Genus 2 Curves in Small Characteristic

Lukas Zobernig

The University of Auckland

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Curves

Curves

If we consider curves of genus g, then

- the only genus 0 curve is the **projective line**,
- genus 1 curves are **elliptic curves**,
- genus 2 curves are **hyperelliptic curves**,
- genus ≥ 3 curves are more complicated.

Genus 2 Curves

As those are **hyperelliptic**, and assuming **characteristic** $\neq 2$, we find models of the form

$$y^2 = f(x) = c_6x^6 + c_5x^5 + \cdots + c_1x + c_0,$$

where c_6 and c_5 are not both zero such that $\deg(f)$ is 5 or 6.

Invariants (1)

Isomorphisms

Two genus 2 curves C and C' are isomorphic if and only if their associated **sextic forms** $f(x)$ and $g(x)$ are **conjugate** via GL_2-action.

Invariants

We can use this to define **isomorphism invariants**.

- Consider the weighted projective space $S_k = \mathbb{P}(2, 4, 6, 8, 10)$ over some field k.
- Igusa gave a set of invariants $[J_2 : J_4 : J_6 : J_8 : J_{10}] \in S_k$ for each genus 2 curve defined over k.
- Two genus 2 curves are isomorphic over k^{alg} if and only if their Igusa-invariants are the same in S_k.

Invariants (2)

Absolute Invariants

- If $\mathrm{char}(k) \neq 2$, Cardona, Quer, Nart, and Pujolàs [CQ05; CNP05] gave the **absolute "G2"-invariants** $(g_1, g_2, g_3) \in \mathbb{A}^3(k)$.
- Explicitly:

$$(g_1, g_2, g_3) = \begin{cases} \left(\frac{J_2^5}{J_{10}}, \frac{J_2^3 J_4}{J_{10}}, \frac{J_2^2 J_6}{J_{10}}\right) & \text{if } J_2 \neq 0, \\ \left(0, \frac{J_4^5}{J_{10}^2}, \frac{J_4 J_6}{J_{10}}\right) & \text{if } J_2 = 0, J_4 \neq 0, \\ \left(0, 0, \frac{J_6^5}{J_{10}^3}\right) & \text{if } J_2 = J_4 = 0. \end{cases} \quad (1)$$

- The k-points $\mathcal{M}_2(k)$ of the coarse moduli space $\mathcal{M}_2$ of genus 2 curves are in bijection with the "G2"-invariants.
- If $\mathrm{Br}_2(k)$ is trivial, then **fields of moduli** is **fields of definition**.

p-Rank and a-Number

p-Torsion in Characteristic p

Consider an algebraically closed field k of characteristic p containing $\mathbb{F}_p$. Recall the finite group schemes $\alpha_p \cong \operatorname{Spec} k[X]/X^p$ and $\mu_p \cong \operatorname{Spec} k[X]/(X^p - 1)$. Let A/k be an abelian variety of dimension g.

- The **p-rank** of A is given by $f = \dim_{\mathbb{F}_p} \operatorname{Hom}(\mu_p, A[p])$,
- the **a-number** of A is given by $a = \dim_k \operatorname{Hom}(\alpha_p, A[p])$.
- It holds that $0 \leq f \leq g$ and $1 \leq a + f \leq g$,
- and hence, geometrically, $A[p](k) \cong (\mathbb{Z}/p\mathbb{Z})^f$.

For Curves

Define the p-rank and the a-number of a genus g curve C as the **corresponding invariants** of its Jacobian $\operatorname{Jac}(C)$ as an abelian variety.

Computing the p-Rank and a-Number

- Assume $\mathrm{char}(k) \neq 2$ and consider a genus g hyperelliptic curve C defined by an equation $y^2 = f(x)$ for $f(x) \in k[x]$ of degree $2g+1$ or $2g+2$.
- Let c_i denote the coefficient of x^i in the expansion of $f(x)^{(p-1)/2}$, and define for $\ell = 0, \ldots, g-1$ the $g \times g$ matrix A_ℓ with entries $(A_\ell)_{i,j} = (c_{ip-j})^{p^\ell}$.
- We call A_0 the **Cartier-Manin** matrix of the curve C and define the matrix $M = A_{g-1} \cdots A_1 A_0$.

