Determining Aschbacher classes using characters

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Abstract

Abstract. Let $\Delta\colon G \to \mathrm{GL}(n, K)$ be an absolutely irreducible representation of an arbitrary group G over an arbitrary field K; let $\chi\colon G \to K\colon g \mapsto \mathrm{tr}(\Delta(g))$ be its character. In this paper, we assume knowledge of χ only, and study which properties of Δ can be inferred. We prove criteria to decide whether Δ preserves a form, is realizable over a subfield, or acts imprimitively on $K^{n\times 1}$. If K is finite, this allows us to decide whether the image of Δ belongs to certain Aschbacher classes.

Keywords. Representations of groups, character theory, Aschbacher classification

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1 Introduction

Let $\Delta\colon G \to \mathrm{GL}(n, K)$ be a representation, where G is an arbitrary group (possibly infinite) and K is an arbitrary field. Denote by $\chi = \chi_\Delta\colon G \to K\colon g \mapsto \mathrm{tr}(\Delta(g))$ its character. In this paper, we assume knowledge of χ only, and study which properties of Δ can be inferred. We will restrict to absolutely irreducible representations; in this case, Δ is uniquely determined by χ, up to equivalence.

The results are motivated by Aschbacher's classification of maximal subgroups of $\mathrm{GL}(n, F)$, where F is a finite field [Asc84]. According to this classification, every subgroup of $\mathrm{GL}(n, F)$ belongs to one of nine classes, often denoted $\mathcal{C}_1$ up to $\mathcal{C}_9$. For example, a subgroup of $\mathrm{GL}(n, F)$ belongs to class $\mathcal{C}_2$ if acts imprimitively on $F^{n\times 1}$; it belongs to class $\mathcal{C}_5$ if it is definable modulo scalars over a subfield of F. We give criteria on χ to decide whether the image H of Δ belongs to one of the classes $\mathcal{C}_2$, $\mathcal{C}_5$, or $\mathcal{C}_8$. While motivated by Aschbacher's classification, most of the results are also valid for arbitrary fields. In most cases, the criterion is of the form that χ has a non-trivial stabilizer under certain actions.

Most of our results are generalizations of results in [PF09], where, Plesken and Fabiańska describe an L_2-quotient algorithm. This algorithm, its generalization [Jam14], and the L_3-U_3-quotient algorithm [Jam12], provide examples where only the character of a representation is known, but not the actual representation. The algorithms take as input a finitely presented group G on two generators and compute all quotients of G which are isomorphic to $\mathrm{PSL}(2, q)$, $\mathrm{PSL}(3, q)$, or $\mathrm{PSU}(3, q)$. Instead of constructing the possible representations into $\mathrm{PSL}(2, q)$ or $\mathrm{PSL}(3, q)$, they construct all possible characters. One advantage of this approach is that the representations recovered from the characters are pairwise non-equivalent, whereas that would not necessarily be the case if we constructed the representations directly. Another advantage is that the minimal splitting field of a representation over finite fields is the field generated by the character values,

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and the latter can be easily determined from the character. The characters are constructed by translating the group relations into arithmetic conditions for the possible character values. This yields characters $\chi\colon F_2 \to R$, where F_2 is the free group on two generators and R is a finitely generated commutative ring. Taking quotients of R yields characters $\chi_q\colon F_2 \to \mathbb{F}_q$, and every such character corresponds to a representation $\Delta_q\colon F_2 \to \mathrm{SL}(n,q)$ which induces a homomorphism $\delta_q\colon G \to \mathrm{PSL}(n,q)$ (with $n=2$ or $n=3$, depending on the algorithm). To decide whether δ_q is surjective, we must decide whether the image of Δ_q lies in one of the Aschbacher classes. This can be done using the criteria on χ_q described in this paper. These criteria are independent of the characteristic of the field, so instead of applying the criteria to every χ_q, they can be applied to χ, thus deciding the membership simultaneously for every quotient of R. Using this approach, the algorithms can handle all prime powers q at once, and determine all possible values of q automatically. The decision whether δ_q maps onto $\mathrm{PSU}(3,q)$ is similar.

