

Forcings with tunable energy, frequency and bandwidth: implications for grazing bifurcations of rocking frames

Ruisong Xue
Civil and Environmental Engineering

and

Hinke M. Osinga
Mathematics

University of Auckland, Private Bag 92019, Auckland 1142, New Zealand

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Abstract

Earthquake-resilience is important for building design in areas prone to earthquakes. Building techniques have started to take advantage of geometric non-linearities so that structures, such as frames or bridges, are able to rock when an earthquake hits. Rocking reduces bending along structural members and limits the transfer of seismic energy, leading to far less post-earthquake damages. Whether the tilt angle of the rocking frame exceeds a prescribed maximum depends on the forcing properties. The boundary of such structural failure is organised by so-called grazing solutions, which reach but do not move beyond the design limit of the frame. Both theoretical and numerical analysis of such geometrically non-linear structures is commonly done based on a harmonic load with a fixed frequency and amplitude. Even though non-linear systems exhibit much more complicated behaviour than linear systems, the harmonic load is thought to comprise a worst-case scenario, provided the frequency and amplitude are chosen appropriately. In this paper, we design more realistic, non-harmonic forcings with broader frequency contents. Importantly, the energy, predominant frequency and bandwidth of these forcings can be controlled independently. We propose an aperiodic non-harmonic load along with a simplified form that super-imposes a very low-frequency amplitude modulation on the harmonic load; this simplified non-harmonic load provides sufficiently similar responses when compared with the broad-frequency aperiodic load, and allows for efficient and easy numerical investigation. We compare failure boundaries for harmonic and non-harmonic loads by computing curves of grazing bifurcation in the (predominant) frequency-amplitude plane for a class of rockable frames that are held together by pre-stressed steel tendons. Our results show that, grazing bifurcations occur earlier under non-harmonic loads. Furthermore, our findings are counter-intuitive: to minimise seismic failure, the frame should be designed as flexible as possible—this insight cannot be obtained by only considering harmonic loads.

1 Introduction

Rockable structures have structural elements that can rotate freely during strong earthquakes, thereby minimising local deformation and allowing stress development to be significantly reduced [10]. The partial and temporary separation between structural members also limits the influx of seismic energy [23]. These earthquake-resilient properties are

favourable features, and a few such structures designed to permit uplift have been built in the world. An early example is the South Rangitikei Viaduct in New Zealand, which is the first bridge designed with the ability to rock [31]. Pioneering earthquake scientist Ivan Skinner (1923–2014) came up with the idea of ‘base isolation’ in the late 1960s. The viaduct, built in the late 1970s has energy-absorbing dampers included in each of the five twin-shafted piers that carry the pre-stressed concrete beams of this 315m long structure [8]. During strong earthquakes, the bridge can partially and temporarily separate from its supports, while the dampers provide additional energy dissipation during this rocking. To date, it has endured more than a thousand earthquakes without damage. Potential application of rocking was also investigated for buildings [2, 7], non-anchored equipment [13, 20], and interlocking structures [3]. Our findings, presented in this paper, suggest that tall, slender, rockable buildings are particularly earthquake-resilient. A good case in point is the rockable chimney at Christchurch airport [30], which has survived, with only very minor damage to the concrete footing, four large earthquakes and numerous aftershocks, including strong aftershocks of magnitudes well above 5 during the 2010–2012 Canterbury earthquake sequence [11].

Despite the increased research interest in rockable structures, their non-linear behaviour is still not well understood. Indeed, rocking can lead to complicated dynamics [6, 36]. Modelling efforts have largely focussed on the accuracy of the rocking by the structure itself, e.g., by considering structural flexibility [1], support flexibility [35], and including re-contact force after rocking [9]. Moreover, theoretical analysis is mainly concerned with stable dynamics, such as bifurcation and co-existence of attracting periodic orbits, which only exists in a setting with periodic (harmonic) forcing, and only if this forcing is applied indefinitely. Here, so-called grazing bifurcations are important, which arise when the periodic orbit interacts, e.g., with a discontinuity or impact boundary. Indeed, the dynamics near grazing bifurcations can be very rich, even for relatively simple models [24, 25, 29, 32]; we refer to [14, 15] for the bifurcation theory specific to such grazing periodic orbits. In contrast to periodic grazing, we are interested in transient grazing, induced when the rocking structure skirts its design limit as a result of an earthquake load that lasts only a finite time; such grazing events tend to occur well before oscillations of the rocking structure converge to a (hypothetical) asymptotic state—if it reaches such a state at all before the earthquake stops. Far fewer studies exist that investigate transient phenomena, especially in the context of rocking structures; the power system presented in [17] is a relevant example.

The influence of excitation characteristics on the rocking behaviour has also received much less attention to date. Many existing studies only analyse rockable structures under harmonic or pulse-like excitations, e.g. [21, 22, 33], while others are directly based on real earthquake records; e.g., see [19]. The former approach allows qualitative studies of rocking dynamics, but may be oversimplified and lacks realism. The latter approach, on the other hand, while ensuring realism, offers little control over distinct load features, such as frequency content, duration and energy. Since each earthquake record only represents a distinct event with an uncontrolled mix of features, it is difficult to isolate the effects of individual features. This calls for model loads that bridge controllability and realism, to enable a more thorough understanding of rockable structures. Several attempts had been made, e.g., by using the *Ricker wavelet* [4], or the *Mavroeidis and Papageorgiou pulse*, proposed by Mavroeidis and Papageorgiou [26]; see also [18]. These loads modify the predominant frequency based on the time-domain amplitude, phase shift, and number of load cycles.

In this paper, we provide an in-depth analysis of two non-harmonic loads as suitable candidates for the analysis of rockable structures. They are improved versions of the loads in [34] and exhibit a broadened frequency spectrum centred on a predominant frequency. Their definition depends on three independent parameters: the predominant frequency, the width of the frequency spectrum, and the energy of the loads (and, therefore, the *Arias intensity*, as defined by Arias [5]). Note that energy is a feature that depends on both frequency and duration characteristics. We derive improved closed-form equations to set the load energy, which were only obtained in transcendental form in [34]. Firstly, we consider a simplified non-harmonic load that contains only three discrete frequencies, amounting to a harmonic load with fixed frequency and a slowly oscillating amplitude. Next, we construct a fully aperiodic load derived from the inverse Fourier transform of a truly continuous spectrum.

As a case study, we consider the model for a class of post-tensioned rockable frames introduced in [2]; see also [27]. Our analysis makes use of the numerical tools from [27, 28], which compute curves of grazing bifurcation as solution families of a suitably posed two-point boundary value problem that we solve with pseudo-arclength continuation in the MATLAB-based software environment CoCo [12]. These earlier studies applied the numerical continuation of grazing curves only to harmonic loads; see again [17] for an example of transient grazing. Hence, our results confirm that these methods can also be applied to non-harmonic and even non-periodic loads in an equally straightforward and computationally efficient manner.

This paper is organised as follows. In the next section, we give the equations and recall specific details of the model, which is derived from first principles in [2]. Section 3 defines the load in terms of a force applied to the non-dimensionalised system (1) and presents the three loads considered in our study: the harmonic load with a specific frequency ω and amplitude A ; the simplified non-harmonic load oscillating at the same frequency ω , but with the non-synchronous amplitude modulation; and the aperiodic load obtained via the inverse Fourier transform of a continuous-frequency signal. We also explain how to derive their explicit expressions in both the frequency and time domains. The effects of applying these three different loads to both the purely linear and post-tensioned rockable frames are presented in Sec. 4, where we compare the frame behaviours under the non-harmonic loads with that for the harmonic load. Our findings show that the effects of the loads on the purely linear frame are not suitable to predict the behaviour of the post-tensioned rockable frame. Moreover, the dynamics of the nonlinear frame changes considerably when non-harmonic loads are considered and, importantly, the analysis based on harmonic loads does not predict whether the frame will fail under different loading conditions. We end with conclusions in Sec. 5.

