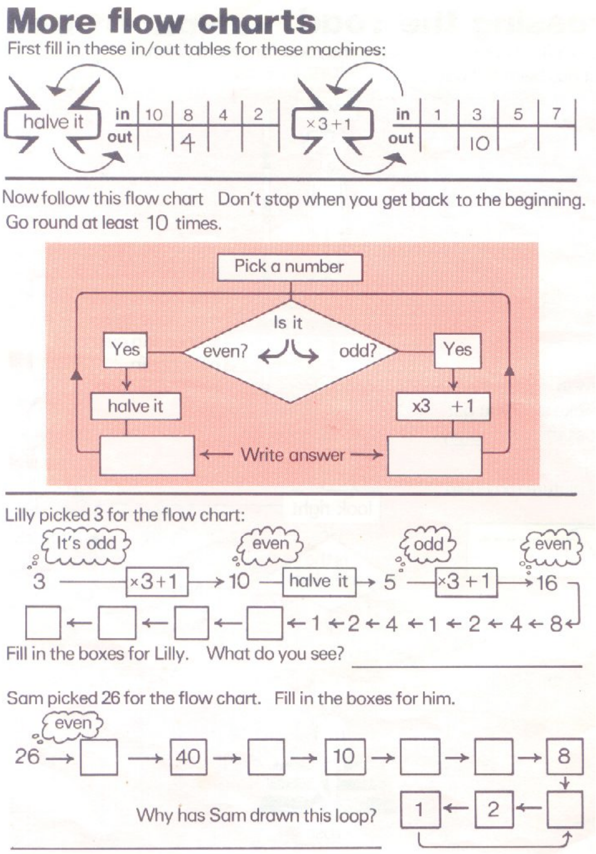
The Collatz Conjecture as a motivator for Complexity and Chaos



A problem made the 'cover piece' of Inner Ring Mathematics the flow chart number cruncher illustrated in figure 2.5. Once again, this is a problem which can be appreciated as a recursive arithmetic decision-making process and even more so for its surprising variety and unpredictability by young children who have no knowledge of abstract algebra, yet, far from being trivial, it remains an unsolved problem in mathematics whether all numbers generate a sequence forming a discrete orbit, which is eventually periodic to the portrayed cyclic sequence $4 \to 2 \to 1$.

The Collatz conjecture is an unsolved conjecture in mathematics. It is named after Lothar Collatz, who first proposed it in 1937. The conjecture is also known as the 3n + 1 conjecture, the Ulam conjecture (after Stanislaw Ulam), the Syracuse problem, as the hailstone sequence or hailstone numbers, or as Wondrous numbers per Gödel, Escher, Bach. It asks whether a certain kind of number sequence always ends in the same way, regardless of the starting number.

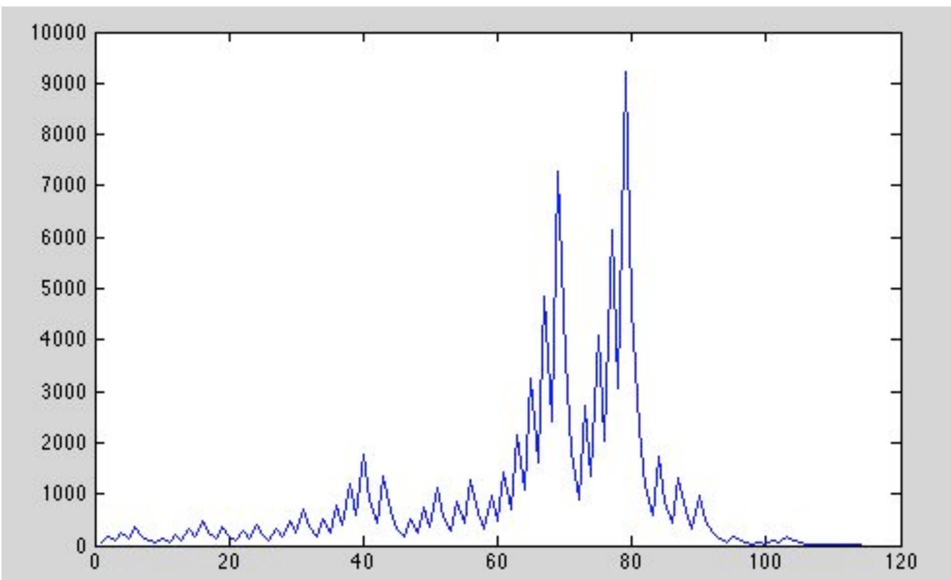
Paul Erdős said about the Collatz conjecture, "Mathematics is not yet ready for such problems." He offered $500 for its solution. (Lagarias 1985)

