A NOTE ON FORT'S THEOREM

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Abstract. Fort's theorem states that if $F : X \to 2^Y$ is an upper (lower) semicontinuous set-valued mapping from a Baire space (X, τ) into the nonempty compact subsets of a metric space (Y, d) then F is both upper and lower semicontinuous at the points of a dense G_δ subset of X. In this paper we show that a variant of Fort's theorem holds, without the assumption of the compactness of the images, provided we restrict the domain space of the mapping to a large class of "nice" Baire spaces.

AMS (2010) subject classification: Primary 54C60, 58C06.

Key words: Upper semicontinuous, lower semicontinuous, set-valued mappings.

Suppose that $F : X \to 2^Y$ is a set-valued mapping acting from a topological space (X, τ) into subsets of a topological space (Y, τ'). We shall say that F is *upper semicontinuous* at a point $x_0 \in X$ if for each $W \in \tau'$ containing $F(x_0)$ there exists a neighbourhood U of x_0 such that $F(U) := \bigcup_{x \in U} F(x) \subseteq W$. Similarly, we shall that F is *lower semicontinuous* at a point x_0 if for each $W \in \tau'$ such that $F(x_0) \cap W \neq \varnothing$ there exists a neighbourhood U of x_0 such that $F(x) \cap W \neq \varnothing$ for all $x \in U$. If F is both upper and lower semicontinuous at a point x_0 then we simply say that F is *continuous* at x_0. Further, if F is upper semicontinuous (lower semicontinuous) [continuous] at each point of X then we say that F is *upper semicontinuous on* X (*lower semicontinuous on* X) [*continuous on* X].

Throughout this paper we shall denote by $C(F) := \{x \in X : F \text{ is continuous at } x\}$ and we shall be particularly interested in set-valued mappings that are defined on Baire spaces. So let us recall that a topological space (X, τ) is called a *Baire space* if for every sequence $\{O_n : n \in \mathbb{N}\}$ of dense open subsets, $\bigcap_{n \in \mathbb{N}} O_n$ is dense in X. Since the family of all dense open subsets on a topological space X is closed under finite intersections it follows that a topological space (X, τ) is a Baire space if $\bigcap_{n \in \mathbb{N}} O_n$ is dense in X for each decreasing sequence (i.e., $O_{n+1} \subseteq O_n$ for all $n \in \mathbb{N}$) of dense open subsets $\{O_n : n \in \mathbb{N}\}$ of X with $O_1 = X$.

A well-known and useful theorem of Fort [2] states that if $F : X \to 2^Y$ is an upper (lower) semi continuous set-valued mapping from a Baire space (X, τ) into the nonempty compact subsets of a metric space (Y, d) [denoted $\mathcal{K}(Y, d)$] then $C(F)$ is dense in X.

Remarks 1 *Let (Y, d) be a metric space. For any nonempty subset A of Y we can always define $d_A : Y \to [0, \infty)$ by, $d_A(x) := \inf\{d(x, a) : a \in A\}$. Then we can define a metric D on $\mathcal{K}(Y, d)$ by, $D(A, B) := \max\{\sup_{x \in A} d_B(x), \sup_{x \in B} d_A(x)\}$. Now, for any mapping $F : (X, \tau) \to \mathcal{K}(Y, d)$,* *emanating from a topological space* (X, τ) *we have that* $C(F) = \bigcap_{n \in \mathbb{N}} O_{1/n}$ *where, for each* $\varepsilon > 0$ $O_\varepsilon := \bigcup\{U \in \tau : D - diam\{F(t) : t \in U\} < \varepsilon\}$. *Thus, we see that for any set-valued mapping* $F : (X, \tau) \to \mathcal{K}(Y, d)$, $C(F)$ *is always a* G_δ *subset of* X.

It is known that Fort's theorem provides a characterisation for the class of Baire spaces in the following sense.

