Separable subspaces of affine function spaces on convex compact sets

Warren B. Moors[1] and Evgenii A. Reznichenko

Abstract. Let K be a compact convex subset of a separated locally convex space (over $\mathbb{R}$) and let $\mathcal{A}_p(K)$ denote the space of all continuous real-valued affine mappings defined on K, endowed with the topology of pointwise convergence on the extreme points of K. In this paper we shall examine some topological properties of $\mathcal{A}_p(K)$. For example, we shall consider when $\mathcal{A}_p(K)$ is monolithic and when separable compact subsets of $\mathcal{A}_p(K)$ are metrizable.

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1 Introduction

This paper is a tentative first step in the study of the continuous real-valued affine functions defined on a compact convex set K, endowed with the topology of pointwise convergence on the extreme points of K.

For this reason we have attempted to make this paper as self-contained as possible. As a consequence, we have included the statements and proofs of several well-known results.

We shall begin with some background material and some basic notation, then in section 2, we shall examine when separable subsets of $\mathcal{A}_p(K)$ are separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$. In section 3 we examine when separable compact subsets of $\mathcal{A}_p(K)$ are metrizable and when $\mathcal{A}_p(K)$ has countable tightness. Recall that a topological space (X, τ) is said to have *countable tightness* if for every subset Y of X and every element $x \in \overline{Y}$ there exists a countable subset C of Y such that $x \in \overline{C}$. Finally, in sections 4 and 5, we give several counter-examples that illuminate the boundaries of our investigations.

A vector space $(X, +, \odot)$ over a field $\mathbb{K}$, endowed with a topology τ, is called a *topological vector space* if the functions $\odot : \mathbb{K} \times X \to X$ (scalar multiplication) and $+ : X \times X \to X$ (addition) are continuous with respect to τ. A subset K of a vector space is said to be *convex* if for each $x, y \in K$ and $0 \le \lambda \le 1$, $\lambda x + (1-\lambda)y \in K$. A topological vector space is said to be *locally convex* if 0 has a local base consisting of convex neighbourhoods.

In this paper we will be exclusively working with separated locally convex spaces over $\mathbb{R}$. Recall that a topological vector space is said to be *separated* if the topology defined on it is Hausdorff. The reason for this restriction is revealed in the following theorem.

Theorem 1.1 [7, page 118] *Suppose that X is a locally convex space over $\mathbb{R}$ and C is a nonempty closed convex subset of X. If $x \notin C$ then there exists a continuous linear functional x^* such that $\sup\{x^*(c) : c \in C\} < x^*(x)$.*

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An immediate consequence of Theorem 1.1 is that in separated locally convex spaces the relative weak topology coincides with the relative linear topology on compact subsets.

We say a subset E of a set K in a vector space X is an *extremal subset of* K if $x, y \in E$ whenever $\lambda x + (1-\lambda)y \in E$, $x, y \in K$ and $0 < \lambda < 1$. A point x in a set K is called an *extreme point of* K if the set $\{x\}$ is an extremal subset of K. For a set K in a vector space X we will denote the set of all extreme points of K by, $\mathrm{Ext}(K)$.

Proposition 1.2 *Let K be a nonempty subset of a vector space. If E is an extremal subset of K then* $\mathrm{Ext}(E) \subseteq \mathrm{Ext}(K)$.

Proof : Suppose that $e \in \mathrm{Ext}(E)$ and that $\lambda x + (1-\lambda)y = e$ for some $x, y \in K$ and $0 < \lambda < 1$. Then, since E is extremal, $x, y \in E$, but since $e \in \mathrm{Ext}(E)$, $x = y = e$. Thus $e \in \mathrm{Ext}(K)$. $\square$

If a convex subset K of a separated locally convex space is also compact then the set $\mathrm{Ext}(K)$ is sufficiently large to recapture the entire set.

Theorem 1.3 (Krein-Milman Theorem) [9] Let K be a compact convex subset of a separated locally convex space. Then $\mathrm{Ext}(K)$ is nonempty and K is the closed convex hull of it's extreme points.

There is also a partial converse to the Krein-Milman Theorem.

Theorem 1.4 (Milman's Theorem) [12, page 8] *Let E be a nonempty subset of a separated locally convex space. If $K := \overline{\mathrm{co}}(E)$ is compact then* $\mathrm{Ext}(K) \subseteq \overline{E}$.

Let K and T be convex subsets of vector spaces. A function $f : K \to T$ is said to be *affine* if for all $x, y \in K$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y).$$

The set of all continuous real-valued affine functions on a compact convex subset K of a topological vector space will be denoted by $\mathcal{A}(K)$. Clearly, all translates of continuous linear functionals are members of $\mathcal{A}(K)$, but not all members of $\mathcal{A}(K)$ are translates of continuous linear functionals [12, page 21]. However, we do have the following relationship.

Proposition 1.5 [12, Proposition 4.5] *Suppose that K is a compact convex subset of a separated locally convex space X then*

$$\{a \in \mathcal{A}(K) : a = r + x^*|_K \text{ for some } x^* \in X^* \text{ and some } r \in \mathbb{R}\}$$

is dense in $(\mathcal{A}(K), \|\cdot\|_\infty)$.

Affine maps acting between convex sets preserve some of the geometrical structure of the underlying convex sets, as the following theorems demonstrate.

Theorem 1.6 *Suppose that $f : K \to T$ is a surjective affine mapping acting between convex subsets K and T of vector spaces. If $t \in T$ then $f^{-1}(t)$ is an extremal subset of K if, and only if, $t \in \mathrm{Ext}(T)$.*

Proof : Firstly, suppose that $f^{-1}(t)$ is an extremal subset of K. If there exists $x, y \in T$ such that $t = \lambda x + (1-\lambda)y$ for some $0 < \lambda < 1$, then $\lambda f^{-1}(x) + (1-\lambda)f^{-1}(y) \subseteq f^{-1}(\lambda x + (1-\lambda)y) = f^{-1}(t)$. Since $f^{-1}(t)$ is extremal, $f^{-1}(x) \subseteq f^{-1}(t)$ and $f^{-1}(y) \subseteq f^{-1}(t)$. Hence $x = y = t$ i.e., $t \in \mathrm{Ext}(T)$.

Now, suppose $t \in \mathrm{Ext}(T)$. If there exists $x, y \in K$ and $0 < \lambda < 1$ such that $\lambda x + (1-\lambda)y \in f^{-1}(t)$, then $t = f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$. However, since $t \in \mathrm{Ext}(T)$, this means $f(x) = f(y) = t$, or $x, y \in f^{-1}(t)$, i.e., $f^{-1}(t)$ is extremal. $\square$

Corollary 1.7 *Suppose that K and T are convex subsets of vector spaces and $f : K \to T$ is a surjective affine map. If $e \in \mathrm{Ext}(T)$, then $\mathrm{Ext}(f^{-1}(e)) \subseteq \mathrm{Ext}(K)$.*

Proof : Since e is an extreme point of T, by Theorem 1.6, $f^{-1}(e)$ is an extremal subset of K. However, by Proposition 1.2, $\mathrm{Ext}(f^{-1}(e)) \subseteq \mathrm{Ext}(K)$. $\square$

Let K and T be compact convex subsets of topological vector spaces and let $f : K \to T$ be a continuous affine map. We define $f^{\#} : \mathcal{A}(T) \to \mathcal{A}(K)$ by, $f^{\#}(a) := a \circ f$. Note that $f^{\#}$ is always an isometric embedding of $(\mathcal{A}(T), \|\cdot\|_\infty)$ into $(\mathcal{A}(K), \|\cdot\|_\infty)$.

If K, S and T are compact convex subsets of separated topological vector spaces and $g : K \to S$ and $h : S \to T$ are continuous affine mappings then $(h \circ g)^{\#} : \mathcal{A}(T) \to \mathcal{A}(K)$ and $(h \circ g)^{\#} = g^{\#} \circ h^{\#}$.

Theorem 1.8 *Let K and T be compact convex subsets of separated topological vector spaces and let $f : K \to T$ be a surjective continuous affine map. Then $g \in f^{\#}(\mathcal{A}(T))$ if, and only if, $g \in \mathcal{A}(K)$ and g is constant on $f^{-1}(t)$ for each $t \in T$.*

Proof : Clearly if $g \in f^{\#}(\mathcal{A}(T))$ then g is constant on $f^{-1}(t)$ for each $t \in T$. Now suppose g is constant on each $f^{-1}(t)$ and define a map $b : T \to \mathbb{R}$ by, $b(t) := g(k)$ for some $k \in f^{-1}(t)$. Since f is surjective and g is constant on $f^{-1}(t)$, for each $t \in T$, b is well defined. Moreover, since f is a perfect mapping b is continuous. Finally, it is easy to check that b is affine and hence $b \in \mathcal{A}(T)$. Now $f^{\#}(b) = b \circ f = g$. Thus $g \in f^{\#}(\mathcal{A}(T))$. $\square$

Next we introduce some notation from topology.

Let the set of all continuous real-valued functions defined on a topological space X be denoted by, $C(X)$. If $Y \subseteq X$ then we shall denote by, $\tau_p(Y)$ the topology on $C(X)$ of pointwise convergence on Y. Furthermore, in the special case when $Y = X$ we shall denote by $C_p(X)$ the set $C(X)$ endowed with the topology $\tau_p(X)$.

If K is a compact convex subset of a topological vector space then will shall write $\mathcal{A}_p(K)$ to indicate the set $\mathcal{A}(K)$ endowed with the topology $\tau_p(\mathrm{Ext}(K))$ and we shall write $B_{\mathcal{A}(K)}$ to indicate the set $\{f \in \mathcal{A}(K) : \|f\|_\infty \leq 1\}$.

Corollary 1.9 *Suppose that K is a compact convex subset of a separated locally convex space and $M \subseteq \mathcal{A}(K)$. Then the mapping $f : K \to (\mathbb{R}^M, \tau_p(M))$ defined by, $f(k)(m) := m(k)$ is a continuous affine mapping onto $T := f(K)$ and $\overline{M}^{\tau_p(K)} \subseteq f^{\#}(\mathcal{A}(T))$. Thus, if $|M| \leq \aleph_0$ then $\overline{M}^{\tau_p(K)}$ is separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$.*

If X is a topological space then we shall call any measure μ defined on the σ-algebra of Borel subsets of X (or their completion with respect to μ) a *Borel measure*. We shall say that μ is a *regular Borel measure* if for each Borel subset S of X,

$$\sup\{\mu(K) : K \subseteq S \text{ and } K \text{ is compact}\} = \mu(S) = \inf\{\mu(U) : S \subseteq U \text{ and } U \text{ is open}\}.$$

We say that μ is a *probability measure* if μ is a positive measure on X and if $\mu(X) = 1$. If μ is a probability measure on X, then we say that μ is *supported by a set* S, or that, μ *is carried on a set* S, if there exists a μ-measurable set $B \subseteq S$ such that $\mu(B) = 1$.

Let K be a compact convex subset of a separated locally convex space, let μ be a regular Borel measure on K and let $\mathscr{F}$ be a subset of $\mathcal{A}(K)$. If k is a point in K, then we say that μ *represents* k *over* $\mathscr{F}$ if,

$$\int_K f \, \mathrm{d}\mu = f(k) \quad \text{for every } f \in \mathscr{F}.$$

If $\mathscr{F} = \mathcal{A}(K)$, then we simply say that μ *represents* k. The restriction to separated locally convex spaces ensures the existence of separating functions for K, which in turn, ensures that each Borel probability measure μ represents at most one point.

Theorem 1.10 [12, page 6] *Suppose that K is a compact convex subset of a separated locally convex space and suppose that μ is a regular Borel probability measure on K. If T is a compact convex subset of K and $\mu(T) = 1$ then there exists a point $k \in T$ such that μ 'represents' k.*

Thus each regular Borel probability measure represents a point in K. Hence it is natural to ask the converse question; namely, does every point in a compact convex subset K of a separated locally convex space have a regular Borel probability measure that represents it? Trivially, every point is represented by it's 'point mass' measure; however, if K is metrizable, then we are able to guarantee the existence of representing measures whose support is carried on $\mathrm{Ext}(K)$.

Theorem 1.11 (Choquet's Representation Theorem) [4] *Suppose K is a metrizable compact convex subset of a separated locally convex space and suppose that $k \in K$. Then $\mathrm{Ext}(K)$ is a G_δ subset and there exists a regular Borel probability measure μ carried on $\mathrm{Ext}(K)$ that 'represents' k.*

If K is non-metrizable then we have the following version of Choquet's Theorem which is less precise as to the support of the representing measures.

Theorem 1.12 (Bishop-de Leeuw Theorem) [12, page 17] *Suppose that K is a compact convex subset of a separated locally convex space and suppose that $k \in K$. Then there exists a regular Borel probability measure μ on K that 'represents' k and which vanishes on every G_δ subset of $K \backslash \mathrm{Ext}(K)$. In particular, if $\mathrm{Ext}(K) \subseteq X \subseteq K$ is universally measurable and Lindelöf (e.g. if X is K-analytic) then $\mu(X) = 1$.*

Although the statement of the above theorem differs slightly to that stated in [12], it's proof is the same. An immediate consequence of the Bishop-de Leeuw theorem is the following geometric result.

