ of compact subsets of βX such that for each $x \in X$ and $y \in \beta X \setminus X$ there exists an $n \in \mathbb{N}$ such that $x \in K_n$ and $y \notin K_n$.

For a metric space (M, d) we shall denote by, $UC(M)$ the bounded real-valued uniformly continuous functions defined on M. We shall denote by, $UC_p(M)$ the set $UC(M)$ endowed with the topology of pointwise convergence on M.

Proposition 2.3 *For any metric space (M, d), $UC_p(M)$ is countably determined.*

Proof: Let (K, ρ) be any metric compactification of $(\mathbb{R}, |\cdot|)$. To prove the proposition it will be sufficient to construct a countable family of compact subsets $\{K_n^m : (m, n) \in \mathbb{Z}^+ \times \mathbb{N}\}$ of K^M, endowed with the product topology, so that if $f, g \in K^M$, $f \in UC(M)$ and $g \notin UC(M)$ then there exists $(m', n') \in \mathbb{Z}^+ \times \mathbb{N}$ such that $f \in K_{n'}^{m'}$ and $g \notin K_{n'}^{m'}$. To this end, we shall define for each $(m, n) \in \mathbb{N}^2$,

$$K_n^m := \{f \in K^M : \rho(f(x), f(y)) \leq 1/m \text{ for all } x, y \in M \text{ with } d(x, y) < 1/n\}$$

and for each $n \in \mathbb{N}$ define $K_n^0 := \{f \in K^M : f(x) \in [-n, n] \text{ for all } x \in M\}$. Now suppose that $f, g \in K^M$, $f \in UC(M)$ and $g \notin UC(M)$. Since $f \in UC(M)$, f is bounded so there exists $n \in \mathbb{N}$ such that $f \in K_n^0$. If $g \notin K_n^0$ then we are done. So let us suppose that $g \in K_n^0$. In particular, g is real-valued (and bounded). However, since $g \notin UC(M)$, g is not uniformly continuous. Therefore, there exists an $m' \in \mathbb{N}$ and sequences $(x_n : n \in \mathbb{N})$ and $(y_n : n \in \mathbb{N})$ in M such that:

(i) $\lim_{n\to\infty} d(x_n, y_n) = 0$ and

(ii) $\rho(g(x_n), g(y_n)) > 1/m'$.

On the other hand, since $f \in UC(M)$ there exists an $n' \in \mathbb{N}$ such that $\rho(f(x), f(y)) < 1/m'$ for all $x, y \in M$ with $d(x, y) < 1/n'$. Clearly then, $f \in K_{n'}^{m'}$, but $g \notin K_{n'}^{m'}$. ☺

Corollary 2.3 *For any metric space (M, d), if $UC_p(M)$ is fragmentable then it is fragmented by a metric whose topology is at least as strong as the topology of pointwise convergence on M.*

Proof: Clearly, every countably determined space has countable separation. Therefore, the result follows directly from [18, Proposition 4.2] which says that every fragmentable space (X, τ) that has countable separation is fragmented by a metric whose topology is at least as strong as τ. ☺

3 Metric-valued function spaces

The following result is a slight generalisation of [18, Theorem 2.1]. Since its proof is essentially the same as that given in [18, Theorem 2.1] we will not repeat it here.

Proposition 3.1 [18, Theorem 1.3] *Let $(Y, \|\cdot\|)$ be a Banach space and suppose that τ is a topology on Y such that (i) for every $0 < r < \infty$ and $x \in Y$, $B[x; r] := x + rB_Y$ is closed in (Y, τ) and (ii) every bounded sequence in Y that converges with respect to the τ-topology, converges with respect to the weak topology on Y. Then for any $X \subseteq Y$, (X, τ) is fragmented by a metric whose topology is at least as strong as the τ topology if, and only if, (X, τ) is fragmented by a metric whose topology is at least as strong as the $\|\cdot\|$-topology on X.*

Let X be a set with two (not necessarily distinct) topologies τ_1 and τ_2. On X we will consider the $\mathscr{G}(X, \tau_1, \tau_2)$*-game* played between two players A and B. Player A goes first (always - life is not always fair) and chooses a nonempty subset A_1 of X. Player B must then respond by choosing a nonempty relatively τ_1-open subset B_1 of A_1. Following this, player A must select another nonempty set $A_2 \subseteq B_1 \subseteq A_1$ and in turn player B must again respond by selecting a nonempty relatively τ_1-open subset $B_2 \subseteq A_2 \subseteq B_1 \subseteq A_1$. Continuing this process indefinitely the players A and B produce a sequence $((A_n, B_n) : n \in \mathbb{N})$ of pairs of nonempty subsets (with B_n relatively τ_1-open in A_n) called a *play* of the $\mathscr{G}(X, \tau_1, \tau_2)$-game. We shall declare that player B *wins* a play $((A_n, B_n) : n \in \mathbb{N})$ if either (i) $\bigcap_{n \in \mathbb{N}} A_n = \varnothing$ or else (ii) $\bigcap_{n \in \mathbb{N}} A_n = \{x\}$ for some $x \in X$ and for every τ_2-open neighbourhood U of x there exists an $n \in \mathbb{N}$ such that $A_n \subseteq U$. Otherwise, the player A is said to have won. By a *strategy* σ for the player B we mean a "rule" that specifies each move of the player B in every possible situation that can occur. Since in general the moves of B may depend upon the previous moves of the player A we shall denote by, $\sigma(A_1, A_2, \ldots, A_n)$ the n^{th}-move of the player B under the strategy σ. We shall call a strategy σ, for the player B, a *winning strategy* if he/she wins every play of the $\mathscr{G}(X, \tau_1, \tau_2)$-game, in which they play according to the strategy σ. For a more precise definition of a strategy see [3].

