ANY SEMITOPOLOGICAL GROUP THAT IS HOMEOMORPHIC TO A PRODUCT OF ČECH-COMPLETE SPACES IS A TOPOLOGICAL GROUP

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Abstract. A semitopological group (topological group) is a group endowed with a topology for which multiplication is separately continuous (multiplication is jointly continuous and inversion is continuous). In this paper we answer [1, Problem 10.4], by showing that if $(G, \cdot, \tau)$ is a semitopological group and (G, τ) is homeomorphic to a product of Čech-complete spaces, then $(G, \cdot, \tau)$ is a topological group.

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A *semitopological group* (*topological group*) is a group endowed with a topology for which multiplication is separately continuous (multiplication is jointly continuous and inversion is continuous). Ever since [22] there has been continued interest in determining topological properties of a semitopological group that are sufficient to ensure that it is a topological group. There have been many significant contributions to this area, see [1–9,12–14,18,19,21–32] to name but a few. Just about all of these results require the semitopological group to be *regular* (i.e., every closed subset and every point not in this set, can be separated by disjoint open sets) and *Baire*, (i.e., the intersection of any countable family of dense open sets is dense) and satisfy some additional completeness properties.

In this paper we answer [1, Problem 10.4], by showing that if $(G, \cdot, \tau)$ is a semitopological group such that (G, τ) is homeomorphic to a product of Čech-complete spaces, then $(G, \cdot, \tau)$ is a topological group. Our approach is based upon topological games.

Let (X, τ) be a topological space and let D be a dense subset of X. The $\mathscr{G}(D)$*-game* is a two player game. An instance of the $\mathscr{G}(D)$-game is a sequence $(A_n, B_n, b_n)_{n\in\mathbb{N}}$ defined inductively in the following way: player β begins by choosing a pair (B_1, b_1) consisting of a nonempty open subset B_1 of X and a point $b_1 \in D$; player α then chooses a nonempty open subset A_1 of B_1. When (A_i, B_i, b_i), $i = 1, 2, \ldots, (n-1)$, have been defined, player β chooses a pair (B_n, b_n) consisting of a nonempty open subset B_n of A_{n-1} and a point $b_n \in A_{n-1} \cap D$. Player α then chooses a nonempty open subset A_n of B_n. Player α is declared the *winner* if:

$$\bigcap_{n\in\mathbb{N}} \overline{\{b_k : k \geq n\}} \cap \bigcap_{n\in\mathbb{N}} B_n \neq \varnothing.$$

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We shall call a topological space (X, τ) *nearly strongly Baire* if it is a regular topological space and there exists a dense subset D of X such that the player β does **not** have a winning strategy in the $\mathscr{G}(D)$-game played on X.

In this paper we also consider another game. Let (X, τ) be a topological space, $a \in X$, and let D be a dense subset of X. The *$\mathscr{G}_p(a, D)$-game* is a two player game. An instance of the $\mathscr{G}_p(a, D)$-game is a sequence $(A_n, b_n)_{n \in \mathbb{N}}$ defined inductively in the following way: player β begins by choosing a point $b_1 \in D$; player α then chooses an open neigbourhood A_1 of a. When (A_i, b_i), $i = 1, 2, \ldots, (n-1)$, have been defined, player β chooses a point $b_n \in A_{n-1} \cap D$. Player α then chooses an open neighbourhood A_n of a. Player α is declared the *winner* if the sequence $(b_n)_{n \in \mathbb{N}}$ has a cluster-point in X. We shall call a point a a *nearly q_D-point* if the player α has a winning strategy in the $\mathscr{G}_p(a, D)$-game played on X. For more information on topological games, see [10].

Lemma 1 *Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is nearly strongly Baire then for each pair of open neighbourboods U and W of identity element $e \in G$ there exists a nonempty open subset V of U such that $V^{-1} \subseteq W \cdot W \cdot W$.*

Proof: Suppose, in order to obtain a contradiction, that there exists a pair of open neighbourhoods U and W of $e \in G$ such that for each nonempty open subset V of U, $V^{-1} \not\subseteq W \cdot W \cdot W$. From this it follows that for each nonempty open subset $V$of U and each dense subset D' of V there exists a point $x \in V \cap D'$ such that $x^{-1} \notin W \cdot W$, because otherwise,

$$V^{-1} \subseteq (\overline{V \cap D'})^{-1} \subseteq W \cdot (V \cap D')^{-1} \subseteq W \cdot W \cdot W.$$

Recall that for any nonempty subset A of a semitopological group $(H, \cdot, \tau)$ and any open neighbourhood W of the identity element $e \in H$, $(\overline{A})^{-1} \subseteq W \cdot A^{-1}$.