For instance, starting with n = 6, one gets the sequence 6, 3, 10, 5, 16, 8, 4, 2, 1.

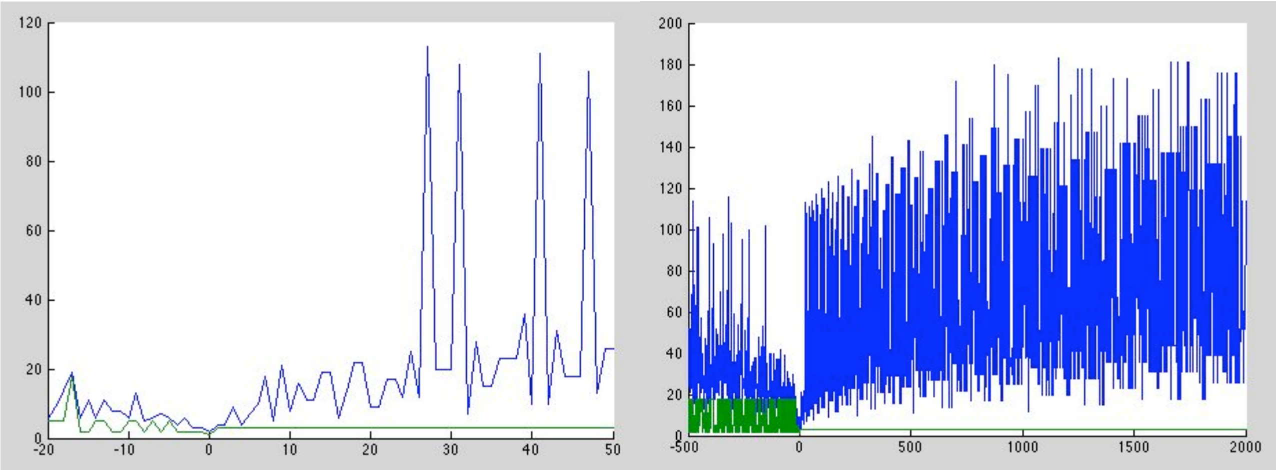
Starting with n = 11, the sequence takes longer to reach 1: 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.

If the starting value n = 27 is chosen, the sequence takes 111 steps, climbing above 9,000 before descending to 1.