Lemma (Yui [Yui78])

1. *The p-rank of C is $f = \mathrm{rank}(M)$.*
2. *The a-number of C is $a = g - \mathrm{rank}(A_0)$.*

Genus 2 Types

Example

Let A/k be of dimension 2, i.e. an abelian surface, then we have the following possible types:

f	a	$A[p]$	Type	Codim.
2	0	L^2	ordinary	0
1	1	$L \oplus I_{1,1}$	non-ordinary	1
0	1	$I_{2,1}$	supersingular	2
0	2	$I_{1,1} \oplus I_{1,1}$	superspecial	3

Here $L = \mathbb{Z}/p\mathbb{Z} \oplus \mu_p$ is the p-torsion of an ordinary elliptic curve, and $I_{1,1}$ is the p-torsion of a supersingular elliptic curve. Similarly, $I_{2,1}$ is the unique BT_1 group scheme of rank p^2 with p-rank 0 and a-number 1.

Genus 2 Curves in Characteristic 2 and 3

Maisner and Nart [MN07], and Howe [How08] studied **supersingular genus 2 curves in characteristic 2 and 3**, respectively. In characteristic 3 we have that:

- The coarse moduli space $\mathcal{S}_2$ of supersingular genus 2 curves is isomorphic to the affine line $\mathbb{A}^1$.
- The absolute invariants $(0, 0, g_3)$ correspond to the curve $y^2 = x^6 + g_3^2 x^3 + g_3^3 x + g_3^4$ if $g_3 \neq 0$ and the point $(0, 0, 0)$ corresponds to $y^2 = x^5 + 1$.
- If $q = 3^r$, then the subspace $\mathcal{S}_2(\mathbb{F}_q)$ of $\mathcal{M}_2(\mathbb{F}_q)$ corresponding to supersingular genus 2 curves contains q elements.

Non-Ordinary Curves in Characteristic 3

We show that over the finite field $\mathbb{F}_q$ where $q = 3^r$, there are

- $q^2(q-1)$ many ordinary,
- $q(q-1)$ many non-ordinary, and
- q many supersingular genus 2 curves.

Theorem

Let k be a finite field of characteristic 3. Let C be a genus 2 curve defined over k and let (g_1, g_2, g_3) be its absolute invariants. Then C is non-ordinary if and only if $g_1 = 0$, $g_2 \in k^\times$, and $g_3 \in k$.

Corollary

Let k be a finite field of characteristic 3. Every ordinary genus 2 curve defined over k has absolute invariants (g_1, g_2, g_3) with $g_1 \in k^\times$ and $g_2, g_3 \in k$.

The Coarse Moduli Space $\mathcal{M}_2$ in Characteristic 3

Corollary

Let k be a finite field of characteristic 3. For $g_2 \in k^\times$ and $g_3 \in k$, the curve

$$y^2 = x^6 + \sqrt[6]{1 + g_2 - g_2 g_3}\, x^3 + \sqrt[3]{g_2}\, x^2 + 1$$

defined over k^{alg} is non-ordinary and has absolute invariants $(0, g_2, g_3)$.

For a finite field $k = \mathbb{F}_q$ with $q = p^r$, denote by $\mathcal{S}_2(k)$, $\mathcal{N}_2(k)$, and $\mathcal{O}_2(k)$ the supersingular, non-ordinary, and ordinary strata of the coarse moduli space $\mathcal{M}_2(k)$.

Theorem

Let k be a finite field of characteristic 3 with q elements. Then

1. $\#\mathcal{O}_2(k) = q^2(q-1)$,
2. $\#\mathcal{N}_2(k) = q(q-1)$,
3. $\#\mathcal{S}_2(k) = q$.