Now, let D be any dense subset of G such that β does not have a winning strategy in the $\mathscr{G}(D)$-game played on G. We will define a (necessarily non-winning) strategy t for β in the $\mathscr{G}(D)$-game played on G, but first we set, for notational reasons, $A_0 := U$ and $b_0 := e$.

Step 1. Choose $b_1 \in A_0 \cap D$ so that $(b_0^{-1} \cdot b_1)^{-1} = b_1^{-1} \notin W \cdot W$. Then choose U_1 to be any open neighbourhood of e, contained in $U \cap W$, such that $b_1 \cdot \overline{U_1} \subseteq A_0$. Then define $t(\emptyset) := (b_1 \cdot U_1, b_1)$.

Now, suppose that b_j, U_j and $t(A_1, \ldots, A_{j-1})$ have been defined for each $1 \leq j \leq n$ so that:

(i) $b_j \in A_{j-1} \cap D$ and $(b_{j-1}^{-1} \cdot b_j)^{-1} \notin W \cdot W$;

(ii) U_j is an open neighbourhood of e, contained in $U \cap W$, such that $b_j \cdot \overline{U_j} \subseteq A_{j-1}$;

(iii) $t(A_1, \ldots, A_{j-1}) := (b_j \cdot U_j, b_j)$.

Step $n+1$. Choose $b_{n+1} \in A_n \cap D$ so that $(b_n^{-1} \cdot b_{n+1})^{-1} \notin W \cdot W$. Note that this is possible since $b_n^{-1} \cdot (A_n \cap D)$ is a dense subset of $b_n^{-1} \cdot A_n$ and

$$b_n^{-1} \cdot A_n \subseteq b_n^{-1} \cdot (b_n \cdot U_n) = U_n \subseteq U.$$

Then choose U_{n+1} to be any neighbourhood of e, contained in $U \cap W$, such that $b_{n+1} \cdot \overline{U_{n+1}} \subseteq A_n$. Finally, define $t(A_1, \ldots, A_n) := (b_{n+1} \cdot U_{n+1}, b_{n+1})$. Note that:

(i) $b_{n+1} \in A_n \cap D$ and $(b_n^{-1} \cdot b_{n+1})^{-1} \notin W \cdot W$;

(ii) U_{n+1} is an open neighbourhood of e, contained in $U \cap W$, such that $b_{n+1} \cdot \overline{U_{n+1}} \subseteq A_n$;

(iii) $t(A_1, \ldots, A_n) := (b_{n+1} \cdot U_{n+1}, b_{n+1})$.

This completes the definition of t. Since t is not a winning strategy for β there exists a play $(A_n, t(A_1, \ldots, A_{n-1}))_{n\in\mathbb{N}}$ where α wins. Let $b_\infty \in \bigcap_{n\in\mathbb{N}} \overline{\{b_k : k \geq n\}} \cap \bigcap_{n\in\mathbb{N}} B_n$. Choose $k \in \mathbb{N}$ so that

$$b_k \in b_\infty \cdot W \subseteq A_{k+1} \cdot W \subseteq b_{k+1} \cdot U_{k+1} \cdot W \subseteq b_{k+1} \cdot W \cdot W.$$

Therefore, $(b_k^{-1} \cdot b_{k+1})^{-1} = b_{k+1}^{-1} \cdot b_k \in W \cdot W$. However, this contradicts the way b_{k+1} was chosen. This completes the proof. $\square$

Let X, Y and Z be topological spaces. We will say that a function $f : X \times Y \to Z$ is *strongly quasi-continuous, with respect to the second variable,* at $(x, y) \in X \times Y$, if for each neighbourhood W of $f(x, y)$ and each product of open sets $U \times V \subseteq X \times Y$ containing (x, y) there exists a nonempty open subset $U' \subseteq U$ and a neighbourhood V' of y such that $f(U' \times V') \subseteq W$, [25]. Further, a function $f : X \times Y \to Z$ is said to be *separately continuous* on $X \times Y$ if for each $x_0 \in X$ and $y_0 \in Y$ the functions $y \mapsto f(x_0, y)$ and $x \mapsto f(x, y_0)$ are both continuous on Y and X respectively.

Variations of the following result are well-known, see [5, 6, 12, 18, 21].

Lemma 2 *Let X be a nearly strongly Baire space, Y be topological space and Z a regular space. If $f : X \times Y \to Z$ is a separately continuous function and D is a dense subset of Y, then for each nearly q_D-point $y_0 \in Y$ the function f is strongly quasi-continuous, with respect to the second variable, at each point of $X \times \{y_0\}$.*

If $f : (X, \tau) \to (Y, \tau')$ is a surjection acting between topological spaces (X, τ) and (Y, τ') then we say that f is *feebly continuous* on X if for each nonempty open subset V of Y, $\text{int}[f^{-1}(V)] \neq \varnothing$, [9, 15].