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1



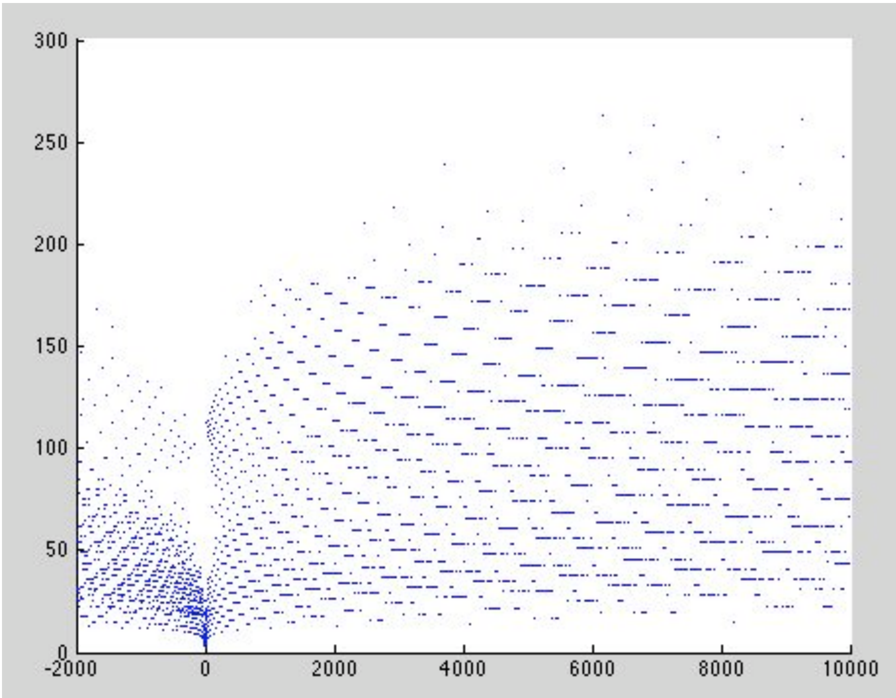
collatziter(55, 200, 0)



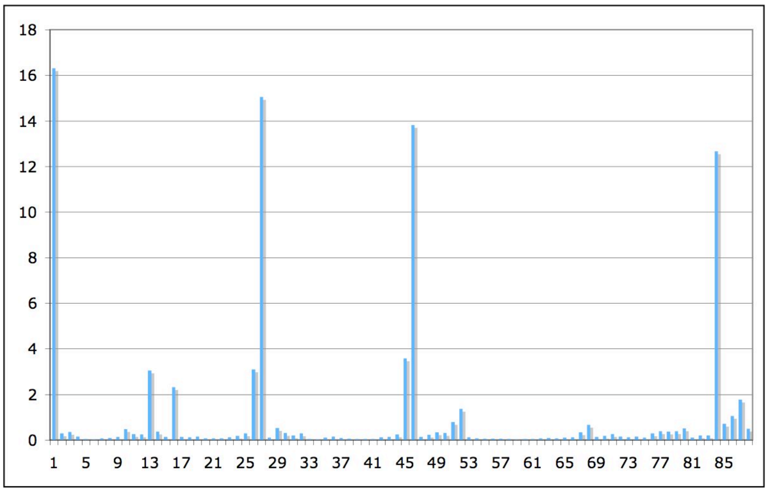
Left: cruncher(2,3,1,-20,50,0) Right: cruncher(2,3,1,-500,2000,0)
Orbits lengths are all 3 for positive integers, but 2, 5 and 18 for negative integers.

```
function cruncher(sm,bg,ad,nums,nums2,myfl)
%the function is either iterating n/sm (even) or bg*n+ad (odd)
%usually sm=2 bg=3 ad=1 to give n/2 and 3n+1
%nums and nums2 give start and finish integers
%blue chart shows orbit length to get to 1
%green chart shows period length
%if myfl=1 instead does scatter plot of maxima
newplot
huge=10000000;
huge2=100000;
myv=zeros(2,nums2-nums);
loopv=zeros(1,huge2);
for i=nums:nums2-1
    iter=i;
    loop=1;
    loopf=0;
    loopv(loop)=iter;
    while loopf==0
        loop=loop+1;
        loopj=0;
        if mod(iter,sm)~=0
            iter=bg*iter+ad;
            if iter>huge
                loopf=1;
                loopj=1;
            end
        else
            iter=iter/sm;
        end
        loopv(loop)=iter;
        j=1;
        while loopj==0
            if iter~=loopv(j);
                j=j+1;
            else
                loopf=1;
                loopj=1;
            end
            if j==loop
                loopj=1;
            end
         end
    end
    myv(1,i-nums+1)=loop;
    if iter>huge
        myv(2,i-nums+1)=0;
```

```
    else
        myv(2,i-nums+1)=loop-j;
    end
end
if myfl
    mym=myv';
    mym=mym(:,1);
    scatter(linspace(nums,nums2,length(mym)),mym,1);
else
    s=size(myv);
    plot(linspace(nums,nums2,s(2)),myv');
end
```



cruncher(2,3,1,-2000,10000,1)



Divergences of the (3x+1)OR(x/2) iteration from ($m(r)/r^2$) for successive path records.