Corollary 1.13 *Let K be a nonempty compact convex subset of a separated locally convex space. If T and $\{C_n : n \in \mathbb{N}\}$ are compact convex subsets of K whose union cover $\mathrm{Ext}(K)$ then for each $x \in K \setminus T$ there exist elements $y \in K$, $z \in \bigcup_{n \in \mathbb{N}} C_n$ and $\lambda \in [0, 1)$ such that $x = \lambda y + (1 - \lambda) z$*

Proof : By Theorem 1.12 there exists a regular Borel probability measure μ carried on $T \cup \bigcup_{n \in \mathbb{N}} C_n$ that 'represents' x. Now, $\mu(T) < 1$ since if $\mu(T) = 1$ then by Theorem 1.10, $x \in T$. Therefore, $0 < \mu(\bigcup_{n \in \mathbb{N}} C_n) \leq 1$. In particular, this implies that for some $k \in \mathbb{N}$, $0 < \mu(C_k) \leq 1$. Note that if

$\mu(C_k) = 1$ then by Theorem 1.10, $x \in C_k$ and so the Corollary follows by letting $y := z := x$ and $\lambda := 1/2$. So we will assume that $0 < \mu(C_k) < 1$. For each Borel set $B \subseteq K$ let

$$\mu_1(B) := \frac{1}{\mu(K \setminus C_k)}\mu(B \cap [K \setminus C_k]) \quad \text{and} \quad \mu_2(B) := \frac{1}{\mu(C_k)}\mu(B \cap C_k).$$

Then both μ_1 and μ_2 are regular Borel probability measures on K. Therefore, by Theorem 1.10 there exist elements $y, z \in K$ such that μ_1 'represents' y and μ_2 'represents' z. Moreover, since $\mu_2(C_k) = 1$ it follows from Theorem 1.10 that $z \in C_k$. If we set $0 < \lambda := \mu(K \setminus C_k) < 1$ then $\mu(C_k) = (1 - \lambda)$ and $\mu(B) = \lambda\mu_1(B) + (1 - \lambda)\mu_2(B)$ for each Borel subset B of K and so $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$. In particular, this means that for every $a \in \mathcal{A}(K)$

$$\begin{aligned} a(x) = \int_K a \, d\mu &= \int_K a \, d(\lambda\mu_1 + (1-\lambda)\mu_2) \\ &= \lambda \int_K a \, d\mu_1 + (1-\lambda) \int_K a \, d\mu_2 \\ &= \lambda a(y) + (1-\lambda)a(z) = a(\lambda y + (1-\lambda)z). \end{aligned}$$

Since the elements of $\mathcal{A}(K)$ separate the points of K, $x = \lambda y + (1 - \lambda)z$. $\square$

For a set X we shall denote by $\Delta_X := \{(x, x) \in X \times X : x \in X\}$.

Theorem 1.14 *Let K be a nonempty compact convex subset of a separated locally convex space. Then the following are equivalent:*

(i) $\mathrm{Ext}(K)$ *is a G_δ-subset of K;*

(ii) $\Delta_{\mathrm{Ext}(K)}$ *is a G_δ-subset of $K \times K$;*

(iii) $\Delta_{\mathrm{Ext}(K)}$ *is a G_δ-subset of* $\mathrm{Ext}(K) \times \mathrm{Ext}(K)$.

Proof : (i) $\Longrightarrow$ (ii). Let $m : K \times K \to K$ be defined by, $m(x, y) := 1/2(x + y)$. Then m is continuous and $m^{-1}(\mathrm{Ext}(K)) = \Delta_{\mathrm{Ext}(K)}$. Therefore, if $\mathrm{Ext}(K)$ is a G_δ-subset of K then $\Delta_{\mathrm{Ext}(K)}$ is a G_δ-subset of $K \times K$. (ii) $\Longrightarrow$ (iii) is obvious. (iii) $\Longrightarrow$ (i). Let us begin by noticing that if $(K \times K) \setminus \Delta_K \subseteq F \subseteq K \times K$ and $\Delta_{\mathrm{Ext}(K)} \cap F = \varnothing$ then $m(F) = K \setminus \mathrm{Ext}(K)$. So to prove this implication it will be sufficient to show that there exists an F_σ set $F \subseteq K \times K$ such that $(K \times K) \setminus \Delta_K \subseteq F$ and $\Delta_{\mathrm{Ext}(K)} \cap F = \varnothing$. To this end, suppose that $\Delta_{\mathrm{Ext}(K)} = \bigcap_{n\in\mathbb{N}} U_n$, where each $U_n \subseteq \mathrm{Ext}(K) \times \mathrm{Ext}(K)$ is an open neighbourhood of $\Delta_{\mathrm{Ext}(K)}$. For each $n \in \mathbb{N}$, let $C_n := \overline{\mathrm{co}}[\mathrm{Ext}(K) \times \mathrm{Ext}(K) \setminus U_n]$. By a Theorem 1.4, $C_n \cap \Delta_{\mathrm{Ext}(K)} = \varnothing$ for each $n \in \mathbb{N}$ and so

$$[\mathrm{Ext}(K) \times \mathrm{Ext}(K)] \setminus \Delta_{\mathrm{Ext}(K)} = [\mathrm{Ext}(K) \times \mathrm{Ext}(K)] \cap \bigcup_{n\in\mathbb{N}} C_n.$$

Since $\mathrm{Ext}(K \times K) = \mathrm{Ext}(K) \times \mathrm{Ext}(K)$ we have that $\mathrm{Ext}(K \times K) \subseteq \Delta_K \cup \bigcup_{n\in\mathbb{N}} C_n$.

Thus it follows from Corollary 1.13 that if we define

$$F_{(m,n)} := \{\lambda y + (1-\lambda)z : y \in K \times K, z \in C_n \text{ and } 0 \le \lambda \le (1 - 1/m)\} \text{ for each } (m, n) \in \mathbb{N}^2$$

then (i) $F := \bigcup_{(m,n)\in\mathbb{N}^2} F_n$ is an F_σ-set (ii) $[K \times K] \setminus \Delta_K \subseteq F$ and $F \cap \Delta_{\mathrm{Ext}(K)} = \varnothing$. $\square$

Let K be a compact Hausdorff space and let $P(K)$ denote the set of all regular Borel probability measures on K endowed with the weak* topology. If T is also a compact Hausdorff space and $f : K \to T$ is a continuous surjection then $f^{\#\#} : P(K) \to P(T)$ is defined by,

$$f^{\#\#}(\mu)(B) := \mu(f^{-1}(B)) \text{ for each Borel subset } B \text{ of } T.$$

One can easily check that $f^{\#\#}$ does indeed map regular Borel probability measures on K to regular Borel probability measures on T.

Theorem 1.15 *Let K and T be compact Hausdorff spaces and suppose that $f : K \to T$ is a continuous surjection. Let μ be a regular Borel probability measure on K and let $\nu := f^{\#\#}(\mu)$. If g is a bounded Borel measurable function on T then*

$$\int_K g \circ f \, \mathrm{d}\mu = \int_T g \, \mathrm{d}\nu.$$

Proof : Consider first the case when g is a simple function on T, i.e., $g = \sum_{i=1}^n c_i \chi_{E_i}$ where $c_i \in \mathbb{R}$, $\{E_i : 1 \le i \le n\}$ is a partition of T by Borel subsets and χ_S denotes the characteristic function of a Borel set S. Furthermore, we note that if E is a Borel subset of T then $f^{-1}(E)$ is also Borel in K. Now,

$$\begin{aligned}
\int_T g \, \mathrm{d}\nu &= \int_T \sum_{i=1}^n c_i \chi_{E_i} \, \mathrm{d}\nu = \sum_{i=1}^n c_i \int_T \chi_{E_i} \, \mathrm{d}\nu = \sum_{i=1}^n c_i \nu(E_i) \\
&= \sum_{i=1}^n c_i \mu(f^{-1}(E_i)) = \sum_{i=1}^n c_i \int_K \chi_{f^{-1}(E_i)} \, \mathrm{d}\mu = \sum_{i=1}^n c_i \int_K \chi_{E_i} \circ f \, \mathrm{d}\mu \\
&= \int_K \sum_{i=1}^n c_i (\chi_{E_i} \circ f) \, \mathrm{d}\mu = \int_K \left(\sum_{i=1}^n c_i \chi_{E_i} \right) \circ f \, \mathrm{d}\mu = \int_K g \circ f \, \mathrm{d}\mu.
\end{aligned}$$

In the general case, when g is a bounded Borel measurable function on T, the result follows from the fact that g can be uniformly approximated by simple functions. $\square$

Next, we state the well-known characterisation of the extreme points of $P(K)$.

Theorem 1.16 [5, page 422] *Let K be a compact Hausdorff space. Then* $\mathrm{Ext}(P(K)) = \{\delta_k : k \in K\}$ *where δ_k is the 'point mass' measure for k, i.e.,*

$$\delta_k(A) := \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{if } k \notin A \end{cases} \quad \text{for each Borel subset } A \text{ of } K.$$

In fact the mapping $k \mapsto \delta_k$ is a homeomorphism from K onto $\mathrm{Ext}(P(K))$.

The next theorem relies heavily upon this characterisation.

Theorem 1.17 *Let K and T be compact Hausdorff spaces and let $f : K \to T$ be a continuous surjection. Then $f^{\#\#} : P(K) \to P(T)$ is a continuous affine surjection.*

Proof : Firstly, from Theorem 1.15, it is clear that $f^{\#\#}$ is continuous. However, it is also clear that $f^{\#\#}$ is affine. Hence it remains to show that $f^{\#\#}$ is onto. Now, $f^{\#\#}(P(K))$ is a compact convex subset of $P(T)$. Thus if $\mathrm{Ext}(P(T)) \subseteq f^{\#\#}(P(K))$ then by the Krein-Milman Theorem,

$P(T) \subseteq f^{\#\#}(P(K))$.

By Theorem 1.16, $\mathrm{Ext}(P(T)) = \{\delta_t : t \in T\}$ where δ_t is the 'point mass' measure for t on T. Choose $\delta_t \in \mathrm{Ext}(P(T))$. Since f is onto, $t = f(k)$ for some $k \in K$. Then $f^{\#\#}(\delta_k)(A) = \delta_k(f^{-1}(A))$ for each Borel subset A of T, i.e.,

$$f^{\#\#}(\delta_k)(A) = \begin{cases} 1 & \text{if } k \in f^{-1}(A) \\ 0 & \text{if } k \notin f^{-1}(A). \end{cases}$$

That is, $f^{\#\#}(\delta_k)(A) = 1 \iff t \in A$. Thus, $f^{\#\#}(\delta_k) = \delta_t$. Therefore $\mathrm{Ext}(P(T)) \subseteq f^{\#\#}(P(K))$ and so $f^{\#\#}$ is onto. $\square$

Corollary 1.18 *Let X be a compact Hausdorff space. Then there exists a linear topological isomorphism from $C_p(X)$ onto $\mathcal{A}_p(P(X))$.*

Proof : Consider the mapping $T : C_p(K) \to \mathcal{A}_p(P(K))$ defined by, $[T(f)](\mu) := \int_K f \, \mathrm{d}\mu$. It is easy to see that T does indeed map into $\mathcal{A}(P(K))$ and is linear. Let $\mu \in \mathrm{Ext}(P(K))$ then by Theorem 1.16, $\mu = \delta_k$ for some $k \in K$. Therefore,

$$[T(f)](\mu) = \int_K f \, \mathrm{d}\delta_k = f(k) \text{ for all } f \in C(K).$$

Hence it follows that T is a topological embedding of $C_p(K)$ into $\mathcal{A}_p(P(K))$. Let $a \in \mathcal{A}(P(K))$ and define $f : K \to \mathbb{R}$ by, $f(k) := a(\delta_k)$. Then $f \in C(K)$ as $k \mapsto \delta_k$ is continuous. Moreover, $T(f) = a$ and so T is surjective. $\square$

In this way, we see that the study of $\mathcal{A}_p(K)$, for K a compact convex subset of a separated locally convex space, includes the study of $C_p(X)$, for X a compact Hausdorff space. Let us also note that since $\mathcal{A}(K) \subseteq C(K)$ we have by Riesz's representation theorem [for the dual of $(C(K), \|\cdot\|_\infty)$] and Theorem 1.10 that the weak topology on $(\mathcal{A}(K), \|\cdot\|_\infty)$ coincides with $(\mathcal{A}(K), \tau_p(K))$.

This completes the introduction.

2 Separability in $(\mathcal{A}(K), \|\cdot\|_\infty)$

We begin this section by describing a family of sets that includes the set of extreme points. Let K be a compact convex subset of a separated locally convex space. We shall call a subset B of K a *boundary* for K if for each $a \in \mathcal{A}(K)$ there exists a $b \in B$ such that $a(b) = \max\{a(k) : k \in K\}$. Clearly, if B is a boundary for K then for each $a \in \mathcal{A}(K)$ there exists a $b \in B$ such that $a(b) = \min\{a(k) : k \in K\}$. Hence for any boundary B of K, $B_{\mathcal{A}(K)}$ is closed in the $\tau_p(B)$-topology. The prototypical example of a boundary for K is $\mathrm{Ext}(K)$. However, there are many other examples. For example, if B is any pseudo-compact dense subset of $\mathrm{Ext}(K)$ then B is also a boundary for K.