2 Model of a post-tensioned rockable frame

For our case study, we use the analytical model for a class of post-tensioned rockable frames that was proposed in [2] and verified by shake-table experiments. As shown in Fig. 1, the frame joints can open under strong earthquakes, allowing beams and columns to rock. Here, it is assumed that the frame movement can be described by a universal angle θ , independent of the number of storeys and bays; in other words, the frame is dynamically equivalent to a rocking block [22]. For the case of a post-tensioned rockable frame, as illustrated in Fig. 1, the joints can open as soon as θ exceeds a given critical angle θ_{opening} . We express the dynamics of the rotation angle in terms of the non-dimensionalised quantity $\varphi = \theta/\theta_{\text{opening}}$,

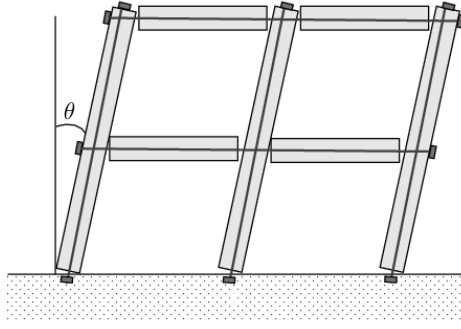


Figure 1: Post-tensioned rockable frame with geometrically non-linear beam-column connections that can open during horizontal ground excitation when the deformation angle θ exceeds a critical value. The number of storeys and bays may vary.

which means that the frame joints open as soon as $\varphi > 1$. As explained in detail in [2], the equation of motion for this class of rockable frames is then given by

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \mu(\varphi) = F(t), \quad (1)$$

where the derivatives are with respect to a rescaled time representing original time multiplied by the natural frequency of the frame. The parameter γ is the damping ratio, and $\mu(\varphi)$ is a piecewise-smooth non-linear function, which is derived from first principles as

$$\mu(\varphi) = \begin{cases} \varphi, & \text{if } |\varphi| \leq 1, \\ \left(\frac{3}{\beta} + \frac{12}{\beta^2} + \frac{8}{\beta^3}\right)\varphi + \left(3 + \frac{9}{\beta} + \frac{6(1-\sqrt{\psi})}{\beta^2} - \frac{6\sqrt{\psi}}{\beta^3} - \frac{2\psi\sqrt{\psi}}{\beta^3\varphi^2}\right)\text{sgn}(\varphi), & \text{if } |\varphi| > 1, \end{cases}$$

with $\psi = \psi(\varphi) = (1 + \beta)(\varphi^2 + \beta|\varphi|)$. Here, β is the dimensionless ratio of contact-to-cable stiffness. As can be checked, $\mu(\varphi)$ is continuously differentiable, also at $|\varphi| = 1$. This C^1 -smooth property ensures that the numerical continuation of grazing bifurcation works seamlessly across the transition from linear to non-linear states; we refer to [28] for details.

Throughout this paper, we fix $\gamma = 0.05$ and $\beta = 85$, which are the same parameter values that were used when performing the experimental verification of the model [2]. The right-hand side $F(t)$ is the earthquake load. To ensure that the rockable structure remains functional after major earthquakes, we impose a serviceability limit state, determined as $|\varphi| = 10$ [2]. Any structural response with $|\varphi| \leq 10$ will be considered safe, while exceedance of this range indicates failure of the frame. If the response reaches the design limit $|\varphi| = 10$ without exceeding it, we say that it is a *grazing solution* [27, 28]; see already Sec. 4.

3 Load definition

As mentioned, the right-hand side $F(t)$ of Eq. (1) is the earthquake load. In this paper, we wish to compare responses under different earthquake loads. More precisely, we compare responses under harmonic loads for fixed, but arbitrary, frequencies and amplitudes, with loads that have broader frequency contents and, thus, lead to non-harmonic oscillations in the time domain.

For fair comparison between loads, we define two loads as equivalent if they have the same total energy over a pre-specified (scaled) time interval $[0, t_{\text{total}}]$, which one can think of

as the duration of the earthquake measured relative to the natural frequency of the frame; in this paper, we fix $t_{\text{total}} = 50$, that is, the earthquake lasts a total of 50 times the natural period of the rocking frame. Recall that the energy (or power) of a continuous-time signal $F(t)$ is formally defined as

$$\int_{-\infty}^{\infty} |F(t)|^2 dt \approx \int_0^{t_{\text{total}}} |F(t)|^2 dt,$$

if we assume that $F(t) \approx 0$ outside the interval $[0, t_{\text{total}}]$.

We consider two different non-harmonic loads that are both based on the same idea of broadening the single peak of the harmonic load in the frequency domain. To this end, we first recall properties for a harmonic earthquake load.

3.1 Harmonic load

Alexander *et al.* [2] showed that, under harmonic load, the right-hand side of Eq. (1) can be expressed as

$$F(t) = F_{\text{harm}}(t) := A \sin(\omega t), \quad (2)$$

where the dimensionless amplitude of the sinusoidal load $A = \frac{\varrho \|\ddot{u}_g\|_{\infty}}{h \theta_{\text{opening}}}$ is given by the product of peak ground acceleration $\|\ddot{u}_g\|_{\infty}$ of the earthquake and the modal participation factor ϱ , divided by the storey height h and the critical joint opening angle θ_{opening} of the frame. The parameter ω is the ratio between the predominant earthquake frequency and the natural frequency of the frame. Here, the time t is again rescaled, so that the duration of one load cycle in original time is with respect to the predominant earthquake frequency, as intended.

The Fourier transform $\widehat{F}_{\text{harm}}(\chi)$ of the infinitely long sinusoidal load is defined as

$$\widehat{F}_{\text{harm}}(\chi) = \int_{-\infty}^{\infty} F(t)_{\text{harm}} e^{-i\chi t} dt = \frac{1}{2i} (\delta(\chi - \omega) - \delta(\chi + \omega)), \quad (3)$$

where the angular frequencies χ are scaled in the same way as ω . Hence, the Fourier amplitude spectrum (which plots $|\widehat{F}_{\text{harm}}(\chi)|$ versus χ) comprises two (infinitely high) peaks at $\chi = \pm\omega$.

The total energy over the time interval $[0, t_{\text{total}}]$ is given by

$$\begin{aligned} E_{\text{harm}}(\omega, A) &:= \int_0^{t_{\text{total}}} |A \sin(\omega t)|^2 dt = A^2 \int_0^{t_{\text{total}}} \frac{1}{2} (1 - \cos(2\omega t)) dt \\ &= A^2 \frac{2\omega t_{\text{total}} - \sin(2\omega t_{\text{total}})}{4\omega}. \end{aligned} \quad (4)$$

3.2 Non-Harmonic load

To make the load more realistic, we broaden the singular peaks in the Fourier amplitude spectrum of the harmonic load into two bell-shaped curves centred on the predominant frequencies $\pm\omega$. More precisely, we define a fully aperiodic load F_{aper} such that its Fourier transform $\widehat{F}_{\text{aper}}$ satisfies

$$|\widehat{F}_{\text{aper}}(\chi)| = \widehat{A}_{\text{peak}} e^{-b(\chi - \omega)^2}. \quad (5)$$

Figure 2 shows an example of the one-sided Fourier amplitude spectrum, where we chose $\widehat{A}_{\text{peak}} = 2.25$ and $b = 20$. Note that the maximal Fourier amplitude $\widehat{A}_{\text{peak}}$ of the aperiodic load must be smaller than that of an equivalent harmonic load if both loads are to have

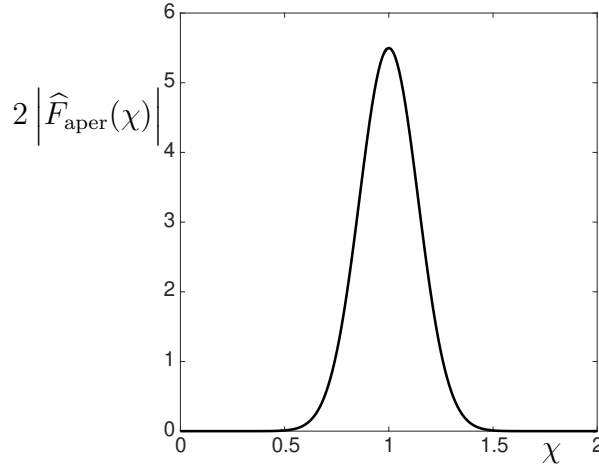


Figure 2: Fourier amplitude spectrum of the proposed fully aperiodic load for parameters $\hat{A}_{\text{peak}} = 2.25$ and $b = 20$.