Our game-theoretic approach requires the use of the following three facts, all of which are proven in [18].

Theorem 3.1 [18, Theorem 1.2] *Let τ_1, τ_2 be two (not necessarily distinct) topologies on a set X. The space (X, τ_1) is fragmentable by a metric whose topology is at least as strong as τ_2 if, and only if, the player B has a winning strategy in the $\mathscr{G}(X, \tau_1, \tau_2)$-game played on X.*

Proposition 3.2 [18, Proposition 3.1] *Let (X, τ) be a topological space that is fragmented by a metric d whose topology is at least as strong as the topology generated by some other metric ρ defined on X. Then (X, τ) is σ-fragmented by ρ.*

Proposition 3.3 [18, Proposition 3.2] *If a regular Hausdorff topological space (X, τ) is σ-fragmented by a metric ρ whose topology is at least as strong as τ then (X, τ) is fragmented by some metric ρ' whose topology is at least as strong as τ.*

Next we need to describe a simultaneous generalisation of both the pointwise topology and uniform topology on a $C(K)$-space.

Let (X, τ) be a topological space and let $\mathcal{F} \subseteq 2^X$. We shall say that $\mathcal{F}$ is a *compact cover collection* or (*ccc* for short) of X if:

(i) every member of $\mathcal{F}$ is a nonempty compact subset of X;

(ii) $\mathcal{F}$ is a cover of X, i.e., $\bigcup_{F \in \mathcal{F}} F = X$;

(iii) $\mathcal{F}$ is closed under finite unions, i.e., if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cup F_2 \in \mathcal{F}$.

Given a compact cover collection of a completely regular space X we can define a topology $\tau_{\mathcal{F}}$ on $C(X)$ by saying that a subset U of $C(X)$ is $\tau_{\mathcal{F}}$*-open* if for every $f \in U$ there exists a $F \in \mathcal{F}$ and $\varepsilon > 0$ such that $N(f, F, \varepsilon) := \{g \in C(X) : \max\{|g(x) - f(x)| : x \in F\} < \varepsilon\} \subseteq U$. It is easy to check that this does indeed define a topology on $C(X)$ and that for each $f \in C(X)$, $F \in \mathcal{F}$ and $\varepsilon > 0$, $N(f, F, \varepsilon)$ is $\tau_{\mathcal{F}}$-open.

Some special extremal cases of this topology on $C(X)$ are well-known. For example, if $\mathcal{F}$ comprises of all the finite subsets of X then $\tau_{\mathcal{F}}$ coincides with τ_p - the topology of pointwise convergence on

X. In the other extreme, if X is compact and $\mathcal{F} = \{X\}$ then $\tau_{\mathcal{F}}$ coincides with τ_u - the topology of uniform convergence on X. Note that for any ccc $\mathcal{F}$ of a compact space X, the $\tau_{\mathcal{F}}$-topology on $C(X)$ always lies somewhere between the topology of pointwise convergence on X and the topology of uniform convergence on X.

Our interest in this simultaneous generalization of both the topology of pointwise convergence and the topology of uniform convergence comes from considering product spaces. Suppose that X and Y are completely regular topological spaces and suppose also that $\mathcal{F}_1$ is a ccc of X and $\mathcal{F}_2$ is a ccc of Y. We may then define $\mathcal{F}_1 \times \mathcal{F}_2$ to be the smallest collection of nonempty compact subsets of $X \times Y$ that contains $\{F_1 \times F_2 : F_1 \in \mathcal{F}_1 \text{ and } F_2 \in \mathcal{F}_2\}$ and is closed under finite unions. Again the extremal cases are of interest. If $\mathcal{F}_1$ comprises of all the finite subsets of X and $\mathcal{F}_2$ comprises of all the finite subsets of Y then $\mathcal{F}_1 \times \mathcal{F}_2$ consists of all the finite subsets of $X \times Y$ and so $\tau_{\mathcal{F}_1 \times \mathcal{F}_2} = \tau_p$. At the other extreme (assuming X and Y are compact), if $\mathcal{F}_1 = \{X\}$ and $\mathcal{F}_2 = \{Y\}$ then $\mathcal{F}_1 \times \mathcal{F}_2 = \{X \times Y\}$ and so $\tau_{\mathcal{F}_1 \times \mathcal{F}_2} = \tau_u$. Of particular interest to us is the case when $\mathcal{F}_1$ consists of all the finite subsets of X and $\mathcal{F}_2 = \{Y\}$.

Suppose that X and Y are compact spaces. Let $f \in C(X \times Y)$ and for each $x \in X$, let $f_{(x,\cdot)} \in C(Y)$ be defined by, $f_{(x,\cdot)}(y) := f(x, y)$ for all $y \in Y$. Similarly, for each $y \in Y$, let $f_{(\cdot,y)} \in C(X)$ be defined by, $f_{(\cdot,y)}(x) := f(x, y)$ for all $x \in X$. For a subset $A \subseteq C(X \times Y)$ let us denote by, $A_{(x,\cdot)} := \{f_{(x,\cdot)} \in C(Y) : f \in A\}$ and $A_{(\cdot,y)} := \{f_{(\cdot,y)} \in C(X) : f \in A\}$ for each $(x, y) \in X \times Y$.

