Parameter Continuation with COCO

Lecture given during the
2016 NZMRI Summer School on
Continuation Methods in Dynamical Systems

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January 12, 2016

Outline

Principles of Continuation

The *extended continuation problem*

$$\begin{pmatrix} \Phi(u) \\ \Psi(u) - \mu \end{pmatrix} = 0$$

is defined in terms of a collection of *zero functions* $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^m$, *monitor functions* $\Psi : \mathbb{R}^n \mapsto \mathbb{R}^r$, *continuation variables* $u \in \mathbb{R}^n$, and *continuation parameters* $\mu \in \mathbb{R}^r$.

A *restricted continuation problem*

$$\left.\begin{pmatrix} \Phi(u) \\ \Psi(u) - \mu \end{pmatrix}\right|_{\mu_{\mathbb{I}}=\mu_{\mathbb{I}}^*} = 0$$

is obtained by selecting a subset $\mathbb{I}$ of $\{1, \ldots, r\}$.

Principles of Continuation

We *construct* a restricted continuation problem by defining the functions Φ and Ψ, and by choosing the index set $\mathbb{I}$ corresponding to *inactive continuation parameters*.

We *initialize* a restricted continuation problem by choosing an initial solution guess u_0 for u, and letting

$$\mu_{\mathbb{I}}^* = \Psi_{\mathbb{I}}(u_0)$$

and $\Psi_{\mathbb{J}}(u_0)$ be an initial solution guess for $\mu_{\mathbb{J}}$.

The *dimensional deficit* of the restricted continuation problem equals $n - m - |\mathbb{I}|$. Continuation along a d-dimensional solution manifold requires that $d = n - m - |\mathbb{I}|$.

A Forced Linear Oscillator

Consider the non-autonomous dynamical system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2 - px_1 + \cos t \end{pmatrix}$$

For every p,

$$x_1(t) = \frac{\sin t + (p-1)\cos t}{p^2 - 2p + 2}, \; x_2(t) = \frac{\cos t - (p-1)\sin t}{p^2 - 2p + 2}$$

is a unique, asymptotically stable periodic orbit with $\mathcal{L}_2$ norm

$$\sqrt{\frac{2\pi}{p^2 - 2p + 2}}.$$

A Forced Linear Oscillator

In COCO, the commands

```
>> prob = coco_prob();
>> prob = coco_set(prob, 'ode', 'autonomous', false);
>> coll_args = { @linode, t0, x0, 'p', p0 };
>> prob = ode_isol2po(prob, '', coll_args{:});
```

- construct a single-segment collocation zero problem in terms of the vector field `@linode` on a default mesh with 10 intervals and 5 base points and 4 collocation nodes in each interval,
- initialize the continuation problem using `t0`, `x0`, and `p0`,
- associate p with an inactive continuation parameter `'p'`,
- append the boundary conditions $\upsilon_f - \upsilon_i = 0$,
- and associate T_0 and T with inactive continuation parameters `'po.tinit'` and `'po.period'`.

A Forced Linear Oscillator

The dimensional deficit of the restricted continuation problem is 0. Continuation along a one-dimensional solution manifold requires that one index be reassigned from $\mathbb{I}$ to $\mathbb{J} = \{1, \ldots, r\} \setminus \mathbb{I}$.

The command

```
>> coco(prob, 'run', [], 1, 'p', [0.2 2]);
```

identifies the desired manifold dimension as 1, reassigns the index of the continuation parameter `'p'` to $\mathbb{J}$, and restricts continuation to the domain `'p'` $\in [0.2, 2]$.

As an alternative, releasing the continuation parameter `'po.tinit'` results in continuation along a family of phase-shifted versions of the periodic orbit.

Nonlinear Vibrations

Consider the non-autonomous dynamical system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -dx_2 - x_1 - x_1^3 + A\cos(2\pi t/\tau) \end{pmatrix}$$

in terms of the vector of state variables $x = (x_1, x_2) \in \mathbb{R}^2$ and the vector of problem parameters $p = (A, \tau, d) \in \mathbb{R}^3$.

This represents the response of a hardening nonlinear oscillator to harmonic excitation with period τ.