the same total energy. The parameter b controls the ‘width’ of $\hat{F}_{\text{aper}}$, which we defined as the difference $\Delta\chi$ between the two roots of the equation

$$|\hat{F}_{\text{aper}}(\chi)| = \varepsilon,$$

for some value $0 < \varepsilon \ll \hat{A}_{\text{peak}}$. Then

$$\hat{A}_{\text{peak}} e^{-b(\chi-\omega)^2} = \varepsilon \Leftrightarrow \chi = \omega \pm \sqrt{\frac{\ln(\hat{A}_{\text{peak}}) - \ln(\varepsilon)}{b}}.$$

Throughout this paper, we assume $\varepsilon = 0.01 \hat{A}_{\text{peak}}$. Then the example shown in Fig. 2 satisfies $\Delta\chi = 0.2\sqrt{5 \ln(100)} \approx 0.9597$. In what follows, we stipulate $\Delta\chi = 1$, that is, we set

$$\Delta\chi = 2\sqrt{\frac{\ln(\varepsilon) - \ln(\hat{A}_{\text{peak}})}{-b}} = 1 \Leftrightarrow b = 4(\ln(\hat{A}_{\text{peak}}) - \ln(\varepsilon)) = 4 \ln(100) \approx 18.4207. \quad (6)$$

3.2.1 Simplified non-harmonic load in time domain

Before converting Eq. (5) directly to a complicated expression for F_{aper} in the time domain, let us first consider a simplified approximation of the aperiodic load obtained from the summation of only a finite number of frequencies; for example, consider the predominant sinusoidal load with two additional sub-loads, that is,

$$F_{\text{aper}}(t) \approx F_{\text{simple}}(t) := A_{\text{simple}} \sin(\omega t) + A_{\text{sub}} [\sin((\omega + d)t) + \sin((\omega - d)t)]. \quad (7)$$

These two additional sub-loads are centred symmetrically on either side of the predominant frequency ω at distance d , which is assumed to be small. To mimic the frequency profile defined in Eq. (5), we also assume that

$$\frac{A_{\text{sub}}}{A_{\text{simple}}} = e^{-bd^2}. \quad (8)$$

Using the angle addition rules from trigonometry, Eq. (7) can then be rewritten as

$$F_{\text{simple}}(t) = [A_{\text{simple}} + 2A_{\text{sub}} \cos(dt)] \sin(\omega t) = A_{\text{simple}} \left[1 + 2(e^{-b})^{d^2} \cos(dt) \right] \sin(\omega t). \quad (9)$$

In the case with b as in Eq. (6), we find

$$F_{\text{simple}}(t) = A_{\text{simple}} \left[1 + 2(0.01)^{4d^2} \cos(dt) \right] \sin(\omega t).$$

Hence, F_{simple} oscillates with frequency ω and a modulated amplitude that slowly varies with frequency d . The idea is that d is chosen such that the amplitude modulation has a large period, beyond $t_{\text{total}} = 50$, and ideally this small frequency is incommensurate with ω . We chose $d = 0.1$ for this paper, so the amplitude modulation has period $2\pi/d \approx 62.8319 > t_{\text{total}}$; note that d and ω are rationally related, which means that F_{simple} is actually periodic, but the load cycle does not repeat within the chosen time interval of $[0, t_{\text{total}}]$.

We can now determine how A_{simple} depends on the amplitude A of the harmonic load so that both loads have the same total energy on $[0, t_{\text{total}}]$, as defined by $E_{\text{harm}}(\omega, A)$ in Eq. (4). Setting $C = 2e^{-bd^2}$, for convenience, we have

$$\begin{aligned} E_{\text{harm}}(\omega, A) = E_{\text{simple}}(\omega, A) &= \int_0^{t_{\text{total}}} |A_{\text{simple}} [1 + C \cos(dt)] \sin(\omega t)|^2 dt \\ &= A_{\text{simple}}^2 (I_0 + 2C I_1 + C^2 I_2), \end{aligned}$$

where we define,

$$I_k := \int_0^{t_{\text{total}}} \cos(dt)^k \sin(\omega t)^2 dt$$

for $k = 0, 1, 2$. Trigonometric rules for angle addition lead to

$$\begin{aligned} I_0 &= \frac{1}{2} \left[t_{\text{total}} - \frac{\sin(2\omega t_{\text{total}})}{2\omega} \right], \\ I_1 &= \frac{\sin(dt_{\text{total}})}{2d} - \frac{1}{4} \left[\frac{\sin((2\omega + d)t_{\text{total}})}{2\omega + d} + \frac{\sin((2\omega - d)t_{\text{total}})}{2\omega - d} \right], \quad \text{and} \\ I_2 &= \frac{1}{2} I_0 + \frac{\sin(2dt_{\text{total}})}{8d} - \frac{1}{16} \left[\frac{\sin(2(\omega + d)t_{\text{total}})}{2(\omega + d)} + \frac{\sin(2(\omega - d)t_{\text{total}})}{2(\omega - d)} \right]. \end{aligned}$$

Since $E_{\text{harm}}(\omega, A) = A^2 I_0$, we have

$$A_{\text{simple}} = A \sqrt{\frac{I_0}{I_0 + 2C I_1 + C^2 I_2}} = A \sqrt{\frac{I_0 e^{2bd^2}}{I_0 e^{2bd^2} + 4 I_1 e^{bd^2} + 4 I_2}}.$$

Note that I_0 , I_1 , and I_2 are all functions of ω that oscillate around averages given by

$$\hat{I}_0 = \frac{t_{\text{total}}}{2}, \quad \hat{I}_1 = \frac{\sin(dt_{\text{total}})}{2d}, \quad \text{and} \quad \hat{I}_2 = \frac{t_{\text{total}}}{4} + \frac{\sin(dt_{\text{total}})}{8d},$$

We approximate A_{simple} further by ignoring the ω -dependent oscillations. With our choices of $b = 4 \ln(100)$ and $d = 0.1$, we obtain

$$A_{\text{simple}} \approx A \sqrt{\frac{\hat{I}_0 e^{2bd^2}}{\hat{I}_0 e^{2bd^2} + 4 \hat{I}_1 e^{bd^2} + 4 \hat{I}_2}} \approx \frac{A}{1.2924} \approx 0.7738 A \quad (10)$$

We checked that this simple proportional approximation for A_{simple} in the definition for F_{simple} as given by Eq. (7) leads to no more than a 3.26% energy difference from the equivalent harmonic load over the range $\omega \in [0.2, 2]$; for $\omega < 0.2$, the exact expression for A_{simple} should be used.

3.2.2 Aperiodic load in time domain

We now proceed and derive a more realistic representation F_{aper} of seismic excitation that is directly comparable to the harmonic load. Unfortunately, Eq. (5) only defines the Fourier amplitude of $\hat{F}_{\text{aper}}$, while the transformation back to the time domain requires knowledge of $\hat{F}_{\text{aper}}$ in terms of both real and imaginary parts. To overcome this issue, we assume that F_{aper} is an odd-symmetric function, just like the harmonic load. This means that, in complete analogy with Eq. (3), the Fourier transform $\hat{F}_{\text{aper}}$ and its complex conjugate $[\hat{F}_{\text{aper}}]^*$ satisfy the relation

$$\hat{F}_{\text{aper}}(\chi) = \frac{1}{2i} \left(\hat{F}_{\text{aper}}(\chi) - [\hat{F}_{\text{aper}}]^*(-\chi) \right) = \frac{1}{2i} \hat{A}_{\text{peak}} \left(e^{-b(\chi-\omega)^2} - e^{-b(\chi+\omega)^2} \right).$$

Hence, the inverse Fourier transformation gives

$$\begin{aligned} F_{\text{aper}}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}_{\text{aper}}(\chi) e^{i\chi t} d\chi = \frac{\hat{A}_{\text{peak}}}{2\pi} \frac{1}{2i} \int_{-\infty}^{\infty} \left(e^{-b(\chi-\omega)^2} - e^{-b(\chi+\omega)^2} \right) e^{i\chi t} d\chi \\ &= \frac{\hat{A}_{\text{peak}}}{2\pi} \frac{1}{2i} \int_{-\infty}^{\infty} e^{-([\sqrt{b}(\chi-\omega)]^2 - i\chi t)} - e^{-([\sqrt{b}(\chi+\omega)]^2 - i\chi t)} d\chi. \end{aligned}$$