By contrast, with the situation for the extreme points of a compact convex set, there are in general no integral representations for the points of a compact convex subset in terms of the regular Borel probability measures supported on their boundaries, see [2, page 330]. Despite this, we still have the following version of Rainwater's theorem.

Theorem 2.1 (Rainwater-Simons Theorem) [15] *Suppose that K is a compact convex subset of a separated locally convex space and B is a boundary for K. If $(a_n : n \in \mathbb{N})$ is a bounded sequence in $(\mathcal{A}(K), \|\cdot\|_\infty)$ then $(a_n : n \in \mathbb{N})$ converges to 0 with respect to $\tau_p(B)$ if, and only if, $(a_n : n \in \mathbb{N})$ converges to 0 with respect to $\tau_p(K)$ i.e., converges to 0 with respect to the weak topology on $\mathcal{A}(K)$.*

We now examine the question of when a separable subset of $(\mathcal{A}(K), \tau_p(B))$, for a boundary B of K, is separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$.

Our first result in this direction follows directly from [3].

Theorem 2.2 *Let X be a compact Hausdorff space and let B be any boundary for $P(X)$. Then every separable subset of $(\mathcal{A}(P(X)), \tau_p(B))$ is separable in $(\mathcal{A}(P(X)), \|\cdot\|_\infty)$.*

Proof : It is shown in [3] that for any countable subset $\{f_n : n \in \mathbb{N}\}$ of $C(X)$ and any $x \in X$ there exists a $\mu \in B$ such that

$$f_n(x) = \int_X f_n \, \mathrm{d}\mu \quad \text{ for all } n \in \mathbb{N}.$$

Consequently, for any countable subset $\{a_n : n \in \mathbb{N}\}$ of $\mathcal{A}_p(P(X))$ we have that $\overline{\{a_n : n \in \mathbb{N}\}}^{\tau_p(B)} \subseteq \overline{\{a_n : n \in \mathbb{N}\}}^{\tau_p(\mathrm{Ext}(P(X)))}$. The result then follows from Corollary 1.18 and Corollary 1.9. $\square$

This result might lead one to speculate that the following is always true: "Let B be an arbitrary boundary for a compact convex subset K of a separated locally convex space. Then every separable subset of $(\mathcal{A}(K), \tau_p(B))$ is separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$."

However, there are many examples to show that this naive conjecture is false (e.g., Example 4.5).

To further explore the problem of when a separable subset of $(\mathcal{A}(K), \tau_p(B))$, for a boundary B of K, is separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$ we need to consider some further notions from topology.

The *weight* $w(X, \tau)$ of a topological space (X, τ) is the minimal infinite cardinality of any base for X, while the *network weight* $nw(X, \tau)$ of X is the smallest infinite cardinality of any network in X. Recall that a *network* for a topological space X is a family $\mathscr{N}$ of subsets of X such that for any point $x \in X$ and any open neighbourhood U of x there is a $N \in \mathscr{N}$ such that $x \in N \subseteq U$. Let us also recall that the *density* $d(X, \tau)$ of a topological space X is the minimal infinite cardinality of any everywhere dense set in X.

We can now define monolithicity in terms of these preliminary concepts. A topological space X is called *τ-monolithic* if $nw(\overline{A}) \leq \tau$ for every set $A \subseteq X$ such that $|A| \leq \tau$. In particular, a topological space X is *$\aleph_0$-monolithic* if the closure of every countable set is a space with a countable network. A topological space X is called *monolithic* if it is τ-monolithic for every infinite cardinal τ, i.e., if for every $Y \subseteq X$ we have $d(Y, \tau) = nw(Y, \tau)$. A related notion to monolithicity is that of stability. A subset Y of a topological space X is said to be *τ-stable in X* if for every pair of continuous functions $f : X \to S$ and $g : S \to T$, $nw(f(Y)) \leq \tau$ whenever g separates the points of $f(Y)$ and $w(T) \leq \tau$. A subset Y of a topological space X is said to be *stable in X* if it is τ-stable in X for every infinite cardinal τ.

It can be shown that every Lindelöf Σ-subspace of a topological space X is stable in X, (see [1, Theorem II.6.21]). Recall that a space X is a Lindelöf Σ-space if it is the continuous image of a space Y that can be perfectly mapped onto a space with a countable base. It can also be shown (see [1, Proposition II.6.2]) that a pseudo-compact subspace of a topological space X is $\aleph_0$-stable in X.

The following theorem, which may be deduced by modifying the proof of [1, Theorem II.6.8], reveals the relationship between monolithicity and stability.

Theorem 2.3 *Let τ be an infinite cardinal and let Y be a subset of a completely regular topological space X. Then $(C(X), \tau_p(Y))$ is τ-monolithic if, and only if, Y is τ-stable in X.*

To create a more diverse range of τ-stable spaces we can use the following theorem. For subsets X and Y of a set Z we let $X\Delta Y := (X \setminus Y) \cup (Y \setminus X)$.

Theorem 2.4 *Suppose that X and Y are subsets of a completely regular space Z. If $|X\Delta Y| \leq \aleph_0$, τ is an infinite cardinal and $M \subseteq C(Z)$ then:*

(i) $nw(M, \tau_p(X)) = nw(M, \tau_p(Y))$;

(ii) *X is τ-stable in Z if, and only if, Y is τ-stable in Z.*

Proof : (i) Let $W := X \cap Y$. Then clearly $nw(M, \tau_p(W)) \leq \min\{nw(M, \tau_p(X)), nw(M, \tau_p(Y))\}$. So we need to show that $\max\{nw(M, \tau_p(X)), nw(M, \tau_p(Y))\} \leq nw(M, \tau_p(W))$. To this end, let $\mathscr{N}$ be a network for $(M, \tau_p(W))$ such that $|\mathscr{N}| \leq nw(M, \tau_p(W))$ and let $\mathscr{N}'$ be a network for $(M, \tau_p(X \setminus W))$ such that $|\mathscr{N}'| \leq nw(M, \tau_p(X \setminus W)) = \aleph_0$ since $|X \setminus W| \leq \aleph_0$. Define $\mathscr{N}'' := \{N \cap N' : (N, N') \in \mathscr{N} \times \mathscr{N}'\}$. Then $\mathscr{N}''$ is a network for $(M, \tau_p(X))$ and $|\mathscr{N}''| \leq |\mathscr{N} \times \mathscr{N}'| = nw(M, \tau_p(W))$. This shows that $nw(M, \tau_p(X)) \leq nw(M, \tau_p(W))$. A similar argument can be used to show that $nw(M, \tau_p(Y)) \leq nw(M, \tau_p(W))$.

(ii) Suppose that X is τ-stable in Z. Let $W := X \cap Y$. We will first show that W is τ-stable in Z. Suppose that $f : Z \to S$ and $g : S \to T$ are continuous mappings such that $w(T) \leq \tau$ and g separates the points of $f(W)$. Let $G := \{x \in X \setminus W : x \text{ is a } G_\tau\text{-point relative to } X\}$. [Recall that a point $x \in X$ is called a G_τ-point if there is a family of open subsets $\{O_\alpha : \alpha \in A\}$ such that $\{x\} = \bigcap_{\alpha \in A} O_\alpha$ and $|A| \leq \tau$.] Then there exists a topological space R with $w(R) \leq \tau$ and a continuous mapping $h : Z \to R$ such that $f(x) \notin f(X \setminus \{x\})$ for each $x \in G$. Let $f' : Z \to S \times R$ be defined by, $f' := f\Delta h$ [i.e., $f'(x) = (f(x), h(x))$ for each $x \in X$], let $S' := f'(Z) \subseteq S \times R$ and let $g' := (g \times \mathrm{id}_R)|_{S'}$. Finally, set $T' := g'(S') \subseteq T \times R$. Then $w(T') \leq w(T \times R) \leq \tau$. We claim that g' separates the points of $f'(X)$. To justify this we first observe that $f(X \setminus G) = f(W)$, since if $f(x) \notin f(W)$ then x is a G_τ-point relative to X. Next, suppose that $x, y \in X$ and $g'(f'(x)) = g'(f'(y))$, (i.e., $g(f(x)) = g(f(y))$ and $h(x) = h(y)$). If either of x or y are members of G then $h(x) = h(y)$ implies that $x = y$ which in turn implies that $f'(x) = f'(y)$. So suppose that $x, y \in X \setminus G$. Then $f(x), f(y) \in f(W)$ and since g separates the points of $f(W)$ we must have that $f(x) = f(y)$, and so $f'(x) = f'(y)$. This shows that g' separates the points of $f'(X)$. Since X is τ-stable in Z, $nw(f'(W)) \leq nw(f'(X)) \leq \tau$. However, since $f(W)$ is a continuous image of $f'(W)$, $nw(f(W)) \leq nw(f'(W)) \leq \tau$. Hence, W is τ-stable in Z. Now, we show that Y is τ-stable in Z. To this end, suppose that $f : Z \to S$ and $g : S \to T$ are continuous mappings such that $w(T) \leq \tau$ and g separates the points of $f(Y)$. Note that in particular, g separates the points of $f(W)$ and so $nw(f(W)) \leq \tau$. However, $f(Y) = f(W) \cup f(Y \setminus W)$ and $|f(Y \setminus W)| \leq \aleph_0$, therefore $nw(f(Y)) = nw(f(W)) \leq \tau$. This shows that Y is τ-stable in Z.

A similar argument shows that if Y is stable in Z then X is stable in Z. $\square$

Corollary 2.5 *Suppose that X and Y are subsets of a completely regular space Z. If $|X\Delta Y| \leq \aleph_0$ and X is $\aleph_0$-stable in Z then for any countable set $M \subseteq C(Z)$, $nw(\overline{M}^{\tau_p(X \cap Y)}, \tau_p(Y)) = \aleph_0$.*

To deduce separability results in $(\mathcal{A}(K), \|\cdot\|_\infty)$ from the above theorems we need to be able to relate the network weight in $(\mathcal{A}(K), \tau_p(B))$ to the network weight in $(\mathcal{A}(K), \|\cdot\|_\infty)$.

Theorem 2.6 *Let K be a compact convex subset of a separated locally convex space and let B be a boundary for K. If $M \subseteq \mathcal{A}(K)$ and $nw(M, \tau_p(B)) = \aleph_0$ then M is separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$. In particular, if $(\mathcal{A}(K), \tau_p(B))$ is $\aleph_0$-monolithic and $d(M, \tau_p(B)) = \aleph_0$, then M is separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$.*

Proof : Without loss of generality we may assume that M is bounded in $(\mathcal{A}(K), \|\cdot\|_\infty)$. Let $\mathscr{N}$ be a countable network for M. By possibly making $\mathscr{N}$ larger (but still countable) we may assume that $\mathscr{N}$ has the finite intersection property (i.e. if $\{N_k : 1 \leq k \leq n\} \subseteq \mathscr{N}$ and $\bigcap_{1\leq k\leq n} N_k \neq \varnothing$ then $\bigcap_{1\leq k\leq n} N_k \in \mathscr{N}$.) For each $N \in \mathscr{N}$ choose $a_N \in N$. We claim that

$$M \subseteq \overline{\{a_N : N \in \mathscr{N}\}}^{\tau_p(K)}$$

and the latter set is separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$, by Corollary 1.9. To see this, consider $a \in M$ and let $\{N_k : k \in \mathbb{N}\} \subseteq \mathscr{N}$ be a decreasing sequence of sets such that (i) $a \in N_k$ for all $k \in \mathbb{N}$ and (ii) for every $N \in \mathscr{N}$ such that $a \in N$, there exists a $k \in \mathbb{N}$ such that $a \in N_k \subseteq N$. It now follows that $(a_{N_k} : k \in \mathbb{N})$ converges to a with respect to $\tau_p(B)$ and so by Theorem 2.1 $(a_{N_k} : k \in \mathbb{N})$ converges to a with respect to $\tau_p(K)$. Thus, $M \subseteq \overline{\{a_N : N \in \mathscr{N}\}}^{\tau_p(K)}$. $\square$

Theorem 2.7 *Let B be a boundary for a compact convex subset K of a separated locally convex space. If X is a co-countable subset of B and B is $\aleph_0$-stable in K then for any countable subset M of $\mathcal{A}(K)$, $\overline{M}^{\tau_p(X)}$ is separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$.*

Note: there are examples where $(\mathcal{A}(K), \tau_p(\text{Ext}(K)))$ is monolithic but $\text{Ext}(K)$ is not $\aleph_0$-stable in K (i.e., examples where $(\mathcal{A}(K), \tau_p(\text{Ext}(K)))$ is monolithic but $(C(K), \tau_p(\text{Ext}(K)))$ is not even $\aleph_0$-monolithic, e.g., see Example 5.7).

We can use Theorem 2.7 to deduce some metrizability theorems for compact convex subsets in terms of some topological properties of their boundaries, but first we need to consider a closure-type operation.

Let X be a nonempty set. If $A \subseteq B \subseteq \mathbb{R}^X$ and Y is a subset of X then the 2*-point closure of A, relative to B, over Y*, denoted $\overline{A}^{\tau_p^2(B;Y)}$, is defined by,

$$\{g \in B : \text{ for all } y, y' \in Y \text{ and all } \varepsilon > 0 \text{ there exists an } f \in A \text{ such that } |g(y) - f(y)| < \varepsilon \text{ and } |g(y') - f(y')| < \varepsilon\}.$$

Sometimes when it is clear from the context, e.g., when $B = C(X)$ or $B = \mathcal{A}(X)$, we shall simply denote $\overline{A}^{\tau_p^2(B;Y)}$ by $\overline{A}^{\tau_p^2(Y)}$.