Larger excitation amplitudes are associated with the onset of *bistability*, i.e., intervals in excitation frequency with coexisting stable steady-state responses.

The commands

```
>> prob = coco_prob();
>> prob = coco_set(prob, 'ode', 'autonomous', false);
>> pnames = { 'A' 'tau' 'd' };
>> coll_args = { @bistable, t0, x0, pnames, p0 };
>> prob = ode_isol2po(prob, '', coll_args{:});
```

- construct a single-segment collocation zero problem in terms of the vector field `@bistable` on a default mesh with 10 intervals and 5 base points and 4 collocation nodes in each interval,
- initialize the continuation problem using `t0`, `x0`, and `p0`,
- associate A, τ, and d with inactive continuation parameters `'A'`, `'tau'`, and `'d'`,
- append the boundary conditions $\upsilon_f - \upsilon_i = 0$,
- and associate T_0 and T with inactive continuation parameters `'po.tinit'` and `'po.period'`.

Nonlinear Vibrations

The dimensional deficit of the restricted continuation problem equals 0.

A frequency-response diagram is obtained by continuation along a family of periodic orbits under variations in the excitation frequency $(= 2\pi/\tau)$. Such a family is obtained by appending the zero function

$$u \mapsto T - \tau$$

and releasing the continuation parameters `'tau'` and `'po.period'`, as shown in the commands below.

```
>> [data uidx] = coco_get_func_data(prob, ...
     'po.orb.coll', 'data', 'uidx');
>> maps = data.coll_seg.maps;
>> prob = coco_add_glue(prob, 'glue', ...
     uidx(maps.T_idx),  uidx(maps.p_idx(2)));
>> coco(prob, 'freq_resp', [], 1, {'po.period' 'tau'});
```

Saddle-node bifurcations found along the frequency-response curve may be used as starting points for continuation along a family of such points, as shown in the commands below.

```
>> bd   = coco_bd_read('freq_resp');
>> labs = coco_bd_labs(bd, 'SN');
>> prob = coco_prob();
>> prob = ode_SN2SN(prob, '', 'freq_resp', labs(1));
>> [data uidx] = coco_get_func_data(prob, ...
     'po.orb.coll', 'data', 'uidx');
>> maps = data.coll_seg.maps;
>> prob = coco_add_glue(prob, 'glue', ...
     uidx(maps.T_idx), uidx(maps.p_idx(1)));
>> cont_args = { 1,  { 'po.period' 'T' 'A' } };
>> coco(prob, 'saddle-node', [], cont_args{:});
```

The fold observed along this family corresponds to a cusp bifurcation and the onset of bistability in the nonlinear frequency response of the hardening oscillator.

Nonlinear Vibrations

The *backbone curve* for the nonlinear oscillator is the one-dimensional family of periodic orbits obtained for $A = d = 0$ and emanating from the limit of zero response amplitude with period equal to 2π.

The following sequence of commands construct a corresponding periodic orbit zero problem.

```
>> t0 = (0:0.01:2*pi)';
>> x0 = 2e-2*[sin(t0) cos(t0)];
>> p0 = [0; 2*pi; 0];
>> prob = coco_prob();
>> prob = coco_set(prob, 'ode', 'autonomous', false);
>> pnames =  { 'A' 'tau' 'd' };
>> coll_args = { @bistable, t0, x0, pnames, p0 };
>> prob = ode_isol2po(prob, '', coll_args{:});
```

Nonlinear Vibrations

Although the dimensional deficit of the continuation problem encoded thus far equals 0, this problem is degenerate, since arbitrary shifts in time are still solutions.

We restrict attention to a particular phase by holding fixed the value of x_1 on the initial point on the periodic orbit.

```
>> [data uidx] = coco_get_func_data(prob, ...
     'po.orb.coll', 'data', 'uidx');
>> maps = data.coll_seg.maps;
>> prob = coco_add_pars(prob, 'section', ...
     uidx(maps.x0_idx(1)), 'y0');
>> cont_args = { 1, { 'po.period'  'd' } };
>> coco(prob, 'backbone', [], cont_args{:});
```

Note that the value of `'d'` remains approximately 0 throughout continuation, since periodic orbits with nonzero amplitude exist for $A = 0$ only if $d = 0$.

