Continuation Methods in Dynamical Systems

Basic Tutorial

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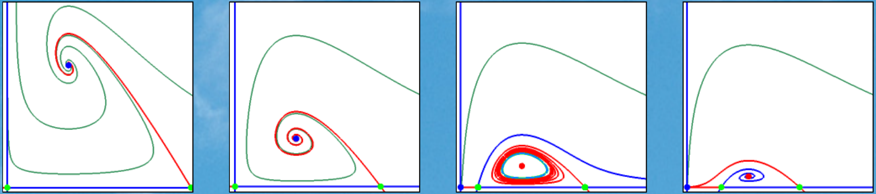
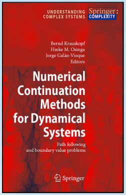


Acknowledgement

This lecture is heavily based on lecture notes written by Eusebius Doedel, Concordia University, Montreal.

For more information, see

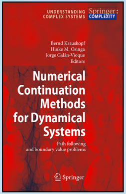
`http://cmvl.cs.concordia.ca/auto`

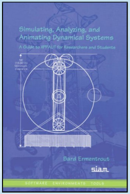


[Krauskopf, Osinga & Galán-Vioque (Eds.) Springer, 2007]

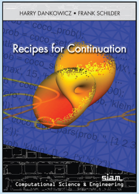
Continuation Packages

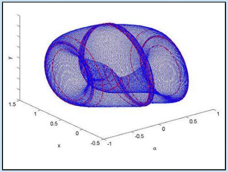
Numerical methods in dynamical systems and bifurcation theory are based on *continuation*





- AUTO by Eusebius Doedel (Concordia University)
- CoCo by Harry Dankowicz (UIUC, Champaign) and Frank Schilder (DTU, Copenhagen)
- MatCont by Willy Govaerts (Ghent University) and Yuri Kuznetsov (Utrecht University)
- XPPAUT by Bard Ermentrout (University of Pittsburgh)





Continuation of equilibria

Consider a predator-prey model

$$\begin{cases} u_1' &= 3u_1(1-u_1) - 2u_1u_2, \\ u_2' &= -u_2 + 3u_1u_2. \end{cases}$$

We can think of u_1 as 'fish' and u_2 as 'sharks'

The equilibria are

$$\left.\begin{array}{rcl} 3u_1(1-u_1) - 2u_1u_2 &=& 0 \\ -u_2 + 3u_1u_2 &=& 0 \end{array}\right\} \Rightarrow (u_1, u_2) = (0,0),\ (1,0),\ (\tfrac{1}{3}, 1).$$

Continuation of equilibria

Consider a predator-prey model with fishing

$$\begin{cases} u_1' &= 3u_1(1-u_1) - 2u_1u_2 - \lambda(1 - e^{-5u_1}), \\ u_2' &= -u_2 + 3u_1u_2. \end{cases}$$

We can think of u_1 as 'fish' and u_2 as 'sharks'

The equilibria are

$$\left.\begin{array}{rcl} 3u_1(1-u_1) - 2u_1u_2 &=& 0 \\ -u_2 + 3u_1u_2 &=& 0 \end{array}\right\} \Rightarrow (u_1, u_2) = (0,0),\ (1,0),\ (\tfrac{1}{3}, 1).$$

What happens to the equilibria if we introduce a 'fishing-quota' $\lambda > 0$?

Persistence for $\lambda > 0$?

Theorem (Implicit Function Theorem (IFT))

Consider $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *with* $\mathbf{f}(\mathbf{u}_0, \lambda_0) = \mathbf{0}$ *for* $\mathbf{u}_0 \in \mathbb{R}^n$ *and* $\lambda_0 \in \mathbb{R}$. *Suppose the following holds:*

- *The Jacobian matrix* $\mathbf{f}_{\mathbf{u}}(\mathbf{u}_0, \lambda_0)$ *is nonsingular;*
- $\mathbf{f}$ *and* $\mathbf{f}_{\mathbf{u}}$ *are Lipschitz continuous (in both* $\mathbf{u}$ *and* λ*)*

Then there exists $\delta > 0$ *and interval* $\Lambda_\delta = (\lambda_0 - \delta, \lambda_0 + \delta)$, *with a unique function* $\mathbf{u}(\lambda)$ *continuous on* Λ_δ, *such that*

$$\begin{aligned} \mathbf{u}(\lambda_0) &= \mathbf{u}_0 \\ \mathbf{f}(\mathbf{u}(\lambda), \lambda) &= \mathbf{0}, \ \textit{for all } \|\lambda - \lambda_0\| < \delta. \end{aligned}$$

