Genus 2 Curves in Small Characteristic

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Abstract

We study genus 2 curves over finite fields of small characteristic. The p-rank f of a curve induces a stratification of the coarse moduli space $\mathcal{M}_2$ of genus-2 curves up to isomorphism. We are interested in the size of those strata for all $f \in \{0, 1, 2\}$. In characteristic 2 and 3, previous results show that the *supersingular* $f = 0$ stratum has size q. We show that for $q = 3^r$, over $\mathbb{F}_q$ the *non-ordinary* $f = 1$ and *ordinary* $f = 2$ strata are of size $q(q - 1)$ and $q^2(q - 1)$, respectively. We give results found from computer calculations which suggest that these formulas hold for all $p \leq 7$ and break down for $p > 7$.

1 Introduction

Maisner and Nart [MN07] studied supersingular genus 2 curves in characteristic 2. Howe [How08] showed that over an algebraically closed field k characteristic 3, the supersingular locus $\mathcal{S}_2$ of the coarse moduli space $\mathcal{M}_2$ is in bijection with k via certain invariants of genus 2 curves; we will discuss these invariants in Section 1.2.

In Section 2, we prove a similar result for the non-ordinary stratum of $\mathcal{M}_2$ in characteristic 3, again in terms of certain *absolute* invariants. From this we obtain a complete classification of isomorphism classes of supersingular, non-ordinary, and ordinary genus 2 curves defined over a finite field $\mathbb{F}_q$ with $q = 3^r$ in terms of absolute invariants, and their counts. In Section 3 we present and discuss further computational results for $p > 3$.

Let us briefly recall some background on the p-torsion structure of abelian varieties, moduli and fields of definition of genus 2 curves, and previous results for those curves over finite fields of characteristic 2 and 3.

1.1 The p-rank and a-number of Abelian Varieties and Curves

Fix a prime p, and an algebraically closed field k of characteristic p containing $\mathbb{F}_p$. Consider the *additive group scheme* $\mathbb{G}_a = \operatorname{Spec} k[X]$ and the *multiplicative group scheme* $\mathbb{G}_m = \operatorname{Spec} k[X, X^{-1}]$. The kernel of the relative Frobenius on these yields the finite group schemes $\alpha_p \cong \operatorname{Spec} k[X]/X^p$ and $\mu_p \cong \operatorname{Spec} k[X]/(X^p - 1)$, respectively. The Cartier dual of α_p is itself, and the Cartier dual of μ_p is the constant group scheme $\mathbb{Z}/p\mathbb{Z}$.

Let A/k be an abelian variety of genus g and consider its p-torsion $A[p]$ as a group scheme. The *p-rank* of A is given by $f = \dim_{\mathbb{F}_p} \operatorname{Hom}(\mu_p, A[p])$. Similarly, the a-number of A is given by $a = \dim_k \operatorname{Hom}(\alpha_p, A)$. Thus, geometrically, $A[p](k) \cong (\mathbb{Z}/p\mathbb{Z})^f$. It holds that $0 \leq f \leq g$ and $1 \leq a + f \leq g$.

Example 1 (Pries [Pri08])**.** *Let A/k be of genus 2, i.e. an abelian surface, then we have the following possible types:*

f	a	$A[p]$	Type	Codim.
2	0	L^2	*ordinary*	*0*
1	1	$L \oplus I_{1,1}$	*non-ordinary*	*1*
0	1	$I_{2,1}$	*supersingular*	*2*
0	2	$I_{1,1} \oplus I_{1,1}$	*superspecial*	*3*

Here $L = \mathbb{Z}/p\mathbb{Z} \oplus \mu_p$ is the p-torsion of an ordinary elliptic curve, whereas $I_{1,1}$ is the p-torsion of a supersingular elliptic curve. It is also the unique BT_1 group scheme of rank p and fits into the following non-split exact sequence: $0 \to \alpha_p \to I_{1,1} \to \alpha_p \to 0$. Similarly, $I_{2,1}$ is the unique BT_1 group scheme of rank p^2 with p-rank 0 and a-number 1. Note that a superspecial abelian surface is in particular supersingular. The codimension of the associated strata in the full moduli space of abelian surfaces $\mathcal{A}_2$ is given as well.

We define the p-rank and the a-number of a genus g curve C as the corresponding invariants of its Jacobian $\mathrm{Jac}(C)$ as an abelian variety.