Proposition 1 *Let $(G, \cdot, \tau)$ be a semitopological group. If multiplication, $(h, g) \mapsto h \cdot g$, is feebly continuous on $G \times G$ then for each nonempty open subset U of G and $n \in \mathbb{N}$ there exist a point x in U and an open neighbourhood V of the identity element $e \in G$ such that:*

$$x \cdot \underbrace{V \cdot V \cdot V \cdots V}_{n-\text{times}} \subseteq U \quad \textit{and} \quad \underbrace{V \cdot V \cdot V \cdots V}_{n-\text{times}} \cdot x \subseteq U.$$

Proof: The proof of this follows from a simple induction argument and the fact that for each $g \in G$, both $\{g \cdot U : U \text{ is a neighbourhood of } e\}$ and $\{U \cdot g : U \text{ is a neighbourhood of } e\}$ are local bases for τ at the point $g \in G$. $\square$

Remarks 1 *It follows from Proposition 1 that the multiplication operation on a semitopological group $(G, \cdot, \tau)$ is feebly continuous on $G \times G$ if, and only if, it is strongly quasi-continuous, with respect to the second variable, at the point $(e, e) \in G \times G$.*

Lemma 3 *Let $(G, \cdot, \tau)$ be a semitopological group and let D be a dense subset of G. If (G, τ) is nearly strongly Baire and the identity element $e \in G$ is a nearly q_D-point then the multiplication operation, $(h, g) \mapsto h \cdot g$, is continuous on $G \times G$.*

Proof: Since $(G, \cdot, \tau)$ is a semitopological group it is sufficient to show that multiplication is jointly continuous at (e, e). So, in order to obtain a contradiction, we will assume that multiplication is not jointly continuous at (e, e). Therefore, by the regularity of (G, τ), there exists an open neighbourhood W of e so that for every neighbourhood U of e, $U \cdot U \not\subseteq \overline{W}$. Since (G, τ) is a nearly strongly Baire space there exists a dense subset D_G of G such that β does not possess a winning strategy in the $\mathscr{G}(D_G)$-game played on G.

We will now inductively define a (necessarily non-winning) strategy t for the player β in the $\mathscr{G}(D_G)$-game played on G.
Step 1. We may choose a point $x \in G$ and an open neighbourhood U of $e \in G$ such that

$$x \cdot U \subseteq x \cdot U \cdot U \cdot U \subseteq G.$$

Next, we may pick $y, z \in U$ such that $y \cdot z \notin \overline{W}$ (i.e., $y \notin \overline{W} \cdot z^{-1}$ and so $U \backslash (\overline{W} \cdot z^{-1}) \neq \varnothing$). By Lemma 2 and Proposition 1 we may select a point $y' \in U \backslash (\overline{W} \cdot z^{-1})$ and an open neighbourhood V of e, contained in U, such that

$$V \cdot V \cdot V \cdot V \cdot y' \subseteq U \backslash (\overline{W} \cdot z^{-1}).$$

Then, $(V \cdot V \cdot V \cdot x^{-1}) \cdot (x \cdot V) \cdot y' \cdot z \cap \overline{W} = \varnothing$. By Lemma 1 there exists a nonempty open subset B_1 of $x \cdot V \subseteq x \cdot U \subseteq G$ such that $(B_1)^{-1} \subseteq V \cdot V \cdot V \cdot x^{-1}$. Thus, $(B_1)^{-1} \cdot B_1 \cdot y' \cdot z \cap \overline{W} = \varnothing$. Choose

$$b_1 \in (B_1 \cdot y' \cdot z) \cap D_G \subseteq B_1 \cdot U \cdot U \subseteq x \cdot V \cdot U \cdot U \subseteq x \cdot U \cdot U \cdot U \subseteq G.$$

Then define $t(\emptyset) := (B_1, b_1)$. Note that: $(B_1)^{-1} \cdot b_1 \cap \overline{W} = \varnothing$ — $(*_1)$.

Now suppose that $t(A_1, \ldots, A_{j-1})$ has been defined for each $1 \leq j \leq n$.