Further, if we suppose that $\varnothing \neq \Lambda_1 \subseteq C(X)$ and $\varnothing \neq \Lambda_2 \subseteq C(Y)$ then we may define:

$$C^{\Lambda_1 \times \Lambda_2}(X \times Y) := \{f \in C(X \times Y) : f_{(\cdot,y)} \in \Lambda_1 \text{ and } f_{(x,\cdot)} \in \Lambda_2 \text{ for each } (x, y) \in X \times Y\}.$$

Let us now apologize, in advance, for the complicated notation in the following theorem. We hope that we are eventually vindicated by the subsequent corollaries that require the extra complication. However, on first reading, it is perhaps better to just consider the case when $\Lambda_1 = C(X)$, $\Lambda_2 = C(Y)$ and $\tau_{\mathcal{F}_1} = \tau_{\mathcal{F}_2} = \tau_p$.

A special case of the following theorem was proven in [24] and also independently by N. K. Ribarska. A proof of this special case was eventually published in [29]. However, we would like to acknowledge here, that the proof in [24] was completely inspired by the corresponding result for co-Namioka spaces given in [2].

Theorem 3.2 *Let X and Y be compact Hausdorff spaces and suppose that*

$$\varnothing \neq \Lambda_1 \subseteq C(X) \text{ and } \varnothing \neq \Lambda_2 \subseteq C(Y).$$

Suppose also that $\mathcal{F}_1$ is a ccc of X and $\mathcal{F}_2$ is a ccc of Y. If both $(\Lambda_1, \tau_{\mathcal{F}_1})$ and $(\Lambda_2, \tau_{\mathcal{F}_2})$ are fragmented by metrics whose topologies are at least as strong as the norm topologies on $C(X)$ and $C(Y)$ respectively then $(C^{\Lambda_1 \times \Lambda_2}(X \times Y), \tau_{\mathcal{F}_1 \times \mathcal{F}_2})$ is fragmented by a metric whose topology is at least as strong as the norm topology on $C(X \times Y)$.

Proof: Let d_1 be a fragmenting metric on $(\Lambda_1, \tau_{\mathcal{F}_1})$ whose topology is at least as strong as the $\|\cdot\|_\infty$-topology on Λ_1 and let d_2 be a fragmenting metric on $(\Lambda_2, \tau_{\mathcal{F}_2})$ whose topology is at least as strong as the $\|\cdot\|_\infty$-topology on Λ_2. We will construct a winning strategy σ for the player B in the $\mathscr{G}(C^{\Lambda_1 \times \Lambda_2}(X \times Y), \tau_{\mathcal{F}_1 \times \mathcal{F}_2}, \tau_u)$-game played on $C^{\Lambda_1 \times \Lambda_2}(X \times Y)$.

Suppose that player A chooses a nonempty subset $A_1 \subseteq C^{\Lambda_1 \times \Lambda_2}(X \times Y)$ as their first move of the game. Note that by, [18, Proposition 2.1] we may assume that A_1 is bounded. Player B's response to this move is to first arbitrarily choose points $x_0 \in X$ and $y_0 \in Y$ and then define

$$\alpha_1 := \sup\{\|f_{(x,\cdot)} - f_{(x_0,\cdot)}\|_\infty : f \in A_1,\ x \in X\}.$$

He/she then chooses $x_1 \in X$ and $f \in A_1$ so that

$$\|f_{(x_1,\cdot)} - f_{(x_0,\cdot)}\|_\infty > \alpha_1 - 1/3.$$

Player B then selects a nonempty relatively $\tau_{\mathcal{F}_1 \times \mathcal{F}_2}$-open subset B_1' of A_1 so that

(i) $\inf\{\|f_{(x_1,\cdot)} - f_{(x_0,\cdot)}\|_\infty : f \in B_1'\} > \alpha_1 - 1/2$ and

(ii) $d_2 - \operatorname{diam}(B_1')_{(x_j,\cdot)} < 1/2$ for each $0 \leq j \leq 1$.

Next, he/she defines

$$\beta_1 := \sup\{\|f_{(\cdot,y)} - f_{(\cdot,y_0)}\|_\infty : f \in B_1',\ y \in Y\}.$$

Similarly, to above, player B finds a point $y_1 \in Y$ and a nonempty relatively $\tau_{\mathcal{F}_1 \times \mathcal{F}_2}$-open subset B_1 of B_1' so that:

(i) $\inf\{\|f_{(\cdot,y_1)} - f_{(\cdot,y_0)}\|_\infty : f \in B_1\} > \beta_1 - 1/2$ and

(ii) $d_1 - \operatorname{diam}(B_1)_{(\cdot,y_j)} < 1/2$ for each $0 \leq j \leq 1$.

Finally, player B defines $\sigma(A_1) := B_1$.

In general, suppose that the players A and B have chosen nonempty sets

$$B_n \subseteq A_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1 \subseteq A_1.$$

so that $\{(A_j, B_j) : 1 \leq j \leq n\}$ is a partial play of the $\mathscr{G}(C^{\Lambda_1\times\Lambda_2}(X \times Y), \tau_{\mathcal{F}_1 \times \mathcal{F}_2}, \tau_u)$-game.