The problem of determining the Aschbacher class of a given matrix group $G \leq \mathrm{GL}(n,q)$ is central in the matrix group recognition project; see [NP98, HLGOR96, GLGO06, CNRD09] for algorithms dealing with Aschbacher classes $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_5$, and $\mathcal{C}_8$. The results in this paper do not aim to replace any of these algorithms; while they could be applied to matrix group recognition, the resulting runtime would certainly be worse than that of the existing algorithms. The algorithmic value of our results is rather that they can be applied to characters, without the knowledge of the full representation, and that they work for arbitrary fields. Furthermore, our results have a purely theoretical value, linking Aschbacher classes to characters admitting certain stabilizers (see Theorem 6.1).

The results in this paper are based on the following proposition, which is a special case of [Car94, Théorème 1] and [Nak00, Theorem 6.12]. We include a short proof for the convenience of the reader.

Proposition 1.1. *Let G be a group, K a field, and let $\Delta_i\colon G \to \mathrm{GL}(n,K)$ be representations for $i=1,2$ with $\chi_{\Delta_1} = \chi_{\Delta_2}$. Assume that Δ_1 is absolutely irreducible. Then Δ_1 and Δ_2 are equivalent; in particular, Δ_2 is absolutely irreducible.*

Proof. Denote the representations of the group algebra whose restriction to G is Δ_i again by Δ_i. Let $\chi = \chi_{\Delta_1} = \chi_{\Delta_2}$, and let $\mathrm{rad}(\chi)$ be the radical of the trace bilinear form $KG \times KG \to K\colon (x,y) \mapsto \chi(xy)$. Then $\ker(\Delta_i) \subseteq \mathrm{rad}(\chi)$, so $\varphi_i\colon \Delta_i(KG) \to KG/\mathrm{rad}(\chi)\colon \Delta_i(x) \mapsto x + \mathrm{rad}(\chi)$ are epimorphisms. But $\Delta_1(KG) = K^{n\times n}$ is simple, so φ_1 is invertible. Comparing dimensions we see that $\varphi_2 \circ \varphi_1^{-1}\colon K^{n\times n} \to K^{n\times n}\colon \Delta_1(x) \mapsto \Delta_2(x)$ is an automorphism, which must be inner by the Skolem-Noether Theorem ([Jac89, Theorem 4.9]). $\square$

2 Actions on characters

Definition 2.1. Let χ be the character of a representation $\Delta\colon G \to \mathrm{GL}(n,K)$.

1. For $\alpha \in \mathrm{Gal}(K)$ define ${}^{\alpha}\chi$ by $({}^{\alpha}\chi)(g) := \alpha(\chi(g))$ for $g \in G$.

2. For $\sigma \in \mathrm{Hom}(G, K^*)$ define ${}^{\sigma}\chi$ by $({}^{\sigma}\chi)(g) := \sigma(g)\chi(g)$ for $g \in G$.

3. Let $\mathrm{C}_2 = \langle\gamma\rangle$ be a cyclic group of order 2 generated by γ. Define ${}^{\gamma}\chi$ by $({}^{\gamma}\chi)(g) := \chi(g^{-1})$ for $g \in G$.

Clearly ${}^{\alpha}\chi$, ${}^{\sigma}\chi$, and ${}^{\gamma}\chi$ are characters of the representation ${}^{\alpha}\Delta\colon G \to \mathrm{GL}(n,K)\colon g \mapsto \alpha(\Delta(g))$, ${}^{\sigma}\Delta\colon G \to \mathrm{GL}(n,K)\colon g \mapsto \sigma(g)\Delta(g)$, and the contragredient representation $\Delta^{-\mathrm{tr}}\colon G \to$

$\mathrm{GL}(n, K)\colon g \mapsto (\Delta(g)^{-1})^{\mathrm{tr}}$, respectively, so we get actions of $\mathrm{Gal}(K)$, $\mathrm{Hom}(G, K^*)$, and $\langle\gamma\rangle$ on the set of all characters.

