Dual Differentiation Spaces

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Abstract. We show that if $(X, \|\cdot\|)$ is a Banach space that admits an equivalent locally uniformly rotund norm and the set of all norm attaining functionals is residual then the dual norm $\|\cdot\|^*$ on X^* is Fréchet at the points of a dense subset of X^*. This answers the main open problem in a 2018 paper by Guiarao, Montesinos and Zizler [Studia Math. **241**, pages 71–86].

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We shall begin with some basic notation and assumed terminology. In this short note all normed linear spaces will be over the field of real numbers (denoted $\mathbb{R}$). The *closed unit ball* in a normed linear space $(X, \|\cdot\|)$ will be denoted by B_X and the *norm closed convex hull* of a subset K of a normed linear space $(X, \|\cdot\|)$ will be denoted by, $\overline{\text{co}}(K)$. If X is a set and $f : X \to (-\infty, \infty]$ is a function then $\text{Dom}(f) := \{x \in X : f(x) < \infty\}$ and we say that f is a *proper* function if $\text{Dom}(f) \neq \varnothing$. Furthermore, we define $\text{argmax}(f) := \{x \in X : f(y) \leq f(x) \text{ for all } y \in X\}$. We shall call a proper function $f : X \to (-\infty, \infty]$, defined on a vector space X, (over the real numbers) a *convex* function if, for each $x, y \in \text{Dom}(f)$ and $0 < \lambda < 1$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.

The main purpose of this note is to answer [7, Problem 3.7]. Many of the results contained in this note were previously known, although perhaps not explicitly published anywhere. Here we shall put these results together in order to solve some open problems concerning norm attaining functionals. The key notion required is that of a dual differentiation space. We shall say that a Banach space $(X, \|\cdot\|)$ is a *dual differentiation space*, [3] (or *DD-space* for short) if every continuous convex function $\varphi : A \to \mathbb{R}$ defined on a nonempty open convex subset A of X^* for which $\{x^* \in A : \partial\varphi(x^*) \cap \widehat{X} \neq \varnothing\}$ is residual in A, is Fréchet differentiable at the points of a dense subset of A.

Recall that a continuous convex function $\varphi : A \to \mathbb{R}$ defined on a nonempty open convex subset A of a normed linear space $(X, \|\cdot\|)$ is said to be *Fréchet differentiable at* $x_0 \in A$ provided there exists a continuous linear functional x^* such that for every $0 < \varepsilon$, there exists a $0 < \delta$ such that

$$|\varphi(x + x_0) - \varphi(x_0) - x^*(x)| \leq \varepsilon\|x\| \quad \text{for all } \|x\| < \delta.$$

For convex functions there is a derivative-like notion that holds even when the function is not differentiable. This is presented next.

Let $\varphi : C \to \mathbb{R}$ be a convex function defined on a nonempty convex subset of a normed linear space $(X, \|\cdot\|)$ and let $x \in C$. Then we define the *subdifferential* $\partial\varphi(x)$ by,

$$\partial\varphi(x) := \{x^* \in X^* : x^*(y - x) \leq \varphi(y) - \varphi(x) \text{ for all } y \in C\}.$$

The relationship between Fréchet differentiability and the subdifferential mapping is revealed in the following proposition.

Proposition 1 ([16]). *Let $\varphi : A \to \mathbb{R}$ be a continuous convex function defined on a nonempty open convex subset A of a normed linear space $(X, \|\cdot\|)$ and let $x_0 \in A$. Then φ is Fréchet differentiable at x_0 if, and only if, $\partial\varphi(x_0)$ is a singleton (i.e., $\partial\varphi(x_0) = \{x_0^*\}$ for some $x_0^* \in X^*$) and for every $0 < \varepsilon$ there exists a $0 < \delta$ such that $\partial\varphi(B(x_0;\delta)) \subseteq B(x_0^*;\varepsilon)$.*

Let $(X, \|\cdot\|)$ be a Banach space. We shall say that an element $x^* \in X^*$ *attains its norm* (or that x^* is *norm-attaining*) if there exists an $x \in B_X$ such that $x^*(x) = \|x^*\|^*$. The set of all norm-attaining functionals on X is denoted by $NA(X, \|\cdot\|)$.