In the course of the game, the player B will have also defined:

(i) some real numbers $0 \leq \alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_1$ and $0 \leq \beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_1$;

(ii) some points $(x_j, y_j) \in X \times Y$ for $0 \leq j \leq n$ and

(iii) some nonempty relatively $\tau_{\mathcal{F}_1 \times \mathcal{F}_2}$-open subsets B_j' of A_j for $1 \leq j \leq n$

such that for each $1 \leq k \leq n$:

(a) $\alpha_k := \sup\{\min_{0 \leq j < k} \|f_{(x,\cdot)} - f_{(x_j,\cdot)}\|_\infty : f \in A_k,\ x \in X\}$;

(b) B_k' is a nonempty relatively $\tau_{\mathcal{F}_1 \times \mathcal{F}_2}$-open subset of A_k chosen so that;

(c) $\inf\{\min_{0 \leq j < k} \|f_{(x_k,\cdot)} - f_{(x_j,\cdot)}\|_\infty : f \in B_k'\} > \alpha_k - 1/k$ and

(d) $d_2 - \operatorname{diam}(B_k')_{(x_j,\cdot)} < 1/k$ for each $1 \leq j \leq k$;

(e) $\beta_k := \sup\{\min_{0 \leq j < k} \|f_{(\cdot,y)} - f_{(\cdot,y_j)}\|_\infty : f \in B_k',\ y \in Y\}$;

(f) B_k is a nonempty relatively $\tau_{\mathcal{F}_1 \times \mathcal{F}_2}$-open subset of B_k' chosen so that;

(g) $\inf\{\min_{0 \leq j < k} \|f_{(\cdot,y_k)} - f_{(\cdot,y_j)}\|_\infty : f \in B_k\} > \beta_k - 1/k$ and

(h) $d_1 - \operatorname{diam}(B_k)_{(\cdot,y_j)} < 1/k$ for each $0 \leq j \leq k$;

(i) $\sigma(A_1, A_2, \ldots, A_k) := B_k$.

Inductive step. Suppose that player A has chosen a nonempty set $A_{n+1} \subseteq B_n$. Player B responds to this by defining

$$\alpha_{n+1} := \sup\left\{\min_{0\leq j\leq n} \|f_{(x,\cdot)} - f_{(x_j,\cdot)}\|_\infty : f \in A_{n+1},\ x \in X\right\}.$$

He/she then chooses $x_{n+1} \in X$ and $f \in A_{n+1}$ so that:

$$\min_{0\leq j\leq n} \|f_{(x_{n+1},\cdot)} - f_{(x_j,\cdot)}\|_\infty > \alpha_{n+1} - 1/(2n+1).$$

Player B then selects a nonempty relatively $\tau_{\mathcal{F}_1\times\mathcal{F}_2}$-open subset B'_{n+1} of A_{n+1} so that:

(i) $\inf\{\min_{1\leq j\leq n} \|f_{(x_{n+1},\cdot)} - f_{(x_j,\cdot)}\|_\infty : f \in B'_{n+1}\} > \alpha_{n+1} - 1/(n+1)$ and

(ii) $d_2 - \operatorname{diam}(B'_{n+1})_{(x_j,\cdot)} < 1/(n+1)$ for each $0 \leq j \leq (n+1)$.

Next, he/she defines

$$\beta_{n+1} := \sup\left\{\min_{0\leq j\leq n} \|f_{(\cdot,y)} - f_{(\cdot,y_j)}\|_\infty : f \in B'_{n+1},\ y \in Y\right\}.$$

Similarly, to above, player B finds a point $y_{n+1} \in Y$ and a nonempty relatively $\tau_{\mathcal{F}_1\times\mathcal{F}_2}$-open subset B_{n+1} of B'_{n+1} so that:

(i) $\inf\{\min_{1\leq j\leq n} \|f_{(\cdot,y_{n+1})} - f_{(\cdot,y_j)}\|_\infty : f \in B_{n+1}\} > \beta_{n+1} - 1/(n+1)$ and

(ii) $d_1 - \operatorname{diam}(B_{n+1})_{(\cdot,y_j)} < 1/(n+1)$ for each $0 \leq j \leq (n+1)$.

Finally, player B defines $\sigma(A_1, A_2, \ldots, A_{n+1}) := B_{n+1}$. This completes the definition of σ.

We claim that $\lim_{n\to\infty} \|\cdot\|_\infty - \operatorname{diam} A_n = 0$ whenever $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. However, to achieve this we must first show that

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$$

whenever $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. Indeed, let us suppose that there exists an $0 < r$ so that $r < \alpha_n$ for all $n \in \mathbb{N}$. (Recall that $\alpha_{n+1} \leq \alpha_n$ for all $n \in \mathbb{N}$). Let x_∞ be any cluster point of $(x_n : n \in \mathbb{N})$ and let $f \in \bigcap_{n\in\mathbb{N}} A_n$. Now, by the continuity of f there exists a neighbourhood U of x_∞ so that $\|f_{(x,\cdot)} - f_{(x_\infty,\cdot)}\|_\infty < r/4$ whenever $x \in U$. On the other hand, there exist $m < n \in \mathbb{N}$ with $2/r < m < n$ so that $x_m, x_n \in U$. However, this is impossible since,

$$\begin{aligned}\frac{r}{2} = \left(r - \frac{r}{2}\right) < \left(r - \frac{1}{n}\right) < \left(\alpha_n - \frac{1}{n}\right) &< \|f_{(x_n,\cdot)} - f_{(x_m,\cdot)}\|_\infty \\ &\leq \|f_{(x_n,\cdot)} - f_{(x_\infty,\cdot)}\|_\infty + \|f_{(x_\infty,\cdot)} - f_{(x_m,\cdot)}\|_\infty \\ &< \frac{r}{4} + \frac{r}{4} = \frac{r}{2}.\end{aligned}$$