Note that

$$\begin{aligned} e^{-([\sqrt{b}(\chi \pm \omega)]^2 - i\chi t)} &= e^{\pm i\omega t} \exp \left[\frac{i(\chi \pm \omega) t}{2\sqrt{b}(\chi \pm \omega)} \right]^2 \exp \left([\sqrt{b}(\chi \pm \omega)]^2 - i(\chi \pm \omega) t + \left[\frac{i(\chi \pm \omega) t}{2\sqrt{b}(\chi \pm \omega)} \right]^2 \right) \\ &= e^{\pm i\omega t} e^{-t^2/(4b)} \exp \left[\frac{2b(\chi \pm \omega) - it}{2\sqrt{b}} \right]^2. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} F_{\text{aper}}(t) &= \frac{\hat{A}_{\text{peak}}}{2\pi} e^{-t^2/(4b)} \frac{1}{2i} e^{i\omega t} \int_{-\infty}^{\infty} \exp \left(- \left[\frac{2b(\chi - \omega) - it}{2\sqrt{b}} \right]^2 \right) - e^{-2i\omega t} \exp \left(- \left[\frac{2b(\chi + \omega) - it}{2\sqrt{b}} \right]^2 \right) d\chi \\ &= \frac{\hat{A}_{\text{peak}}}{2\pi} e^{-t^2/(4b)} \frac{1}{2i} e^{i\omega t} \frac{\sqrt{\pi}}{2\sqrt{b}} \left[\text{erf} \left(\frac{2b(\chi - \omega) - it}{2\sqrt{b}} \right) - e^{-2i\omega t} \text{erf} \left(\frac{2b(\chi + \omega) - it}{2\sqrt{b}} \right) \right]_{\chi \rightarrow -\infty}^{\chi \rightarrow \infty} \\ &= \frac{\hat{A}_{\text{peak}}}{4\sqrt{\pi} b} e^{-t^2/(4b)} \frac{1}{2i} e^{i\omega t} [2(1 - e^{-2i\omega t})] = \frac{\hat{A}_{\text{peak}}}{2\sqrt{\pi} b} e^{-t^2/(4b)} \frac{1}{2i} [e^{i\omega t} - e^{-i\omega t}]. \end{aligned}$$

Then Euler's formula leads to a seismic excitation of the form

$$F_{\text{aper}}(t) = A_{\text{peak}}(t) \sin(\omega t) := \frac{\hat{A}_{\text{peak}}}{2\sqrt{\pi} b} e^{-t^2/(4b)} \sin(\omega t), \quad (11)$$

where the constant $\hat{A}_{\text{peak}}$ still needs to be found by equating energies. As before, we restrict to the time interval $[0, t_{\text{total}}]$, and we have

$$\begin{aligned} E_{\text{aper}}(\omega, A) &= \int_0^{t_{\text{total}}} [F_{\text{aper}}(t)]^2 dt = \int_0^{t_{\text{total}}} \left| \frac{\hat{A}_{\text{peak}}}{2\sqrt{\pi} b} e^{-t^2/(4b)} \sin(\omega t) \right|^2 dt \\ &= \frac{\hat{A}_{\text{peak}}^2}{4\pi b} \int_0^{t_{\text{total}}} e^{-t^2/(2b)} \sin^2(\omega t) dt. \end{aligned}$$

Writing $\sin(\omega t)$ again in complex exponential form, and using the fact that

$$-\frac{t^2}{2b} \pm 2i\omega t = -\frac{(t \mp 2i\omega b)^2}{2b} - 2b\omega^2,$$

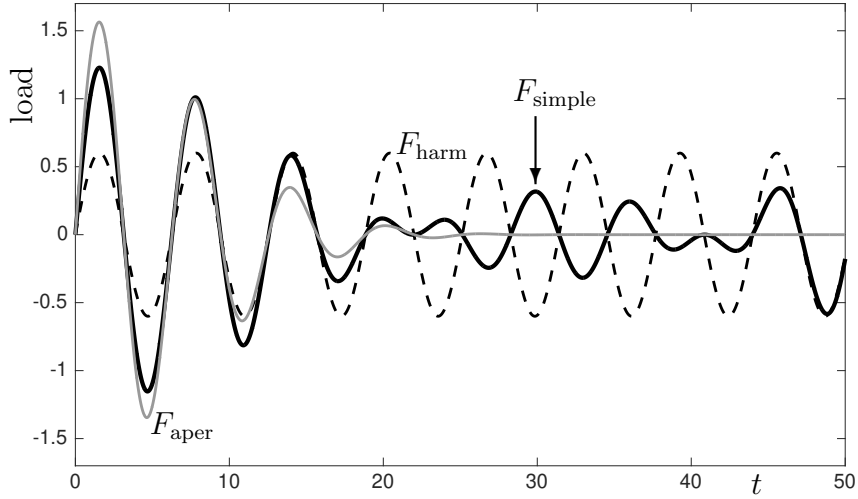


Figure 3: Graphs of the three loads for $\omega = 1$ and $A = 0.6$ over the total time $t_{\text{total}} = 50$. Shown are the harmonic load F_{harm} (dashed black curve), the simplified non-harmonic load F_{simple} (black curve) with $d = 0.1$ and the fully aperiodic load F_{aper} (grey curve) as defined by Eqs. (2), (7), and (11), respectively.

we deduce

$$\begin{aligned} E_{\text{aper}}(\omega, A) &= -\frac{\hat{A}_{\text{peak}}^2}{16\pi b} e^{-2b\omega^2} \int_0^{t_{\text{total}}} \exp - \left[\frac{(t - 2i\omega b)}{\sqrt{2b}} \right]^2 - 2e^{2b\omega^2} \exp - \left[\frac{t}{\sqrt{2b}} \right]^2 + \exp - \left[\frac{(t + 2i\omega b)}{\sqrt{2b}} \right]^2 dt \\ &= \frac{\hat{A}_{\text{peak}}^2}{16\pi b} e^{-2b\omega^2} \frac{\sqrt{\pi}}{2} \sqrt{2b} \left[2e^{2b\omega^2} \operatorname{erf} \left(\frac{t}{\sqrt{2b}} \right) - \operatorname{erf} \left(\frac{t - 2i\omega b}{\sqrt{2b}} \right) - \operatorname{erf} \left(\frac{t + 2i\omega b}{\sqrt{2b}} \right) \right]_{\chi \rightarrow -\infty}^{\chi \rightarrow \infty}. \end{aligned}$$

By applying the same arguments as for the derivation of Eq. (11), this equality implies

$$E_{\text{aper}}(\omega, A) = \frac{\hat{A}_{\text{peak}}^2}{8\sqrt{2\pi} b} (1 - e^{-2b\omega^2}). \quad (12)$$

Equating $E_{\text{aper}}(\omega, A)$ to $E_{\text{harm}}(\omega, A)$, and using $b = 4 \ln(100)$, we derive the value for the constant $\hat{A}_{\text{peak}}$ as

$$\hat{A}_{\text{peak}} := \sqrt{\frac{8\sqrt{2\pi} b E_{\text{harm}}(\omega, A)}{1 - e^{-2b\omega^2}}} = 4 \left(\sqrt{2\pi \ln(100)} \right)^{1/4} \left(1 - e^{-8 \ln(100) \omega^2} \right)^{-1/2} \sqrt{E_{\text{harm}}(\omega, A)}, \quad (13)$$

with $E_{\text{harm}}(\omega, A)$ as given by Eq. (4).