Furthermore, for $\alpha \in \mathrm{Gal}(K)$ and $\sigma \in \mathrm{Hom}(G, K^*)$ define ${}^{\alpha}\sigma \in \mathrm{Hom}(G, K^*)$ by $({}^{\alpha}\sigma)(g) := \alpha(\sigma(g))$ for all $g \in G$, and ${}^{\gamma}\sigma \in \mathrm{Hom}(G, K^*)$ by $({}^{\gamma}\sigma)(g) := \sigma(g^{-1})$ for all $g \in G$. This defines a semi-direct product $\Omega(G, K) := (\langle\gamma\rangle \times \mathrm{Gal}(K)) \ltimes \mathrm{Hom}(G, K^*)$, and it is easy to check that the three actions of Definition 2.1 yield an action of $\Omega(G, K)$ on the set of characters of representations $G \to \mathrm{GL}(n, K)$.

3 Aschbacher class $\mathcal{C}_2$

In this section, K is an arbitrary field, unless specified otherwise. Let $\Delta\colon G \to \mathrm{GL}(n, K)$ be an irreducible representation; denote by $V := K^{n\times 1}$ the induced KG-module. Let $N \trianglelefteq G$ have finite index; denote by V_N the restricted KN-module. Let $W_1, \ldots, W_k \le V_N$ be representatives of the isomorphism classes of simple KN-submodules of V_N, and let V_i be the W_i-homogeneous component of V_N, that is, the sum of all submodules of V_N isomorphic to W_i. By Clifford's Theorem (see for example [LP10, Theorem 3.6.2]), $V_i \cong \bigoplus_{j=1}^{e} W_i$ with e independent of V_i, and $V = \bigoplus_{i=1}^{k} V_i$; furthermore, G acts transitively on the V_i.

If V permits a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_k$ as vector spaces such that G permutes the V_i transitively, then Δ is *imprimitive with blocks* $V_1, \ldots, V_k$, or G *acts imprimitively on the blocks* $V_1, \ldots, V_k$. Define a homomorphism $\psi\colon G \to \mathrm{S}_k$ by $gV_i = V_{\psi(g)(i)}$ for $g \in G$ and $i \in \{1, \ldots, k\}$; then Δ *is imprimitive with block action* ψ. The aim of this section is, given $\psi\colon G \to \mathrm{S}_k$, to provide criteria on χ to decide whether Δ is imprimitive with block action ψ. We start out with general k and ψ, but later restrict to the special case that k is prime and $\operatorname{im}\psi$ is solvable.

We have the following necessary condition on χ.

Lemma 3.1. *Let $\Delta\colon G \to \mathrm{GL}(n, K)$ be an absolutely irreducible imprimitive representation with blocks $V_1, \ldots, V_k$, and let χ be the character of Δ. Let $\psi\colon G \to \mathrm{S}_k$ be a homomorphism such that G acts on the blocks via ψ. Then $\chi(g) = 0$ for all $g \in G$ with $\psi(g)$ fixed-point free.*

Proof. Let $T \le G$ be the stabilizer of V_1. Then V_1 is a KT-module, and $V \cong KG \otimes_{KT} V_1$ by Clifford's Theorem. Let $\Gamma\colon T \to \mathrm{GL}(n/k, K)$ be a representation of V_1, and let $P\colon G \to \mathrm{GL}(k, K)$ be the permutation representation corresponding to ψ. Let $h_1, \ldots, h_k$ be representatives of the cosets of T in G. After conjugation, we may assume that the images of Δ are Kronecker products: that is, $\Delta(g) = P(h_i) \otimes \Gamma(t)$, where $g = h_i t$ with $t \in T$. In particular, $\mathrm{tr}(\Delta(g)) = 0$ if $\psi(g) = \psi(h_i)$ is fixed-point free. □