Hence $\lim_{n\to\infty} \alpha_n = 0$. The proof that $\lim_{n\to\infty} \beta_n = 0$ is analogous. Now, suppose that $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and $\varepsilon > 0$ is given. Then we may choose $n_\varepsilon \in \mathbb{N}$ so that $0 \leq \alpha_{n_\varepsilon} < \varepsilon$ and $0 \leq \beta_{n_\varepsilon} < \varepsilon$. On the other hand we may also choose $n_\varepsilon < m_\varepsilon \in \mathbb{N}$ so that

$$\|\cdot\|_\infty - \operatorname{diam}(A_{m_\varepsilon})_{(x_j,\cdot)} < \varepsilon \text{ and } \|\cdot\|_\infty - \operatorname{diam}(A_{m_\varepsilon})_{(\cdot,y_j)} < \varepsilon \quad \text{for each } 0 \leq j \leq n_\varepsilon.$$

[Note that this is possible since $\bigcap_{n\in\mathbb{N}}(A_n)_{(x_j,\cdot)} \neq \varnothing$ and $\bigcap_{n\in\mathbb{N}}(A_n)_{(\cdot,y_j)} \neq \varnothing$ for each $0 \leq j \leq n_\varepsilon$.]

We claim that $\|\cdot\|_\infty - \operatorname{diam} A_{m_\varepsilon} \leq 7\varepsilon$. Indeed, consider $(x,y) \in X \times Y$ and $f,g \in A_{m_\varepsilon}$. Then there exist $i,j \in \{0,1,2,\ldots,n_\varepsilon - 1\}$ so that $\|f_{(\cdot,y)} - f_{(\cdot,y_j)}\|_\infty \leq \beta_{n_\varepsilon} < \varepsilon$ and $\|g_{(x,\cdot)} - g_{(x_i,\cdot)}\|_\infty \leq \alpha_{n_\varepsilon} < \varepsilon$ since $f,g \in A_{m_\varepsilon} \subseteq B'_{n_\varepsilon} \subseteq A_{n_\varepsilon}$. Hence,

$$\begin{aligned} |f(x,y) - g(x,y)| &\leq |f(x,y) - f(x,y_j)| + |f(x,y_j) - g(x,y_j)| + |g(x,y_j) - g(x_i,y_j)| \\ &+ |g(x_i,y_j) - f(x_i,y_j)| + |f(x_i,y_j) - f(x_i,y)| + |f(x_i,y) - g(x_i,y)| \\ &+ |g(x_i,y) - g(x,y)| < 7\varepsilon. \end{aligned}$$

This shows that $\|\cdot\|_\infty - \operatorname{diam} A_k \leq \|\cdot\|_\infty - \operatorname{diam} A_{m_\varepsilon} \leq 7\varepsilon$ for all $k \geq m_\varepsilon$. Hence σ is indeed a winning strategy for the player B in the $\mathscr{G}(C^{\Lambda_1\times\Lambda_2}(X\times Y), \tau_{\mathcal{F}1\times\mathcal{F}_2}, \tau_u)$-game played on $C^{\Lambda_1\times\Lambda_2}(X\times Y)$.

The result now follows from Theorem 3.1. ☺

Before we can give some of the applications of this result we need to recall two well-known results from functional analysis.

Proposition 3.4 *For every Banach space $(Y, \|\cdot\|)$ there exists a compact Hausdorff space X and an isometry $T : (Y, \|\cdot\|) \to (C(X), \|\cdot\|_\infty)$ such that $T : (Y, \text{weak}) \to C_p(X)$ is a topological embedding.*

Proof: Consider $X := B_{Y^*}$ endowed with the weak* topology. By the Banach-Alaoglu theorem (B_{Y^*}, weak^*) is a compact Hausdorff space. For each $x \in Y$ define $\widehat{x} : B_{Y^*} \to \mathbb{R}$ by, $\widehat{x}(x^*) := x^*(x)$ for all $x^* \in B_{Y^*}$. Next, we define $T : (Y, \|\cdot\|) \to (C(X), \|\cdot\|_\infty)$ by, $T(x) := \widehat{x}$. Clearly, T is linear and from the Hahn-Banach theorem we see that $\|T(x)\|_\infty = \|\widehat{x}\|_\infty = \|x\|$ for all $x \in Y$. It is also easy to see that $T : (Y, \text{weak}) \to C_p(X)$ is a topological embedding. ☺

Proposition 3.5 *For every metric space (M,d) there exists a compact Hausdorff space X and an isometry $T : (M,d) \to (C(X), \|\cdot\|_\infty)$.*

Proof: In light of Proposition 3.4 it is sufficient to show that there is an isometry S from (M,d) into some Banach space $(Y, \|\cdot\|)$. So this is what we do next. Let $\mathscr{L}(M)$ be the set of all real-valued functions on M that vanish at some fixed point $x_0 \in M$ and satisfy $|f(x) - f(y)| \leq Kd(x,y)$ for all $x,y \in M$, for some K (depending on f). Then $(\mathscr{L}(M), \|\cdot\|)$ is a Banach space if we define

$$\|f\| := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

For each $x \in M$ consider the continuous linear functional δ_x defined on $\mathscr{L}(M)$ by, $\delta_x(f) := f(x)$. Then the mapping $S : (M,d) \to (\mathscr{L}(M)^*, \|\cdot\|)$ defined by, $S(x) := \delta_x$ is an isometry. ☺

We may now present our first application of Theorem 3.2. The corresponding property for the Namioka property was established in [28, Theorem A2], but using a completely different argument.

