WIJSMAN HYPERSPACES: SUBSPACES AND EMBEDDINGS

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Dedicated to the memory of Professor Jun-iti Nagata

ABSTRACT. In this paper, topological properties of Wijsman hyperspaces are investigated. We study the existence of isolated points in Wijsman hyperspaces. We show that every Tychonoff space can be embedded as a closed subspace in the Wijsman hyperspace of a complete metric space which is locally $\mathbb{R}$.

1. INTRODUCTION

In this paper, we consider the set 2^X consisting of all non-empty closed subsets of a metric space (X, d), equipped with the *Wijsman topology* $\tau_{w(d)}$. In the following, we denote the *Wijsman hyperspace* $(2^X, \tau_{w(d)})$ of a metric space (X, d) by $2^{(X,d)}$. The easiest way to describe the space $2^{(X,d)}$ is through an identification: identify a non-empty closed subset S of X with the distance function $d(\cdot, S)$. In this way 2^X is identified with a subset of the function space $C(X)$ and the Wijsman topology is the topology of pointwise convergence on this subset.

We refer the reader to [1] and [2] for background on Wijsman hyperspaces.

Lechicki and Levi showed in [8] that the Wijsman hyperspace of a separable metric space is metrizable. There has been a considerable effort to explore completeness properties of Wijsman topologies, and one line of research was completed with the result of Costantini [4] that the Wijsman hyperspace of a Polish metric space is Polish.

Besides metrizability and various completeness properties, other topological properties of Wijsman hyperspaces have not been widely studied. In this paper we give results which show that Wijsman hyperspaces of topologically simple non-separable metric spaces can have very complicated topologies. The first result of this kind in the literature is an example, due to Costantini [5] of an uncountable discrete complete metric space (X, d) such that $2^{(X,d)}$ is not Čech-complete. In Section 3 below, we give a general result which yields Costantini's result as well as some later results as corollaries.

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2. Isolated points of Wijsman hyperspaces

In this section, we give three examples to demonstrate various possibilities on the existence of isolated points in Wijsman hyperspaces.

The first example is due to Chaber and Pol [3, Remark 3.1].

Example 2.1. *Let δ be the 0-1 metric on a set X. Then $2^{(X,\delta)}$ is homeomorphic to $\{0,1\}^X \setminus \{\mathbf{0}\}$, where $\{0,1\}$ is discrete and $\mathbf{0}$ is the constant function with value 0. If X is infinite, then 2^X has no isolated points.*

The above example and well-known properties of the Cantor cube $\{0,1\}^X$ show that the Wijsman hyperspace $2^{(X,\delta)}$ of a 0-1 metric space (X,δ) is locally compact and satisfies the countable chain condition. Moreover, if $|X| \le 2^\omega$, then $2^{(X,\delta)}$ is separable.

The Wijsman hyperspace of an infinite metric space is non-discrete. Nevertheless, Wijsman hyperspaces may have many isolated points.

Example 2.2. *For every set X, there exists a discrete metric d on X such that every singleton subset of X is an isolated point in $2^{(X,d)}$.*

Proof. To avoid a triviality, let X be infinite. Express X as the union of a family of pairwise disjoint two-point subsets, that is, write $X = \bigcup\{X_\alpha : \alpha \in A\}$, where $X_\alpha = \{x_\alpha^0, x_\alpha^1\}$ with $x_\alpha^0 \ne x_\alpha^1$ for each $\alpha \in A$ and $X_\alpha \cap X_\beta = \emptyset$ whenever $\alpha \ne \beta$. Define $d : X \times X \to \{0,1,2\}$ by setting

$$d(x,y) = \begin{cases} 0, & \text{if } x = y\,; \\ 2, & \text{if } \{x,y\} = X_\alpha \text{ for some } \alpha\,; \\ 1, & \text{otherwise.} \end{cases}$$

It can be checked that d is a metric on X. Let $\alpha \in A$ and $i \in \{0,1\}$. Note that $d(x_\alpha^{1-i}, x_\alpha^i) = 2$ and $d(x_\alpha^{1-i}, z) \le 1$ for every $z \ne x_\alpha^i$. As a consequence, we have $\{F \in 2^X : d(x_\alpha^{1-i}, F) > 1\} = \{\{x_\alpha^i\}\}$, and the set $\{\{x_\alpha^i\}\}$ is thus open in $2^{(X,d)}$. □