We call $\mathbf{u}_0$ an *isolated* solution of $\mathbf{f}(\mathbf{u}, \lambda_0) = 0$

Equilibrium branches

The Jacobian matrix is

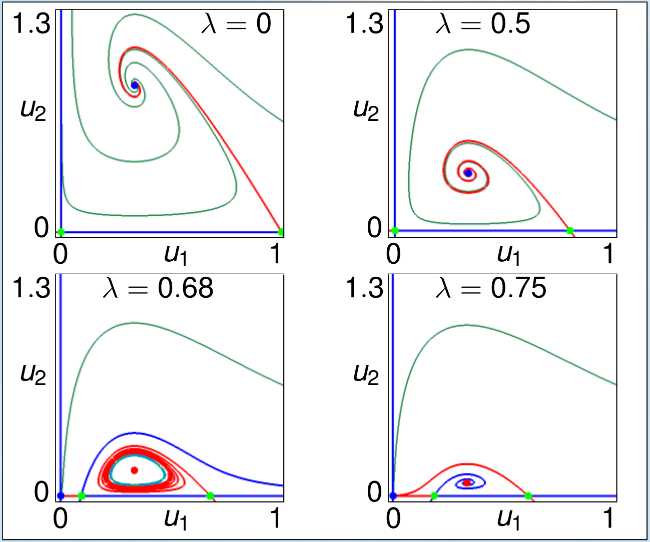
$$J(u_1, u_2; \lambda) = \begin{pmatrix} 3 - 6u_1 - 2u_2 - 5\lambda e^{-5u_1} & -2u_1 \\ u_2 & -1 + 3u_1 \end{pmatrix}$$

$$\begin{aligned} J(0,0;0) &= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J(1,0;0) = \begin{pmatrix} 3 & -2 \\ 0 & 2 \end{pmatrix}, \\ J(\tfrac{1}{3}, 1; 0) &= \begin{pmatrix} -1 & -\frac{2}{3} \\ 6 & 0 \end{pmatrix}, \end{aligned}$$

All three Jacobians at $\lambda = 0$ are nonsingular.

Thus, by the IFT, all three equilibria persist for (small) $\lambda \neq 0$.

Phase portraits for $\lambda \geq 0$

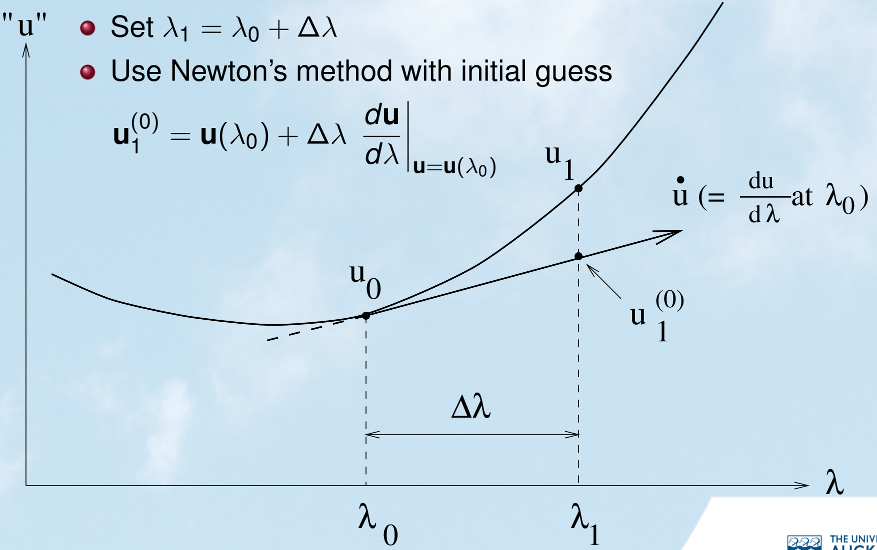


Note:
moderate fishing quotas do not seem to have an effect on the number of fish, but greatly reduce the shark population!