The connection between norm attaining functionals and weak* continuous subderivatives of the dual norm is given in the next proposition.

Proposition 2. *Let $(X, \|\cdot\|)$ be a Banach space and let $x^* \in X^*$. Then $x^* \in NA(X, \|\cdot\|)$ if, and only if, $\partial\|x^*\|^* \cap \widehat{X} \neq \varnothing$.*

Proof. Suppose that $x^* \in NA(X, \|\cdot\|)$. Then there exists an $x \in B_X$ such that $x^*(x) = \|x^*\|^*$. Note also that for every $y^* \in X^*$, $y^*(x) \leq \|y^*\|^*\|x\| \leq \|y^*\|^*$. Hence, if y^* is any element of X^* then

$$\widehat{x}(y^* - x^*) = (y^* - x^*)(x) = y^*(x) - x^*(x) \leq \|y^*\|^* - \|x^*\|^*.$$

Thus, $\widehat{x} \in \partial\|x^*\|^* \cap \widehat{X}$. Conversely, suppose that $\widehat{x} \in \partial\|x^*\|^* \cap \widehat{X}$. Then for any $y^* \in X^*$,

$$\widehat{x}(y^*) = \widehat{x}((y^* + x^*) - x^*) \leq \|y^* + x^*\|^* - \|x^*\|^* \leq \|y^*\|^* \quad \text{(by the triangle inequality)}.$$

Thus, $\|x\| = \|\widehat{x}\|^{**} \leq 1$. That is, $x \in B_X$. On the other hand, if we substitute $y^* = 0$ into the definition of $\widehat{x} \in \partial\|x^*\|^*$ we obtain,

$$-\widehat{x}(x^*) = \widehat{x}(0 - x^*) \leq \|0\|^* - \|x^*\|^* = -\|x^*\|^*$$

and so $\|x^*\|^* \leq \widehat{x}(x^*)$. Therefore,

$$\|x^*\|^* \leq \widehat{x}(x^*) = x^*(x) \leq \|x^*\|^*\|x\| \leq \|x^*\|^*.$$

This shows that $x^*(x) = \widehat{x}(x^*) = \|x^*\|^*$, i.e., $x^* \in NA(X, \|\cdot\|)$. □

Theorem 3. *Let $(X, \|\cdot\|)$ be a dual differentiation space. Then the set $NA(X, \|\cdot\|)$ is residual in $(X^*, \|\cdot\|^*)$ if, and only if, $\|\cdot\|^*$ is Fréchet differentiable at the points of a dense subset of $(X^*, \|\cdot\|^*)$.*

Proof. Suppose that $NA(X, \|\cdot\|)$ is residual in X^*. Then by Proposition 2, $\{x^* \in X^* : \partial\|x^*\|^* \cap \widehat{X} \neq \varnothing\} = NA(X, \|\cdot\|)$ is residual in X^*. Thus, by the definition of a DD-space $\|\cdot\|^*$ is Fréchet differentiable at the points of a dense subset of X^*. Conversely, suppose that $D := \{x^* \in X^* : \|\cdot\|^* \text{ is Fréchet differentiable at } x^*\}$ is dense in X^*. Then, by Proposition 1, $D = \bigcap_{n\in\mathbb{N}} O_n$, where for each $n \in \mathbb{N}$,

$$O_n := \bigcup\{U \subseteq X^* : U \text{ is open and } \operatorname{diam}(\partial\|U\|^*) < 1/n\}.$$