Assume $\mathrm{char}(k) \neq 2$ and consider a genus g hyperelliptic curve C defined by an equation $y^2 = f(x)$ for $f(x) \in k[x]$ of degree $2g+1$ or $2g+2$. Let c_i denote the coefficient of x^i in the expansion of $f(x)^{(p-1)/2}$, and define for $\ell = 0, \ldots, g-1$ the $g \times g$ matrix A_ℓ with entries $(A_\ell)_{i,j} = (c_{ip-j})^{p^\ell}$. Following Yui [Yui78] and heeding Achter and Howe [AH19] we call A_0 the Cartier-Manin matrix of the curve C and define the matrix $M = A_{g-1} \cdots A_1 A_0$. Then we have the following lemma.

Lemma 1. *Let C be a genus g hyperelliptic curve defined by $y^2 = f(x)$.*

1. *The p-rank of C is $f_C = \mathrm{rank}(M)$.*
2. *The a-number of C is $a_C = g - \mathrm{rank}(A_0)$.*

1.2 Moduli Space and Invariants

Points in the coarse moduli space $\mathcal{M}_2$ of genus-2 curves up to isomorphism have various different descriptions. Consider the weighted projective space $S_k = \mathbb{P}(2,4,6,8,10)(k)$ over an arbitrary field k. Igusa [Igu60] gave a set of invariants $[J_2 : J_4 : J_6 : J_8 : J_{10}] \in S_k$ associated to every genus-2 curve C defined over k. Two curves are isomorphic over k^{alg} if any only if their Igusa-invariants are the same in S_k. If $\mathrm{char}(k) \neq 2$, we can assume that C is given by a model of the form $y^2 = f(x)$. Then $2^{12} J_{10}$ is simply the discriminant of $f(x)$.

For fields of characteristic different from 2, Cardona, Quer, Nart, and Pujolàs [CQ05; CNP05] gave the absolute "G2" invariants

$$(g_1, g_2, g_3) = \begin{cases} \left(\frac{J_2^5}{J_{10}}, \frac{J_2^3 J_4}{J_{10}}, \frac{J_2^2 J_6}{J_{10}}\right) & \text{if } J_2 \neq 0, \\ \left(0, \frac{J_4^5}{J_{10}^2}, \frac{J_4 J_6}{J_{10}}\right) & \text{if } J_2 = 0, J_4 \neq 0, \\ \left(0, 0, \frac{J_6^5}{J_{10}^3}\right) & \text{if } J_2 = J_4 = 0. \end{cases} \tag{1}$$

Again, two curves are isomorphic over k^{alg} if any only if their absolute invariants are the same. Hence, the k-points $\mathcal{M}_2(k)$ are in bijection with the set of points $(g_1, g_2, g_3) \in \mathbb{A}^3(k)$ as above.

Remark 1. *Consider a point of moduli $P = [C] \in \mathcal{M}_2(k)$ defined over k. Then Mestre [Mes91], and Cardona and Quer [CQ05, Theorem 2] showed that it is only the generic case $\mathrm{Aut(C)} \cong C_2$ where there exists an obstruction to the point P being represented by a curve C' defined over k, which is an element in*

$\mathrm{Br}_2(k)$ *(i.e. a non-trivial 2-torsion element). If the automorphism group* $\mathrm{Aut}(C) \not\cong C_2$*, then there always exists a curve* C' *defined over* k *with* $C \cong C'$ *(geometrically). Hence, if* k *has trivial Brauer group (for example when* k *is a finite field, or an algebraic closure thereof), then for every* $(g_1, g_2, g_3) \in \mathbb{A}^3(k)$ *there exists a curve* C *defined over* k *with those invariants.*

1.3 Genus 2 Curves in Characteristic 2 and 3

Maisner and Nart [MN07], and Howe [How08] studied supersingular genus 2 curves in characteristic 2 and 3, respectively. [How08, Theorem 2.2] states that in characteristic 3, the coarse moduli space $\mathcal{S}_2$ of supersingular genus 2 curves is isomorphic to the affine line $\mathbb{A}^1$: The absolute invariants $(0, 0, g_3)$ correspond to the curve $y^2 = x^6 + g_3^2 x^3 + g_3^3 x + g_3^4$ if $g_3 \neq 0$ and the point $(0, 0, 0)$ corresponds to $y^2 = x^5 + 1$. Hence, over the finite field $\mathbb{F}_q$ where $q = 3^r$, the subspace $\mathcal{S}_2(\mathbb{F}_q)$ of $\mathcal{M}_2(\mathbb{F}_q)$ corresponding to isomorphism classes of supersingular genus 2 curves contains q elements.

2 Theoretical Results

The goal of this section is to show that over the finite field $\mathbb{F}_q$ where $q = 3^r$, there are $q^2(q-1)$ many ordinary, $q(q-1)$ many non-ordinary, and q many supersingular genus 2 curves.