Parameter continuation

- Set $\lambda_1 = \lambda_0 + \Delta\lambda$
- Use Newton's method with initial guess

$$\mathbf{u}_1^{(0)} = \mathbf{u}(\lambda_0) + \Delta\lambda \left. \frac{d\mathbf{u}}{d\lambda} \right|_{\mathbf{u}=\mathbf{u}(\lambda_0)}$$



Regular solutions

Definition (Regular solutions)

A solution $(\mathbf{u}_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ of $\mathbf{f}(\mathbf{u}, \lambda) = \mathbf{0}$ is called *regular* if

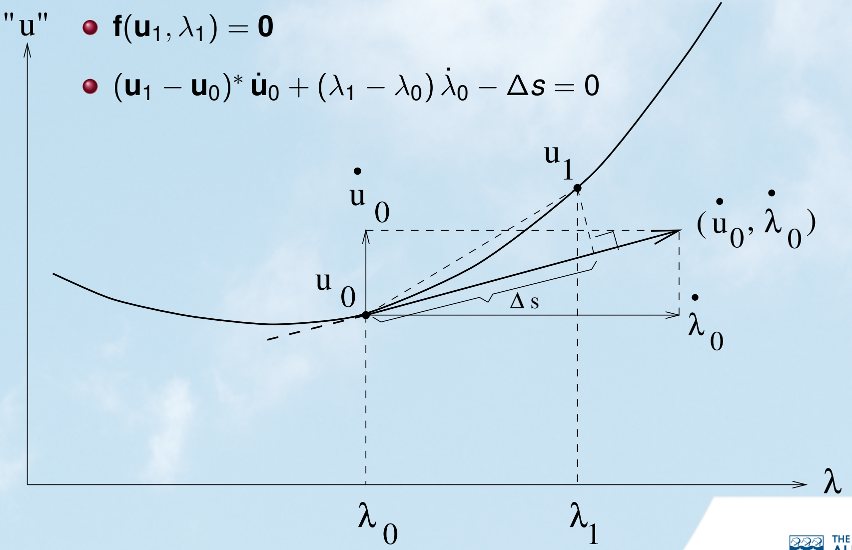
$$\operatorname{Rank}\left(\mathbf{f}_{\mathbf{u}}^0 \;\middle|\; \mathbf{f}_\lambda^0 \right) = n \Leftrightarrow \begin{cases} \text{(i)} \quad \mathbf{f}_{\mathbf{u}}^0 \text{ is nonsingular,} \\ \text{or} \\ \text{(ii)} \quad \begin{cases} \dim \mathcal{N}(\mathbf{f}_{\mathbf{u}}^0) = 1, \\ \text{and} \\ \mathbf{f}_\lambda^0 \notin \mathcal{R}(\mathbf{f}_{\mathbf{u}}^0). \end{cases} \end{cases}$$

Here, $\mathcal{N}(\mathbf{f}_{\mathbf{u}}^0)$ denotes the *null space* of $\mathbf{f}_{\mathbf{u}}^0$, and $\mathcal{R}(\mathbf{f}_{\mathbf{u}}^0)$ denotes the *range* of $\mathbf{f}_{\mathbf{u}}^0$, i.e., the linear space spanned by the n columns of $\mathbf{f}_{\mathbf{u}}^0$.

Regular solutions include fold points with respect to λ

Pseudo-arclength continuation

- $\mathbf{f}(\mathbf{u}_1, \lambda_1) = \mathbf{0}$
- $(\mathbf{u}_1 - \mathbf{u}_0)^* \, \dot{\mathbf{u}}_0 + (\lambda_1 - \lambda_0)\, \dot{\lambda}_0 - \Delta s = 0$



Newton's method in this context

Newton's method for pseudo-arclength continuation becomes

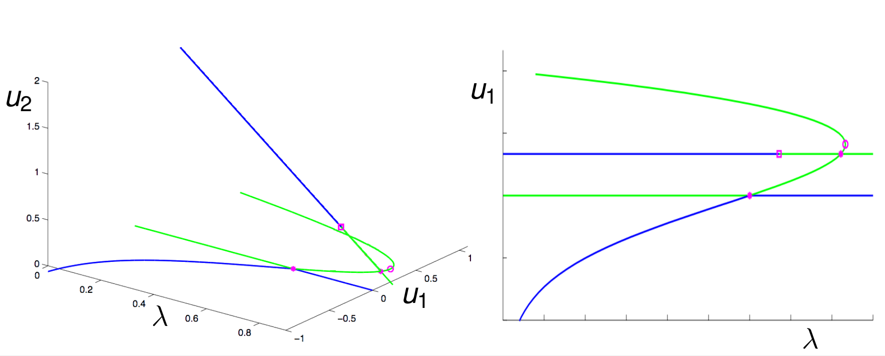
$$\begin{pmatrix} (\mathbf{f}^1_{\mathbf{u}})^{(\nu)} & (\mathbf{f}^1_{\lambda})^{(\nu)} \\ (\dot{\mathbf{u}}_0)^* & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \Delta \mathbf{u}_1^{(\nu)} \\ \Delta \lambda_1^{(\nu)} \end{pmatrix} = - \begin{pmatrix} \mathbf{f}(\mathbf{u}_1^{(\nu)}, \lambda_1^{(\nu)}) \\ (\mathbf{u}_1^{(\nu)} - \mathbf{u}_0)^* \, \dot{\mathbf{u}}_0 + (\lambda_1^{(\nu)} - \lambda_0) \, \dot{\lambda}_0 - \Delta s \end{pmatrix},$$