The converse is not true in general, that is, if $\chi(g) = 0$ for all $g \in G$ with $\psi(g)$ fixed-point free, then there does not necessarily exist a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_k$ such that G acts on the blocks via ψ. For example, let $G = \mathrm{A}_5$, and let $\Delta\colon G \to \mathrm{GL}(5, \mathbb{C})$ be the unique absolutely irreducible representation of degree 5; let $\psi\colon G \to \mathrm{S}_5$ be the embedding. Then $\chi(g) = 0$ for every 5-cycle g, but Δ is primitive. Our aim is to give some conditions which imply the converse.

The following is a partial converse of Lemma 3.1.

Lemma 3.2. *Let $\Delta\colon G \to \mathrm{GL}(n, K)$ be an absolutely irreducible representation with character χ, and let $\psi\colon G \to \mathrm{S}_k$ be a homomorphism with kernel N. If there exists $g \in G$ such that $\chi(gx) = 0$ for all $x \in N$, then $\Delta_{|N}$ is not absolutely irreducible.*

Proof. Suppose $\Delta_{|N}$ is absolutely irreducible, so $\Delta(N)$ contains a basis of $K^{n\times n}$. Let $S\colon K^{n\times n}\times K^{n\times n}\to K$ be the trace bilinear form. Then $S(\Delta(g),\Delta(x))=\chi(gx)=0$ for all $x\in N$, and since S is non-degenerate, this implies $\Delta(g)=0$, which is impossible. □

We now restrict to the case $k=p$, a prime.

Theorem 3.3. *Let K be algebraically closed, and let $\Delta\colon G\to \mathrm{GL}(n,K)$ be irreducible with character χ. Let p be a prime with $(n,p-1)=1$, and let $\psi\colon G\to \mathrm{S}_p$ be a homomorphism such that the image is transitive and solvable. Then Δ is imprimitive with block action ψ if and only if $\chi(g)=0$ for all $g\in G$ with $\psi(g)$ fixed-point free.*

Proof. By Lemma 3.1 the condition is necessary. We prove that it is sufficient. Let $V:=K^{n\times 1}$ be the KG-module induced by Δ, and let $N:=\ker\psi$. By the O'Nan-Scott Theorem (see for example [DM96, Theorem 4.1A]), a transitive subgroup of S_p is either almost simple or isomorphic to a subgroup of $\mathrm{AGL}(1,p)$, where $\mathrm{AGL}(1,p)$ denotes the one-dimensional affine group over $\mathbb{F}_p$, acting on $\mathbb{F}_p$. We identify $\mathrm{AGL}(1,p)$ with $\mathbb{F}_p\rtimes\mathbb{F}_p^*$. The transitive subgroups of $\mathrm{AGL}(1,p)$ are conjugate to $\mathbb{F}_p\rtimes H$ for $H\le\mathbb{F}_p^*$, so $G/N\cong\mathbb{F}_p\rtimes H$ for some $H\le\mathbb{F}_p^*$. Let C be the preimage of $\mathbb{F}_p$ under ψ. Then $N\trianglelefteq C\trianglelefteq G$, and C/N is cyclic of order p. Note that V_C is irreducible by Clifford's Theorem, since G/C is cyclic and $|G/C|$ is coprime to n. On the other hand, V_N is not irreducible by Lemma 3.2, thus $V_N=W_1\oplus\cdots\oplus V_\ell$. Since C/N is cyclic, the W_i are not all pairwise isomorphic, so $V_N=V_1\oplus\cdots\oplus V_k$ with $k>1$ in the notation of Clifford's Theorem above. But C/N is cyclic of order p and acts transitively on the V_i, hence $k=p$. Now G/N acts on the blocks $V_1,\dots,V_p$. Since there is only one conjugacy class of subgroups of G/N of index p, the action of G/N on the blocks is isomorphic to the action induced by ψ. □