Theorem 1 (Theorem 3.1 in [9]) *Let* (X, τ) *be a topological space and* (Y, d) *be a metric space with at least one non-isolated point. Then* (X, τ) *is a Baire space if, and only if, for every upper semicontinuous mapping* $F : X \to \mathcal{K}(Y, d)$, $C(F)$ *is dense in* (X, τ).

Proof: If (X, τ) is a Baire space, then $C(F)$ is dense in (X, τ) by Fort's theorem. So let us consider the converse. Let $\{O_n : n \in \mathbb{N}\}$ be a decreasing sequence of dense open of (X, τ) with $O_1 = X$. We need to show that $\bigcap_{n \in \mathbb{N}} O_n$ is dense in (X, τ). Let $y_\infty \in Y$ be a non-isolated point of Y and let $(y_n : n \in \mathbb{N})$ be a sequence of distinct points of $Y \setminus \{y_\infty\}$ that converge to y_∞. For each $n \in \mathbb{N}$, let $Y_n := \{y_k : k \geq n\} \cup \{y_\infty\}$.

Define $F : X \to \mathcal{K}(Y)$ by, $\quad F(x) := \begin{cases} \{y_\infty\} & \text{if } x \in \bigcap_{n \in \mathbb{N}} O_n \\ Y_n & \text{if } x \in O_n \setminus O_{n+1}. \end{cases}$

Then F is upper semicontinuous on X and $C(F) = \bigcap_{n \in \mathbb{N}} O_n$. Therefore, if $C(F)$ is dense in X then $\bigcap_{n \in \mathbb{N}} O_n$ is dense in X; which implies that X is a Baire space.

Theorem 2 (Theorem 3.2 in [9]) *Let* (X, τ) *be a topological space and* (Y, d) *be a metric space with at least one non-isolated point. Then* (X, τ) *is a Baire space if, and only if, for every lower semicontinuous mapping* $F : X \to \mathcal{K}(Y, d)$, $C(F)$ *is dense in* (X, τ).

Proof: If (X, τ) is a Baire space, then $C(F)$ is dense in (X, τ) by Fort's theorem. So let us consider the converse. Let $\{O_n : n \in \mathbb{N}\}$ be a decreasing sequence of dense open of (X, τ) with $O_1 = X$. We need to show that $\bigcap_{n \in \mathbb{N}} O_n$ is dense in (X, τ). Let $y_\infty \in Y$ be a non-isolated point of Y and let $(y_n : n \in \mathbb{N})$ be a sequence of distinct points of $Y \setminus \{y_\infty\}$ that converge to y_∞. For each $n \in \mathbb{N}$, let $Y_n := \{y_k : 1 \leq k \leq n\}$ and let $Y_\infty := \{y_k : k \in \mathbb{N}\} \cup \{y_\infty\}$.

Define $F : X \to \mathcal{K}(Y)$ by, $\quad F(x) := \begin{cases} Y_\infty & \text{if } x \in \bigcap_{n \in \mathbb{N}} O_n \\ Y_n & \text{if } x \in O_n \setminus O_{n+1}. \end{cases}$

Then F is lower semicontinuous on X and $C(F) = \bigcap_{n \in \mathbb{N}} O_n$. Therefore, if $C(F)$ is dense in X then $\bigcap_{n \in \mathbb{N}} O_n$ is dense in X; which implies that X is a Baire space.

In the literature there have been several generalizations of Fort's theorem see, [5, 6, 7]. Most of these generalizations involve relaxing the notions of upper semicontinuity and lower semicontinuity. However, it is also natural to want to relax the hypothesis that the images are compact. Unfortunately, the following examples show that this is not really possible unless we strengthen some of the other hypotheses of Fort's theorem.