Example 2.3. *A discrete metric d on $\mathbb{N}$ such that every non-empty finite subset of $\mathbb{N}$ is an isolated point in $2^{(\mathbb{N},d)}$.*

Proof. Define d by the formula $d(n,k) = |2^{-n} - 2^{-k}|$ for all $n, k \in \mathbb{N}$, and note that d is a discrete metric on $\mathbb{N}$.

Let $E \subset \mathbb{N}$ be non-empty and finite. Set $m = 2 + \max E$. The set

$$\mathcal{W} = \{F \in 2^{\mathbb{N}} : |d(k,F) - d(k,E)| < 2^{-m-1} \text{ for each } k \le m\}$$

is a neighborhood of E in $2^{(\mathbb{N},d)}$. We show that $\mathcal{W} = \{E\}$.

Let $F \in \mathcal{W}$. To show that $F = E$, we first show that $F \cap [1,m] = E \cap [1,m]$. Let $k \le m$. Since $F \in \mathcal{W}$, we have $|d(k,F) - d(k,E)| < 2^{-m-1}$. Thus, if $k \in E$, then $d(k,F) < 2^{-m-1}$. Since

$$d(k, \mathbb{N} \setminus \{k\}) = d(k, k+1) = 2^{-k} - 2^{-k-1} = 2^{-k-1} \ge 2^{-m-1},$$

it follows that $k \in F$. On the other hand, if $k \notin E$, then

$$d(k,E) \ge d(k, \mathbb{N} \setminus \{k\}) = 2^{-k-1} \ge 2^{-m-1}$$

and it follows that

$$d(k,F) > d(k,E) - 2^{-m-1} \ge 2^{-m-1} - 2^{-m-1} = 0,$$

which implies that $k \notin F$. Hence, we have shown that $F \cap [1,m] = E \cap [1,m]$.

To conclude the proof of $F = E$, it suffices to show that $F \subseteq [1, m]$. Assume on the contrary that there exists $k \in F$ with $k > m$. Then we have that

$$d(m, F) \leq d(m, k) = 2^{-m} - 2^{-k} < 2^{-m}.$$

Since $m = 2 + \max E$, we have that

$$d(m, E) = 2^{-m+2} - 2^{-m} > 2^{-m+1},$$

and it follows that

$$|d(m, F) - d(m, E)| > 2^{-m+1} - 2^{-m} = 2^{-m}.$$

This, however, is a contradiction since $F \in \mathcal{W}$. □

Question 2.4. *Does there exist an uncountable metric space (X, d) such that every non-empty finite subset of X is an isolated point in $2^{(X,d)}$?*

3. On subspaces of Wijsman Hyperspaces

In this section, we investigate those topological spaces which can be embedded as closed subspaces in Wijsman hyperspaces of various metric spaces. We start with a simple observation.

Proposition 3.1. *A T_1-space T is zero-dimensional and locally compact if, and only if, T embeds as a closed subspace in $2^{(X,\delta)}$ for some 0-1 metric space (X, δ).*

Proof. Sufficiency follows immediately from Example 2.1. To prove necessity, let T be a zero-dimensional and locally compact T_1-space. The one-point compactification $T^* = T \cup \{\infty\}$ is zero-dimensional and compact. There exists an embedding $\varphi : T^* \to \{0, 1\}^\kappa$ for some cardinal κ. The compact set $\varphi(T^*)$ is closed in $\{0, 1\}^\kappa$, and it follows that $\varphi|T$ is an embedding of T onto a closed subspace of $\{0, 1\}^\kappa \setminus \{\varphi(\infty)\}$. By homogeneity of $\{0, 1\}^\kappa$ and Example 2.1 again, $\{0, 1\}^\kappa \setminus \{\varphi(\infty)\}$ is homeomorphic to the Wijsman hyperspace $2^{(X,\delta)}$ of a 0-1 metric space (X, δ). □

To obtain some deeper results on embeddings, we first establish the following key lemma.