The last proof used the fact that K is algebraically closed, and in fact the statement is no longer true for arbitrary fields; it already fails for a cyclic action. For example, let $G:=\mathbb{F}_{q^2}^*\rtimes\mathrm{Aut}(\mathbb{F}_{q^2}/\mathbb{F}_q)$, and let $\Delta\colon G\to\mathrm{GL}(2,q)$ be an embedding. Let $\psi\colon G\to\mathrm{S}_2$ be the projection onto $\mathrm{Aut}(\mathbb{F}_{q^2}/\mathbb{F}_q)$, so $N:=\mathbb{F}_{q^2}^*\trianglelefteq G$. Then $\Delta_{|N}$ is irreducible, but not absolutely irreducible; $\mathbb{F}_{q^2}$ is the smallest splitting field for $\Delta_{|N}$. Thus Δ is imprimitive only over the field $\mathbb{F}_{q^2}$, not over the field $\mathbb{F}_q$. However, we can prove a variant of Theorem 3.3 over arbitrary fields with some restrictions on the representation.

Theorem 3.4. *Let K be an arbitrary field. Let p be prime and let $\Delta\colon G\to\mathrm{GL}(p,K)$ be an absolutely irreducible representation with character χ. Let $\psi\colon G\to\mathrm{S}_p$ be a homomorphism such that the image is transitive and solvable, but not cyclic. Then Δ is imprimitive with block action ψ if and only if $\chi(g)=0$ for all $g\in G$ with $\psi(g)$ fixed-point free.*

Proof. We use the notation of the proof of Theorem 3.3. There exists an extension field L/K such that Δ is imprimitive over L, so $L\otimes_K V_N=V_1\oplus\cdots\oplus V_p$ for one-dimensional LN-modules $V_1,\dots,V_p$. We show that we can choose $L=K$. Let $T:=\mathrm{Stab}_G(V_1)$, and let $\Gamma\colon T\to L^*=\mathrm{GL}(1,L)$ be the induced representation. By Clifford's Theorem, $L\otimes_K V\cong(V_1)_T^G=V_1\otimes_{LT}LG$, so $\chi=\Gamma^G$. Note that $\mathrm{AGL}(1,p)$ is a Frobenius group, so the intersection of two distinct stabilizers is trivial. Thus if $t\in T\setminus N$, then $gtg^{-1}\notin T$ for all $g\in G\setminus T$. The formula for induced characters shows $\Gamma(t)=\chi(t)\in K^*$ for all $t\in T\setminus N$. Since T has index p in G and G/N is not cyclic, $T\ne N$. Fix $t\in T\setminus N$, and let $n\in N$. Then $nt\in T\setminus N$, so $\Gamma(n)\Gamma(t)=\Gamma(nt)=\chi(nt)\in K^*$, hence $\Gamma(n)\in K^*$. Thus Γ is realized over K. Let V_1' be the KT-module induced by Γ; let $\Delta'\colon G\to\mathrm{GL}(p,K)$ be the representation induced by $(V_1')_T^G$ and χ' the character of Δ'. Then Δ' is imprimitive, and $\chi'=\Gamma^G=\chi$, so Δ' is equivalent to Δ. □

It would be very interesting to find general criteria which work for actions of non-prime degree or with non-solvable image. However, it seems that other techniques are needed for this. The problem with the current approach is that we do not have control over the action on the blocks; in the proof of Theorem 3.3 we use that V_N decomposes into p blocks, and there is only one transitive action of $\mathrm{AGL}(1, p)$ on a p-element set, but this is no longer true for arbitrary groups.