Corollary 3.1 *Suppose that X is a compact Hausdorff space and (M,d) is a metric space. If $C_p(X)$ is σ-fragmented by its norm then $C_p(X;M)$ is fragmented by a metric whose topology is at least as strong as the D-topology on $C(X;M)$, where $D : C(X;M) \times C(X;M) \to [0,\infty)$ is defined by, $D(F,G) := \max_{x\in X} d(F(x), G(x))$.*

Proof: By Proposition 3.3 and Proposition 3.1, $C_p(X)$ is fragmented by a metric whose topology is at least as strong as the τ_u-topology on $C(X)$. Now, by Proposition 3.5 there exists a compact Hausdorff space Y and an isometry $T : (M,d) \to (C(Y), \|\cdot\|_\infty)$. Clearly, $(C(Y), \tau_u)$ is fragmented by a metric whose topology is at least as strong as the τ_u-topology. For example, just take $\rho : C(Y) \times C(Y) \to [0, \infty)$ to be, $\rho(f, g) := \|f - g\|_\infty$. So by Theorem 3.2, $(C(X \times Y), \tau_{\mathcal{F}_1 \times \mathcal{F}_2})$ is fragmented by a metric whose topology is at least as strong as the $\|\cdot\|_\infty$-topology on $C(X \times Y)$; where $\mathcal{F}_1$ consists of all the finite subsets of X and $\mathcal{F}_2 := \{Y\}$.

Next, consider the mapping $S : C(X; M) \to C(X \times Y)$ defined by,

$$S(F)(x, y) := T(F(x))(y) \text{ for all } (x, y) \in X \times Y.$$

One can check that:

(i) S is well-defined, i.e., for every $F \in C(X; M)$, $S(F) \in C(X \times Y)$;

(ii) for every $F, G \in C(X; M)$, $D(F, G) := \max_{x \in X} d(F(x), G(x)) = \|S(F) - S(G)\|_\infty$;

(iii) S is a topological embedding of $C_p(X; M)$ into $(C(X \times Y), \tau_{\mathcal{F}_1 \times \mathcal{F}_2})$.

The result now follows. ☺

Corollary 3.2 *Let X be a compact Hausdorff space and (M, d) be a metric space. If $C_p(X; M)$ is fragmented by a metric whose topology is at least as strong as the τ_p-topology on $C(X; M)$ then $C_p(X; M)$ is fragmented by a metric whose topology is at least as strong as the D-topology on $C(X; M)$, where $D : C(X; M) \times C(X; M) \to [0, \infty)$ is defined by, $D(F, G) := \max_{x \in X} d(F(x), G(x))$.*

Proof: By Proposition 3.5 there exists a compact Hausdorff space Y and an isometry $T : (M, d) \to (C(Y), \|\cdot\|_\infty)$. As in Corollary 3.1 we consider the mapping $S : C(X; M) \to C(X \times Y)$ defined by,

$$S(F)(x, y) := T(F(x))(y) \text{ for all } (x, y) \in X \times Y$$

and as in Corollary 3.1 we note that:

(i) S is well-defined, i.e., for every $F \in C(X; M)$, $S(F) \in C(X \times Y)$;

(ii) for every $F, G \in C(X; M)$, $D(F, G) := \max_{x \in X} d(F(x), G(x)) = \|S(F) - S(G)\|_\infty$;

(iii) S is a topological embedding of $C_p(X; M)$ into $(C(X \times Y), \tau_{\mathcal{F}_1 \times \mathcal{F}_2})$.

Thus, $(S(C(X; M)), \tau_{\mathcal{F}_1 \times \mathcal{F}_2})$ is fragmented by a metric whose topology is at least as strong as the $\tau_{\mathcal{F}_1 \times \mathcal{F}_2}$-topology on $C(X \times Y)$. Therefore, by Proposition 3.1, $(S(C(X; M)), \tau_{\mathcal{F}_1 \times \mathcal{F}_2})$ is fragmented by a metric whose topology is at least as strong as the $\|\cdot\|_\infty$-topology on $C(X \times Y)$. The result then follows. ☺

In order to be able to state our next corollary we need to introduce some more notation.

Given a Banach space $(Y, \|\cdot\|)$ and a compact Hausdorff space X, we denote by $(C(X; Y), \tau_p(\text{weak}))$ the set $C(X; Y)$ endowed with the topology of pointwise convergence on X, when Y is considered with the weak topology. That is, a net $(F_\alpha : \alpha \in A)$ in $C(X; Y)$ converges to $F \in C(X; Y)$ with respect to the τ_p(weak)-topology if for each $x \in X$, $\lim_{\alpha \in A} F_\alpha(x)$ converges weakly to $F(x)$.

Corollary 3.3 *Suppose that X is a compact Hausdorff space and $(Y, \|\cdot\|)$ is a Banach space. If $C_p(X)$ is σ-fragmented by the $\|\cdot\|_\infty$-norm and $(Y, weak)$ is σ-fragmented by its norm then $(C(X; Y), \tau_p(weak))$ is σ-fragmented by the $\|\cdot\|_\infty$-norm on $C(X; Y)$. In particular, $(C(X; Y), weak)$ is σ-fragmented by the $\|\cdot\|_\infty$-norm on $C(X; Y)$.*

Proof: By Proposition 3.4 there exists a compact Hausdorff space Z and an isometry $T : (Y, \|\cdot\|) \to (C(Z), \|\cdot\|_\infty)$ such that $T : (Y, \text{weak}) \to C_p(Z)$ is a topological embedding. Let $\Lambda_1 := C(X)$ and $\Lambda_2 := T(Y)$. Then by Proposition 3.3 and Proposition 3.1 (Λ_1, τ_p) is fragmented by a metric whose topology is at least as strong as the $\|\cdot\|_\infty$-topology on $C(X)$ and (Λ_2, τ_p) is also fragmented by a metric whose topology is at least as strong as the $\|\cdot\|_\infty$-topology on $C(Z)$. Hence by Theorem 3.2, $(C^{\Lambda_1 \times \Lambda_2}(X \times Z), \tau_p)$ is fragmented by a metric whose topology is at least as strong as the $\|\cdot\|_\infty$-topology on $C(X \times Z)$. Now consider the mapping $S : C(X; Y) \to C(X \times Z)$ defined by,

$$S(F)(x, z) := T(F(x))(z) \text{ for all } (x, z) \in X \times Z.$$

Then

(i) $S(F) \in C^{\Lambda_1 \times \Lambda_2}(X \times Z)$ for each $F \in C(X; Y)$;

(ii) for every $F, G \in C(X; Y)$, $\|F - G\|_\infty = \|S(F) - S(G)\|_\infty$;

(iii) S is a topological embedding of $(C(X; Y), \tau_p(\text{weak}))$ into $C_p(X \times Z)$.