Theorem 1. *Let* k *be a finite field of characteristic* 3*. Let* C *be a genus 2 curve defined over* k *and let* (g_1, g_2, g_3) *be its absolute invariants. Then* C *is non-ordinary if and only if* $g_1 = 0$*,* $g_2 \in k^\times$*, and* $g_3 \in k$*.*

Proof. We can assume that C has a model over k of the form $dy^2 = x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$. Then the matrices A_0 and A_1 of C are given by

$$A_0 = \begin{pmatrix} c_2 & c_1 \\ c_5 & c_4 \end{pmatrix}, \ A_1 = \begin{pmatrix} c_2^3 & c_1^3 \\ c_5^3 & c_4^3 \end{pmatrix}.$$

Thus, the Hasse-Witt matrix M of C becomes

$$M = A_1 A_0 = \begin{pmatrix} c_1^3 c_5 + c_2^4 & c_1^3 c_4 + c_1 c_2^3 \\ c_2 c_5^3 + c_4^3 c_5 & c_1 c_5^3 + c_4^4 \end{pmatrix}.$$

By Lemma 1, C has p-rank 1 if $\mathrm{rank}(M) = 1$. To make analysis easier, we can alternatively require $\mathrm{rank}(A_0) = 1$, and exclude the supersingular cases $dy^2 = x^6 + c_5x^5 + c_3x^3 + c_0$ and $dy^2 = x^6 + c_3x^3 + c_1x + c_0$ we find from the constraint $M = 0$ (again by Lemma 1). The remaining cases correspond to the following seven possible A_0 matrices:

$$\begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & c_4 \end{pmatrix}, \begin{pmatrix} c_2 & 0 \\ c_5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c_1 \\ 0 & c_4 \end{pmatrix}, \begin{pmatrix} c_2 & 0 \\ c_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c_5 \\ 0 & c_5 \end{pmatrix}, \begin{pmatrix} c_2 & c_5 \\ c_2 & c_5 \end{pmatrix}.$$

Both approaches yield the following seven possible types of models:

$$\begin{aligned}
dy^2 &= x^6 + c_3x^3 + c_2x^2 + c_0, \\
dy^2 &= x^6 + c_4x^4 + c_3x^3 + c_0, \\
dy^2 &= x^6 + c_5x^5 + c_3x^3 + c_2x^2 + c_0, \\
dy^2 &= x^6 + c_4x^4 + c_3x^3 + c_1x + c_0, \\
dy^2 &= x^6 + c_3x^3 + c_2(x^2 + x) + c_0, \\
dy^2 &= x^6 + c_5(x^5 + x^4) + c_3x^3 + c_0, \\
dy^2 &= x^6 + c_5(x^5 + x^4) + c_3x^3 + c_2(x^2 + x) + c_0.
\end{aligned}$$

These types are all pairwise isomorphic over k^{alg} either by simply switching affine patches via the transformation $x \mapsto 1/x$ followed by a rescaling of y, or by a more general $\mathrm{GL}_2(k^{\mathrm{alg}})$ transformation of the associated sextic forms.

Without loss of generality, assume that C has a model $dy^2 = x^6 + c_3x^3 + c_2x^2 + c_0$. For its Igusa invariants we find $[0 : c_2^3 : -c_0^3 - c_0c_2^3 + c_3^6 : -c_2^6 : -c_0c_2^6]$. Using Equation (1), the absolute invariants are

$$\left(0, \frac{c_2^3}{c_0^2}, \frac{c_0^3 + c_0c_2^3 - c_3^6}{c_0c_2^3}\right) = (0, g_2, g_3).$$

Since the discriminant $2^{12}J_{10} = -c_0c_2^6$ has to be non-zero for the model to define a smooth curve, we have that $c_0, c_2 \neq 0$ and so $g_2 \in k^\times, g_3 \in k$.

On the other hand, choose $g_2 \in k^\times, g_3 \in k$, and set $c_2 = \sqrt[3]{g_2}$, $c_3 = \sqrt[6]{1 + g_2 - g_2g_3}$. Then by Remark 1 the curve $y^2 = x^6 + c_3x^3 + c_2x^2 + 1$ is geometrically isomorphic to a curve defined over k with absolute invariants $(0, g_2, g_3)$, and is non-ordinary by Lemma 1. □

The proof of Theorem 1 immediately implies the following corollary.

Corollary 1. *Let k be a finite field of characteristic 3. For $g_2 \in k^\times$ and $g_3 \in k$, the curve*

$$y^2 = x^6 + \sqrt[6]{1 + g_2 - g_2g_3}x^3 + \sqrt[3]{g_2}x^2 + 1$$

defined over k^{alg} is non-ordinary and has absolute invariants $(0, g_2, g_3)$.