Example 1 *Let* $X := Y := \mathbb{R}$, *let* τ *be the Sorgenfrey topology on* $\mathbb{R}$ *(i.e., each point* x *of* $\mathbb{R}$ *has a* τ*-neighbourhood base comprising of sets of the form:* $(a, x]$ *where* $a < x$*) and let* $\rho : Y \times Y \to \{0, 1\}$ *be the discrete metric on* Y *(i.e.,* $\rho(x, y) = 1$ *if, and only if,* $x \neq y$*). Let us also consider the following set-valued mappings.* $F_1 : (X, \tau) \to 2^Y$ *defined by* $F_1(x) := (-\infty, x]$ *and* $F_2 : (X, \tau) \to 2^Y$ *defined by* $F_2(x) = [x, \infty)$. *Then* (X, τ) *is a Baire space,* F_1 *is upper semicontinuous but* $C(F_1) = \varnothing$ *and* F_2 *is lower semicontinuous but* $C(F_2) = \varnothing$.

Despite this example, it is possible to relax the hypothesis that the underlying mapping possesses nonempty compact images. However the price one must pay is that one must restrict the class of domain spaces. For example, in [9] the authors proved the following theorems.

Theorem 3 (Theorem 4.2 in [9]) *If $F : X \to 2^Y$ is a upper semicontinuous set-valued mapping from a complete metric space (X, d) into nonempty subsets of a metric space (Y, ρ) then F is lower semicontinuous at the points of a dense G_δ subset of X.*

The corresponding result for lower semicontinuous mappings fails, [9, Example 4.1]. However, by introducing the stronger notion of metric lower semicontinuity we can state the following theorem.

Theorem 4 (Theorem 4.4 in [9]) *If $F : X \to 2^Y$ is a metric lower semicontinuous set-valued mapping from a complete metric space (X, d) into nonempty subsets of a metric space (Y, ρ) then F is metric upper semicontinuous at the points of a dense G_δ subset of X.*

Recall that a set-valued mapping $F : X \to 2^Y$ from a topological space (X, τ) into nonempty subsets metric space (Y, ρ) is said to be *metric lower (upper) semicontinuous at a point* $x_0 \in X$ if for each $\varepsilon > 0$ there exists a neighbourhood U of x_0 such that $F(x_0) \subseteq B(F(x); \varepsilon)$ $(F(x) \subseteq B(F(x_0); \varepsilon))$ for all $x \in U$. [Here we are using the notation $B(A; \varepsilon) := \bigcup_{a \in A} B(a; \varepsilon)$ for all $\varnothing \neq A \subseteq Y$ and $\varepsilon > 0$.] If F is metric lower (upper) semicontinuous at each point of X then we say that F is *metric lower (upper) semicontinuous on* X.

In this paper we extend the class of domain spaces for which Theorems 3 and 4 hold.

Let (X, τ) be a topological space. On X we consider the *$\mathcal{G}_c$-game* played between two players α and β. Player β goes first (always!) and chooses a nonempty open subset $B_1 \subseteq X$. Player α must then respond by choosing a nonempty open subset $A_1 \subseteq B_1$ and a point $a_1 \in A_1$. Following this, player β must select another nonempty open subset $B_2 \subseteq A_1 \subseteq B_1$ and in turn player α must again respond by selecting a nonempty open subset $A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1$ and a point $a_2 \in A_2$. Continuing this procedure indefinitely the players α and β produce a sequence $\{((a_n, A_n), B_n) : n \in \mathbb{N}\}$ called a *play* of the $\mathcal{G}_c$-game. We shall declare that α *wins* a play $\{((a_n, A_n), B_n) : n \in \mathbb{N}\}$ of the $\mathcal{G}_c$-game if:

(i) $I := \bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$ and

(ii) for each open subset W containing I there exists a $k \in \mathbb{N}$ such that $a_k \in W$.