with the new direction vector defined as

$$\begin{pmatrix} \mathbf{f}^1_{\mathbf{u}} & \mathbf{f}^1_{\lambda} \\ \dot{\mathbf{u}}_0^* & \dot{\lambda}_0 \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}}_1 \\ \dot{\lambda}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix},$$

- The orientation of the branch is preserved for Δs small.
- Rescale direction vector: $\| \dot{\mathbf{u}}_1 \|^2 + \dot{\lambda}_1^2 = 1$.

Effects of fishing on equilibria



- Non-trivial equilibrium is stable until $\lambda = \lambda_{\mathrm{H}} \approx 0.6716$
- Fishing does not seem to affect fish population for small λ
- All other stable equilibria are non-physical or correspond to extinction of fish (and shark)

The Hopf bifurcation

Theorem (Hopf bifurcation)

Suppose that along an equilibrium branch $(\mathbf{u}(\lambda), \lambda)$*, of*

$$\mathbf{u}' = \mathbf{f}(\mathbf{u}, \lambda),$$

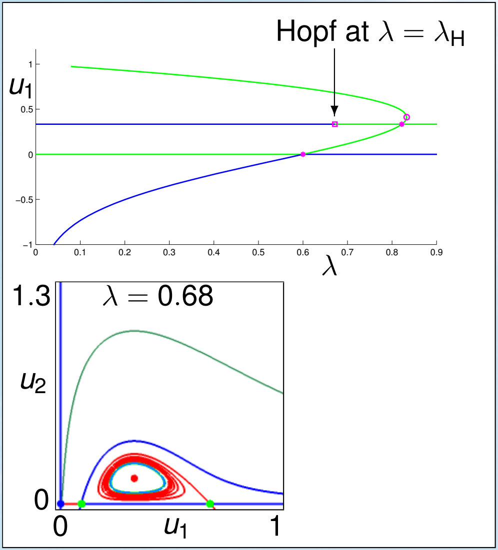
a complex conjugate pair $\alpha(\lambda) \pm i\,\beta(\lambda)$ *of eigenvalues of* $\mathbf{f}_{\mathbf{u}}(\mathbf{u}(\lambda), \lambda)$ *crosses the imaginary axis transversally, i.e., for some* λ_0*,*

$$\alpha(\lambda_0) = 0, \quad \beta(\lambda_0) \neq 0, \quad \textit{and} \quad \dot{\alpha}(\lambda_0) \neq 0.$$

Also assume that there are no other eigenvalues on the imaginary axis.

Then there is a Hopf bifurcation and a family of periodic solutions bifurcates from the stationary solution at (u_0, λ_0)*.*

Continuation of periodic orbits



Bifurcation theory and centre manifold theory are used to determine a first approximation of the periodic orbit and its period

$$\begin{cases} u_1 = a\cos(\beta(\lambda_H)t), \\ u_2 = a\sin(\beta(\lambda_H)t) \end{cases}$$

with $0 \leq t < 2\pi$, amplitude $a = \sqrt{|\lambda - \lambda_H|}$, and period

$$T = \frac{2\pi}{\beta(\lambda)}$$

A BVP approach

Consider the first-order system

$$\mathbf{u}'(t) = \mathbf{f}(\mathbf{u}(t), \lambda), \qquad \mathbf{u}(\cdot), \mathbf{f}(\cdot) \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}.$$

Fix the interval of periodicity by the transformation $t \mapsto \frac{t}{T}$.
Then the equation becomes

$$\mathbf{u}'(t) = T\,\mathbf{f}(\mathbf{u}(t), \lambda), \qquad \mathbf{u}(\cdot), \mathbf{f}(\cdot) \in \mathbb{R}^n, \quad T, \lambda \in \mathbb{R},$$

and we seek an orbit segment **u** subject to the boundary condition

$$\mathbf{u}(0) = \mathbf{u}(1)$$

Note that the intergation time T, the period, is one of the unknowns.