Combining Section 1.3 and Theorem 1 yields the following classification for the absolute invariants of an ordinary genus 2 curve over a finite field of characteristic 3.

Corollary 2. *Let k be a finite field of characteristic 3. Every ordinary genus 2 curve defined over k has absolute invariants (g_1, g_2, g_3) with $g_1 \in k^\times$ and $g_2, g_3 \in k$.*

For a finite field $k = \mathbb{F}_q$ with $q = p^r$, denote by $\mathcal{S}_2(k)$, $\mathcal{N}_2(k)$, and $\mathcal{O}_2(k)$ the supersingular, non-ordinary, and ordinary strata of the coarse moduli space $\mathcal{M}_2(k)$. Finally, we can use [How08, Theorem 2.2] and Theorem 1 to describe the sizes of these p-rank strata defined over $\mathbb{F}_q$ where $q = 3^r$.

Theorem 2. *Let k be a finite field of characteristic 3 with q elements. Then*

1. $\#\mathcal{O}_2(k) = q^2(q - 1)$,
2. $\#\mathcal{N}_2(k) = q(q - 1)$,
3. $\#\mathcal{S}_2(k) = q$.

Proof. The discussion in Section 1.2 implies that there are $\#\mathbb{A}^3(k) = q^3$ many geometric isomorphism classes of curves defined over k. Then the sizes of $\mathcal{O}_2(k)$, $\mathcal{N}_2(k)$, and $\mathcal{S}_2(k)$ immediately follow from [How08, Theorem 2.2] and Theorem 1. □

3 Computational Results for Small Primes

We have computed the p-rank of a curve corresponding to each absolute invariant $(g_1, g_2, g_3) \in \mathbb{A}^3(\mathbb{F}_q)$ for small finite fields $\mathbb{F}_q$ of characteristic 2, 3, 5, and 7. Table 1 summarises the results we have found.

q	$\#\mathcal{S}_2(k)$	$\#\mathcal{N}_2(k)$	$\#\mathcal{O}_2(k)$
2	2	2	4
2^2	2^2	$2^2 \cdot 3$	$2^4 \cdot 3$
3	3	$3 \cdot 2$	$3^2 \cdot 2$
3^2	3^2	$3^2 \cdot 8$	$3^4 \cdot 8$
3^3	3^3	$3^3 \cdot 26$	$3^6 \cdot 26$
5	5	$5 \cdot 4$	$5^2 \cdot 4$
5^2	5^2	$5^2 \cdot 24$	$5^4 \cdot 24$
7	7	$7 \cdot 6$	$7^2 \cdot 6$
7^2	7^2	$7^2 \cdot 48$	$7^4 \cdot 48$

Table 1: Strata sizes of $\mathcal{M}_2$ over finite fields of characteristic $2, 3, 5$, and 7.

As expected, in characteristic 2, $\#\mathcal{S}_2$ follows the results of Maisner and Nart [MN07]. Similary, characteristic 3 behaves as predicted by Theorem 2. Inspired by the computational results, *we conjecture that this behaviour also holds in characteristic 5 and 7 and for the other strata in characteristic 2*.

Denote by $\Delta_0 = \#\mathcal{S}_2(k) - q$, $\Delta_1 = \#\mathcal{N}_2(k) - q(q-1)$, and $\Delta_2 = \#\mathcal{O}_2(k) - q^2(q-1)$ the difference in number of points to what one would expect if the size of the strata followed the small prime behaviour. We have computed Table 2 determining the p-rank of a curve corresponding to each absolute invariant $(g_1, g_2, g_3) \in \mathbb{A}^3(\mathbb{F}_q)$ for finite fields $\mathbb{F}_q$ with $q > 7$.

It would be interesting to determine how exactly the differences Δ_1 and Δ_2 of the non-ordinary and ordinary strata, respectively, depend on q. Alternatively we would like to say something about their distribution depending on the characteristic p. This seems to require some Sato-Tate style argument depending on the reduction of the moduli space $\mathcal{M}_2$ at various primes.

3.1 The Supersingular Locus

In this section we recall some theory about the size of the supersingular locus $\mathcal{S}_2$. Ibukiyama et al. [IKO86], Katsura and Oort [KO87], and Koblitz [Kob75] show that the supersingular locus of $\mathcal{M}_2$ is a union of projective lines with the singular points of this union corresponding to the superspecial points. Denote the finite set of superspecial points by $\mathcal{SP}_2$.