Hence D is always a G_δ set. We claim that $D \subseteq NA(X, \|\cdot\|)$. To confirm this assertion consider any $x^* \in D$ and suppose, for the purpose of obtaining a contradiction, that $x^* \notin NA(X, \|\cdot\|)$. That is, by Proposition 2, $\partial\|x^*\|^* \cap \widehat{X} = \varnothing$. Since $x^* \in D$, we have, by Proposition 1, that there exists an $x^{**} \in X^{**}$ such that $\partial\|x^*\|^* = \{x^{**}\}$. Furthermore, as $x^{**} \notin \widehat{X}$ there exists a $0 < \varepsilon$ such that $B(x^{**};\varepsilon) \cap \widehat{X} = \varnothing$. Then, by Proposition 1 again, there exists a $0 < \delta$ such that $\partial\|B(x^*;\delta)\|^* \subseteq B(x^{**};\varepsilon)$. In particular, $\partial\|B(x^*;\delta)\|^* \cap \widehat{X} = \varnothing$. However, this is impossible since by the Bishop-Phelps theorem (see, [1]) there exists a $y^* \in NA(X, \|\cdot\|) \cap B(x^*;\delta) \neq \varnothing$, and for this y^*, $\partial\|y^*\|^* \cap \widehat{X} \neq \varnothing$ (see Proposition 2). Thus it must be the case that $x^* \in NA(X, \|\cdot\|)$. □

Remark 4. *In [13, Theorem 4.4] it is shown that a Banach space $(X, \|\cdot\|)$ has the Radon Nikodým Property (see [16, page 79] for the definition) if, and only if, for every equivalent norm $|\!|\!|\cdot|\!|\!|$ on X, $NA(X, |\!|\!|\cdot|\!|\!|)$ is residual in X^*. This gives a non-separable version of [7, Theorem 3.4].*

The significance of Theorem 3 depends upon the size of the class of DD-spaces. Fortunately there are many results in this direction. Perhaps the first paper on this topic is [2] where it is shown that every Banach space that can be equivalently renormed to be locally uniformly rotund (see [7] for the definition) is a DD-space. This line of inquiry was pursued further in the papers [4–6, 12] where various generalisations of local uniform rotundity were considered and shown to imply the same conclusion i.e., that the space is a DD-space.

In the paper [3] the first real systematic study of DD-spaces was conducted. In this paper it was shown that every Radon-Nikodým Property space is a DD-space and that every Banach space whose dual is weak Asplund (see [16, page 13] for the definition) is a DD-space. It was also shown that every space that admits an equivalent weak locally uniformly rotund norm is a DD-space, however, it has since been shown that such spaces admit an equivalent locally uniformly rotund norm, [11]. In [3] it is also shown that the class of DD-spaces is stable under passing to subspaces.

Following the paper [3], was the paper [14], where the investigation of DD-spaces continued. In [14] the class of GC-spaces (*Generic continuity spaces*) were considered and it was shown that every GC-space is a DD-space. It was also shown that if a Banach space X admits an equivalent weak mid-point locally uniformly rotund norm, and every weakly continuous function acting from an α-favourable space into X, is norm continuous at the points of a dense subset of its domain, then X is a DD-space (see, [14, p. 249] and [8, p. 2745]).

Finally, in [10] it was shown that every Banach space $(X, \|\cdot\|)$ such that (X, weak) is Lindelöf is a DD-space. Note also that in [9] an example wass given, under the continuum hypothesis, of a DD-space without an equivalent locally uniformly rotund norm.

We shall end this paper with some applications of DD-spaces to optimisation and the geometry of Banach spaces.

We shall say that a function $f : X \to [-\infty, \infty)$ defined on a normed linear space $(X, \|\cdot\|)$ attains a (or has a) *strong maximum at* $x_0 \in X$ if,

$$f(x_0) = \sup_{x\in X} f(x) \quad \text{and} \qquad \lim_{n\to\infty} x_n = x_0$$

whenever $(x_n : n \in \mathbb{N})$ is a sequence in X such that

$$\lim_{n\to\infty} f(x_n) = \sup_{x\in X} f(x) = f(x_0).$$

In order to expedite the phrasing of Theorem 5 we introduce some terminology from optimisation. If $f : X \to (-\infty, \infty]$ is a proper function on a Banach space $(X, \|\cdot\|)$ then the *Fenchel conjugate of* f, denoted $f^* : X^* \to (-\infty, \infty]$, is defined by,

$$f^*(x^*) := \sup_{x\in X}(x^* - f)(x) \;=\; \sup_{x\in \mathrm{Dom}(f)}(x^* - f)(x).$$