A Piecewise Smooth System

The equations

$$\dot{r} = r(1 - r),\ \dot{\theta} = 1$$

and

$$\dot{r} = \alpha r(\beta - r),\ \dot{\theta} = \gamma + \beta - r$$

describe two planar dynamical systems expressed in polar coordinates.

For sufficiently large γ, closed curves may be constructed inside the annulus bounded by $r = 1$ and $r = \beta$ by stitching together trajectory segments for each of the two vector fields. These correspond to periodic orbits of a suitably defined piecewise-smooth dynamical system.

A Piecewise Smooth System

The zero problem for continuation of multi-segment periodic orbits appends the zero functions

$$\left(\{\upsilon_{j,i}\}_{j=1}^{M}, \{\upsilon_{j,f}\}_{j=1}^{M}, p\right) \mapsto \begin{pmatrix} h(\upsilon_{1,f}, p; \mathfrak{e}_1) \\ \upsilon_{2,i} - g(\upsilon_{1,f}, p; \mathfrak{r}_1) \\ h(\upsilon_{2,f}, p; \mathfrak{e}_2) \\ \upsilon_{3,i} - g(\upsilon_{2,f}, p; \mathfrak{r}_2) \\ \vdots \\ h(\upsilon_{M,f}, p; \mathfrak{e}_M) \\ \upsilon_{1,i} - g(\upsilon_{M,f}, p; \mathfrak{r}_M) \end{pmatrix}$$

to the collection of collocation zero problems characterized by the vector fields $f(x, p; \mathfrak{m}_j)$, $j = 1, \ldots, M$.

The sequences $\{\mathfrak{m}_j\}_{j=1}^{M}$, $\{\mathfrak{e}_j\}_{j=1}^{M}$, and $\{\mathfrak{r}_j\}_{j=1}^{M}$ of *mode labels*, *event labels*, and *reset labels* are referred to as the *orbit signature*.

A Piecewise Smooth System

The dimensional deficit of the multi-segment periodic orbit zero problem is q. The commands below

```
>> prob = coco_prob();
>> prob = ode_isol2hspo(prob, '', ...
     {@piecewise, stop, jump}, modes, events, resets, ...
     t0, x0, {'al' 'be' 'ga'}, p0);
>> coco(prob, 'pw1', [], 1, 'be', [0 5]);
```

- construct the multi-segment periodic orbit zero problem in terms of the vector field `@piecewise`, event function `stop`, and reset function `jump` and the associated mode, event, and reset labels `modes`, `events`, and `resets`, respectively.
- associate α, β, and γ with the inactive continuation parameters `'al'`, `'be'`, and `'ga'`,
- perform continuation along a one-dimensional solution manifold under variations in the continuation parameter `'be'` on the computational domain `'be'`$\in [0, 5]$.

An Impact Oscillator

Consider the hybrid dynamical system governed by the vector field

$$F(x,p) = \begin{pmatrix} x_2 \\ -gx_1 - cx_2 + A\cos x_3 \\ \omega \end{pmatrix}$$

the reset maps

$$g(x,p;\mathfrak{bounce}) = \begin{pmatrix} x_1 \\ -ex_2 \\ x_3 \end{pmatrix},\ g(x,p;\mathfrak{phase}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 - 2\pi \end{pmatrix}$$

and the event functions

$$h(x,p;\mathfrak{impact}) = d - x_1,\ h(x,p;\mathfrak{impact}) = \pi - x_3.$$

An Impact Oscillator

The following commands construct a two-segment impacting periodic-orbit continuation problem, with initial solution guess obtained by forward simulation.

```
>> p0     = [1; 0.1; 1; 1; 1; 0.8];
>> modes  = {'free' 'free'};
>> events = {'impact' 'phase'};
>> resets = {'bounce' 'phase'};
>> f      = @(t, x) impact(x, p0, 'free');
>> [t1, x1] = ode45(f, [0 3.2], [-0.98; -0.29; -pi]);
>> [t2, x2] = ode45(f, [0 3.1], [1; -1.36; 0.076]);
>> t0 = {t1 t2};
>> x0 = {x1 x2};
>> funcs = {@impact, @impact_events, @impact_resets};
>> hspo_args = {funcs, modes, events, resets, ...
     t0, x0, {'k' 'c' 'A' 'w' 'd' 'e'}, p0};
>> prob = coco_prob();
>> prob = coco_set(prob, 'hspo', 'bifus', false);
>> prob = ode_isol2hspo(prob, '', hspo_args{:});
```