4 Aschbacher class $\mathcal{C}_5$

The results in this section are only valid for finite fields, so we assume that K is finite. An absolutely irreducible subgroup $H \leq \mathrm{GL}(n, K)$ is in Aschbacher class $\mathcal{C}_5$ if it is conjugate to a subgroup of $\mathrm{GL}(n, F)K^*$ for some subfield $F < K$, where we identify K^* with the scalar matrices in $\mathrm{GL}(n, K)$. Let $\mathrm{N}_{K/F} \colon K^* \to F^*$ be the norm; then $\ker(\mathrm{N}_{K/F})$ is the set of elements of K which have norm 1 over F.

Theorem 4.1. *Let K/F be an extension of finite fields, and let α be a generator of the Galois group of K/F. Let $\Delta \colon G \to \mathrm{GL}(n, K)$ be an absolutely irreducible representation with character χ. The image of Δ is conjugate to a subgroup of $\mathrm{GL}(n, F)K^*$ if and only if ${}^\alpha\chi = {}^\sigma\chi$ for some $\sigma \in \mathrm{Hom}(G, \ker(\mathrm{N}_{K/F}))$.*

Proof. Assume first that the image is conjugate to a subgroup of $\mathrm{GL}(n, F)K^*$. We may assume that it is in fact a subgroup of $\mathrm{GL}(n, F)K^*$, since we are interested only in the traces. For every $g \in G$ there exist $\lambda_g \in K^*$ and $X_g \in \mathrm{GL}(n, F)$ with $\Delta(g) = \lambda_g X_g$. Set $\sigma(g) := \alpha(\lambda_g)\lambda_g^{-1}$. This defines a homomorphism $\sigma \colon G \to \ker(\mathrm{N}_{K/F})$ with the desired properties.

Now assume conversely ${}^\alpha\chi = {}^\sigma\chi$. We show first that we can assume ${}^\alpha\Delta = {}^\sigma\Delta$, by adapting an argument of [GH97]. By Proposition 1.1, ${}^\alpha\Delta$ and ${}^\sigma\Delta$ are equivalent, so $y({}^\sigma\Delta)y^{-1} = {}^\alpha\Delta$ for some $y \in \mathrm{GL}(n, K)$. Let $g \in G$. Then

$$\Delta(g) = \alpha^{\ell-1}(y\sigma(g)\Delta(g)y^{-1}) = \mathrm{N}_{K/F}(\sigma(g))\alpha^{\ell-1}(y)\cdots\alpha(y)y\Delta(g)y^{-1}\alpha(y)^{-1}\cdots\alpha^{\ell-1}(y)^{-1},$$

where $\ell = |\alpha|$. Since g is arbitrary and Δ is absolutely irreducible, Schur's Lemma yields $\alpha^{\ell-1}(y)\cdots\alpha(y)y = \lambda I_n$ for some $\lambda \in K^*$. Applying α to this equation and conjugating with y^{-1}, we see that λ is fixed by α, hence $\lambda \in F$. Choose $\eta \in K^*$ with $\mathrm{N}_{K/F}(\eta) = \lambda$; replacing y by $\eta y \in \mathrm{GL}(n, K)$ we may assume that $\alpha^{\ell-1}(y)\cdots\alpha(y)y = I_n$. Hilbert's Theorem 90 for matrices applies (see [GH97, Proposition 1.3]), so there exists $z \in \mathrm{GL}(n, K)$ with $y = \alpha(z)^{-1}z$. Since ${}^\alpha({}^z\Delta) = {}^\sigma({}^z\Delta)$, for the rest of the proof we may assume ${}^\alpha\Delta = {}^\sigma\Delta$ and we will show that the image of Δ is a subgroup of $\mathrm{GL}(n, F)K^*$.