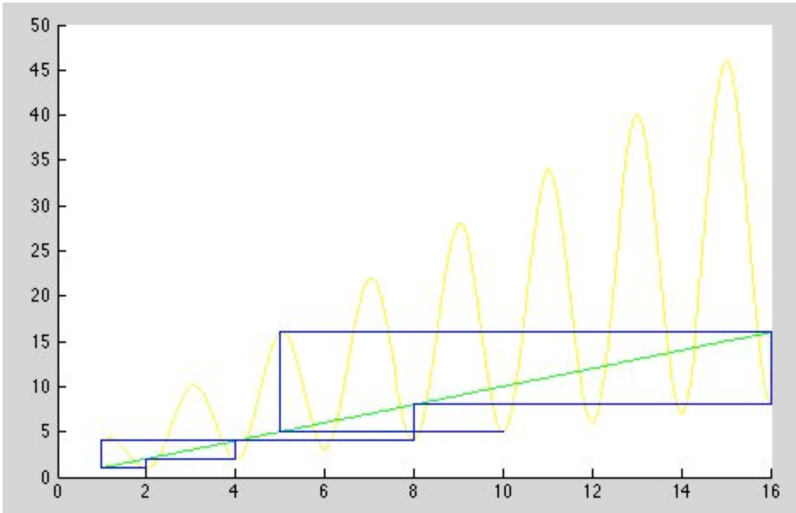
Computational paths records have been established for this iteration[i]. A current highest known, the 88^{th} path record is 1,980976,057694,848447 which reached 64,024667,322193,133530,165877,294264,738020 before eventually entering the 4>2>1 cycle. The successive path records 2(2) 3(16), 7(52). 15(160) 27(9232) etc. vary erratically along:

$$\log_2 m(r) = 2\log_2 r$$

The Collatz map can be viewed as the restriction to the integers of the

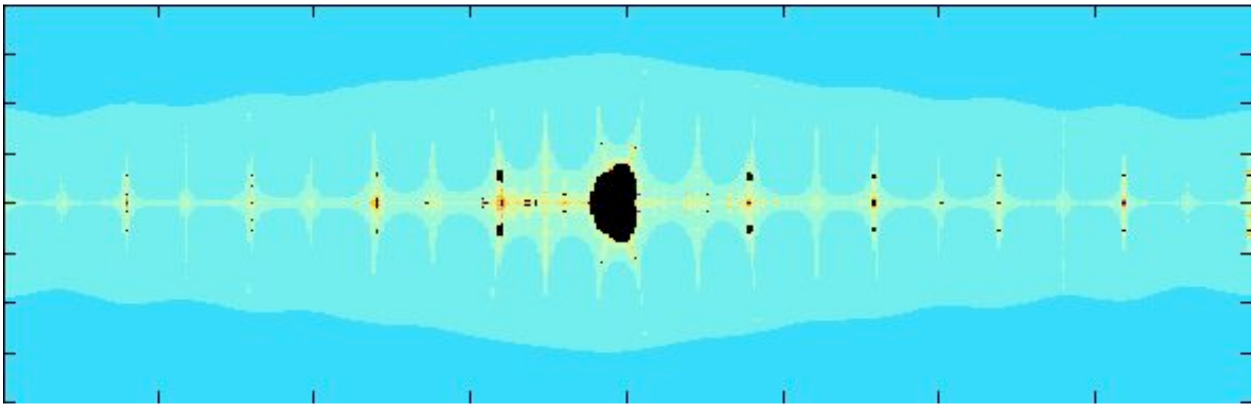
smooth real and complex map $$f(z) := \frac{1}{2} z \cos^2\left(\frac{\pi}{2} z\right) + (3z+1)\sin^2\left(\frac{\pi}{2} z\right)$$