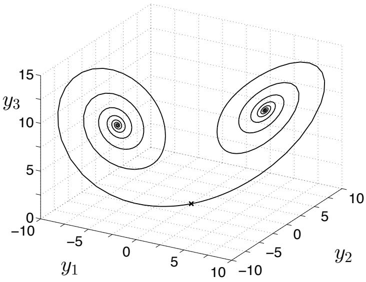
An Impact Oscillator

In order to support detection of grazing contact, we extract the second component of the initial end point of the second trajectory segment and monitor changes in its sign, as shown below.

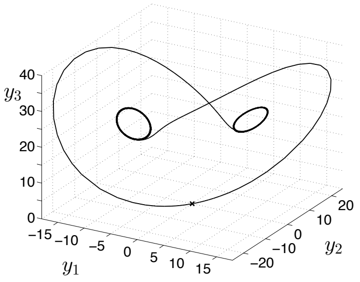
```
>> [data, uidx] = coco_get_func_data(prob, ...
     'hspo.orb.bvp.seg2.coll', 'data', 'uidx');
>> maps = data.coll_seg.maps;
>> prob = coco_add_pars(prob, 'grazing', ...
     uidx(maps.x0_idx(2)), 'graze', 'active');
>> prob = coco_add_event(prob, 'GR', 'graze', 0);
```

As the continuation problem is implemented in terms of a constrained multi-segment boundary-value problem, the analysis produces solutions that are contained entirely in the $x_1 \leq d$ half space, as well as solutions that cross the $x_1 = d$ boundary, even though the latter violate the association of crossings of this boundary with the application of $g(x, p; \mathfrak{bounce})$.

Heteroclinic connections



(a) Manifold for $r = 10$.

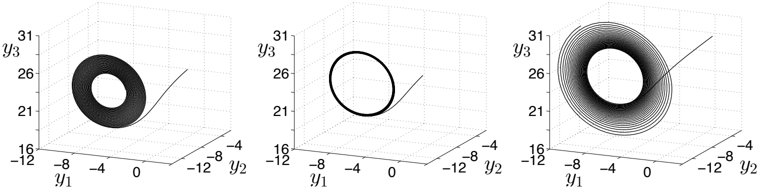


(b) Manifold for $r = 24$.

Figure 10.1. *Orbits of the Lorentz system given by the vector field in Eq. (10.87) starting in the unstable eigenspace of the equilibrium at* 0*, tracing the unstable manifold. The orbits in (a) approach equilibria, while the orbits in (b) seem to approach periodic orbits. This approach is shown in more detail in Fig. 10.2.*

 Note: "Lorentz" should be "Lorenz".

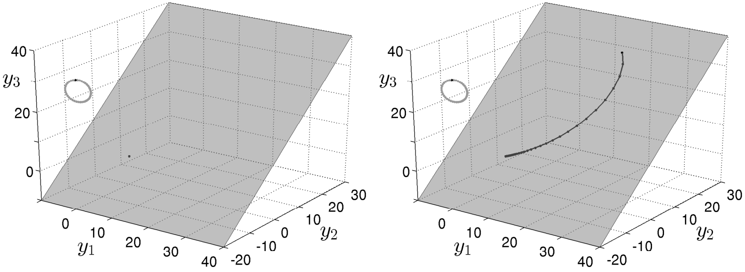
Heteroclinic connections



(a) Approach for $r = 23$. (b) Approach for $r = 24$. (c) Approach for $r = 25$.