Lemma 3.2. *Let $\{(X_\alpha, d_\alpha) : \alpha \in A\}$ be a family of mutually disjoint complete metric spaces such that for each $\alpha \in I$, the set $E_\alpha = d_\alpha(X_\alpha \times X_\alpha)$ is a subset of the closed unit interval $\mathbb{I}$. Then there exists a compatible complete metric d on the free sum $X = \bigoplus_{\alpha \in A} X_\alpha$ such that the product space $\prod_{\alpha \in A} 2^{(X_\alpha, d_\alpha)}$ embeds as a closed subspace in $2^{(X,d)}$. Moreover, $d(X \times X) \subseteq \bigcup_{\alpha \in A} E_\alpha \cup \{2\}$.*

Proof. Set $\mathcal{Y} = \prod_{\alpha \in A} 2^{(X_\alpha, d_\alpha)}$. Equip X with the metric d defined by

$$d(x, y) = \begin{cases} d_\alpha(x, y), & \text{if } x, y \in X_\alpha \text{ for some } \alpha \in A; \\ 2, & \text{otherwise.} \end{cases}$$

It is easy to see that d is a compatible complete metric for X and the inclusion $d(X \times X) \subseteq \bigcup_{\alpha \in A} E_\alpha \cup \{2\}$ holds. To complete the proof, we show that $\mathcal{Y}$ embeds as a closed subspace in $2^{(X,d)}$. To this end, we define a mapping $\varphi : \mathcal{Y} \to 2^{(X,d)}$ by the formula

$$\varphi(\langle F_\alpha \rangle_{\alpha \in A}) = \bigcup \{F_\alpha : \alpha \in A\}.$$

Denote by $\mathcal{Z}$ the subspace $\varphi(\mathcal{Y})$ of $2^{(X,d)}$. Note that we have $\mathcal{Z} = \{F \in 2^X : F \cap X_\alpha \neq \emptyset \text{ for every } \alpha \in A\}$. The mapping φ has an inverse $\varphi^{-1} : \mathcal{Z} \to \mathcal{Y}$ defined by the formula $\varphi^{-1}(F) = \langle F \cap X_\alpha \rangle_{\alpha \in A}$. As a consequence, φ is one-to-one. Next we verify that φ is a homeomorphism $\mathcal{Y} \to \mathcal{Z}$.

The space $\mathcal{Y}$ has a subbase consisting of sets of the form

$$\Gamma_{\nu,x,a,b} = \{\langle F_\alpha \rangle_{\alpha \in A} \in \mathcal{Y} : a < d_\nu(x, F_\nu) < b\},$$

where $\nu \in A$, $x \in X_\nu$ and $a, b \in \mathbb{R}$.

Note that if $x \in X_\nu$ and $F \in \mathcal{Z}$, then $d(x, F) = d(x, F \cap X_\nu) = d_\nu(x, F \cap X_\nu)$. It follows that we have

$$\varphi(\Gamma_{\nu,x,a,b}) = \{F \in \mathcal{Z} : a < d_\nu(x, F \cap X_\nu) < b\} = \{F \in \mathcal{Z} : a < d(x, F) < b\}.$$

The relative Wijsman topology of $\mathcal{Z}$ has a subbase consisting of sets of the form $\{F \in \mathcal{Z} : a < d(x, F) < b\}$, where $x \in X$ and $a, b \in \mathbb{R}$. Hence we have shown that the one-to-one mapping φ transforms a subbase of $\mathcal{Y}$ onto a subbase of $\mathcal{Z}$. As a consequence, φ is a homeomorphism $\mathcal{Y} \to \mathcal{Z}$.