Let X be a nonempty set. If $B \subseteq \mathbb{R}^X$ and Y is a subset of X then:

(i) if $M \subseteq N \subseteq B$ then $\overline{M}^{\tau_p^2(B;Y)} \subseteq \overline{N}^{\tau_p^2(B;Y)}$;

(ii) if $M \subseteq B$ then $\overline{(\overline{M}^{\tau_p^2(B;Y)})}^{\tau_p^2(B;Y)} = \overline{M}^{\tau_p^2(B;Y)}$;

(iii) if $M \subseteq B$ then $\overline{M}^{\tau_p(Y)} \cap B \subseteq \overline{M}^{\tau_p^2(B;Y)}$;

(iv) if $M \subseteq \mathbb{R}^X$ is a lattice then $\overline{M}^{\tau_p(Y)} \cap B = \overline{M}^{\tau_p^2(B;Y)}$.

Note: in general $\overline{M}^{\tau_p^2(B;Y)} \cup \overline{N}^{\tau_p^2(B;Y)} \neq \overline{M \cup N}^{\tau_p^2(B;Y)}$. For example: If $X := [0,1]$, $B := C[0,1]$, M is the set of all non-decreasing functions in B and N is the set of all non-increasing functions in B. Then $\overline{M}^{\tau_p^2(B;X)} \cup \overline{N}^{\tau_p^2(B;X)} = M \cup N$ while $\overline{M \cup N}^{\tau_p^2(B;X)} = B$. This also demonstrates that in general the 2-point closure is distinct from the pointwise closure since $\overline{M \cup N}^{\tau_p(B;X)} = \overline{M}^{\tau_p(B;X)} \cup \overline{N}^{\tau_p(B;X)} = M \cup N$.

It is also routine to show that if A is a linear subspace of $B \subseteq \mathbb{R}^X$ and $1 \in A$ then:

(i) $\overline{A}^{\tau_p^2(B;Y)} = \overline{\mathrm{alg}(A)}^{\tau_p(Y)} \cap B$, where $\mathrm{alg}(A)$ is the algebra generated by A in $\mathbb{R}^X$;

(ii) $\overline{A}^{\tau_p^2(B;Y)} = \overline{\mathrm{lat}(A)}^{\tau_p(Y)} \cap B$, where $\mathrm{lat}(A)$ is the lattice generated by A in $\mathbb{R}^X$;

(iii) if A separates the points of Y then $\overline{A}^{\tau_p^2(B;Y)} = B$.

Corollary 2.8 *Suppose that X is a subset of a topological space Z. If X is $\aleph_0$-stable in Z and M is a countable subset of $C(Z)$ then $nw(\overline{M}^{\tau_p^2(X)}, \tau_p(X)) \leq \aleph_0$.*

Corollary 2.9 *Let B be a boundary for a compact convex subset K of a separated locally convex space. If B is $\aleph_0$-stable in K then K is metrizable if, and only if, there a countable family in $\mathcal{A}(K)$ that separates the points of a co-countable subset of B.*

The following example demonstrates that even if K has a Lindelöf boundary B and there exists a countable family in $\mathcal{A}(K)$ that separates all the points of B then K is still not obliged to be metrizable.

Example 2.10 *There exists a non-metrizable compact convex subset K of a separated locally convex space and a Lindelöf space X such that $\mathrm{Ext}(K) \subseteq X \subseteq K$ and $B_{\mathcal{A}(K)}$ is separable with respect to $\tau_p(X)$. In particular, $(B_{\mathcal{A}(K)}, \tau_p(X))$ is not $\aleph_0$-monolithic.*

For the justification of this, see Example 4.5

Question 2.1 *Let K be a compact convex subset of a separated locally convex space and let $B \subseteq K$ be a boundary for K. Characterise, in terms of B, when $(\mathcal{A}(K), \tau_p(B))$ is $\aleph_0$-monolithic.*

In the remainder of this section we shall only consider the case when the boundary of K consists of the extreme points of K.

Theorem 2.11 (Lift Theorem) *Let $f : K \to T$ be a continuous affine surjection acting between compact convex subsets of separated locally convex spaces and let $\varnothing \neq X \subseteq \mathrm{Ext}(K)$. If $f(X) \subseteq \mathrm{Ext}(T)$, $Y := f^{-1}(f(X)) \cap \mathrm{Ext}(K)$ and $g := f|_X$ then $g^\#$ is a topological embedding of $C_p(f(X))$ into $C_p(X)$ and*

$$\{h \in C(X) : h = a|_X \textit{ for some } a \in \overline{f^\#(\mathcal{A}(T))}^{\tau_p^2(Y)}\} \subseteq g^\#(C(f(X))) \subseteq C(X).$$

In particular, if $X = \mathrm{Ext}(K)$, then $g^\#$ is a topological embedding of $C_p(\mathrm{Ext}(T))$ into $C_p(\mathrm{Ext}(K))$ and

$$\{h \in C(\mathrm{Ext}(K)) : h = a|_{\mathrm{Ext}(K)} \textit{ for some } a \in \overline{f^\#(\mathcal{A}(T))}^{\tau_p^2(\mathrm{Ext}(K))}\} \subseteq g^\#(C(\mathrm{Ext}(T)) \subseteq C(\mathrm{Ext}(K)).$$

Proof : Clearly $g^\#$ is a topological embedding of $C_p(f(X))$ into $C_p(X)$. So it remains to show that

$$\{h \in C(X) : h = a|_X \text{ for some } a \in \overline{f^\#(\mathcal{A}(T))}^{\tau_p^2(Y)}\} \subseteq g^\#(C(f(X))) \subseteq C(X).$$

Let $a \in \overline{f^\#(\mathcal{A}(T))}^{\tau_p^2(Y)}$ and let $h := a|_X$. We claim that a is constant on $f^{-1}(e)$ for each $e \in f(X)$. To this end, let $e \in f(X)$. Then by Corollary 1.7, $\mathrm{Ext}(f^{-1}(e)) \subseteq Y \subseteq \mathrm{Ext}(K)$. Now since each member of $f^\#(\mathcal{A}(T))$ is constant over $f^{-1}(e)$, and in particular, over $\mathrm{Ext}(f^{-1}(e))$, it follows that a is constant over $\mathrm{Ext}(f^{-1}(e))$. However, since a is continuous and affine, a is constant over $\overline{\mathrm{co}}(\mathrm{Ext}(f^{-1}(e))) = f^{-1}(e)$. Next, since $f|_{f^{-1}(f(X))}$ is a perfect map and a is constant over the fibres of $f|_{f^{-1}(f(X))}$ there exists a function $k \in C(f(X))$ such that $a(x) = (k \circ f)(x)$ for all $x \in f^{-1}(f(X))$. Thus $h = a|_X = g^\#(k) \in g^\#(C(f(X))$. $\square$

Theorem 2.12 [13, Theorem 2.10] *Let $f : K \to T$ be a continuous affine surjection acting between compact convex subsets of separated locally convex spaces. If $w(T) \leq \tau$ and $X \subseteq \mathrm{Ext}(K)$ is Lindelöf then there exists a compact convex subset S, of a separated locally convex space, and continuous surjective affine maps $g : K \to S$ and $h : S \to T$ such that $f = h \circ g$, $w(S) \leq \tau$ and $g(Z) \subseteq \mathrm{Ext}(S)$. In particular, if $X = \mathrm{Ext}(K)$, then $g(X) = \mathrm{Ext}(S)$.*

Lemma 2.13 [1, Theorem I.1.3] *Let X be a completely regular space. Then $nw(C_p(X)) = nw(X)$.*

Theorem 2.14 *Let K be a compact convex subset of a separated locally convex space. If $\mathrm{Ext}(K)$ is Lindelöf and $M \subseteq \mathcal{A}_p(K)$ is infinite then $nw(\overline{M}^{\tau_p^2(\mathrm{Ext}(K))}) \leq |M|$. In particular, $\mathcal{A}_p(K)$ is monolithic.*

Proof : Suppose $M \subseteq \mathcal{A}(K)$ and $|M| = \tau$. Consider the mapping $f : K \to \mathbb{R}^M$ defined by, $[f(k)](m) := m(k)$ for all $m \in M$. Let $T := f(K)$ then T is a compact convex subset with $w(T) \leq \tau$. Therefore by Theorem 2.12 there exists a compact convex set S with $w(S) \leq \tau$ and continuous affine surjections $g : K \to S$ and $h : S \to T$ such that $f = h \circ g$ and $g(\mathrm{Ext}(K)) = \mathrm{Ext}(S)$. Then by Corollary 1.9, $M \subseteq f^{\#}(\mathcal{A}(T)) = g^{\#}(h^{\#}(\mathcal{A}(T)) \subseteq g^{\#}(\mathcal{A}(S))$. Hence by Theorem 2.11

$$H := \{h \in C(\mathrm{Ext}(K)) : h = a|_{\mathrm{Ext}(K)} \text{ for some } a \in \overline{M}^{\tau_p^2(\mathrm{Ext}(K))}\} \subseteq (g|_{\mathrm{Ext}(K)})^{\#}(C(\mathrm{Ext}(S)))$$

and so $nw(H) \leq nw(C(\mathrm{Ext}(S))) = nw(\mathrm{Ext}(S)) \leq \tau$. The result now follows since the mapping $a \mapsto a|_{\mathrm{Ext}(K)}$ is a homeomorphism from $(\overline{M}^{\tau_p^2(\mathrm{Ext}(K))}, \tau_p(\mathrm{Ext}(K)))$ onto $(H, \tau_p(\mathrm{Ext}(K)))$. $\square$

It is easy to see that if K is a compact convex subset of a separated locally convex space and $\mathrm{Ext}(K)$ is Lindelöf then $\mathrm{Ext}(K)$ has a G_δ-diagonal in $\mathrm{Ext}(K) \times \mathrm{Ext}(K)$ if, and only if, there is a countable family in $\mathcal{A}(K)$ that separates the points of $\mathrm{Ext}(K)$. Hence we may use Theorem 2.14 and Theorem 2.6 to deduce the following well-known result

Corollary 2.15 *Let K be a compact convex subset of a separated locally convex space. If $\mathrm{Ext}(K)$ is Lindelöf and has a G_δ-diagonal in $\mathrm{Ext}(K) \times \mathrm{Ext}(K)$ then K is metrizable.*

The following example shows that although $\mathrm{Ext}(K)$ being Lindelöf is enough to ensure that $\mathcal{A}_p(K)$ is monolithic, separability of $\mathrm{Ext}(K)$ is not.

Example 2.16 *There exists a non-metrizable compact convex subset K of a separated locally convex space such that both $\mathrm{Ext}(K)$ and $(B_{\mathcal{A}(K)}, \tau_p(\mathrm{Ext}(K)))$ are separable. In particular, $(B_{\mathcal{A}(K)}, \tau_p(\mathrm{Ext}(K)))$ is not $\aleph_0$-monolithic.*

For the justification of this see Example 4.6.

The following theorem extends Choquet's representation theorem since if K is a metrizable compact convex subset of a separated locally convex space then $(\mathcal{A}(K), \|\cdot\|_\infty)$ is separable and so there exists a countable family in $\mathcal{A}(K)$ that separates the points of $\mathrm{Ext}(K)$. On the other hand there are many examples of non-metrizable compact convex spaces K for which there is a countable family in $\mathcal{A}(K)$ that separates the points of $\mathrm{Ext}(K)$, (see, Example 4.5).

Theorem 2.17 (Separable Representation Theorem) *Let K be a compact convex subset of a separated locally convex space. If there exists a countable family in $\mathcal{A}(K)$ that separates the points of $\mathrm{Ext}(K)$, then for any separable subspace M of $(\mathcal{A}(K), \|\cdot\|_\infty)$ and any $k \in K$, there exists a regular Borel probability measure μ carried on $\mathrm{Ext}(K)$ that 'represents' k over M.*

Proof : Choose $\{f_i : i \in \mathbb{N}\} \subseteq \mathcal{A}(K)$ so that (i) $\{f_i : i \in \mathbb{N}\}$ separates the points of $\mathrm{Ext}(K)$ and (ii) $\{f_i : i \in \mathbb{N}\} \cap M$ is dense in M with respect to the norm topology on $\mathcal{A}(K)$. Define $f : K \to \mathbb{R}^{\mathbb{N}}$ by,

$$f(k) := (f_1(k), f_2(k), \ldots, f_n(k), \ldots).$$

Let $T := f(K)$ and choose $k \in K$. By Choquet's representation theorem there exists a regular Borel probability measure ν carried on $\mathrm{Ext}(T)$ that 'represents' $f(k)$.

By Theorem 1.17, there exists a regular Borel probability measure μ on K such that $f^{\#\#}(\mu) = \nu$. We will show that μ is carried on $\mathrm{Ext}(K)$ and that μ 'represents' k over M.