Computational Results (1)

q	$\#\mathcal{S}_2(k)$	$\#\mathcal{N}_2(k)$	$\#\mathcal{O}_2(k)$
2	2	2	4
2^2	2^2	$2^2 \cdot 3$	$2^4 \cdot 3$
3	3	$3 \cdot 2$	$3^2 \cdot 2$
3^2	3^2	$3^2 \cdot 8$	$3^4 \cdot 8$
3^3	3^3	$3^3 \cdot 26$	$3^6 \cdot 26$
5	5	$5 \cdot 4$	$5^2 \cdot 4$
5^2	5^2	$5^2 \cdot 24$	$5^4 \cdot 24$
7	7	$7 \cdot 6$	$7^2 \cdot 6$
7^2	7^2	$7^2 \cdot 48$	$7^4 \cdot 48$

Conjecture

We conjecture that the strata sizes in characteristic 5 and 7 behave the same as in characteristic 2 and 3: $\#\mathcal{O}_2(k) = q^2(q-1)$, $\#\mathcal{N}_2(k) = q(q-1)$, $\#\mathcal{S}_2(k) = q$.

Computational Results (2)

Let $\Delta_0 = \#\mathcal{S}_2(k) - q$, $\Delta_1 = \#\mathcal{N}_2(k) - q(q-1)$, and $\Delta_2 = \#\mathcal{O}_2(k) - q^2(q-1)$.

q	$\#\mathcal{S}_2(k)$	Δ_0	$\#\mathcal{N}_2(k)$	Δ_1	$\#\mathcal{O}_2(k)$	Δ_2
11	9	-2	101	-9	1221	11
11^2	117	-4	14403	-117	1757041	121
13	20	7	149	-7	2028	0
13^2	330	161	28231	-161	4798248	0
17	25	8	264	-8	4624	0
19	26	7	335	-7	6498	0
23	36	13	494	-12	11637	-1
29	49	20	851	39	23489	-59
...	...	...	...	...	...	...
127	570	443	17592	1590	2030221	-2033
131	409	278	20931	3901	2226751	-4179
137	576	439	18198	-434	2552579	-5
139	516	377	20745	1563	2664358	-1940

The Supersingular Locus (1)

Let k be a finite field of characteristic p. Thanks to Ibukiyama et al. [IKO86], Katsura and Oort [KO87], and Koblitz [Kob75] we know various facts about $\mathcal{S}_2$ and the **superspecial locus** $\mathcal{SP}_2$:

- Every **principally polarised superspecial abelian surface** A over k^{alg} arises from choosing a principal polarisation on a product $E \times E$ for a supersingular elliptic curve E.
- The **number of principal polarisations** on $E \times E$ over k^{alg} up to automorphisms is equal to $H_2(p, 1)$ (a certain class number related to E).
- As a principally polarised abelian variety, either $A = E_1 \times E_2$ for two supersingular elliptic curves E_1 and E_2, or $A = \text{Jac}(C)$ for a superspecial genus 2 curve C.
- Hence the **number of superspecial genus 2 curves** over k^{alg} is given by $H_p = H_2(p, 1) - h_p(h_p + 1)/2$, where $h_p = H_1(p, 1)$ is the number of isomorphism classes of supersingular elliptic curves over k^{alg}.

The Supersingular Locus (2)

The number H_p is **finite**; every superspecial genus 2 curve can be defined over $\mathbb{F}_{p^2}$. This allows us to determine the **finite contribution** of superspecial genus 2 curves to $\#\mathcal{S}_2(k)$ when k contains $\mathbb{F}_{p^2}$. More interesting are the $\mathbb{F}_p$-points:

p	11	13	17	19	23	29	31	37	41
$\#\mathcal{S}_2(\mathbb{F}_p)$	9	20	25	26	36	49	54	102	70
H_p	2	3	5	7	10	18	20	31	40
$\#\mathcal{SP}_2(\mathbb{F}_p)$	2	3	5	5	8	12	12	9	22

Some Open Questions

- Determine the possible **automorphism groups** in the individual strata of $\mathcal{M}_2$.
- Determine the possible **Weil polynomials** of non-ordinary genus 2 curves in characteristic 3.
- How does the distribution of the **defects** Δ_i depend on the characteristic p? This seems to require some Sato-Tate style argument depending on the reduction of the moduli space $\mathcal{M}_2$ at various primes.

Questions?

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