Figure 10.2. *A close-up of the potential approach and exit of $\mathcal{W}_0^u$ to and from a periodic orbit, as observed in Fig. 10.1. In panel (a), the manifold exits spiraling inward, while the exit is outward in (c). This corresponds to a switch of approach from an orbit inside a stable manifold of a periodic orbit to an approach from an orbit outside. This suggests that, in between these parameter values, there exists an orbit approaching* on *the stable manifold—a heteroclinic connection between the equilibrium at* 0 *and a periodic orbit of saddle type.*

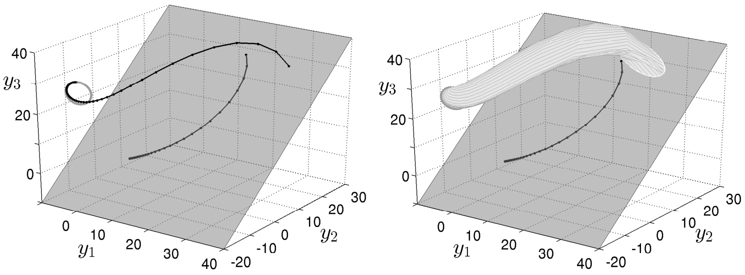
Heteroclinic connections



(a) Initial three-segment solution. (b) After growing the unstable manifold.

Figure 10.3. *Construction of an initial approximation of an orbit connecting an equilibrium and a periodic orbit in the Lorentz system given by the vector field in Eq. (10.87) following the homotopy approach described in Sect. 10.2.2. A state-space representation of the three-segment solution, consisting of a periodic orbit (gray) and two zero-length segments (black dots), that is used to initialize Stage II of the homotopy is shown in panel (a), together with the hyperplane Σ that separates the periodic orbit from the equilibrium. In Stage II, we grow an orbit in $\mathcal{W}_0^u$ until it terminates on Σ, as shown in (b). In the subsequent Stage III we grow an orbit in $\mathcal{W}_{\mathrm{per}}^s$ in a similar way; see Fig. 10.4.*

Heteroclinic connections



(a) Growing one orbit in the stable manifold. (b) After sweeping the stable manifold.

Figure 10.4. *Panel (a) shows the three-segment solution after completing Stage III, i.e., growing an orbit in $\mathcal{W}^s_{\text{per}}$ starting at the solution shown in Fig. 10.3(b). Here, the end point of the orbit segment in $\mathcal{W}^u_0$ and the starting point of the orbit segment in $\mathcal{W}^s_{\text{per}}$ both lie in Σ. Although $\mathcal{W}^s_{\text{per}}$ is 2-dimensional, the connecting orbit is unique. To obtain an initial approximation of the connecting orbit, we first sweep $\mathcal{W}^s_{\text{per}}$ in Stage IV and compute a set of orbit segments that cover the manifold sufficiently densely (b). From this family of orbits we select the one that terminates closest to the end-point of the segment in $\mathcal{W}^u_0$; see Fig. 10.5.*

Heteroclinic connections

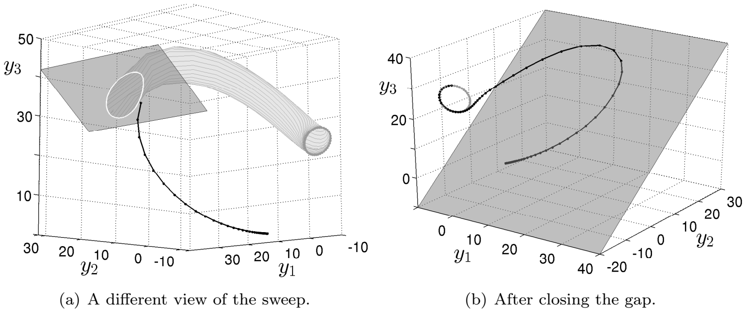


Figure 10.5. *Panel (a) shows a different view of Fig. 10.4(b), the result of a sweep of* $\mathcal{W}^s_{\mathrm{per}}$. *The intersection with* Σ *is highlighted. We compute the point of the intersecting curve that is closest to the end point of the segment in* $\mathcal{W}^u_0$ *and initialize Stage V of the homotopy, i.e., the closing of the Lin gap. The resulting connecting orbit after closing the gap is shown in panel (b).*

Additional resources

- *Recipes for Continuation*, SIAM, 2013: principles of continuation, vectorization, and collocation.
- *The 'ep' and 'po' toolboxes*, SourceForge, 2015: tutorial documentation and demos for production-ready COCO toolbox.
- *Thursday's lecture*, NZMRI, 2016: Atlas algorithms for one- and multidimensional manifolds, recent developments, open problems.