We know that ν is carried on $\mathrm{Ext}(T)$. Suppose $e \in \mathrm{Ext}(T)$. Then by Corollary 1.7, $\mathrm{Ext}(f^{-1}(e)) \subseteq \mathrm{Ext}(K)$. Now, f separates the points of $\mathrm{Ext}(K)$ and hence $\mathrm{Ext}(f^{-1}(e))$ must be a singleton. It then follows from the Krein-Milman Theorem that $f^{-1}(e) = \overline{\mathrm{co}}(\mathrm{Ext}(f^{-1}(e)))$ is also a singleton. Therefore $f^{-1}(\mathrm{Ext}(T)) \subseteq \mathrm{Ext}(K)$. However, since $\mu(f^{-1}(\mathrm{Ext}(T))) = \nu(\mathrm{Ext}(T)) = 1$, μ is carried on $\mathrm{Ext}(K)$.

For each $n \in \mathbb{N}$, let π_n be the n^{th} coordinate projection from T into $\mathbb{R}$. Then $f_n = \pi_n \circ f$, and so by Theorem 1.15

$$\int_K f_n \, \mathrm{d}\mu = \int_K \pi_n \circ f \, \mathrm{d}\mu = \int_T \pi_n \, \mathrm{d}\nu = \pi_n(f(k)) = (\pi_n \circ f)(k) = f_n(k).$$

Therefore μ 'represents' k over $\{f_i : i \in \mathbb{N}\}$ and so over M. $\square$

Theorem 2.18 *Let K be a compact convex subset of a separated locally convex space such that (i) every regular Borel probability measure carried on* $\mathrm{Ext}(K)$ *is atomic and (ii) there exists a countable family in $\mathcal{A}(K)$ that separates the point of* $\mathrm{Ext}(K)$. *Then every bounded separable subset M of $\mathcal{A}_p(K)$ is separable in $(\mathcal{A}(K), \|\cdot\|_\infty)$. In particular, $(B_{\mathcal{A}(K)}, \tau_p(\mathrm{Ext}(K)))$ is $\aleph_0$-monolithic.*

Proof : Without loss of generality we may assume that $M := \overline{\{f_i : i \in \mathbb{N}\}}^{\tau_p(\mathrm{Ext}(K))} \subseteq B_{\mathcal{A}(K)}$. We will show that $M = \overline{\{f_i : i \in \mathbb{N}\}}^{\tau_p(K)}$; which is norm separable by Corollary 1.9. Choose any $g \in M$, let $\{k^i : 1 \leq i \leq n\}$ be an arbitrary finite subset of K and let $\varepsilon > 0$ be given. By Theorem 2.17, for each k^i there exists a regular Borel probability measure μ^i carried on $\mathrm{Ext}(K)$ that 'represents' k^i over $\{g\} \cup \{f_i : i \in \mathbb{N}\}$. Since every measure carried on $\mathrm{Ext}(K)$ is atomic, $\mu^i = \sum_{j \in \mathbb{N}} \lambda^i_j \delta_{e^i_j}$ for some sequence $\{e^i_j : j \in \mathbb{N}\}$ in $\mathrm{Ext}(K)$ and some sequence $\{\lambda^i_j : j \in \mathbb{N}\}$ in $[0,1]$ with $\sum_{j=1}^\infty \lambda^i_j = 1$. For each $1 \leq i \leq n$, let N_i be chosen so that $1 - \varepsilon/4 < \sum_{j=1}^{N_i} \lambda^i_j$. Then, choose $m \in \mathbb{N}$ so that for each $\{1 \leq i \leq n$ and $1 \leq j \leq N_i\}$, $|f_m(e^i_j) - g(e^i_j)| < \varepsilon/2$. Then, for each $1 \leq i \leq n$,

$$\begin{aligned}
|g(k^i) - f_m(k^i)| &\leq \int_{\mathrm{Ext}(K)} |(g - f_m)| \, \mathrm{d}\mu^i \\
&= \sum_{j=1}^{\infty} \lambda^i_j \left[\int_{\mathrm{Ext}(K)} |(g - f_m)| \, \mathrm{d}\delta_{e^i_j} \right] = \sum_{j=1}^{\infty} \lambda^i_j |(g - f_m)(e^i_j)| \\
&= \sum_{j=1}^{N_i} \lambda^i_j |(g - f_m)(e^i_j)| + \sum_{j > N_i}^{\infty} \lambda^i_j |(g - f_m)(e^i_j)| \\
&\leq (\varepsilon/2) \sum_{j=1}^{N_i} \lambda^i_j + \|g - f_m\|_\infty \sum_{j > N_i}^{\infty} \lambda^i_j \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Therefore, $g \in \overline{\{f_i : i \in \mathbb{N}\}}^{\tau_p(K)}$ and hence M is norm separable. $\square$

Theorem 2.18 may be used to deduce a further metrizability theorem for compact convex sets.

Theorem 2.19 *Let K be a compact convex subset of a separated locally convex space such that every regular Borel probability measure carried on* $\text{Ext}(K)$ *is atomic. Then K is metrizable if, and only if, $(B_{\mathcal{A}(K)}, \tau_p(\text{Ext}(K)))$ is separable.*

Question 2.2 *Let K be a compact convex subset of a separated locally convex space.*

(i) *Characterise, in terms of* $\text{Ext}(K)$*, when $\mathcal{A}_p(K)$ is τ-monolithic;*

(ii) *Characterise, in terms of* $\text{Ext}(K)$*, when $(B_{\mathcal{A}(K)}, \tau_p(\text{Ext}(K)))$ is $\aleph_0$-monolithic.*

3 Compactness and Tightness in $\mathcal{A}_p(K)$

In this section we shall first examine the question of when a separable compact subset of $\mathcal{A}_p(K)$ is metrizable. This is connected with our earlier work since a separable compact Hausdorff space is metrizable if, and only if, it is $\aleph_0$-monolithic. Secondly, we will examine the question of when $\mathcal{A}_p(K)$ is countably tight.

We begin by considering some sufficient conditions for a separable compact subset of $\mathcal{A}_p(K)$ to be metrizable.

Theorem 3.1 *Let K be a compact convex subset of a separated locally convex space. If* $\text{Ext}(K)$ *contains a dense Lindelöf subset X then every compact subset of $\mathcal{A}_p(K)$ is monolithic. In particular, every separable compact subset of $\mathcal{A}_p(K)$ is metrizable.*

Proof : Let C be a compact subset of $\mathcal{A}_p(K)$ and let M be an infinite subset of C with $|M| = \tau$. Consider the mapping $f : K \to \mathbb{R}^M$ defined by, $[f(k)](m) := m(k)$ for all $m \in M$. Let $T := f(K)$ then T is a compact convex subset with $w(T) \leq \tau$. Therefore by Theorem 2.12 there exists a compact convex set S with $w(S) \leq \tau$ and continuous affine surjections $g : K \to S$ and $h : S \to T$ such that $f = h \circ g$ and $g(X) \subseteq \text{Ext}(S)$. Then by Corollary 1.9, $M \subseteq f^{\#}(\mathcal{A}(T)) = g^{\#}(h^{\#}(\mathcal{A}(T)) \subseteq g^{\#}(\mathcal{A}(S))$. Hence by Theorem 2.11

$$H := \{h \in C(X) : h = a|_X \text{ for some } a \in \overline{M}^{\tau_p(\text{Ext}(K))}\} \subseteq (g|_X)^{\#}(C(g(X)))$$

and so $nw(H) \leq nw(C(g(X))) = nw(g(X)) \leq \tau$. The result now follows from the fact that the mapping $a \mapsto a|_X$ is a homeomorphism from $(\overline{M}^{\tau_p(\text{Ext}(K))}, \tau_p(\text{Ext}(K)))$ onto $(H, \tau_p(X))$. $\square$

The next theorem is the father of all compactness results in $\mathcal{A}_p(K)$.

Theorem 3.2 [8] *Let K be a compact convex subset of a separated locally convex space. If C is a norm bounded relatively countably compact subset of $\mathcal{A}_p(K)$ then C is relatively compact in $(\mathcal{A}(K), \tau_p(K))$. In particular, every norm bounded compact subset of $\mathcal{A}_p(K)$ is an Eberlein compacta i.e., is homeomorphic to a weakly compact subset of some Banach space.*

From this it follows that each countably compact subset of $\mathcal{A}_p(K)$ is compact, and in fact, is Lindelöf with respect to $\tau_p(K)$. Thus each countably compact subset of $\mathcal{A}_p(K)$ has countable tightness. Finally, since weakly compact subsets of Banach spaces are fragmented by their norm, every compact subset of $\mathcal{A}_p(K)$ is a Radon-Nikodým compacta (see [10] for the definition of Radon-Nikodým compact).

Next, let us recall that a space Y is *surlindelöf* if Y can be embedded in $C_p(X)$ for some Lindelöf space X.

Theorem 3.3 *Let K be a compact convex subset of a separated locally convex space. If $X \subseteq K$ is a Lindelöf boundary for K and*

(i) *the PFA holds or*

(ii) *the $\mathsf{MA}(\omega_1)$ holds and $\mathrm{Ext}(K) \subseteq X \subseteq K$*

then each separable compact subset of $(\mathcal{A}(K), \tau_p(X))$ is metrizable.

Proof : Suppose that Y is a separable compact subset of $(\mathcal{A}(K), \tau_p(X))$. Since $(\mathcal{A}(K), \tau_p(X))$ embeds into $C_p(X)$ and X is Lindelöf, Y is surlindelöf.
(i) If the PFA holds then every surlindelöf separable compact space is metrizable (see [11, Theorem 1.8]).
(ii) The compact space Y can be embedded in $A_p(K)$. Hence Y has countable tightness. If the $\mathsf{MA}(\omega_1)$ holds then every surlindelöf separable compact space of countable tightness is metrizable (see [11, Corololary 1.6]). $\square$

By contrast we have the following counter-examples.

Example 3.4

(i) *There exists a compact convex subset K of a separated locally convex space such that $\mathcal{A}_p(K)$ contains a non-metrizable separable compact subset;*

(ii) *If we assume that the continuum hypothesis holds then there exists a compact convex subset K of a separated locally convex space and a Lindelöf subset $\mathrm{Ext}(K) \subseteq X \subseteq K$ such that $(\mathcal{A}(K), \tau_p(X))$ contains a non-metrizable separable compact subset.*

Thus the existence of separable non-metrizable compact subsets of $(\mathcal{A}(K), \tau_p(X))$, for X a Lindelöf boundary of K, is independent of ZFC.

Question 3.1 *Let K be a compact convex subset of a separated locally convex space. Characterise, in terms of $\mathrm{Ext}(K)$, when every separable compact subset of $\mathcal{A}_p(K)$ is metrizable.*

We now consider the question of when $\mathcal{A}_p(K)$ is countably tight. Our first result in this direction may be deduced by modifying the proof of Theorem II.1.1. in [1].

Theorem 3.5 *Let K be a compact convex subset of a separated locally convex space. Then $\mathcal{A}_p(K)$ is countably tight if $[\mathrm{Ext}(K)]^n$ is Lindelöf for each $n \in \mathbb{N}$.*

On the other hand, we have the following example.

Example 3.6 [Example 5.7] *There exists a compact convex subset K of a separated locally convex space such that $\mathrm{Ext}(K)$ is Lindelöf but $\mathcal{A}_p(K)$ is not countably tight.*

4 Examples

All our examples in this section of the paper are based upon the following theorem.

Theorem 4.1 *Let S and T be compact convex subsets of separated locally convex spaces. Suppose that* $\mathrm{Ext}(S)$ *is compact,* $0_S \in \mathrm{Ext}(S)$ *and* $X := \mathrm{Ext}(S) \setminus \{0_S\}$. *If* $f : X \to T$ *is continuous,* $E_0 := \mathrm{Ext}(T) \times \{0_S\}$, $E_1 := \{(f(x), x) : x \in X\}$ *and* $E := E_0 \cup E_1$, *then* $\mathrm{Ext}(K) = E$ *where K is the convex closed hull of E in $T \times S$.*

Proof : By Milman's Theorem, $\mathrm{Ext}(K) \subseteq \overline{E_0 \cup E_1} = \overline{E_0} \cup \overline{E_1} \subseteq (T \times \{0_S\}) \cup \overline{E_1}$. Now, $\overline{E_1} \subseteq E_1 \cup (T \times \{0_S\})$. So $\mathrm{Ext}(K) \subseteq E_1 \cup (T \times \{0_S\})$. However, $\mathrm{Ext}(K) \cap [T \times \{0_S\}] \subseteq E_0$. Therefore, $\mathrm{Ext}(K) \subseteq E$. So it remains to show that $E \subseteq \mathrm{Ext}(K)$. To do this, it is sufficient to show that $E_1 \subseteq \mathrm{Ext}(K)$, since $T \times \{0_S\}$ is an extremal subset of K and hence by Proposition 1.2 $E_0 = \mathrm{Ext}(T) \times \{0_S\} = \mathrm{Ext}(T \times \{0_S\}) \subseteq \mathrm{Ext}(K)$. Let $x \in X$ and let $\pi_S : K \to S$ be the natural projection of K onto S. Since $\pi_S(K) = S$ and $x \in \mathrm{Ext}(S)$ it follows from Theorem 1.3 and Corollary 1.7 that $\pi_S^{-1}(x) \cap \mathrm{Ext}(K) \neq \varnothing$. Since $\mathrm{Ext}(K) \subseteq E$ and $\pi_S^{-1}(x) \cap E = \{(f(x), x)\}$ it follows that $\pi_S^{-1}(x) \cap \mathrm{Ext}(K) = \{(f(x), x)\}$, i.e., $(f(x), x) \in \mathrm{Ext}(K)$. $\square$

We shall apply this construction in a slightly more specialised setting.