As an example, Fig. 3 shows the graphs over the chosen interval $[0, t_{\text{total}}] = [0, 50]$ of the harmonic load F_{harm} (dashed black curve) and equivalent simplified and aperiodic loads F_{simple} (black curve) and F_{aper} (grey curve) as defined by Eqs. (7) and (11), respectively. Here, we fixed $\omega = 1$ and $A = 0.6$. Observe that the graphs for both F_{simple} and F_{aper} initially show significantly higher amplitudes that rapidly reduce to values well below $A = 0.6$. In particular, F_{aper} converges to 0 rather quickly. In contrast, the oscillations for F_{simple} are merely suppressed in the second half of the interval $[0, t_{\text{total}}] = [0, 50]$, and the amplitudes are, in fact, increasing again towards the end of this interval. We stress again that comparison between loads only makes sense on the chosen interval $[0, t_{\text{total}}] = [0, 50]$, for which all three loads are defined to have the same total energy. To be precise, the energy for the simplified load F_{simple} is 1.08% higher than the equal total energy 9.0456 for F_{harm} and F_{harm} .

4 Analysis of a post-tensioned rockable frame under different loading conditions

When the post-tensioned rockable frame is subjected to ground excitations, for a given predominant frequency and amplitude of the load, the frame may stay inside the serviceability limit state, whereas for other choices it may fail. Since the system is C^1 -smooth, there must be load conditions that result in $|\varphi(t)| = 10$ at a time t , say, $t = t_g$ such that $\dot{\varphi}(t_g) = 0$. At this moment in time, the standardised angle φ is just touching the design limit. Hence, the structural response approaches the design limit at a local maximum (or minimum); this phenomenon is called a *grazing bifurcation*. Grazing can occur when either $\varphi(t_g) = 10$ or $\varphi(t_g) = -10$, which we call right- and left-grazing bifurcation, respectively. We are interested in the special (ω, A) -parameter pairs for which there exists $t_g \geq 0$ such that the initial condition $\varphi(0) = \dot{\varphi}(0) = 0$ (the frame is standing still before the earthquake strikes) exhibits a right- or left-grazing bifurcation when $t = t_g$. Continuity arguments imply that such grazing bifurcations occur as families or curves of right- and left-grazing bifurcation in the (ω, A) -plane. These combined right- and left-grazing bifurcation families determine a *failure boundary* that separates the safe and failure load conditions to the structure; see [28] for details.

We are able to compute curves of right- and left-grazing bifurcation directly by pseudo-arclength continuation. More precisely, we consider the two-point boundary value problem that defines solutions of system (1) with initial conditions $\varphi(0) = \dot{\varphi}(0) = 0$ and the third boundary condition $\dot{\varphi}(t_g) = 0$, in dependence on t_g as well as the frequency ω and amplitude A . While not restricted to periodic loads, to date, this computational tool has only been applied for the harmonic load [27, 28]. From these papers, we know that the failure boundary for the harmonic load is composed of curve segments of alternately right- and left-grazing bifurcation; the other curve segments of grazing bifurcation correspond to load conditions under which the frame exceeds the serviceability limit before grazing occurs.

In this paper, we compute curves of right- and left-grazing bifurcation for the non-harmonic loads F_{simple} and F_{aper} . We implemented the two-point boundary value problem and computed the solution families in the MATLAB-based software environment CoCo [12], instead of the Python/Fortran package Auto [16] that was used for the results reported in [27, 28].

4.1 Frame under harmonic load

For completeness, we recall the findings for the harmonic load reported in [28]. We also investigate the behaviour under harmonic loads for a purely linear frame that is not allowed to rock, i.e., the system modelled by (1) with $\mu(\varphi) = \varphi$ for $|\varphi| > 1$. As explained in [28], the failure boundary in the (ω, A) -plane is determined by two families of curves, denoted $\{R_n\}_{n \in \mathbb{N}}$ and $\{L_n\}_{n \in \mathbb{N}}$, of right- and left-grazing bifurcation, respectively. The subscript n indicates that the solution will graze the serviceability limit $|\varphi(t)| = 10$ (approximately) during the n th cycle of vibration, for the given load conditions ω and A . Hence, the larger n , the longer it takes for the frame to graze.

4.1.1 Purely linear frame

The equations for the purely linear frame can be solved exactly, and it is straightforward to extract the following expected behaviour. Since the linear oscillator is underdamped, it will *eventually* oscillate with the same frequency ω as the harmonic load and maximum amplitude

$$\max_{t \rightarrow \infty} |\varphi_{\text{harm}}^{\text{Lin}}(t)| = \frac{A}{\sqrt{(1 - \omega^2)^2 + (2\gamma\omega)^2}}.$$

The notation $\varphi_{\text{harm}}^{\text{Lin}}$ emphasises that we mean the rotation angle for the purely linear frame. Assuming the same serviceability limit $|\varphi(t)| = 10$, we do not expect the linear frame to fail for (ω, A) -pairs that lie below the *limiting failure boundary* $\mathcal{B}_{\infty}^{\text{Lin}}$ defined as

$$\mathcal{B}_{\infty}^{\text{Lin}} := \left\{ (\omega, A) \mid A = 10 \sqrt{(1 - \omega^2)^2 + (2\gamma\omega)^2} \right\}. \quad (14)$$

Hence, $\mathcal{B}_{\infty}^{\text{Lin}}$ is the graph in the (ω, A) -plane with an absolute minimum at $\omega = \sqrt{1 - 2\gamma^2} \approx 0.9975$, where $A = 20\gamma\sqrt{1 - \gamma^2} \approx 0.9987$. Note that the minimum-amplitude bound as a function of ω is valid for *eventual* oscillations. *Initial* oscillations may fail for amplitudes below this bound, and typically do so when the frequency ratio ω lies away from the resonance at $\sqrt{1 - 2\gamma^2}$. The actual failure boundary, which takes into account the behaviour of initial oscillations, must be determined in terms of right- and left-grazing bifurcations. Unfortunately, it is not straightforward to determine the (ω, A) -pairs that lead to grazing, even for the purely linear frame. Therefore, we employ the same approach as for the post-tensioned rockable frame, and numerically compute families $\{R_n^{\text{Lin}}\}_{n \in \mathbb{N}}$ and $\{L_n^{\text{Lin}}\}_{n \in \mathbb{N}}$ of right- and left-grazing bifurcations, respectively. We find that the curves R_n^{Lin} and L_n^{Lin} both accumulate on the limiting failure boundary $\mathcal{B}_{\infty}^{\text{Lin}}$ as n increases; failures below $\mathcal{B}_{\infty}^{\text{Lin}}$ only occur for relatively large values of A and frequencies ω that lie significantly away from resonance.

4.1.2 Post-tensioned rockable frame

The post-tensioned rockable frame exhibits a similar accumulating minimum with respect to A as n increases. However, this minimum occurs away from $\omega = \sqrt{1 - 2\gamma^2}$. Figure 4 shows the curves of grazing bifurcation that determine the failure boundaries in the (ω, A) -plane for both the rockable and purely linear frames; we also plot the limiting failure boundary $\mathcal{B}_{\infty}^{\text{Lin}}$ (thick black curve). We labelled the curves R_n (brown curves) and L_n (orange curves), with $n = 1, \dots, 4$, in the families of, respectively, right- and left-grazing bifurcations for the rockable frame, but computed them up to $n = 12$. The curve R_1 of right-grazing bifurcation at which solutions fail within the first load cycle lies above the upper boundary of A -values shown in Fig. 4. Furthermore, only the curves $R_1, L_1, R_2, L_2, R_3, L_3, R_4, L_4$, and R_5 correspond (roughly) to load cycles within the admissible time interval $[0, t_{\text{total}}] = [0, 50]$; we used a lighter colour shade for the other curves of left- and right-grazing bifurcation. Similarly, the two families of right- and left-grazing bifurcation for the linear system are labelled R_n^{Lin} (dark-grey curves) and L_n^{Lin} (light-grey curves), for $n = 1, \dots, 4$, where the grazing curves $R_1^{\text{Lin}}, L_1^{\text{Lin}}, R_2^{\text{Lin}}$, and L_2^{Lin} lie above the region shown in Fig. 4. In total, we computed curves up to $n = 20$, with those that correspond to load cycles beyond $t = t_{\text{total}}$, that is, $n > 8$ for the linear frame, shown in a lighter shade.