Otherwise the player β is said to have won this play. By a *strategy* t for the player β we mean a *'rule'* that specifies each move of the player β in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for β is a sequence of τ-valued functions such that

$$\varnothing \neq t_1(\emptyset) \quad \text{and} \quad \varnothing \neq t_{n+1}((a_1, A_1), (a_2, A_2), \ldots, (a_n, A_n)) \subseteq A_n \text{ for each } n \in \mathbb{N}.$$

The domain of t_1 is $\{\emptyset\}$, (where $\emptyset$ denotes the sequence of length 0) and the domain of t_2 is $\{(a, A) \in X \times \tau : a \in A \subseteq t_1(\emptyset)\}$. For $n \geq 3$ the domain of each function t_n is precisely the set of all finite sequences $\{(a_1, A_1), (a_2, A_2), \ldots, (a_{n-1}, A_{n-1})\}$ of length $n - 1$ in $X \times \tau$ such that

$$a_1 \in A_1 \subseteq t_1(\emptyset) \quad \text{and} \quad a_j \in A_j \subseteq t_j(A_1, \ldots, A_{j-1}) \text{ for all } 2 \leq j \leq n - 1.$$

Such a finite sequence $\{(a_1, A_1), (a_2, A_2), \ldots, (a_{n-1}, A_{n-1})\}$ or infinite sequence $\{(a_n, A_n) : n \in \mathbb{N}\}$ is called a *t-sequence*. A strategy $t := (t_n : n \in \mathbb{N})$ for the player β is called a *winning strategy* if each t-sequence is won by β. We will call a topological space (X, τ) *nearly complete* if the player β does **not** have a winning strategy in the $\mathcal{G}_c$-game played on X. Note that nearly complete spaces are very similar to the σ-β défavorable spaces considered in [8]. In fact, the only difference is that in the $\mathcal{G}_c$-game, the player α is obliged to choose the points $a_n \in A_n$, whereas, in the game considered in [8], α is free to choose the points a_n anywhere in X. Hence, it follows that every nearly complete space is σ-β défavorable. It follows from [8, Theorem 1] that every nearly complete space is a Baire space. However, from Theorem 5 and Example 1 we see that not every Baire space is nearly complete. The class of nearly complete spaces is quite large, as it includes all the strongly Baire spaces [3]. In particular, it includes all Baire metric spaces, all Čech-complete spaces and all regular, locally countably compact spaces.

Theorem 5 *If $F : X \to 2^Y$ is a metric upper semicontinuous set-valued mapping from a nearly complete space (X, τ) into nonempty subsets of a metric space (Y, d) then F is metric lower semi-continuous at the points of a dense G_δ subset of X.*

Proof: For each $x \in X$, let $\mathcal{N}(x)$ denote the family of all open subsets of X that contain the point x and for each $\varepsilon > 0$ let,

$$O_\varepsilon := \{x \in X : \exists\ 0 < \varepsilon' < \varepsilon \text{ and } \exists\ U \in \mathcal{N}(x) \text{ such that } F(x) \subseteq \textstyle\bigcap_{y \in U} B(F(y); \varepsilon')\}.$$

[Note that if $x \notin O_\varepsilon$ then for all $0 < \varepsilon' < \varepsilon$ and all $U \in \mathcal{N}(x)$, $F(x) \not\subseteq \bigcap_{y \in U} B(F(y); \varepsilon')$.]

It is easy to check, using the metric upper semicontinuity of F, that each O_ε is an open subset of X. It is also easy to see that F is metric lower semicontinuous at each point of $\bigcap_{n \in \mathbb{N}} O_{1/n}$. So to prove the theorem it is sufficient to show that each O_ε is dense in X. To this end, let us fix $0 < \varepsilon < \infty$. In order to obtain a contradiction let us assume that there exists a nonempty open subset W of X such that $O_\varepsilon \cap W = \varnothing$. We shall use this assumption to inductively construct a (necessarily non-winning) strategy for the player β in the $\mathcal{G}_c$-game played on X.

Step 1. Let $t_1(\emptyset) := W$, $\varepsilon' := (2/3)\varepsilon$ and $\delta := (1/3)\varepsilon$.

Step 2. Suppose that α has chosen $(a_1, A_1) \in X \times \tau$ such that $a_1 \in A_1 \subseteq t_1(\emptyset)$ as their first move. Since $a_1 \notin O_\varepsilon$, $0 < \varepsilon' < \varepsilon$ and $A_1 \in \mathcal{N}(a_1)$, $F(a_1) \not\subseteq \bigcap_{y \in A_1} B(F(y); \varepsilon')$. Therefore, there exists a $b_1 \in A_1$ such that $F(a_1) \not\subseteq B(F(b_1); \varepsilon')$. Let $t_2((a_1, A_1)) := \operatorname{int}\{x \in A_1 : F(x) \subseteq B(F(b_1); \delta)\}$.