(Families of) orbit segments

One-parameter family of solutions to BVP

$$\begin{aligned} \dot{\mathbf{u}}(t) &= T\mathbf{f}(\mathbf{u}(t), \lambda), \\ \mathbf{u}(0) &\in L(\theta), \end{aligned}$$

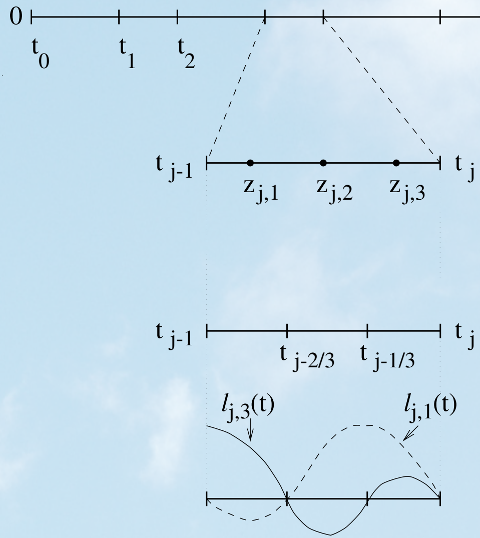
with $L(\theta)$ curve on linear approximation, and one additional boundary condition, e.g.:

- Fixed integration time: $T = T_0$
- Fixed arclength:

$$\int_0^1 T \, \|\mathbf{f}(\mathbf{u}(s))\| \, ds - L = 0$$

- Constrained end point: $g(u(1), \theta, T) - \alpha = 0$

Orthogonal Collocation



- Solutions $\mathbf{u}(t)$ are *piecewise polynomials*

$$\mathcal{P}_h^m = \{\mathbf{p}_h \in C[0,1] \mid \mathbf{p}_h|_{[t_{j-1},t_j]} \in \mathcal{P}^m\},$$

where $\mathcal{P}^m$ is the space of (vector-valued) polynomials of degree $\leq m$.

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Nonuniqueness of periodic orbit

Assume that we have computed

$$(\mathbf{u}_{k-1}(\cdot), T_{k-1}, \lambda_{k-1})$$

and we want to compute the next solution

$$(\mathbf{u}_{k}(\cdot), T_{k}, \lambda_{k})$$

Then $\mathbf{u}_k(t)$ can be translated freely in time:

If $\mathbf{u}_k(t)$ is a periodic solution, then so is $\mathbf{u}_k(t+\sigma)$, for any σ.

We define $(\mathbf{u}_k(\cdot), T_k, \lambda_k)$ uniquely using a *phase condition*

Pseudo-Arclength Continuation

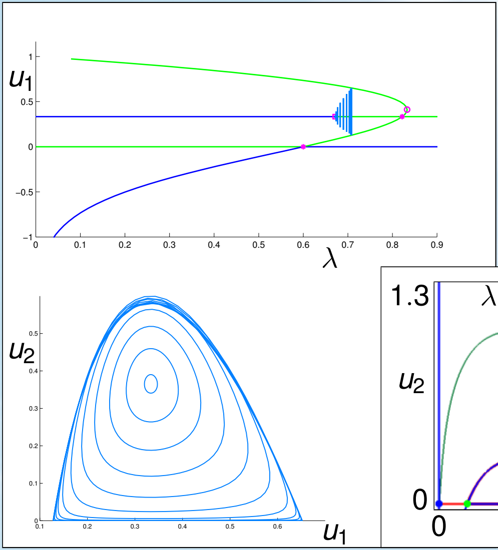
For the continuation of periodic solutions, we solve the system