It now follows that $(C(X; Y), \tau_p(\text{weak}))$ is fragmented by a metric whose topology is at least as strong as the $\|\cdot\|_\infty$-topology in $C(X; Y)$. The fact that $(C(X; Y), \tau_p(\text{weak}))$ is σ-fragmented by the $\|\cdot\|_\infty$-norm on $C(X; Y)$ now follows from Proposition 3.2. ☺

Remarks 1 *If we are given a compact Hausdorff space X and a non-zero Banach space $(Y, \|\cdot\|)$ then we may consider the following two mappings. $R : (Y, \|\cdot\|) \to (C(X; Y), \|\cdot\|_\infty)$ defined by, $R(y)(x) := y$ for all $x \in X$ and $Q : (C(X), \|\cdot\|_\infty) \to (C(X; Y), \|\cdot\|_\infty)$ defined by $Q(f)(x) := f(x)y_0$ for all $x \in X$, where $y_0 \in Y$ is some fixed element of Y with $\|y_0\| = 1$. In this way we can see that if $(C(X; Y), \tau_p(\text{weak}))$ is σ-fragmented by the $\|\cdot\|_\infty$ - norm on $C(X; Y)$ then $C_p(X)$ is σ-fragmented by the $\|\cdot\|_\infty$ - norm on $C(X)$ and (Y, weak) is σ-fragmented by the norm on Y.*

Our final application is an extension of Theorem 3.2 from finite products of compact Hausdorff spaces to arbitrary products of compact Hausdorff spaces. As in Theorem 3.2, the proof is modeled off the corresponding result for co-Namioka spaces given in [1].

Theorem 3.3 *Let $\{T_i : i \in I\}$ be an infinite family of nonempty compact Hausdorff spaces. If each $(C(T_i), \tau_p)$ is σ-fragmentable by the $\|\cdot\|_\infty$-norm then so is $(C(\prod_{i \in I} T_i), \tau_p)$.*

Proof: In order to expedite the latter part of this proof we shall take this opportunity to introduce a slew of definitions and notation. Firstly, let $T := \prod_{i \in I} T_i$ and let t be any fixed element of T. For each $\varnothing \neq J \subseteq I$ we define:

(i) $T_J := \prod_{j \in J} T_j$;

(ii) $x_J \in T$, by $x_J(i) := \begin{cases} x(i) & \text{if } i \in J \\ t(i) & \text{if } i \notin J \end{cases}$ for each $x \in T$;

(iii) $\sigma_J : T_J \to T$ by, $\sigma_J(x)(i) := \begin{cases} x(i) & \text{if } i \in J \\ t(i) & \text{if } i \notin J. \end{cases}$

(iv) $S_J : C_p(T) \to C(T_J)$ by, $S_J(f) := f \circ \sigma_J$.

Clearly, S_J is continuous with respect to the τ_p-topology on both $C(T)$ and $C(T_J)$. Furthermore, for each finite subset $\varnothing \neq J \subseteq I$ we let d_J be a fragmenting metric on $C_p(T_J)$ whose topology is at least as strong as the $\|\cdot\|_\infty$-topology on $C(T_J)$. Of course such a fragmenting metric is guaranteed by Theorem 3.2.

We will construct a winning strategy σ for the player B in the $\mathscr{G}(T, \tau_p, \tau_u)$-game played on $C(T)$.

Suppose that the player A chooses a nonempty subset A_1 of $C(T)$ as their first move of the game. Note that by, [18, Proposition 2.1] we may assume that A_1 is bounded. Player B's response to this move is to first define

$$J_0 := \varnothing \quad \text{and} \quad s_1 := \sup\{|f(x) - f(y)| : f \in A_1,\ x, y \in T\}$$

and then choose $(x_1, y_1) \in T \times T$ and $f \in A_1$ so that:

(i) $|f(x_1) - f(y_1)| > s_1 - 1/3$ and

(ii) $J_1 := \{i \in I : x_1(i) \neq y_1(i)\}$ is finite.

Next, player B selects a nonempty relatively τ_p-open subset B_1 of A_1 so that:

(i) $\inf\{|f(x_1) - f(y_1)| : f \in B_1\} > s_1 - 1$ and

(ii) $d_{J_1} - \operatorname{diam} S_{J_1}(B_1) < 1$.

Finally, player B defines $\sigma(A_1) := B_1$.