which simplifies to $$\frac{1}{4}(2 + 7z - (2+5z)\cos(\pi z))$$



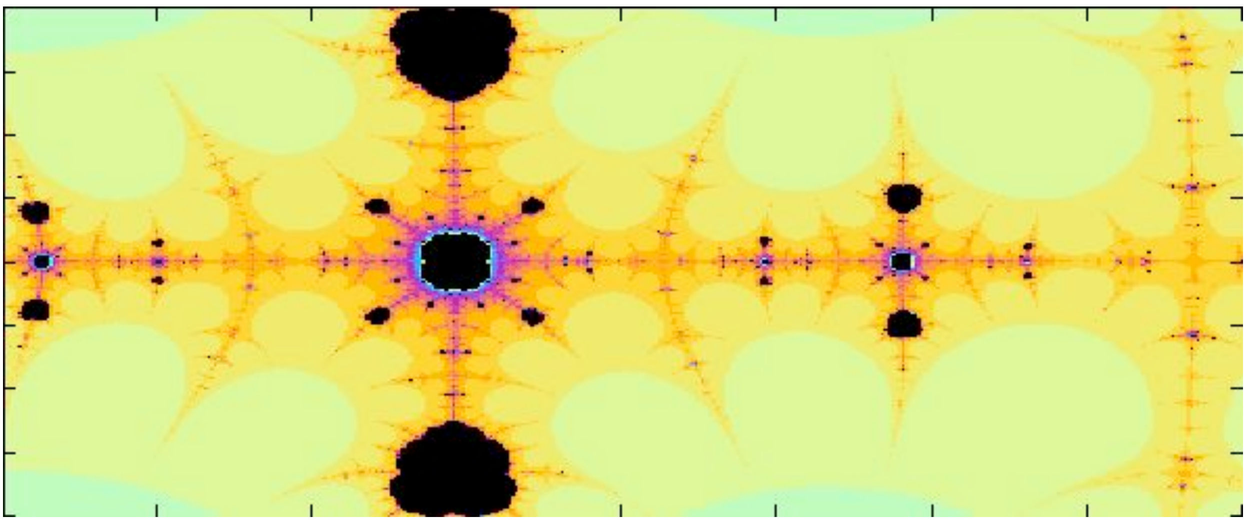
collatziter(10,20,1)

Cobweb plot of the orbit 10-5-16-8-4-2-1-4-2-1- etc. in the real extension of the Collatz map



ccollatz(-10,10,-2,2,1000)

Complex number views of the Collatz problem as fractal dynamics



ccollatz(-0.75,-0.65,-.03,.03,1000)

```matlab
function ccollatz(xmin,xmax,ymin,ymax, maxiter);
%Example ccollatz(-2,2,-2,2,300)
%Fractal ccollatz(-1,-0.5,-.1,.1,1000)

nx = 400;
ny = 400;
ColorMset = zeros(ny,nx,3);
wb = waitbar(0,'Please wait...');
for iy = 1:ny
  cy = ymin + iy*(ymax - ymin)/(ny - 1);
  for ix= 1:nx
    cx = xmin + ix*(xmax - xmin)/(nx - 1);
    k = Mlevel(cx,cy,maxiter);
    if k == 0
        ColorMset(iy,ix,:) = 0;
    else
        ColorMset(iy,ix,1) = abs(sin(2*k/10));
        ColorMset(iy,ix,2) = abs(sin(2*k/10+pi/4));
        ColorMset(iy,ix,3) = abs(cos(2*k/10));
    end
  end
  waitbar(iy/ny,wb)
end
close(wb);
image(ColorMset);

function [potential] = Mlevel(cx,cy,maxiter)
z = complex(cx,cy);
iter = 0;
while (iter < maxiter)&(abs(z) < 100)
   z = (2+7*z-(2+5*z)*cos(pi*z))/4;
   iter = iter+1;
end
if iter < maxiter
    potential = iter;
else
    potential = 0;
end
```

[i] http://www.ericr.nl/wondrous/pathrecs.html