If X is a locally compact, non-compact topological space then we shall denote by, $\alpha(X)$ the one-point compactification of X and we shall let α_X denote the point at infinity of $\alpha(X)$, i.e., $\alpha(X) := X \cup \{\alpha_X\}$.

For a subset A of a topological space X we shall denote by X_A the topological space obtained from X by retaining the topology at each point of $X \setminus A$ and by declaring that the points of A are isolated.

Let us also denote by, $\mathscr{K}$ the class of all triples (T, X, f), where T is a compact convex set (of some separated locally convex space), X is a locally compact, non-compact space and $f : X \to T \setminus \mathrm{Ext}(T)$ is continuous injection.

Given a compact convex subset K of a separated locally space we shall that a subset $M \subseteq K$ is *strongly affinely independent* if for any finite subset $\varnothing \neq F \subseteq M$ and map $f : F \to (-1, 1)$ there exists an $a \in \mathcal{A}(K)$ such that $a|_F = f$ and $\|a\|_\infty \leq 1$. The motivation for this definition comes from the fact that a subset $M \subseteq K$ is affinely independent if, and only if, for any finite subset $\varnothing \neq F \subseteq M$ and map $f : F \to (-1, 1)$ there exists an $a \in \mathcal{A}(K)$ such that $a|_F = f$. Then, with this terminology, we may present the following proposition.

Proposition 4.2 *Let* $(T, X, f) \in \mathscr{K}$ *and let* $S := P(\alpha(X))$. *Suppose that* $E_0 := \mathrm{Ext}(T) \times \{\delta_{\alpha_X}\}$, $E_1 := \{(f(x), \delta_x) : x \in X\}$, $\mathrm{Ext}(T) \subseteq M \subseteq T \setminus f(X)$, $Z := E_1 \cup (M \times \{\delta_{\alpha_X}\})$ *and* $Y := M \cup f(X)$. *Then we have:*

(i) $E := E_0 \cup E_1 = \mathrm{Ext}(K)$ *where K is the closed convex hull of E in $T \times S$;*

(ii) $\pi^{\#}(\mathcal{A}(T))$ *separates the points of E, where $\pi : K \to T$ is the projection of K onto T;*

(iii) *if Y is affinely independent then* $\pi^{\#}(\mathcal{A}(T))$ *is dense in* $(\mathcal{A}(K), \tau_p(Z))$;

(iv) *if Y is strongly affinely independent then* $\pi^{\#}(B_{\mathcal{A}(T)})$ *is dense in* $(B_{\mathcal{A}(K)}, \tau_p(Z))$;

(v) *Z is a continuous injective image of* $Y_{f(X)}$.

(vi) *if X has the discrete topology then Z is homeomorphic to $Y_{f(X)}$ and $Z \setminus E_0$ is homeomorphic to $[Y \setminus \mathrm{Ext}(T)]_{f(X)}$.*

Proof : The proofs of (ii), (iii) and (iv) following easily from the definitions. Moreover, (vi) follows from the proof of (v). So it remains to justify (i) and (v).

(i) This follows from Theorem 4.1 and the fact that (a) $\mathrm{Ext}(S)$ is homeomorphic to $\alpha(X)$; which is compact, (b) $\delta_{\alpha_X} \in \mathrm{Ext}(S)$ and (c) the mapping $\delta_x \mapsto f(x)$ is continuous mapping from $\mathrm{Ext}(S) \setminus \{\delta_{\alpha_X}\}$ into T.

(v) Consider the mapping $h : Y_{f(X)} \to Z$ by,

$$h(x) := \begin{cases} (x, \delta_{\alpha_X}) & \text{if } x \in M \\ (x, \delta_{f^{-1}(x)}) & \text{if } x \in f(X). \end{cases}$$

Clearly h is one-to-one and onto. It is also clear that h is continuous at each point of $f(X)$. So let us consider $x \in M$. Let $f^* : \mathrm{Ext}(S) \setminus \{\delta_{\alpha_X}\} \to T$ be defined by, $f^*(\delta_x) := f(x)$. As mentioned previously, f^* is continuous. Let U and V be open neighbourhoods of x and δ_{α_X} respectively, i.e., $h(x) \in U \times V$. Now, $\mathrm{Ext}(S) \setminus V$ is compact and hence $C := f^*(\mathrm{Ext}(S) \setminus V)$ is compact subset of $Y \setminus \{x\}$. Thus, $U \setminus C$ is a neighbourhood of x and $h(U \setminus C) \subseteq V$. $\square$

Given $(T, X, f) \in \mathscr{K}$, let (i) $E_0(T, X, f) := \mathrm{Ext}(T) \times \{\delta_{\alpha_X}\}$; (ii) $E_1(T, X, f) := \{(f(x), \delta_x) : x \in X\}$; (iii) $E(T, X, f) := E_0(T, X, f) \cup E_1(T, X, f)$ and (iv) $K(T, X, f)$ denote the compact convex set constructed in Proposition 4.2. Moreover, if $\mathrm{Ext}(T) \subseteq M \subseteq T \setminus f(X)$ let us define (v) $Z(T, X, f, M) := E_1(T, X, f) \cup M \times \{\delta_{\alpha_X}\}$ and (vi) $Y(T, X, f, M) := M \cup f(X)$.

It is easy to see that $\mathrm{Ext}(T)$ is homeomorphic $E_0(T, X, f)$ and X is homeomorphic to $E_1(T, X, f)$ Thus, if both X and $\mathrm{Ext}(T)$ are separable (Lindelöf) then $E(T, X, f)$ is separable (Lindelöf). Furthermore, $K(T, X, f)$ is metrizable if, and only if, T is metrizable and X is Lindelöf or, equivalently, T is metrizable and α_X is a G_δ-point of $\alpha(X)$ (see Corollary 2.15). In particular, if X has the discrete topology then $K(T, X, f)$ is metrizable if, and only if, T is metrizable and X is countable.

Before we can give our first concrete example we need a couple of elementary facts from analysis.

Lemma 4.3 *Let K be a compact Hausdorff space, $\{t_k : 1 \le k \le n\} \subseteq K$, $\{r_k : 1 \le k \le n+1\} \subseteq (-1, 1)$ and let m be a non-atomic regular Borel probability measure on K. Then there exists a $f \in C(K)$ such that $\|f\|_\infty \le 1$, $f(t_k) = r_k$ for all $1 \le k \le n$ and $\int_K f \, \mathrm{d}m = r_{n+1}$.*

Proof : Let $\varepsilon := 1 - |r_{n+1}| > 0$ and let V and U be open subsets of K such that:
(i) $\{t_k : 1 \le k \le n\} \subseteq V \subseteq \overline{V} \subseteq U$ and (ii) $m(U) < \varepsilon/2$.
Choose $h : K \to [-1, 1]$ such that:
(i) h is continuous; (ii) $h \equiv 0$ on $K \setminus V$ and (iii) $h(t_k) = r_k$ for each $1 \le k \le n$.
Then choose $g : K \to [0, 1]$ such that:
(i) g is continuous; (ii) $g \equiv 0$ on $\overline{V}$ and (iii) $g \equiv 1$ on $K \setminus U$.
Note that

$$0 < 1 - \varepsilon/2 \le \int_K g \, \mathrm{d}m \le 1 \quad \text{and} \quad \left| \int_K h \, \mathrm{d}m \right| \le \varepsilon/2.$$

Let $\lambda := \dfrac{r_{n+1} - \int_K h \, \mathrm{d}m}{\int_K g \, \mathrm{d}m}$. Then,

$$|\lambda| \le \frac{|r_{n+1}| + |\int_K h \, \mathrm{d}m|}{\int_K g \, \mathrm{d}m} \le \frac{[(1-\varepsilon) + \varepsilon/2]}{1 - \varepsilon/2} = 1$$

and $(h+\lambda g)(t_k) = h(t_k) = r_k$ for each $1 \le k \le n$. Moreover,

$$\int_K (h+\lambda g)\ \mathrm{d}m = \int_K h\ \mathrm{d}m + \lambda \int_K g\ \mathrm{d}m = r_{n+1}$$

and $|(h+\lambda g)(t)| = |h(t)| \le 1$ if $t \in V$ and $|(h+\lambda g)(t)| = |\lambda||g(t)| \le |\lambda| \le 1$ if $t \notin V$. Hence if $f := h + \lambda g$ then f satisfies the conclusions of the lemma. $\square$

Corollary 4.4 *For each $x \in [0,1]$, let μ_x be the Lebesgue measure on $\{x\}\times[0,1]$. Then the mapping $g : [0,1] \to P([0,1]^2)$ defined by, $g(x) := \mu_x$ is a topological embedding and $\mathrm{Ext}(P([0,1]^2)) \cup g([0,1])$ is strongly affinely independent in $P([0,1]^2)$.*

Proof : To show that g is a topological embedding it is sufficient to show that g is continuous since $[0,1]$ is compact and g is 1-to-1. To see that g is continuous consider the following. Let $f \in C([0,1]^2)$ and for each $x \in [0,1]$ define $f_x : [0,1] \to \mathbb{R}$ by, $f_x(y) := f(x,y)$. Then $x \mapsto f_x$ is a continuous map from $[0,1]$ into $(C[0,1], \|\cdot\|_\infty)$. Now for any $x, y \in [0,1]$

$$\left|\int_{[0,1]^2} f\ \mathrm{d}\mu_x - \int_{[0,1]^2} f\ \mathrm{d}\mu_y\right| = \left|\int_{[0,1]} f_x\ \mathrm{d}\mu - \int_{[0,1]} f_y\ \mathrm{d}\mu\right| = \left|\int_{[0,1]} (f_x - f_y)\ \mathrm{d}\mu\right| \le \|f_x - f_y\|_\infty$$

where μ is the Lebesgue measure on $[0,1]$. Therefore, g is continuous. The fact that $\mathrm{Ext}(P([0,1]^2)) \cup g([0,1])$ is strongly affinely independent in $P([0,1]^2)$ follows from Lemma 4.3 in conjunction with Tietze's extension theorem and Corollary 1.18. $\square$

Example 4.5 *There exists a non-metrizable compact convex subset K of a separated locally convex space and a Lindelöf space Z such that $\mathrm{Ext}(K) \subseteq Z \subseteq K$ and $B_{\mathcal{A}(K)}$ is separable with respect to $\tau_p(Z)$. In particular, $(B_{\mathcal{A}(K)}, \tau_p(Z))$ is not $\aleph_0$-monolithic.*

Proof : Let (i) A be an uncountable subset of $[0,1]$ that does not contain any uncountable compact subsets (e.g., A could be a Bernstein set or a perfectly meagre set); (ii) μ_x be the Lebesgue measure on $\{x\} \times [0,1]$ for each $x \in [0,1]$; (iii) $T := P([0,1]^2)$; (iv) $g : [0,1] \to T$ be defined by, $g(x) := \mu_x$; (v) X be the set A endowed with the discrete topology; (vi) $f := g|_X$ and (vii) $M := \mathrm{Ext}(T) \cup g([0,1] \setminus A)$.

Then $(T, X, f) \in \mathscr{K}$ and $K(T, X, f)$ is not metrizable since X is uncountable. Furthermore, $B_{\mathcal{A}(K(T,X,f))}$ is separable with respect to $\tau_p(Z(T, X, f, M))$ since (i) $Y(T, X, f, M) = \mathrm{Ext}(T) \cup g([0,1])$; which is by Corollary 4.4, strongly affinely independent in T and (ii) $(\mathcal{A}(T), \|\cdot\|_\infty)$ is separable.

To show that $Z(T, X, f, M)$ is Lindelöf it is sufficient to show that $Z(T, X, f, M)\setminus E_0(T, X, f)$ is Lindelöf. Hence by Proposition 4.2 part(vi) it is sufficient to show that $[Y(T, X, f, M) \setminus \mathrm{Ext}(T)]_{f(X)} = [g([0,1])]_{f(X)}$ is Lindelöf. However, g is a homeomorphism from $[0,1]_A$ onto $[g([0,1])]_{f(X)}$ and $[0,1]_A$ is Lindelöf if, and only if, A does not contain any uncountable compact subsets. $\square$

We can now use Proposition 4.2 to construct another counter-example that was mentioned in Section 2.