Since $\sigma(g)$ has norm 1, there exists $\lambda_g \in K^*$ with $\sigma(g) = \alpha(\lambda_g)\lambda_g^{-1}$ by Hilbert's Theorem 90. Set $X_g := \lambda_g^{-1}\Delta(g)$. Then $\alpha(X_g) = \alpha(\lambda_g)^{-1}\sigma(g)\Delta(g) = X_g$, so $X_g \in \mathrm{GL}(n, F)$. $\square$

5 Aschbacher class $\mathcal{C}_8$

We study absolutely irreducible representations preserving a symmetric, alternating or Hermitian form. If the field K is finite, then these representations lie in Aschbacher class $\mathcal{C}_8$, but the results are valid for arbitrary fields K.

Proposition 5.1. *Let G be a group, K a field of characteristic $\neq 2$, and $\Delta \colon G \to \mathrm{GL}(n, K)$ an absolutely irreducible representation with character χ. Then Δ fixes a symmetric or alternating form modulo scalars if and only if ${}^{\sigma\gamma}\chi = \chi$ for some $\sigma \in \mathrm{Hom}(G, K^*)$. The restriction "modulo scalars" can be removed if and only if we can choose $\sigma = 1$.*

Proof. Assume first that Δ preserves a symmetric or alternating form modulo scalars. That is, there exists a symmetric or skew-symmetric matrix $y \in K^{n\times n}$ such that $\Delta(g)^{\mathrm{tr}} y \Delta(g) = \sigma(g) y$ for all $g \in G$, where $\sigma(g) \in K^*$. It is easy to verify that $\sigma\colon G \to K^*\colon g \mapsto \sigma(g)$ defines a homomorphism. Let $z \in \overline{K}^{n\times n}$ such that $z^{\mathrm{tr}} z = y$, where $\overline{K}$ is an algebraic closure of K. Then $z\Delta(g)z^{-1} = \sigma(g)(z\Delta(g)z^{-1})^{-\mathrm{tr}}$, so $\chi(g) = \mathrm{tr}(z\Delta(g)z^{-1}) = \sigma(g)\,\mathrm{tr}((z\Delta(g)z^{-1})^{-\mathrm{tr}}) = {}^{\sigma\gamma}\chi(g)$.

Now assume that ${}^{\sigma\gamma}\chi = \chi$. The representations Δ and ${}^{\sigma}\Delta^{-\mathrm{tr}} = (g \mapsto \sigma(g)\Delta(g)^{-\mathrm{tr}})$ are absolutely irreducible with the same traces, so by Proposition 1.1 they are equivalent. Let $y \in \mathrm{GL}(n, K)$ such that $y\Delta(g)y^{-1} = \sigma(g)\Delta(g)^{-\mathrm{tr}}$ for all $g \in G$. Then $\Delta(g) = (\Delta(g)^{-\mathrm{tr}})^{-\mathrm{tr}} = y^{-\mathrm{tr}} y \Delta(g) y^{-1} y^{\mathrm{tr}}$ for all $g \in G$, so $y^{-\mathrm{tr}} y$ lies in the centralizer of Δ by Schur's Lemma. Hence $y^{-\mathrm{tr}} y = \lambda I_n$ for some $\lambda \in K$. But $y^{\mathrm{tr}} = \lambda y = \lambda^2 y^{\mathrm{tr}}$, so either $\lambda = 1$, in which case y is symmetric, or $\lambda = -1$, in which case y is skew-symmetric. □

Proposition 5.2. *Let G be a group and K a field which has an automorphism α of order 2; let $\Delta\colon G \to \mathrm{GL}(n, K)$ be an absolutely irreducible representation with character χ. Then Δ fixes an α-Hermitian form modulo scalars if and only if ${}^{\sigma\alpha\gamma}\chi = \chi$ for some $\sigma \in \mathrm{Hom}(G, K^*)$. The restriction "modulo scalars" can be removed if and only if we can choose $\sigma = 1$.*