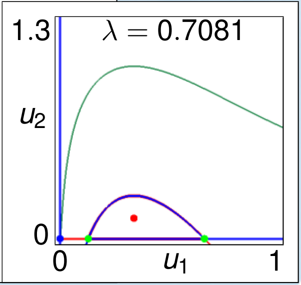
$$\begin{aligned}
\mathbf{u}_k'(t) &= T\,\mathbf{f}(\mathbf{u}_k(t), \lambda_k), \\
\mathbf{u}_k(0) &= \mathbf{u}_k(1), \\
\int_0^1 \mathbf{u}_k(t)^*\,\mathbf{u}_{k-1}'(t)\,dt &= 0, \\
\int_0^1 (\mathbf{u}_k(t) - \mathbf{u}_{k-1}(t))^*\,\dot{\mathbf{u}}_{k-1}(t)\,dt &+ (T_k - T_{k-1})\,\dot{T}_{k-1} \\
+(\lambda_k - \lambda_{k-1})\,\dot{\lambda}_{k-1} &= \Delta s,
\end{aligned}$$

where

$$\mathbf{u}(\cdot),\ \mathbf{f}(\cdot) \in \mathbb{R}^n, \qquad \lambda,\ T \in \mathbb{R}.$$

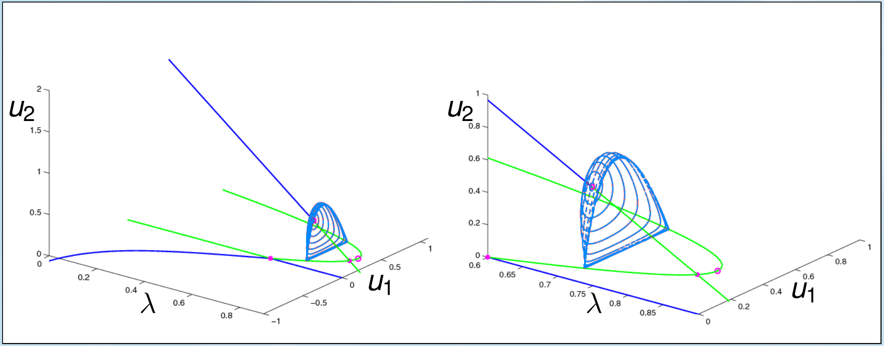
Oscillating fish





- Attracting periodic orbits exist as λ increases
- The period increases and becomes infinite at $\lambda \approx 0.7$
- This final orbit is called a *heteroclinic cycle*

Oscillating fish in 3D



- Attracting periodic orbits exist as λ increases
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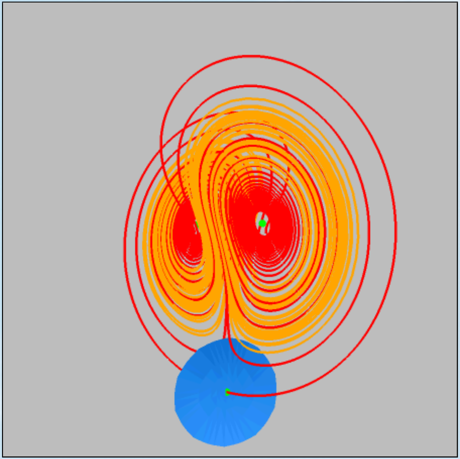
Global manifold computations

Famous test example: the Lorenz system

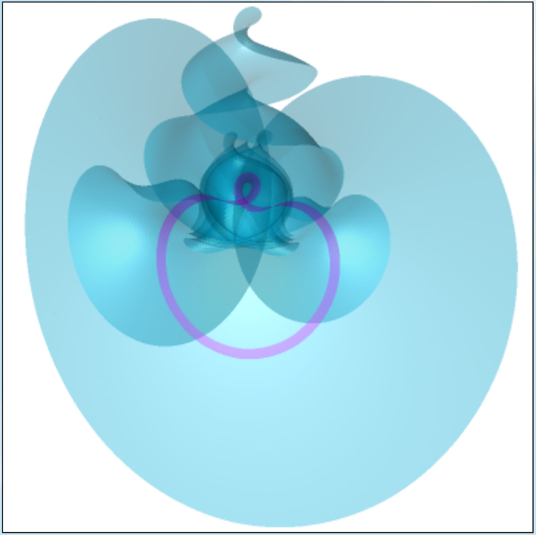


$$\begin{cases} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \varrho x - y - xz \\ \dot{z} &= -\beta z + xy \end{cases}$$

Classical parameters:

$$\begin{cases} \sigma &= 10 \\ \varrho &= 28 \\ \beta &= 2\frac{2}{3} \end{cases}$$

The Lorenz manifold



Manifold is viewed as family of geodesic level sets (GLS)

- Points on a new GLS found by continuation of 2PBVP
- Curve $L(\theta)$ is boundary of manifold computed so far