Step $n+1$. By Lemma 2 and Proposition 1 we may choose a point $x \in A_n$ and an open neighbourhood U of $e \in G$ such that

$$x \cdot U \subseteq x \cdot U \cdot U \cdot U \subseteq A_n.$$

Next, we may pick $y, z \in U$ such that $y \cdot z \notin \overline{W}$ (i.e., $y \notin \overline{W} \cdot z^{-1}$ and so $U \backslash (\overline{W} \cdot z^{-1}) \neq \varnothing$). Again by Lemma 2 and Proposition 1 we may select a point $y' \in U \backslash (\overline{W} \cdot z^{-1})$ and an open neighbourhood V of e, contained in U, such that

$$V \cdot V \cdot V \cdot V \cdot y' \subseteq U \backslash (\overline{W} \cdot z^{-1}).$$

Then, $(V \cdot V \cdot V \cdot x^{-1}) \cdot (x \cdot V) \cdot y' \cdot z \cap \overline{W} = \varnothing$. By Lemma 1 there exists a nonempty open subset B_{n+1} of $x \cdot V \subseteq x \cdot U \subseteq A_n$ such that $(B_{n+1})^{-1} \subseteq V \cdot V \cdot V \cdot x^{-1}$. Thus, $(B_{n+1})^{-1} \cdot B_{n+1} \cdot y' \cdot z \cap \overline{W} = \varnothing$. Choose

$$b_{n+1} \in (B_{n+1} \cdot y' \cdot z) \cap D_G \subseteq B_{n+1} \cdot U \cdot U \subseteq x \cdot V \cdot U \cdot U \subseteq x \cdot U \cdot U \cdot U \subseteq A_n.$$

Then define $t(A_1, \ldots, A_n) := (B_{n+1}, b_{n+1})$. Note that: $(B_{n+1})^{-1} \cdot b_{n+1} \cap \overline{W} = \varnothing$ — $(*_{n+1})$.

This completes the definition of t. Since t is not a winning strategy for β there exists a play $(A_n, t(A_1, \ldots, A_{n-1}))_{n \in \mathbb{N}}$ where α wins. Let $b_\infty \in \bigcap_{n \in \mathbb{N}} \overline{\{b_k : k \geq n\}} \cap \bigcap_{n \in \mathbb{N}} B_n \neq \varnothing$. Fix $n \in \mathbb{N}$, then by equation $(*_n)$, $b_\infty^{-1} \cdot b_n \notin \overline{W}$. Therefore, $e = b_\infty^{-1} \cdot b_\infty \notin W$. However, this contradicts the fact that W is an open neighbourhood of e. Hence the multiplication operation on G is jointly continuous. □

If $f : (X, \tau) \to (Y, \tau')$ is a function acting between topological spaces (X, τ) and (Y, τ') and $x \in X$ then we say that f is *quasi-continuous* at x if for each neighbourhood W of $f(x)$ and neighbourhood U of x there exists a nonempty open subset $V \subseteq U$ such that $f(V) \subseteq W$, [17].

Theorem 1 *Let $(G, \cdot, \tau)$ be a semitopological group and let D be a dense subset of G. If (G, τ) is nearly strongly Baire and the identity element $e \in G$ is a nearly q_D-point then $(G, \cdot, \tau)$ is a topological group.*

Proof: From Lemma 3 we know that the multiplication operation on G is continuous. Therefore, by Lemma 1, we see that inversion is quasi-continuous at e. The result now follows from [18, Lemma 4] where it is shown that each semitopological group with continuous multiplication and inversion that is quasi-continuous at the identity element is a topological group. $\Box$

Example 1 *Suppose that $\{X_s : s \in S\}$ is a family of nonempty Čech-complete spaces. Then $X := \prod_{s\in S} X_s$ is nearly strongly Baire and each point of X is a nearly q_D-point with respect to some dense subset D of X.*

Proof: For each $a \in X = \prod_{s\in S} X_s$ the Σ-*product of* $\{X_s : s \in S\}$ *with base point* a, denoted $\Sigma_{s\in S} X_s(a)$, is the set of all $x \in X$ such that $\{s \in S : x(s) \neq a(s)\}$ is at most countable. Obviously, for each $a \in X$, $\Sigma_{s\in S} X_s(a)$ is dense in X. It follows by making a small modification of the proof of [11, Proposition 4.2] that for an arbitrary $a \in X$, the player α has a winning strategy in the $\mathscr{G}(\Sigma_{s\in S} X_s(a))$-game played on X. Furthermore, it follows in a similar way to [16, Theorem 4.6] or [20, Theorem 2.5] that for each $a \in X$, the player α has a winning strategy in the $\mathscr{G}_p(a, \Sigma_{s\in S} X_s(a))$-game played on X. $\Box$

Remarks 2 *It is easy to show that every strongly Baire space X (see, [18]) is a nearly strongly Baire space and has at least one nearly q_D-point for some dense subset D of X. Hence Theorem 1 improves the main result of [18]. Furthermore, there exist nearly strongly Baire spaces that are not strongly Baire. For example, by above, $\mathbb{R}^{\mathbb{R}}$ is nearly strongly Baire and every point of $\mathbb{R}^{\mathbb{R}}$ is a nearly q_D-point for some dense subset of $\mathbb{R}^{\mathbb{R}}$. However, $\mathbb{R}^{\mathbb{R}}$ is not a strongly Baire space as it has no q_D-points (see, [18]).*

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