We interpret this data as follows. The post-tensioned rockable frame will fail within the first cycle of vibration for any (ω, A) -pair that lies above the curve L_1 , because the

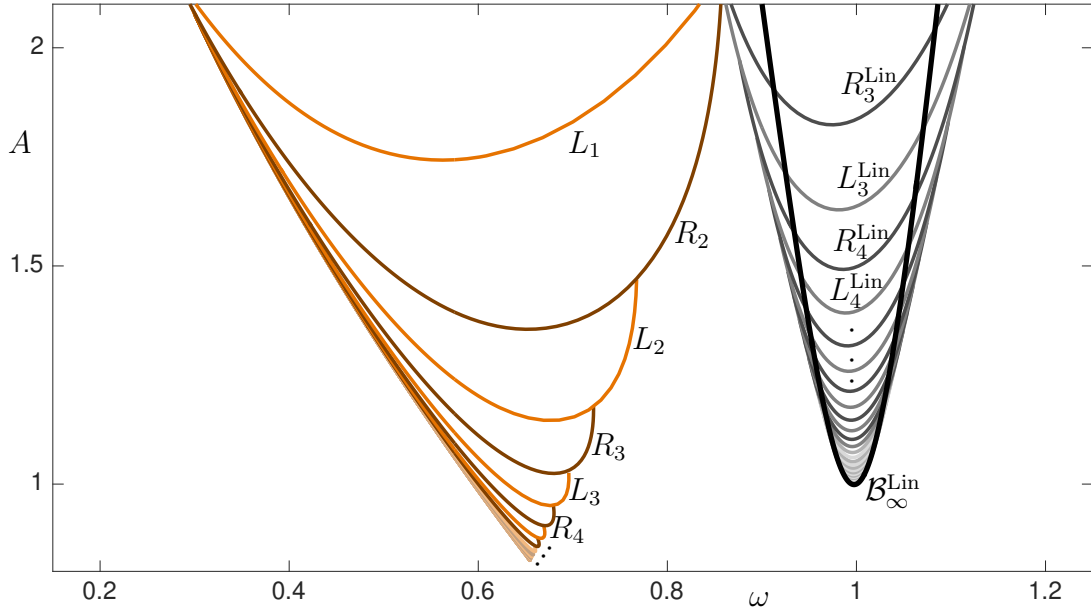


Figure 4: Curves of grazing bifurcation in the (ω, A) -plane for the harmonic load F_{harm} shown with the limiting failure boundary $\mathcal{B}_{\infty}^{\text{Lin}}$ (thick black curve) for the purely linear frame. The cuves of right- and left-grazing bifurcation for the non-linear rockable frame are labelled R_n (brown) and L_n (orange), and those for the linear frame R_n^{Lin} (dark grey) and L_n^{Lin} (light grey), respectively. Here, the subscript $n = 1, 2, \dots$, indicates the integer multiple of the cycle of vibration during which a grazing bifurcation occurs; the curves associated with cycles beyond $t = t_{\text{total}}$ are coloured in lighter shades.

rotation angle will then cross $\varphi = -10$ within the first cycle. For points (ω, A) below L_1 , but above R_2 , the frame only remains safe in the first cycle, and the rotation angle will cross $\varphi = 10$ within the second cycle of vibration; this pattern repeats alternately as n increases. Hence, the failure boundary for the harmonic load is given by the lower-most bound of all grazing bifurcations: points (ω, A) that lie below all curves of grazing bifurcation correspond to harmonic loads during which the post-tensioned rockable frame is safe for all time. This behaviour is very similar to that for the linear frame, but note that all failures during a particular cycle in the post-tensioned rockable case occur for much lower values of the amplitude A and for significantly lower values of the frequency ratio ω . For example, a left-grazing bifurcation in the fourth cycle of vibration for the purely linear frame occurs along the curve L_4^{Lin} , which achieves its minimum at $A \approx 1.3921$ (for $\omega \approx 0.9874$); a similar left-grazing bifurcation for the post-tensioned rockable frame corresponds to the minimum on L_4 (not labelled) at the much lower value $A \approx 0.8757$ (for $\omega \approx 0.6652$). Even if we account for this shift in ω , and compare L_4 with L_8^{Lin} , which has its minimum at $A \approx 1.0860$ for $\omega \approx 0.9961$, the difference of 0.1204 in A -values is still significant.

Figure 4 also illustrates that the accumulation process for the post-tensioned rockable frame is much faster than for the purely linear frame: the minimum on L_4 is already very close to the limit $(\omega, A) \approx (0.6546, 0.8252)$, while the minimum on L_4^{Lin} is still well away from the limit $(\omega, A) \approx (0.9975, 0.9987)$. Since the linear frame will undergo resonance and have the largest response near $\omega = 1$, design specifications commonly suggested to move the frequency ratio ω away from 1. However, for the post-tensioned rockable frame, the design

should avoid $\omega \approx 0.6552$ instead of $\omega \approx 0.9975$. Moreover, when $\omega \approx 0.6552$, a forcing amplitude below 0.9987 can already lead to failure of the post-tensioned rockable frame. A physical explanation is that a post-tensioned rockable frame exhibits larger deformations than the purely linear frame due to additional rocking deformations. In other words, while rocking eliminates stress-induced failure of structures, it does have a potential to increase the deformation demand, which should be considered when the building design has maximum displacement restrictions for serviceability. We conclude that harmonic loads with, roughly, frequencies $\omega \in [0.55, 0.75]$ are most dangerous for the post-tensioned rockable frame.

4.2 Frame under simplified non-harmonic load

We now compute the curves of right- and left-grazing bifurcation when the post-tensioned rockable frame is subjected to the simplified non-harmonic excitation F_{simple} as defined in Eq. (7); here, we set $d = 0.1$ and we vary the same three parameters t_g , ω and A as for the harmonic load; recall from Sec. 3.2.1 that F_{simple} then exhibits the same predominant frequency ω as well as the two frequencies $\omega \pm d$ using the adjusted amplitudes $A_{\text{simple}} \approx A/1.2924$ and $A_{\text{sub}} = e^{-bd^2} A_{\text{simple}}$ as defined in Eqs. (10) and (8), respectively. Since both the harmonic and simplified loads have (approximately) the same energy over the total time $t_{\text{total}} = 50$, it makes sense to compare their respective failure boundaries in the (ω, A) -plane. We also compare the results for the post-tensioned rockable frame with that for the purely linear frame under the same load F_{simple} .

4.2.1 Post-tensioned rockable frame

Figure 5 shows the families $\{R_n\}_{n \in \mathbb{N}}$ (brown) and $\{L_n\}_{n \in \mathbb{N}}$ (orange) of right- and left-grazing bifurcation for the simplified non-harmonic load F_{simple} , respectively, overlaid on the curves from Fig. 4 for the harmonic load, which are now shown in dark- and light-grey, respectively. Here, the subscript n indicates that the frame will graze the serviceability limit (approximately) during the n th cycle of predominant vibration, i.e., with respect to the predominant frequency ω . As for the harmonic load, the two types of grazing bifurcation form two families of curves. Only R_1 , R_2 and L_1 are shown, because the curve L_2 already comprises only grazing bifurcations of solutions that exceed the serviceability limit *before* grazing occurs. More precisely, the computed segments of all right-grazing bifurcations R_n with $n \geq 3$ lie entirely above the failure boundary, and the same holds for all curves L_n of left-grazing bifurcation with $n \geq 2$. These findings are dramatically different from the harmonic load, where each curve of grazing bifurcation contributes a new segment to the failure boundary.

Observe in Fig. 5 that F_{simple} still elicits a resonance-like response; it occurs almost at the same point as for the harmonic load, namely, at $(\omega, A) \approx (0.6557, 0.8475)$. However, when F_{simple} is applied, the minimal load amplitude is achieved by the curve R_2 of right-grazing bifurcation. Hence, the post-tensioned rockable frame realises the resonance-like response for $n = 2$, within the second predominant vibration cycle, instead of as an accumulating limit with $n \rightarrow \infty$.