Let B be the definite quaternion algebra over $\mathbb{Q}$ with discriminant D and let $\mathcal{O}$ be a maximal order of B. Write $D = D_1 D_2$ for two positive integers D_1, D_2 and set $L_n(D_1, D_2)$ the set of left $\mathcal{O}$-lattices in B^n which are equivalent to $(\mathcal{O} \otimes \mathbb{Z}_p)^n$ at p if p does not divide D_2, and otherwise to the other local equivalence class if p divides D_2, see Ibukiyama et al. [IKO86]. Denote by $H_n(D_1, D_2)$ the number of global equivalence classes in $L_n(D_1, D_2)$. In particular, for a prime number p, denote by $H_n(p, 1)$ the class number of the *principal genus* and by $H_n(1, p)$ the class number of the *non-principal genus*.

Let k be a finite field of characteristic p. On the one hand, it is known that every principally polarised superspecial abelian surface A over k^{alg} arises from choosing a principal polarisation on a product $E \times E$ for a supersingular elliptic curve E. Let $B = \mathrm{End}(E) \otimes \mathbb{Q}$ the endomorphism algebra of E, which is the definite quaternion algebra ramified at p. Then the number of principally polarisations on $E \times E$ over k^{alg} up to automorphisms is equal to $H_2(p, 1)$ for B.

On the other hand, as a principally polarised abelian variety, either $A = E_1 \times E_2$ for two supersingular elliptic curves E_1 and E_2, or $A = \mathrm{Jac}(C)$ for a superspecial genus 2 curve C. Hence the number of superspecial genus 2 curves over k^{alg} is given by $H_p = H_2(p, 1) - h_p(h_p + 1)/2$, where $h_p = H_1(p, 1)$ is the number of isomorphism classes of supersingular elliptic curves over

q	$\#\mathcal{S}_2(k)$	Δ_0	$\#\mathcal{N}_2(k)$	Δ_1	$\#\mathcal{O}_2(k)$	Δ_2
11	9	-2	101	-9	1221	11
11^2	117	-4	14403	-117	1757041	121
13	20	7	149	-7	2028	0
13^2	330	161	28231	-161	4798248	0
17	25	8	264	-8	4624	0
19	26	7	335	-7	6498	0
23	36	13	494	-12	11637	-1
29	49	20	851	39	23489	-59
31	54	23	970	40	28767	-63
37	102	65	1229	-103	49322	38
41	70	29	1794	154	67057	-183
47	109	62	2308	146	101406	-208
53	155	102	2843	87	145879	-189
61	186	125	3775	115	223020	-240
67	210	143	4093	-329	296460	186
71	146	75	5770	800	351995	-875
73	269	196	4949	-307	383799	111
79	216	137	6838	676	485985	-813
83	259	176	7529	723	563999	-899
89	226	137	9053	1221	695690	-1358
97	408	311	8726	-586	903539	275
101	347	246	11784	1684	1018170	-1930
103	443	340	11394	888	1080890	-1228
107	357	250	12443	1101	1212243	-1351
109	412	303	11750	-22	1282867	-281
113	417	304	12834	178	1429646	-482
127	570	443	17592	1590	2030221	-2033
131	409	278	20931	3901	2226751	-4179
137	576	439	18198	-434	2552579	-5
139	516	377	20745	1563	2664358	-1940

Table 2: Strata sizes and differences of $\mathcal{M}_2$ over finite fields of small characteristic.

p	11	13	17	19	23	29	31	37	41
$\#\mathcal{S}_2(\mathbb{F}_p)$	9	20	25	26	36	49	54	102	70
H_p	2	3	5	7	10	18	20	31	40
$\#\mathcal{SP}_2(\mathbb{F}_p)$	2	3	5	5	8	12	12	9	22

Table 3: Number of $\mathbb{F}_p$ points of $\mathcal{S}_2$ and $\mathcal{SP}_2$ in small characteristic.

k^{alg}, see Ibukiyama et al. [IKO86, Corollary 2.12].

The number H_p is finite; every superspecial genus 2 curve can be defined over $\mathbb{F}_{p^2}$. This allows us to determine the finite contribution of superspecial genus 2 curves to $\#\mathcal{S}_2(k)$ when k contains $\mathbb{F}_{p^2}$. For small primes p we find Table 3; we have also determined the number of superspecial curves defined over $\mathbb{F}_p$.

The number of irreducible components of the supersingular locus of $\mathcal{M}_2$, i.e. the aforementioned projective lines, is equal to the class number of the non-principal genus $H_2(1, p)$, see Katsura and Oort [KO87, Theorem 5.7].

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