Example 4.6 *There exists a non-metrizable compact convex subset K of a separated locally convex space such that both $\mathrm{Ext}(K)$ and $(B_{\mathcal{A}(K)}, \tau_p(\mathrm{Ext}(K)))$ are separable. In particular, $(B_{\mathcal{A}(K)}, \tau_p(\mathrm{Ext}(K)))$ is not $\aleph_0$-monolithic.*

Proof : Let $C \subseteq [0,1]$ be the usual Cantor set and let D be a countable discrete subset of $[0,1] \setminus C$ such that C is the set of all limits of points of D. For each $c \in C$, let $\xi_c := \{d_n^c : n \in \mathbb{N}\} \subseteq D$ be chosen so that $\lim_{n\to\infty} d_n^c = c$. Let $X := C \cup D$ and let us define a base for the topology on X by:

$$\mathscr{B} := \{\{d\} : d \in D\} \cup \{\{c\} \cup \xi_c \setminus F : c \in C \text{ and } F \text{ is a finite subset of } D\}.$$

Then X is locally compact, separable (since D is dense in X) and Hausdorff. However, X is neither metrizable nor compact. Let $T := P([0,1]^2)$ and let $f : X \to T$ be defined by, $f(x) := \mu_x$ where μ_x is the Lebesgue measure on $\{x\} \times [0,1]$. Then $(T, X, f) \in \mathscr{K}$ since f is a continuous injection into T. Now $K(T, X, f)$ is not metrizable since X is not metrizable. Furthermore, since both X and $\mathrm{Ext}(T)$ are separable $E(T, X, f)$ is separable and as before $B_{\mathcal{A}(K(T,X,f))}$ is separable with respect to $\tau_p(E(T, X, f))$ since (i) $Y(T, X, f, \varnothing) = \mathrm{Ext}(T) \cup f(X)$; which is by Lemma 4.4, strongly affinely independent in T and (ii) $(\mathcal{A}(T), \|\cdot\|_\infty)$ is separable.

Next we provide the required counter-examples from Section 3.

Lemma 4.7 *Let X be a compact Hausdorff space and let $x \in X$ be an isolated point. Then $a_x \in \mathcal{A}(P(X))$ where, $a_x : P(X) \to [0,1]$ is defined by, $a_x(\mu) := \mu(\{x\})$.*

Proof : Let $f_x : X \to \{0,1\}$ be defined by, $f_x(y) := 1$ if, and only if, $x = y$. Then $f_x \in C(X)$ and $a_x(\mu) = \int_X f_x \, \mathrm{d}\mu$ for all $\mu \in P(X)$. It now follows, as in the proof of Corollary 1.18, that $a_x \in \mathcal{A}(P(X))$. $\square$

Note: if $x \in X$ is a G_δ-point then by the same argument as above it can be shown that a_x is of the first Baire class.

Example 4.8 *There exists a compact convex subset K of a separated locally convex space such that $\mathcal{A}_p(K)$ contains a non-metrizable separable compact subset.*

Proof : Let (i) $0_\mathbb{N} \in \mathbb{R}^\mathbb{N}$ be the zero function on $\mathbb{N}$; (ii) $e_n \in \{0,1\}^\mathbb{N}$ be defined by, $e_n(m) := 1$ if, and only if, $m = n$, for each $n \in \mathbb{N}$; (iii) $T := \overline{\mathrm{co}}\,(\{0_\mathbb{N}\} \cup \{e_n : n \in \mathbb{N}\}) \subseteq [0,1]^\mathbb{N} \subseteq \mathbb{R}^\mathbb{N}$.

For an uncountable almost disjoint family $\mathscr{A}$ of subsets of $\mathbb{N}$ we write $\Psi(\mathscr{A}) := \mathscr{A} \cup \mathbb{N}$ and define a base for the topology on $\Psi(\mathscr{A})$ by,

$$\mathscr{B} := \{\{n\} : n \in \mathbb{N}\} \cup \{\{M\} \cup M \setminus F : M \in \mathscr{A} \text{ and } F \text{ is a finite subset of } \mathbb{N}\}.$$

Then $\Psi(\mathscr{A})$ is locally compact, separable (since $\mathbb{N}$ is dense in $\Psi(\mathscr{A})$) and Hausdorff. On the other hand, $\Psi(\mathscr{A})$ is neither compact nor metrizable. However, if $\mathscr{A}$ is a maximal family of almost disjoint subsets of $\mathbb{N}$ then $\Psi(\mathscr{A})$ is pseudo-compact.

We shall denote by X the set $\mathscr{A}$ endowed with the discrete topology and we shall define $f : X \to T$ by,

$$[f(M)](n) := \begin{cases} 2^{-n} & \text{if } n \in M \\ 0 & \text{if } n \notin M \end{cases} \quad \text{for all } M \in \mathscr{A}.$$

Next, set $K := K(T, X, f) \subseteq T \times P(\alpha(X))$ and define $\pi : \alpha(\Psi(\mathscr{A})) \to \mathcal{A}(K)$ by,

$$[\pi(x)](g, \mu) := \begin{cases} 2^x g(x) & \text{if } x \in \mathbb{N} \\ \mu(\{x\}) & \text{if } x \in \mathscr{A} \\ 0_K & \text{if } x = \alpha_{\Psi(\mathscr{A})} \end{cases} \quad \text{for all } (g, \mu) \in K.$$

where 0_K is the zero function on K. It is easy to check that π is well-defined (i.e., $\pi(x) \in \mathcal{A}(K)$ for each $x \in \alpha(\Psi(\mathscr{A}))$) and injective. Moreover, it is routine to check that $\pi : \alpha(\Psi(\mathscr{A})) \to \mathcal{A}_p(K)$ is continuous and hence a topological embedding. $\square$

Remark 4.1 *In the previous example $\mathcal{A}_p(K)$ is not $\aleph_0$-monolithic, but by Theorem 2.18 $(B_{\mathcal{A}(K)}, \tau_p(\mathrm{Ext}(K)))$ is $\aleph_0$-monolithic. Also if $\mathscr{A}$ is a maximal almost disjoint family of subsets of $\mathbb{N}$ then $\pi(\Psi(\mathscr{A}))$ is a non-compact, pseudo-compact subset of $\mathcal{A}_p(K)$.*

Let X be a topological space. Then we will say that a subset A of X *concentrates around* a subset B of X if for each open subset U of X with $B \subseteq U$, $|A \setminus U| \leq \aleph_0$.

Let $\mathscr{F} := \{h \in \{0,1\}^{\mathbb{N}} : |\{n \in \mathbb{N} : h(n) \neq 0\}| < \aleph_0\}$ and for any $A \subseteq \mathbb{N}$ let, $h_A \in \{0,1\}^{\mathbb{N}}$ be defined by, $h_A(n) := 1$ if, and only if, $n \in A$. Finally, for any family $\mathscr{A}$ of subsets of $\mathbb{N}$ let, $\mathscr{A}^* := \{h_A : A \in \mathscr{A}\}$.

Proposition 4.9 *If the continuum hypothesis holds then there exists an uncountable maximal almost disjoint family $\mathscr{A}$ of subsets of $\mathbb{N}$ such that $\mathscr{A}^*$ concentrates around $\mathscr{F}$, with respect to $(\{0,1\}^{\mathbb{N}}, \tau_p(\mathbb{N}))$.*

Proof : Let $\mathcal{G} := \{\mathscr{G} \subseteq \{0,1\}^{\mathbb{N}} : \mathscr{G}$ is a G_δ-set and $\mathscr{F} \subseteq \mathscr{G}\}$ and $\mathscr{M} := \{M \subseteq \mathbb{N} : |\mathbb{N} \setminus M| = \omega\}$. Since $|\mathcal{G}| = |\mathscr{M}| = 2^\omega = \omega_1$, enumerate these families by countable ordinals: $\mathcal{G} = \{\mathscr{G}_\alpha : \alpha < \omega_1\}$, $\mathscr{M} = \{M_\alpha : \alpha < \omega_1\}$. For $\alpha < \omega_1$, let $\mathscr{P}_\alpha := \bigcap_{\beta \leq \alpha} \mathscr{G}_\alpha$. For $M \subseteq \mathbb{N}$, denote $S(M) := \{h_L : L \subseteq M\}$. Induction by $\alpha < \omega_1$, we will construct $\{A_\alpha \subseteq \mathbb{N} : \alpha < \omega_1\}$. Assume, we constructed $\{A_\beta \subseteq \mathbb{N} : \beta < \alpha\}$. Put $A_\alpha := \varnothing$ if there exist a finite $C \subset \alpha$ such that $M_\alpha \setminus \bigcup_{\beta \in C} A_\beta$ is finite. Otherwise, there exist an infinite $M \subseteq M_\alpha$ such that $M \cap A_\beta$ is finite for any $\beta < \alpha$. Since $\mathscr{F} \cap S(M)$ is dense in $S(M)$, $G := (S(M) \cap \mathscr{P}_\alpha) \setminus \mathscr{F}$ is nonempty. Take $A_\alpha \subseteq \mathbb{N}$ such that $h_{A_\alpha} \in G$. One can see that (1) A_α is infinite; (2) $A_\alpha \cap A_\beta$ is finite for any $\beta < \alpha$; (3) $A_\alpha \subseteq M_\alpha$; (4) $h_{A_\alpha} \in \mathscr{P}_\alpha$.
Put $\mathscr{A} := \{A_\alpha : A_\alpha \neq \varnothing, \alpha < \omega_1\}$. (1) and (2) implies that $\mathscr{A}$ is almost disjoint family. (3) imply that $\mathscr{A}$ is maximal and hence uncountable. (4) imply that $\mathscr{A}^*$ concentrates around $\mathscr{F}$. $\square$

Example 4.10 *If we assume the continuum hypothesis holds then there exists a compact convex subset K of a separated locally convex space and a Lindelöf subset $\mathrm{Ext}(K) \subseteq Z \subseteq K$ such that $(\mathcal{A}(K), \tau_p(Z))$ contains a non-metrizable separable compact subset.*

Proof : Let $\mathscr{A}$ be an uncountable maximal almost disjoint family of subsets of $\mathbb{N}$ such that $\mathscr{A}^*$ concentrates around $\mathscr{F}$. Let T, X, f, K and π be as in Example 4.8. Define $g : \{0,1\}^{\mathbb{N}} \to T$ by,

$$[g(h)](n) := \begin{cases} 2^{-n} & \text{if } h(n) \neq 0 \\ 0 & \text{if } h(n) = 0 \end{cases} \quad \text{for all } h \in \{0,1\}^{\mathbb{N}}.$$

Then g is a topological embedding of $\{0,1\}^{\mathbb{N}}$ into T. Let $M := g(\mathscr{F})$. Then one can check, as in Example 4.8, that $\pi : \alpha(\Psi(\mathscr{A})) \to (\mathcal{A}(K), \tau_p(Z(T, X, f, M)))$ is continuous.

So it remains to show that $Z(T, X, f, M)$ is Lindelöf. However, by Proposition 4.2 part(vi) this is equivalent to showing that $[Y(T, X, f, M)]_{f(X)}$ is Lindelöf. Let $\mathscr{H} := \mathscr{A}^* \cup \mathscr{F}$. Then $g|_{\mathscr{H}}$ is a homeomorphism from $\mathscr{H}$ onto $Y(T, X, f, M)$ and so $g|_{\mathscr{H}}$ is a homeomorphism from $[\mathscr{H}]_{\mathscr{A}^*}$ onto $[Y(T, X, f, M)]_{f(X)}$. However, since $\mathscr{A}^*$ concentrates around $\mathscr{F}$, $[\mathscr{H}]_{\mathscr{A}^*}$ is Lindelöf. In fact, $[\mathscr{H}]_{\mathscr{A}^*}$ is Lindelöf if, and only if, $\mathscr{A}^*$ concentrates around $\mathscr{F}$. $\square$

5 Choquet Boundaries

For a nonempty set X we shall denote by, $\mathbf{1}_X$ the mapping $\mathbf{1}_X : X \to \mathbb{R}$ defined by, $\mathbf{1}_X(x) := 1$ for all $x \in X$. If it is clear from the context then we shall simply write $\mathbf{1}$ for $\mathbf{1}_X$.

Suppose that X is a compact Hausdorff space and M is a linear (not necessarily closed) subspace of $(C(X), \|\cdot\|_\infty)$ containing the constant functions. The *state space* $K(M)$ of M is:

$$\{x^* \in M^* : \|x^*\| = 1 \text{ and } x^*(\mathbf{1}) = 1\}.$$

Then $(K(M), \text{weak}^*)$ is a compact convex subset of the separated locally convex space (M^*, weak^*).

For each $x \in X$, let $\varphi(x) \in K(M)$ be defined by, $\varphi(x)(f) := f(x)$ for all $f \in M$. Note that φ is a continuous mapping from X into the weak* topology on $K(M)$. Moreover, if M separates the points of X then φ is 1-to-1 and hence an embedding of X into $K(M)$. By applying Theorem 1.1 one can show that $K(M) = \overline{\text{co}}^{\text{weak}^*}[\varphi(X)]$ and so by Milman's Theorem, (see Theorem 1.4), $\text{Ext}(K) \subseteq \varphi(X)$. Hence one can formulate the following definition.

If X is a compact Hausdorff space and $\mathbf{1} \in M \subseteq C(X)$ is a linear subspace then the *Choquet boundary* $B(M)$ for M is: $\{x \in X : \varphi(x) \in \text{Ext}(K(M))\}$, i.e., $\text{Ext}(K(M)) = \varphi(B(M))$.

Given a normed linear space $(X, \|\cdot\|)$ over $\mathbb{R}$ and an element $x \in X$ we define $\widehat{x} : X^* \to \mathbb{R}$ by, $\widehat{x}(x^*) := x^*(x)$ for all $x^* \in X^*$. Then $(X^*, \text{weak}^*)^* = \{\widehat{x} : x \in X\}$, [6, Theorem 3.17].