Theorem 5 ([15, Theorem 5.6])**.** *Let $f : X \to (-\infty, \infty]$ be a proper function on a dual differentiation space $(X, \|\cdot\|)$. If there exists a nonempty open subset A of $\mathrm{Dom}(f^*)$ and a dense and G_δ subset R of A such that $\mathrm{argmax}(x^* - f) \neq \varnothing$ for each $x^* \in R$, then there exists a dense and G_δ subset R' of A such that $(x^* - f) : X \to [-\infty, \infty)$ has a strong maximum for each $x^* \in R'$. In addition, if $0 \in A$ and $0 < \varepsilon$ then there exists an $x_0^* \in X^*$ with $\|x_0^*\| < \varepsilon$ such that $(x_0^* - f) : X \to [-\infty, \infty)$ has a strong maximum.*

For our application to the geometry of Banach spaces we need the notion of a strongly exposed point.

Let C be a nonempty closed and bounded convex subset of a normed linear space $(X, \|\cdot\|)$. We shall say that a point $x_0 \in C$ is a *strongly exposed point of* C if there exists an $x^* \in X^*$ such that $x^*|_C$ has a strong maximum at x_0 and we shall denote by $\mathrm{Exp}(C)$ the set of all strongly exposed points of C. Note that if $f : X \to (-\infty, \infty]$ is defined by, $f(x) := 0$ if $x \in C$ and $f(x) := \infty$ otherwise, then we have the following:

If $x^* \in X^*$ and $x^* - f$ has a strong maximum at $x_0 \in X$ then $x_0 \in C$ and, x_0 is in fact a strongly exposed point of C.

Theorem 6. *If C is a nonempty closed and bounded convex subset of a DD-space $(X, \|\cdot\|)$ and $\{x^* \in X^* : x^*$ attains its supremum over $C\}$ is residual in X^* then $C = \overline{co}(Exp(C))$.*

Proof. Let $f : X \to (-\infty, \infty]$ be defined by, $f(x) := 0$ if $x \in C$ and by $f(x) := \infty$ otherwise. Then, by assumption

$$R := \{x^* \in X^* : \mathrm{argmax}(x^*|_C) \neq \varnothing\} = \{x^* \in X^* : \mathrm{argmax}(x^* - f) \neq \varnothing\}$$

is residual in X^*. Therefore, by Theorem 5, there exists a dense and G_δ subset R' of X^* such that $(x^* - f)$ has a strong maximum for each $x^* \in R'$. Now suppose, in order to obtain a contradiction, that $C \neq \overline{\mathrm{co}}(\mathrm{Exp}(C))$. Then there exists an $x_0 \in C \setminus \overline{\mathrm{co}}(\mathrm{Exp}(C))$ and an $x^* \in X^*$ such that

$$\sup\{x^*(c) : c \in \overline{\mathrm{co}}(\mathrm{Exp}(C))\} < x^*(x_0).$$

Since C is bounded and R' is dense in X^* we can assume, without loss of generality, that $x^* \in R'$. But then $\mathrm{argmax}(x^*|_C) = \mathrm{argmax}(x^* - f) =: \{x\}$ is a strong maximum of $(x^* - f)$, and hence a strongly exposed point of C. On the other hand,

$$\sup\{x^*(c) : c \in \overline{\mathrm{co}}(\mathrm{Exp}(C))\} < x^*(x_0) \leq x^*(x);$$

which implies that $x \notin \mathrm{Exp}(C)$. Thus, it must be the case that $C = \overline{\mathrm{co}}(\mathrm{Exp}(C))$. □

We end this short note with the most important open question in the area.

Question 7. *Is every Banach space $(X, \|\cdot\|)$ a dual differentiation space?*

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