In general, suppose that the players A and B have chosen nonempty sets

$$B_n \subseteq A_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1 \subseteq A_1$$

so that $\{(A_j, B_j) : 1 \leq j \leq n\}$ is a partial play of the $\mathscr{G}(C(T), \tau_p, \tau_u)$-game. In the course of the game, the player B will have also defined:

(i) some real numbers $0 \leq s_n \leq s_{n-1} \leq \cdots \leq s_1$;

(ii) some points $(x_j, y_j) \in T \times T$ for $1 \leq j \leq n$;

(iii) some finite sets $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq I$

such that for each $1 \leq k \leq n$:

(a) $s_k := \sup\{|f(x) - f(y)| : f \in A_k,\ x, y \in T \text{ and } x(j) = y(j) \text{ for all } j \in J_{k-1}\}$;

(b) $(x_k, y_k) \in T \times T$ is chosen so that $x_k(j) = y_k(j)$ for all $j \in J_{k-1}$ and

(c) $J_k := \{i \in I : x_k(i) \neq y_k(i)\} \cup J_{k-1}$ is finite;

(d) B_k is a nonempty relatively τ_p-open subset of A_k chosen so that;

(e) $\inf\{|f(x_k) - f(y_k)| : f \in B_k\} > s_k - 1/k$ and

(f) $d_{J_j} - \operatorname{diam} S_{J_j}(B_k) < 1/k$ for all $1 \leq j \leq k$;

(g) $\sigma(A_1, A_2, \ldots, A_k) := B_k$.

Inductive step. Suppose that player A has chosen a nonempty set $A_{n+1} \subseteq B_n$. Player B responds to this by defining

$$s_{n+1} := \sup\{|f(x) - f(y)| : f \in A_{n+1},\ x, y \in T \text{ and } x(j) = y(j) \text{ for all } j \in J_n\}.$$

He/she then chooses $(x_{n+1}, y_{n+1}) \in T \times T$ and $f \in A_{n+1}$ so that:

(i) $x_{n+1}(j) = y_{n+1}(j)$ for all $j \in J_n$, $|f(x_{n+1}) - f(y_{n+1})| > s_{n+1} - 1/(2n+1)$ and

(ii) $J_{n+1} := \{i \in I : x_{n+1}(i) \neq y_{n+1}(i)\} \cup J_n$ is finite.

Next, player B selects a nonempty relatively τ_p-open subset B_{n+1} of A_{n+1} so that:

(i) $\inf\{|f(x_{n+1}) - f(y_{n+1})| : f \in B_{n+1}\} > s_{n+1} - 1/(n+1)$ and

(ii) $d_{J_j} - \operatorname{diam} S_{J_j}(B_{n+1}) < 1/(n+1)$ for each $1 \leq j \leq n+1$.

Finally, player B defines $\sigma(A_1, A_2, \ldots, A_{n+1}) := B_{n+1}$.

We claim that $\lim_{n\to\infty} \|\cdot\|_\infty - \operatorname{diam} A_n = 0$ whenever $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. However first we show that $\lim_{n\to\infty} s_n = 0$ whenever, $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. To this end, let us suppose that there exists an $0 < r$ so that $r < s_n$ for all $n \in \mathbb{N}$ and let $J := \bigcup_{n\in\mathbb{N}} J_n \subseteq I$. Let (x_∞, y_∞) be any cluster point of $((x_n, y_n) : n \in \mathbb{N})$. Now for any $i \in I \setminus J$, $x_n(i) = y_n(i)$ for all $n \in \mathbb{N}$ and so $x_\infty(i) = y_\infty(i)$. Furthermore, for each $j \in J$ there exists an $n_0 \in \mathbb{N}$ so that $j \in J_{n_0}$ and so $x_n(j) = y_n(j)$ for all $n > n_0$. Therefore, $x_\infty(j) = y_\infty(j)$ and hence $x_\infty = y_\infty$. Select any $f \in \bigcap_{n\in\mathbb{N}} A_n$, then

$$|f(x_n) - f(y_n)| > s_n - 1/n > r - 1/n \quad \text{for all } n \in \mathbb{N};$$

which contradicts the continuity of the function, $(x, y) \mapsto |f(x) - f(y)|$ at $(x_\infty, y_\infty) = (x_\infty, x_\infty)$. Hence, $\lim_{n\to\infty} s_n = 0$. Now suppose that $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$ and $\varepsilon > 0$ is given. Then we may choose $n_\varepsilon \in \mathbb{N}$ so that $s_{n_\varepsilon} < \varepsilon$. On the other hand we may select $n_\varepsilon < m_\varepsilon \in \mathbb{N}$ so that $\|\cdot\|_\infty - \operatorname{diam} S_{J_{n_\varepsilon}}(A_{m_\varepsilon}) < \varepsilon$. [Note that this is possible since $\bigcap_{n\in\mathbb{N}} S_{J_{n_\varepsilon}}(A_n) \neq \varnothing$].

We claim that $\|\cdot\|_\infty - \operatorname{diam} A_{m_\varepsilon} < 3\varepsilon$. To see this, consider any $x \in T$ and $f, g \in A_{m_\varepsilon}$. Then,

$$\begin{aligned} |f(x) - g(x)| &\leq |f(x) - f(x_{J_{n_\varepsilon}})| + |(f - g)(x_{J_{n_\varepsilon}})| + |g(x_{J_{n_\varepsilon}}) - g(x)| \\ &\leq s_{n_\varepsilon} + \|S_{J_{n_\varepsilon}}(f - g)\|_\infty + s_{n_\varepsilon} < 3\varepsilon. \end{aligned}$$

This shows that $\|\cdot\|_\infty - \operatorname{diam} A_k \leq \|\cdot\|_\infty - \operatorname{diam} A_{m_\varepsilon} < 3\varepsilon$ for all $k \geq m_\varepsilon$. Hence σ is indeed a winning strategy for the player B in the $\mathscr{G}(C(T), \tau_p, \tau_u)$-game played on $C(T)$. The result now follows from Theorem 3.1 and Proposition 3.2. ☺

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