Figure 5 also illustrates that the addition of only two further frequencies has a big effect on the overall size of the safe zone: failure may occur for $A \leq 1$ roughly when $\omega \in [0.45, 0.8]$, which is almost double the size of the dangerous frequency ratios $\omega \in [0.55, 0.75]$ for the

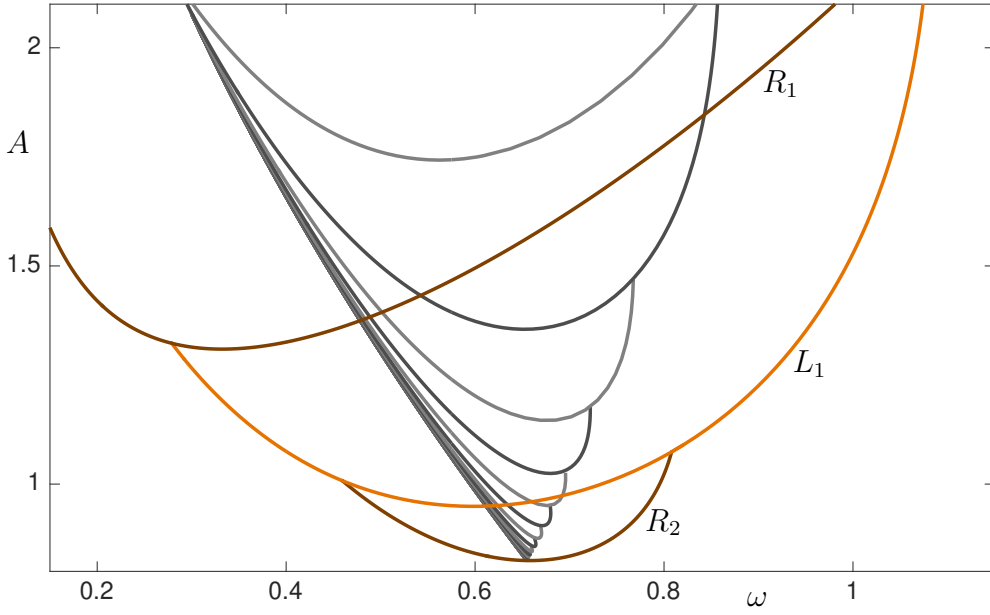


Figure 5: Curves of right- and left-grazing bifurcation (brown and orange, respectively) in the (ω, A) -plane for the simplified non-harmonic load F_{simple} , overlaid on those for the harmonic load (dark- and light-grey curves, respectively). The subscript n for the labels R_n and L_n , with $n = 1, 2$, indicates the integer multiple of the cycle of predominant vibration during which grazing bifurcation occurs.

harmonic load. Although the most dangerous load condition still occurs near $\omega \approx 0.6552$, the safe region is much reduced when the earthquake load is given by F_{simple} .

4.2.2 Comparison with purely linear frame

The harmonic load is commonly claimed to provide worst-case estimates for the purely linear frame, that is, system (1) with $\mu(\varphi) = \varphi$ for all rotation angles φ and *any* load $F(t)$ is not expected to fail if the frequency ratio and amplitude of the predominant wave form lies below the limiting failure boundary $\mathcal{B}_{\infty}^{\text{Lin}}$. We test this claim for the simplified non-harmonic load F_{simple} , as well as the fully aperiodic load F_{aper} . Recall that the simplified non-harmonic load is a superposition of three harmonic waves that can be expressed in the form (9). Consequently, the equations for the purely linear frame with right-hand side equal to F_{simple} can again be solved exactly, and the *eventual* behaviour is that of a beating oscillator $\{\varphi_{\text{simple}}^{\text{Lin}}(t) \mid t \in \mathbb{R}\}$, with maximum amplitude equal to the sum of the maximum amplitudes for each of the three load frequencies, that is,

$$\begin{aligned} \max_{t \rightarrow \infty} |\varphi_{\text{simple}}^{\text{Lin}}(t)| &= \frac{A_{\text{simple}}}{\sqrt{(1 - \omega^2)^2 + (2\gamma\omega)^2}} + \frac{A_{\text{sub}}}{\sqrt{(1 - [\omega + d]^2)^2 + (2\gamma[\omega + d])^2}} \\ &\quad + \frac{A_{\text{sub}}}{\sqrt{(1 - [\omega - d]^2)^2 + (2\gamma[\omega - d])^2}}. \end{aligned}$$

Here, $A_{\text{simple}} \approx A/1.2924$, $A_{\text{sub}} = e^{-bd^2} A_{\text{simple}}$, and $b = 4 \ln(100) \approx 18.4207$ are given by Eqs. (10), (8), and (6), respectively; recall that we fixed the distance $d = 0.1$ from the main frequency ratio ω . Therefore, this beating oscillator will graze the serviceability limit along

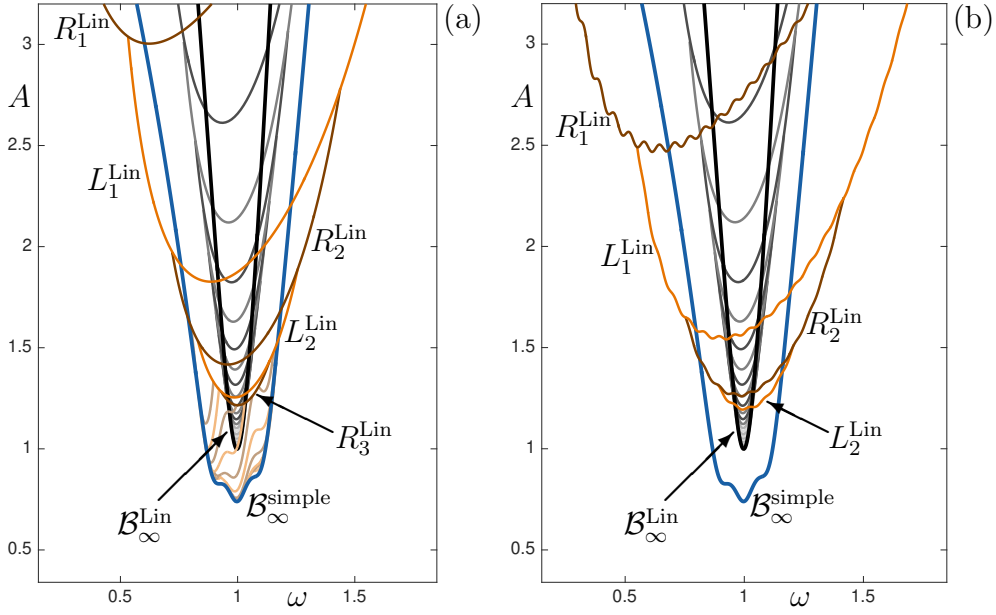


Figure 6: Curves R_n^{Lin} (brown) and L_n^{Lin} (orange) of right- and left-grazing bifurcation, respectively, in the (ω, A) -plane for the purely linear frame subjected to F_{simple} in panel (a) and F_{aper} in panel (b); see Fig. 5 for the color scheme. We also plot the limiting failure boundaries $\mathcal{B}_{\infty}^{\text{Lin}}$ (thick black curve) and $\mathcal{B}_{\infty}^{\text{simple}}$ (thick blue curve).

a curve $\mathcal{B}_{\infty}^{\text{simple}}$ defined as

$$\mathcal{B}_{\infty}^{\text{simple}} := \left\{ (\omega, A) \mid \max_{t \rightarrow \infty} (|\varphi_{\text{simple}}^{\text{Lin}}(t)|) = 10 \right\}. \quad (15)$$

When $d = 0.1$, this limiting failure boundary $\mathcal{B}_{\infty}^{\text{simple}}$ has an absolute minimum at $(\omega, A) \approx (0.9969, 0.7394)$.

Figure 6(a) shows the families $\{R_n^{\text{Lin}}\}_{n \in \mathbb{N}}$ (brown) and $\{L_n^{\text{Lin}}\}_{n \in \mathbb{N}}$ (orange) of right- and left-grazing bifurcation for the simplified load F_{simple} , respectively; they are overlaid on the curves for the harmonic load, using the same color scheme as in Fig. 5. Only R_n^{Lin} for $n = 1, 2, 3$ and L_n^{Lin} for $n = 1, 2$ are labelled, because the other curves, which are coloured in lighter shades, achieve the moment of grazing $\dot{\varphi}_{\text{simple}}^{\text{Lin}}(t_g) = 0$ only at times t_g with $t > t_{\text{total}}$. In fact, similar to the case for the post-tensioned rockable frame, the curves of grazing bifurcation for $n = 3, 4, \dots, 8$ lie entirely above the labelled curves, outside the range shown in the figure. For $n > 8$, both $\{R_n^{\text{Lin}}\}_{n \in \mathbb{N}}$ and $\{L_n^{\text{Lin}}\}_{n \in \mathbb{N}}$ accumulate on $\mathcal{B}_{\infty}^{\text{simple}}$ (thick blue curve) as $n \rightarrow \infty$; observe that $\mathcal{B}_{\infty}^{\text{simple}}$ lies significantly below the curve $\mathcal{B}_{\infty}^{\text{Lin}}$ (thick black curve) associated with F_{harm} . However, the purely linear frame with harmonic load does provide worst-case estimates for the first few cycles of vibration, provided we restrict investigations to near the predominant frequency $\omega \approx 0.9975$. Away from this resonance frequency, the curves R_n^{Lin} and L_n^{Lin} , for $n = 1, 2$, lie well below either $\mathcal{B}_{\infty}^{\text{simple}}$ or $\mathcal{B}_{\infty}^{\text{Lin}}$ and, importantly, significantly broaden the range of frequency ratios ω that could be considered dangerous.