Finally, we verify that $\mathcal{Z}$ is a closed subspace of $2^{(X,d)}$. Let $F \in 2^X \setminus \mathcal{Z}$. Then $F \cap X_\alpha = \emptyset$ for some $\alpha \in A$. Pick a point $x_0 \in X_\alpha$, and put

$$\mathcal{U} = \left\{B \in 2^X : d(x_0, B) > 1\right\}.$$

Then $\mathcal{U}$ is an open neighborhood of F in $2^{(X,d)}$ such that $\mathcal{U} \cap \mathcal{Z} = \emptyset$. □

Lemma 3.3. *If d is a finite-valued metric on a nonempty set X, then $2^{(X,d)}$ is zero-dimensional.*

Proof. Let $d(X \times X) = E$. Then $2^{(X,d)}$ embeds in $C_p(X, E) \subseteq E^X$, where E is equipped with the discrete topology. Thus, the conclusion follows. □

A metric space (X, d) is *uniformly discrete* if X is ε-discrete for some $\varepsilon > 0$.

Question 3.4. *Can we replace "finite-valued" in Lemma 3.3 by "discrete" or "uniformly discrete"?*

We only have the following partial answer to Question 3.4. Following [6], we call a topological space X *totally disconnected* if every quasi-component of X is a singleton.

Proposition 3.5. *Let (X, d) be a discrete metric space. Then $2^{(X,d)}$ is totally disconnected.*

Proof. Let $x \in X$. Let $r_x = \frac{1}{2} d(x, X \setminus \{x\})$ and note that $r_x > 0$. The set

$$\mathcal{G}_x = \{F \in 2^X : d(x, F) < r_x\} = \{F \in 2^X : x \in F\}$$

is open in $2^{(X,d)}$, and it is also closed, because $2^X \setminus G_x = \{F \in 2^X : d(x, F) > r_x\}$.

The family $\{\mathcal{G}_x : x \in X\}$ of clopen sets separates the points of 2^X and hence $2^{(X,d)}$ is totally disconnected. □

Theorem 3.6. *A T_1-space T is zero-dimensional if, and only if, T embeds as a closed subspace in the Wijsman hyperspace $2^{(X,d)}$ of a metric space (X, d) with a 3-valued metric d on X.*

Proof. Sufficiency follows immediately from Lemma 3.3. To prove necessity, suppose that T is zero-dimensional. Take a base $\mathfrak{B} = \{B_\alpha : \alpha < \kappa\}$ consisting of clopen subsets. It is well-known that T embeds in $\{0,1\}^\kappa$ by the mapping $\varphi : T \to \{0,1\}^\kappa$ defined by $\varphi(x) = \chi_{A_x}$, where χ_{A_x} is the characteristic function of the set $A_x = \{\alpha < \kappa : x \in B_\alpha\}$. So, we can assume that $T \subseteq \{0,1\}^\kappa$. Let

$\widetilde{T} = \{0,1\}^\kappa \setminus T$. For every $y \in \widetilde{T}$, let $Y_y = \{0,1\}^\kappa \setminus \{y\}$. By homogeneity of $\{0,1\}^\kappa$ and Example 2.1, for each $y \in \widetilde{T}$, there exists a 0-1 metric space (X_y, d_y) such that Y_y is homeomorphic to $2^{(X_y,d_y)}$. We can choose the sets X_y, $y \in \widetilde{T}$, to be mutually disjoint. It follows from Lemma 3.2 that there exists a 3-valued metric d (with values 0, 1, 2) on the free sum $X = \bigoplus\{X_y : y \in \widetilde{T}\}$ such that the product space $\prod\{Y_y : y \in \widetilde{T}\}$ embeds as a closed subspace in $2^{(X,d)}$. Finally, it is routine to check that the diagonal

$$\Delta = \left\{ \langle a_y \rangle_{y\in\widetilde{T}} \in \prod \left\{ Y_y : y \in \widetilde{T} \right\} : a_y = a_{y'} \text{ for all } y, y' \in \widetilde{T} \right\}$$

is a closed subspace of $\prod\{Y_y : y \in \widetilde{T}\}$ which is homeomorphic to T. Therefore, we conclude that T embeds as a closed subspace in $2^{(X,d)}$. □

Note that Proposition 3.1 and Theorem 3.6 explain why Costantini was able to use a 3-valued but not 2-valued metric in his example of a complete metric space whose Wijsman hyperspace is not Čech-complete. Let us also note that as a consequence of Theorem 3.6, we can give a "3-valued solution" to Zsilinszky's problem in [10]. The original solution, by Chaber and Pol [3], used a non-discrete metric space.