Proof. Again, only the 'if' part is non-trivial. As in Proposition 5.1, there exists $y \in \mathrm{GL}(n, K)$ with $y\Delta(g)y^{-1} = \sigma(g)\alpha(\Delta(g)^{-\mathrm{tr}})$ for all $g \in G$, and $y = \lambda\alpha(y)^{\mathrm{tr}}$ for some $\lambda \in K$. Applying α to this last equation and transposing gives $\alpha(\lambda)^{-1} = \lambda$, so λ has norm 1 over the fixed field. By Hilbert's Theorem 90, there exists $\mu \in K$ with $\lambda = \alpha(\mu)/\mu$, and replacing y by μy we may assume that y is Hermitian. □

6 Actions on characters revisited

Let $\Omega(G, K) = (\langle\gamma\rangle \times \mathrm{Gal}(K)) \ltimes \mathrm{Hom}(G, K^*)$ be the group defined in Section 2, acting on the set of characters of representations $G \to \mathrm{GL}(n, K)$. The results in this paper show that characters with non-trivial stabilizers usually come from special types of representations. The most precise statement can be made if K is finite. Let π_i be the projection onto the ith factor of $\Omega(G, K)$, for $i = 1, 2, 3$.

Theorem 6.1. *Let K be a finite field and χ the character of an absolutely irreducible representation $\Delta\colon G \to \mathrm{GL}(n, K)$. Assume that χ has a non-trivial stabilizer; let $\rho \in \mathrm{Stab}(\chi)$ be an element of prime order.*

1. *If $\pi_1(\rho) = 1$ and $\pi_2(\rho) = 1$, then Δ is imprimitive over an extension field of K, with cyclic block action.*

2. *If $\pi_1(\rho) = 1$ and $\pi_2(\rho) \neq 1$, then Δ is realizable modulo scalars over a proper subfield of K.*

3. *If $\pi_1(\rho) \neq 1$ and $\pi_2(\rho) = 1$, and if $\mathrm{char}(K) \neq 2$, then Δ fixes an alternating or symmetric form modulo scalars.*

4. *If $\pi_1(\rho) \neq 1$ and $\pi_2(\rho) \neq 1$, then Δ fixes a Hermitian form modulo scalars.*

The restriction "modulo scalars" in the last three statements can be removed if $\pi_3(\rho) = 1$.

Proof. Let $p = |\rho|$.

Assume first $\pi_1(\rho) = 1$ and $\pi_2(\rho) = 1$, so $\rho = \sigma \in \mathrm{Hom}(G, K^*)$. Then $\sigma \in \mathrm{Hom}(G, \langle\zeta\rangle)$, where $\zeta \in K$ is a primitive pth root of unity. Identifying ζ with a p-cycle in S_p we may regard σ as a homomorphism $\psi\colon G \to \mathrm{S}_p$. Since $\chi(g) = {}^{\sigma}\chi(g) = \sigma(g)\chi(g)$ for all $g \in G$, we see $\chi(g) = 0$

whenever $\psi(g) = \sigma(g) \neq 1$, that is, $\psi(g)$ is fixed-point free. Thus the first claim follows by Theorem 3.3.

Now assume $\pi_1(\rho) = 1$ and $\pi_2(\rho) \neq 1$. If $\pi_3(\rho) = 1$, then $\rho = \alpha \in \mathrm{Gal}(K)$, so all character values lie in the fixed field F of α. By Wedderburn's Theorem, a representation over a finite field is realizable over its character field, so the claim follows for $\pi_3(\rho) = 1$. If $\pi_3(\rho) \neq 1$, then $\rho = (\alpha, \sigma)$ for $\alpha \in \mathrm{Gal}(K)$ and $\sigma \in \mathrm{Hom}(G, K^*)$. But $(1, 1) = (\alpha, \sigma)^p = (\alpha^p, \sigma \cdot {}^{\alpha}\sigma \cdots {}^{\alpha^{p-1}}\sigma)$, so $\sigma \in \mathrm{Hom}(G, \ker(\mathrm{N}_{K/F}))$, where F is the fixed field of α. The second claim now follows by Theorem 4.1.

The last two claims are reformulations of Propositions 5.1 and 5.2. $\square$

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