Proposition 5.1 *Let X be a compact Hausdorff space and let $\mathbf{1} \in M \subseteq C(X)$ be a closed linear subspace. Then the mapping $T : (M, \tau_p(B(M))) \to \mathcal{A}_p(K(M))$ defined by, $T(m) := \widehat{m}|_{K(M)}$ is a linear topological isomorphism that also preserves the norm (i.e., $\|T(m)\|_\infty = \|m\|_\infty$ for all $m \in M$).*

Proof : It is easy to see that T does indeed map into $\mathcal{A}(K(M))$ and is linear. It is also easy to see that T is a topological embedding from $(M, \tau_p(B(M)))$ into $\mathcal{A}_p(K(M))$ since

$$T(m)(\varphi(x)) = [\varphi(x)](m) = m(x) \text{ for all } x \in B(M) \text{ and } m \in M.$$

Moreover since $\text{Ext}(K(M)) \subseteq \varphi(X)$ it follows that $\|T(m)\|_\infty = \|m\|_\infty$ for each $m \in M$. So it remains to show that T is surjective. As T is an isometry, $T(M)$ is a closed linear subspace of $(\mathcal{A}(K(M)), \|\cdot\|_\infty)$ that also contains $\mathbf{1}_{K(M)}$. Therefore, by Proposition 1.5, $T(M) = \mathcal{A}(K(M))$. $\square$

The main result that we shall use concerning Choquet boundaries is the following.

Theorem 5.2 [12, Proposition 6.2] *Suppose that X is a compact Hausdorff space and $\mathbf{1} \in M$ is a linear subspace of $C(X)$ that separates the points of X. Then $x \in B(M)$ if, and only if, $\mu(\{x\}) = 1$ for each regular Borel probability measure μ on X such that*

$$\int_X f \, \mathrm{d}\mu = f(x) \quad \textit{for all } f \in M.$$

Corollary 5.3 *Suppose that X is a compact Hausdorff space and $\mathbf{1} \in M$ is a linear subspace of $C(X)$ that separates the points of X. If $f \in M$, $x \in X$ and $f(y) < f(x)$ for all $y \in X \setminus \{x\}$ then $x \in B(M)$, i.e., $B(M)$ contains all the "peak-points" with respect to M.*

Example 5.4 *For each $(a,b,c) \in \mathbb{R}^3$ let $p_{(a,b,c)} : [0,1] \to \mathbb{R}$ be defined by, $p_{(a,b,c)}(x) := ax^2 + bx + c$. Let $Q := \{p_{(a,b,c)} : (a,b,c) \in \mathbb{R}^3\}$. Then $\mathbf{1} \in Q$, Q separates the points of $[0,1]$ and $B(Q) = [0,1]$.*

Proof : Suppose that $x \in [0,1]$. Define $(a,b,c) \in \mathbb{R}^3$ by, $a := -1$, $b := 2x$ and $c := 1 - x^2$ then $p_{(a,,b,c)}(x) = 1$ and $p_{(a,b,c)}(y) < 1$ for all $y \in [0,1] \setminus \{x\}$ since $p_{(a,b,c)}(y) = 1 - (y-x)^2$. $\square$

Example 5.5 *Let X be a compact Hausdorff space and let μ be a non-atomic regular Borel probability measure on X. For each $x_0 \in X$ define*

$$M^{\mu}_{x_0} := \left\{ f \in C(X) : \int_X f \ \mathrm{d}\mu = f(x_0) \right\}.$$

Then $\mathbf{1} \in M^{\mu}_{x_0}$, $M^{\mu}_{x_0}$ separates the points of X and $B(M^{\mu}_{x_0}) = X \setminus \{x_0\}$.

Proof : Clearly $M^{\mu}_{x_0}$ is a closed linear subspace of $C(X)$ that contains all the constant functions and separates the points of X (see, Lemma 4.3). Thus, by Theorem 5.2, $B(M) \subseteq X \setminus \{x_0\}$. Now, suppose that $x \in X \setminus \{x_0\}$ and m is any regular Borel probability measure on X such that

$$\int_X f \ \mathrm{d}m = f(x) \quad \text{for all } f \in M^{\mu}_{x_0}.$$

Define $x^* : C(X) \to \mathbb{R}$ and $y^* : C(X) \to \mathbb{R}$ by,

$$x^*(f) := \int_X f \ \mathrm{d}(m - \delta_x) \quad \text{and} \quad y^*(f) := \int_X f \ \mathrm{d}(\mu - \delta_{x_0}).$$

Then $x^*, y^* \in (C(X), \|\cdot\|_\infty)^*$ and $M^{\mu}_{x_0} = \mathrm{Ker}(y^*) \subseteq \mathrm{Ker}(x^*)$ and so $x^* = \lambda y^*$ for some $\lambda \in \mathbb{R}$. Therefore, by Riesz's representation theorem $(m - \delta_x) = \lambda(\mu - \delta_{x_0})$. In particular, this implies that

$$m(\{x\}) - 1 = (m - \delta_x)(\{x\}) = \lambda(\mu - \delta_{x_0})(\{x\}) = \lambda\mu(\{x\}) - \lambda\delta_{x_0}(\{x\}) = 0 - 0 = 0.$$

Therefore, $m(\{x\}) = 1$ and so by Theorem 5.2, $x \in B(M^{\mu}_{x_0})$. $\square$

Remark 5.1 *If X is a compact Hausdorff space that is not scattered then there exists a continuous surjection $f : X \to [0,1]$, [14, §8.5.4, (i) $\Rightarrow$ (ii)]. Therefore, $f^{\#\#}(P(X)) \to P([0,1])$ is also a surjection. In particular there is some measure $m \in P(X)$ that maps onto the restriction of the Lebesgue measure on $[0,1]$. This measure m is necessarily non-atomic. Hence every compact Hausdorff space that is not scattered possesses a non-atomic regular Borel probability measure.*

For a topological space X be shall denote by $\mathscr{B}(X)$ the family of all Borel sets on X.

Theorem 5.6 *Suppose that X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous surjection. If M is a linear subspace of $C(X)$ that separates the points of X and N is a linear subspace of $C(Y)$ that (i) contains all the constant functions; (ii) separates the points of Y; (iii) $f^{\#}(N) \subseteq M$ and (iv) $B(N) = Y$ then $B(M) = \bigcup\{B(M_y) : y \in Y\}$, where for each $y \in Y$, $M_y := \{h \in C(f^{-1}(\{y\})) : h = g|_{f^{-1}(\{y\})}$ and $g \in M\}$.*

Proof : Consider $x \in B(M)$. We claim that $x \in B(M_{f(x)})$. To justify this let m be any regular Borel probability measure on $f^{-1}(f(x))$ such that

$$\int_{f^{-1}(f(x))} g \ \mathrm{d}m = g(x) \quad \text{for all } g \in M_{f(x)}.$$

Define $\widetilde{m} : \mathscr{B}(X) \to [0,1]$ by, $\widetilde{m}(B) := m(B \cap f^{-1}(f(x)))$ for every $B \in \mathscr{B}(X)$. Then $\widetilde{m}$ is a well-defined regular Borel probability measure on X and

$$\int_X g \ \mathrm{d}\widetilde{m} = \int_{f^{-1}(f(x))} g|_{f^{-1}(f(x))} \ \mathrm{d}m = g|_{f^{-1}(f(x))}(x) = g(x) \text{ for all } g \in M.$$

Since $x \in B(M)$, $m(\{x\}) = \widetilde{m}(\{x\}) = 1$. Thus, $x \in B(M_{f(x)}) \subseteq \bigcup_{y \in Y} B(M_y)$.

Conversely, suppose that $x \in \bigcup_{y \in Y} B(M_y)$ and m is a regular Borel probability measure on X such that $h(x) = \displaystyle\int_X h \, \mathrm{d}m$ for all $h \in M$. Let $\nu := f^{\#\#}(m) \in P(Y)$. Then for all $g \in N$, $(g \circ f) = f^{\#}(g) \in M$ and

$$g(f(x)) = (g \circ f)(x) = \int_X (g \circ f) \, \mathrm{d}m = \int_Y g \, \mathrm{d}\nu.$$

Since $f(x) \in B(N)$, $m(f^{-1}(f(x))) = \nu(\{f(x)\}) = 1$. Define $m^* : \mathscr{B}(f^{-1}(f(x))) \to [0,1]$ by, $m^*(B) := m(B)$ for each $B \in \mathscr{B}(f^{-1}(f(x)))$. Then m^* is a well-defined regular Borel probability measure on $f^{-1}(f(x))$ and

$$\int_{f^{-1}(f(x))} g|_{f^{-1}(f(x))} \, \mathrm{d}m^* = \int_X g \, \mathrm{d}m = g(x) = g|_{f^{-1}(f(x))}(x) \text{ for all } g \in M.$$

Therefore, since $x \in B(M_{f(x)})$, $m(\{x\}) = m^*(\{x\}) = 1$. This shows that $x \in B(M)$. $\square$

Let A be an arbitrary subset of $[0,1]$ and let $Z_A := [0,1] \times \{-1\} \cup A \times [0,1]$. We shall equip Z_A with a topology as follows. First, let $\mathscr{B}$ denote a base for the usual topology on $[0,1]$ and let $f : Z_A \to [0,1]$ be defined by, $f(x,y) := x$. Then a sub-base for a topology on Z_A is:

$$\mathscr{B}_A := \{f^{-1}(U) : U \in \mathscr{B}\} \cup \{Z_A \setminus (\{a\} \times [0,1]) : a \in A\} \cup \{\{a\} \times U : U \in \mathscr{B}, a \in A\}.$$

The space Z_A endowed with this topology is compact and Hausdorff. Moreover, the projection mapping $f : Z_A \to [0,1]$ defined above is continuous with respect to this topology and for each $a \in A$, $\{a\} \times [0,1]$ is a clopen subset that is homeomorphic to $[0,1]$.

For each $a \in A$, define $m_a : \mathscr{B}(Z_A) \to [0,1]$ by, $m_a(B) := \lambda(B')$ where, $B' := \{t \in [0,1] : (a,t) \in B\}$ and λ is the Lebesgue measure on $\mathbb{R}$.

Let us now denote

$$M_A := \left\{ g \in C(Z_A) : g((a,-1)) = \int_{Z_A} g \, \mathrm{d}m_a \text{ for all } a \in A \right\}.$$

Then from Theorem 5.6 and Example 5.5 we see that $B(M_A) = Z_A \setminus (A \times \{-1\})$.

Example 5.7 *Let B be a Bernstein subset of $[0,1/2]$ and let $A := B \cup [1/2 + (0,1/2) \setminus B]$. Then A is a Bernstein subset of $[0,1]$ and $B(M_A)$ is Lindelöf. Let L be the set of all functions $f \in M_A$ such that*

$$|\{b \in B : \text{ for some } y \in [0,1], f(b,y) > 1/2 \text{ and } f(b+1/2,-1) > 1/2\}| < \aleph_0.$$

Then $\mathbf{1} \in \overline{L}^{\tau_p(B(M_A))}$ but there are no countable subsets C of L such that $\mathbf{1} \in \overline{C}^{\tau_p^2(B(M_A))}$.

Finally, let us show that in Example 5.7, $\mathrm{Ext}(K(M_A))$ is not $\aleph_0$-stable in $K(M_A)$. Note that since $(K(M_A), \text{weak}^*)$ is a normal topological space it will sufficient to show that $\mathrm{Ext}(K(M_A))$ is not $\aleph_0$-stable in $\varphi(Z_A)$. In fact, since φ is a homeomorphism it will be sufficient to show that $B(M_A)$ is not $\aleph_0$-stable in Z_A. This is what we do next.

Let A be an arbitrary subset of $[0,1]$ and let $Y_A := [0,1] \times \{-1\} \cup A \times \{0\}$. We shall equip Y_A with a topology as follows. First, let $\mathscr{B}$ denote a base for the usual topology on $[0,1]$ and let $g : Y_A \to [0,1]$ be defined by, $g(x,y) := x$. Then a sub-base for a topology on Y_A is:

$$\mathscr{B}_A := \{g^{-1}(U) : U \in \mathscr{B}\} \cup \{Z_A \setminus \{(a,0)\} : a \in A\} \cup \{\{(a,0)\} : a \in A\}.$$

The space Y_A endowed with this topology is compact and Hausdorff. Moreover, the projection mapping $g : Y_A \to [0,1]$ defined above is continuous with respect to this topology and g separates the points $([0,1] \setminus A) \times \{-1\} \cup A \times \{0\}$.

Let $f : Z_A \to Y_A$ be defined by,

$$f(x,y) := \begin{cases} (x,y) & \text{if } y = -1 \\ (x,0) & \text{if } y \in [0,1]. \end{cases}$$

Then f is continuous and g separates the points of $f(B(M_A)) = ([0,1] \setminus A) \times \{-1\} \cup A \times \{0\}$, however, $nw(f(B(M_A))) = |A| > \aleph_0$. Therefore, $B(M_A)$ is not $\aleph_0$-stable in Z_A.

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Warren B. Moors,
Department of Mathematics,
The University of Auckland,
Private Bag 92019, Auckland,
New Zealand.
Email: moors@math.auckland.ac.nz
Phone: (NZ)(9) 3737-999 ext. 84746
Fax: (NZ)(9) 3737-457.