Corollary 3.7. *The space $\mathbb{Q}$ of rationals embeds as a closed subspace in $2^{(X,d)}$ for some 3-valued metric space (X,d). Consequently, there exists a 3-valued metric space whose Wijsman hyperspace is not hereditarily Baire.*

We close the paper with an embedding result for Tychonoff spaces. It provides a generic solution to problems dealing with closed-hereditary properties of Wijsman hyperspaces, and hence it extends some earlier results such as those by Costantini, Chaber and Pol mentioned above.

Theorem 3.8. *Every Tychonoff space can be embedded as a closed subspace in the Wijsman hyperspace of a complete metric space which is locally $\mathbb{R}$.*

Proof. Let T be a Tychonoff space. By a classical result, there exists an infinite cardinal κ such that T can be embedded into the Tychonoff cube $\mathbb{I}^\kappa$. Hence, we may assume that $T \subseteq \mathbb{I}^\kappa$.

We show that for every $y \in \mathbb{I}^\kappa$, there is a locally $\mathbb{R}$ and complete metric space (Z_y, d_y) such that $\mathbb{I}^\kappa \setminus \{y\}$ embeds as a closed set in $2^{(Z_y,d_y)}$. Since $\mathbb{I}^\kappa$ is homogeneous (see [7] and [9]), it suffices to show that the assertion holds for $y = \mathbf{0}$. Let I be the open interval $(0,2)$ in $\mathbb{R}$. Since I is completely metrizable, it admits a compatible complete metric ρ which is bounded by 1. Consider the set $Z_\mathbf{0} = \bigcup_{\alpha<\kappa} (I \times \{\alpha\})$, and define a metric $d_\mathbf{0}$ of $Z_\mathbf{0}$ by the formula

$$d_\mathbf{0}(\langle x, \alpha\rangle, \langle y, \beta\rangle) = \begin{cases} \rho(x,y), & \text{if } \alpha = \beta; \\ 2, & \text{otherwise.} \end{cases}$$

Then $(Z_\mathbf{0}, d_\mathbf{0})$ is a complete metric space, which is locally $\mathbb{R}$.

We show that $\mathbb{I}^\kappa \setminus \{\mathbf{0}\}$ embeds as a closed subspace in $2^{(Z_\mathbf{0},d_\mathbf{0})}$. Consider the mapping $\varphi : \mathbb{I}^\kappa \setminus \{\mathbf{0}\} \to 2^{Z_\mathbf{0}}$ defined by the formula

$$\varphi(y) = \bigcup \{(0, y_\alpha] \times \{\alpha\} : \alpha < \kappa\}.$$

It is easy to see that φ is one-to-one and the set $\varphi(\mathbb{I}^\kappa \setminus \{\mathbf{0}\})$ is closed in $2^{(Z_\mathbf{0},d_\mathbf{0})}$. Like in the proof of Lemma 3.2, we can show here that φ transforms a subbase of

$\mathbb{I}^\kappa \setminus \{\mathbf{0}\}$ onto a subbase of the subspace $\varphi(\mathbb{I}^\kappa \setminus \{\mathbf{0}\})$ of $2^{(Z_{\mathbf{0}}, d_{\mathbf{0}})}$. As a consequence, φ is an embedding.

Let $\widetilde{T} = \mathbb{I}^\kappa \setminus T$. By the foregoing, we know that for every $y \in \widetilde{T}$, there exists a locally $\mathbb{R}$ and complete metric space (Z_y, d_y) such that $Y_y = \mathbb{I}^\kappa \setminus \{y\}$ embeds as a closed subspace in $2^{(Z_y, d_y)}$. Applying Lemma 3.2, we see that there exists a locally $\mathbb{R}$ and complete metric space (X, d) such that the product space $\prod\{Y_y : y \in \widetilde{T}\}$ embeds as a closed subspace in $2^{(X,d)}$. As in the proof of Theorem 3.6, we see that T is homeomorphic with the closed subspace Δ of $\prod\{Y_y : y \in \widetilde{T}\}$. This completes the proof. $\square$

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