Figure 6(b) shows similar results when the purely linear frame is subjected to the fully aperiodic load F_{aper} ; we also plot $\mathcal{B}_{\infty}^{\text{simple}}$ (thick blue curve) and $\mathcal{B}_{\infty}^{\text{Lin}}$ (thick black curve) for reference, but no such limiting curve is known for the fully aperiodic load F_{aper} . As before, the curves R_n^{Lin} (brown) and L_n^{Lin} (orange), for $n = 1, 2$, are overlaid on the results for the

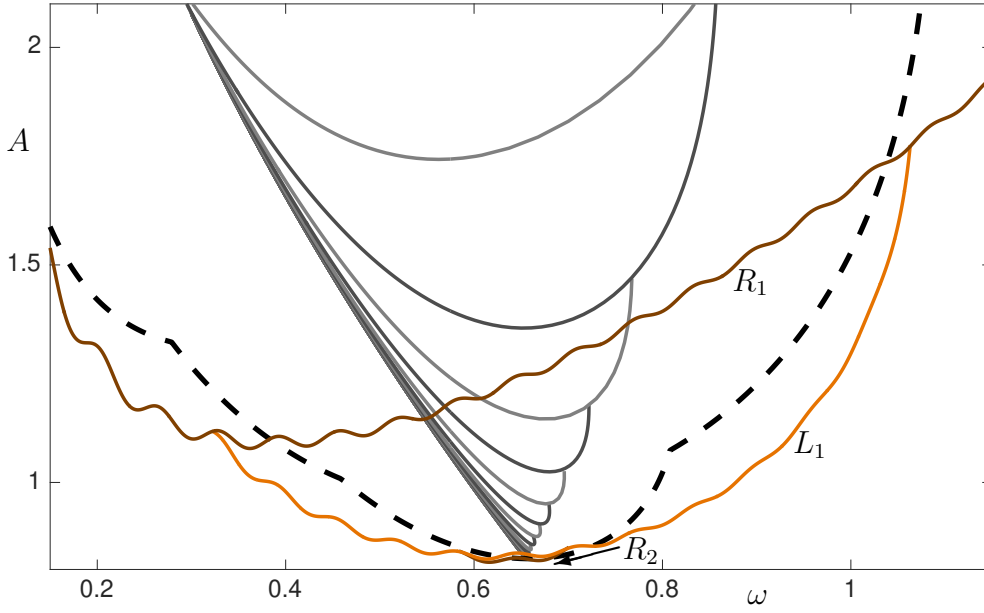


Figure 7: Curves of right- and left-grazing bifurcation (brown and orange, respectively) in the (ω, A) -plane for the fully aperiodic load F_{aper} , overlaid on the grazing bifurcation curves for the harmonic load (dark- and light-grey curves, respectively) and the boundary of the safe zone for the simplified non-harmonic load (dashed black curve). The subscript n for labels R_n and L_n , with $n = 1, 2$, indicates the integer multiple of the cycle of predominant vibration during which grazing occurs.

harmonic load using the same colour scheme as in Fig. 5. Surprisingly, all other curves of grazing bifurcation lie well above R_1^{Lin} and L_1^{Lin} , meaning that the purely linear frame exceeds the serviceability limit within the first cycle, before grazing can possibly occur at times past the second cycle of predominant vibration. This behaviour is significantly different from the simplified case, but similar to what we observe for the post-tensioned rockable frame, when it is subjected to F_{simple} ; compare with Fig. 5. Note that, as in Fig. 6(a), the curves R_n^{Lin} and L_n^{Lin} , for $n = 1, 2$, significantly broaden the range of frequency ratios ω that could be considered dangerous.

4.3 Frame under aperiodic load

Next, we compare the effects caused by the simplified non-harmonic load F_{simple} with those caused by the fully aperiodic excitation F_{aper} as defined in Sec. 3.2.2. As done in Fig. 5 for F_{simple} , Fig. 7 shows curves of (brown) right- and (orange) left-grazing bifurcation for F_{aper} in the (ω, A) -plane. More precisely, the predominant frequency of F_{aper} with the amplitude A_{peak} and ‘width’ of the bell-shaped curve are considered as functions of A ; see Eqs. (5) and (13). We find that, again, the failure boundary is entirely determined by the first two curves R_1 and R_2 of right-grazing bifurcation, and the first curve L_1 of left-grazing bifurcation; these curves are roughly associated with the first and second cycle of the predominant vibration. As before, the curves of right- and left-grazing bifurcation from Fig. 4 are also shown (in dark- and light-grey, respectively); for comparison with Fig. 5, we plot the outlines of the failure boundary associated with F_{simple} as a black dashed curve.

Observe from Fig. 7 that both curves of right- and left-grazing bifurcation have an oscillatory pattern. This is because the plot is with respect to the equivalent harmonic amplitude A . According to Eq. (13), the amplitudes of the aperiodic and harmonic loads are related by oscillating sine functions; this effect is not present in Fig. 5, because approximated the value of A_{simple} for F_{simple} by a constant proportion of A . Despite its oscillatory shape, the failure boundary for F_{aper} also exhibits a resonance-like response within the first two cycles of predominant vibration; the minimum now occurs at $(\omega, A) \approx (0.6163, 0.8166)$, which is again close to the resonance point for the harmonic load.

Note that the aperiodic load F_{aper} causes an even larger reduction of the size of the safe zone, which means that frequencies further away from $\omega \approx 0.6163$ have a significant effect on the response, even when their Fourier amplitudes are small.

5 Conclusions

We presented a comparative case study for the responses of a post-tensioned rockable frame under both harmonic and non-harmonic loads; we used the model from [2], where we changed the right-hand side, the load, from purely harmonic to a non-harmonic load with a broader frequency content centred around a predominant frequency. We derived an explicit expression for this aperiodic non-harmonic load as a function of time. We also considered a simpler non-harmonic load that uses only three fixed frequencies with amplitudes based on the same bell-shaped frequency profile as that for the fully aperiodic load with distributed frequencies.

Our proposed non-harmonic loads are more realistic to simulate seismic excitation for rockable structures, and allow for independent control of the predominant frequency, the width of the frequency spectrum, and the energy of the loads. We asked the important question whether analysis under harmonic and non-harmonic loads would provide consistent insight into possible design specifications for the non-linear rockable frame. We found that analysis under harmonic loads correctly identifies a shift away from the linear resonance frequency $\omega \approx 0.9987$ to the much lower value $\omega \approx 0.655$. However, the failure boundary for the harmonic load does not represent a worst-case scenario for the non-linear, post-tensioned rockable frame. Our results show that the frame can more easily fail under more realistic non-harmonic loads. The fully aperiodic load turned out to be the most dangerous, but the simplified non-harmonic load gives reasonably similar results and may be a more suitable candidate for the efficient numerical investigation of rockable structures.

In terms of design specifications, we conclude from the analysis under non-harmonic loads that the post-tensioned rockable frame should be designed as flexible as possible. This design recommendation cannot be obtained from the analysis under harmonic loads. Based solely on harmonic load, it appears that the failure potential of the structure can significantly be reduced by designing frames with a frequency ratio away from $\omega \approx 0.655$. However, based on the simplified non-harmonic and aperiodic loads, frequency ratios larger than 1 are more likely able to protect the frame from seismic failure.

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