In general, suppose that α has chosen

$$\{(a_1, A_1), (a_2, A_2), \ldots, (a_n, A_n)\}$$

and β has defined

$$t_{j+1}((a_1, A_1), (a_2, A_2), \ldots, (a_j, A_j)) \quad \text{for all } 1 \leq j < n \quad \text{and defined} \quad \{b_1, b_2, \ldots, b_{n-1}\}$$

so that:

(i) $a_j \in A_j$ for all $1 \leq j \leq n$;

(ii) $b_j \in A_j$ and $F(a_j) \not\subseteq B(F(b_j); \varepsilon')$ for all $1 \leq j < n$;

(iii) $t_{j+1}((a_1, A_1), (a_2, A_2), \ldots, (a_j, A_j)) := \operatorname{int}\{x \in A_j : F(x) \subseteq B(F(b_j); \delta)\}$ for all $1 \leq j < n$.

Step n+1. Since $a_n \notin O_\varepsilon$, $0 < \varepsilon' < \varepsilon$ and $A_n \in \mathcal{N}(a_n)$, $F(a_n) \nsubseteq \bigcap_{y \in A_n} B(F(y); \varepsilon')$. Therefore, there exists a $b_n \in A_n$ such that $F(a_n) \nsubseteq B(F(b_n); \varepsilon')$. Let

$$t_{n+1}((a_1, A_1), (a_2, A_2), \ldots, (a_n, A_n)) := \text{int}\{x \in A_n : F(x) \subseteq B(F(b_n); \delta)\}.$$

This completes the definition of $t := (t_n : n \in \mathbb{N})$. Now, since t is **not** a winning strategy for the player β in the $\mathcal{G}_c$-game played on X there exists a t-sequence $\{(a_n, A_n) : n \in \mathbb{N}\}$ [and an accompanying sequence $(b_n : n \in \mathbb{N})$] such that:

(i) $I := \bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$;

(ii) there exists a $k \in \mathbb{N}$ such that $a_k \in \bigcup_{x \in I} \text{int}\{F^{-1}(B(F(x); \delta))\} \subseteq \bigcup_{x \in I} F^{-1}(B(F(x); \delta))$.

Here we have used the notation $F^{-1}(A)$ to denote $\{x \in X : F(x) \subseteq A\}$ for any subset A of Y.

Therefore, for some $x \in I$, $F(a_k) \subseteq B(F(x); \delta) \subseteq B(F(I); \delta)$. On the other hand,

$$F(I) \subseteq F(A_{k+1}) \subseteq B(F(b_k); \delta).$$

Thus,

$$F(a_k) \subseteq B(F(I); \delta) \subseteq B(F(b_k); 2\delta) = B(F(b_k); \varepsilon');$$

which contradicts the way b_k was chosen. This completes the proof.

Theorem 6 *If $F : X \to 2^Y$ is a metric lower semicontinuous set-valued mapping from a nearly complete space (X, τ) into nonempty subsets of a metric space (Y, d) then F is metric upper semicontinuous at the points of a dense G_δ subset of X.*

Proof: The proof of this theorem is almost identical to that of Theorem 5, once one replaces the definition of the O_ε set by,

$$O_\varepsilon := \{x \in X : \exists\ 0 < \varepsilon' < \varepsilon \text{ and } \exists\ U \in \mathcal{N}(x) \text{ such that } \textstyle\bigcup_{y \in U} F(y) \subseteq B(F(x); \varepsilon')\}.$$

Interest in Fort's theorem stems from the fact that it has many applications in optimization, variational inequalities, game theory, generic differentiation, selection theorems and the geometry of Banach spaces, see [4, 9] and the many references within. For more information on topological games see [1].

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