Delay Differential Equations

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A Scalar State-Dependent DDE (with two delays)

A Model DDE with two State-Dependent Delays

$$\varepsilon\dot{u}(t) = -\gamma u(t) - \kappa_1 u\big(t - a_1 - c_1 u(t)\big) - \kappa_2 u\big(t - a_2 - c_2 u(t)\big), \; u(t) \in \mathbb{R},$$

Positive Parameters $\varepsilon, \gamma, \kappa_i, a_i, c_i > 0$.

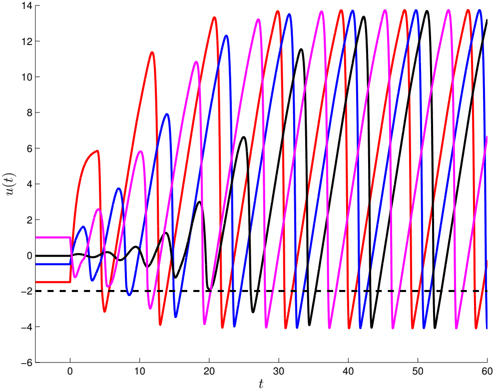
- Scalar State-Dependent DDE with negative feedback.
- 'Delays' are linearly state-dependent $\alpha_i = t - \tau_i = t - a_i - c_i u(t)$
- $c_1 = c_2 = 0 \implies$ linear constant delay DDE $\implies$ boring.
- *NO* nonlinearity in model except for the state-dependency; *interesting dynamics driven by the state-dependency of delays*.
- Will (usually) fix $c_1 = c_2 = c > 0$ & wlog $a_2 > a_1$ then

  $$\alpha_1 - \alpha_2 = a_2 - a_1 = const > 0.$$

Delayed or advanced-retarded??

Modified Problem

$\dot{u}(t) = -\gamma u(t) - \kappa_1 u(t_1) - \kappa_2 u(t_2), \quad t_i = \min\{t, t - a_i - c_i u(t)\}$
$\kappa_1, a_i, c_i > 0$ and $\kappa_2 > \gamma > 0$.



$1 = a_1 = \gamma < \kappa_2 = \kappa_1 = a_2 = 2, c_1 = 0.5, c_2 = 0.4.$

- $\alpha_i = t - a_i - c_i u(t) < t$ for $u(t) > -a_i/c_i$
- Stable periodic orbit enters region where $t < t - a_1 - c_1 u(t)$
- Solution of Original Problem Terminates
- ddesd modifies all state-dep DDEs this way

State-Dependency Bounds Solutions!

$$\dot{u}(t) = -\gamma u(t) - \kappa_1 u(t - a_1 - c_1 u(t)) - \kappa_2 u(t - a_2 - c_2 u(t))$$

Wlog order delays: $0 > -a_1/c_1 \geqslant -a_2/c_2$
$\implies t - a_i - c_i u(t) < t$ if $u(t) > -a_1/c_1$

Theorem (Existence/Boundedness: generalises to N delays)

If $\gamma > \kappa_2$ *&* $u(t) \in [-\frac{a_1}{c_1}, \frac{Ka_1}{\gamma c_1}]\ \forall t \in [-T, 0]$ *where* $K = \kappa_1 + \kappa_2$ *&* $T = \max_i\{a_i + Kc_i a_1/(\gamma c_1)\}$ *then* $u(t) \in [-\frac{a_1}{c_1}, \frac{Ka_1}{\gamma c_1}]\ \forall t \geqslant 0$.

Notice $u = 0$ is only steady-state

Linearization

[Györi & Hartung 07] showed linearization about steady state is
$$\dot{u}(t) = -\gamma u(t) - \kappa_1 u(t - a_1) - \kappa_2 u(t - a_2)$$
(ie freeze delay to its value at steady-state).
This determines stability for state-dependent DDE

Bifurcations

$$\varepsilon\dot{u}(t) = -\gamma u(t) - \kappa_1 u\big(t - a_1 - cu(t)\big) - \kappa_2 u\big(t - a_2 - cu(t)\big),\ u(t) \in \mathbb{R},$$
$$\varepsilon > 0,\ \gamma > \kappa_2 \geqslant 0,\ \kappa_1 > 0,\ a_2 > a_1 > 0,\ c > 0.$$

- $u = 0$ is only steady state
- Stable if $\kappa_1 + \kappa_2 \leqslant \gamma$,
- Require $\kappa_2 < \gamma$ to ensure well-posed, so
- Use $\kappa_1 > 0$ as bifurcation parameter;
  use $\kappa_2 \in [0, \gamma]$ as secondary bifurcation parameter
- Varying other parameters (esp. $A = a_2/a_1 > 1$) interesting

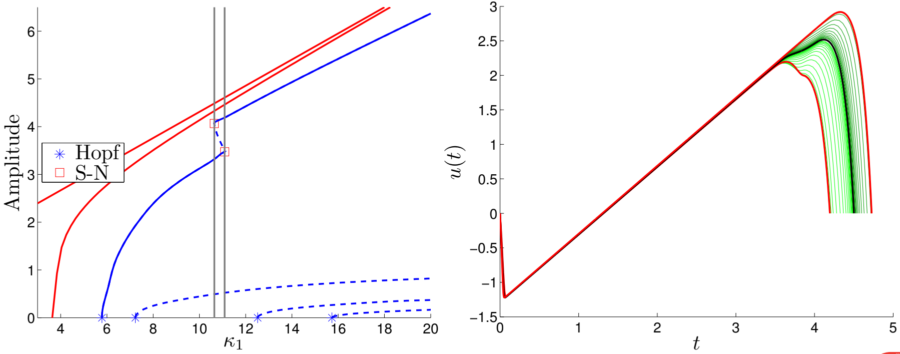
State-Dependent Hopf Bifurcations

Hopf bifurcations [Eichmann 06, Sieber 12, Hu&Wu 10] lead to periodic orbits

Two Delays: Bistability for $a_2 \gg a_1$

$$\dot{u}(t) = -\gamma u(t) - \kappa_1 u(t - a_1 - u(t)) - u(t - a_2 - u(t))$$

$\gamma = 4.75$, $\kappa_2 = \varepsilon = c_1 = c_2 = 1$ and $a_2 = 6 \gg a_1 = 1.3 > 0$:
$[t - a_1 - cu(t)] - [t - a_2 - cu(t)] = a_2 - a_1 = 4.7 > T^*$ constant.

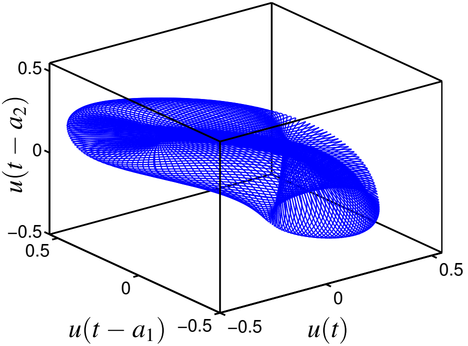


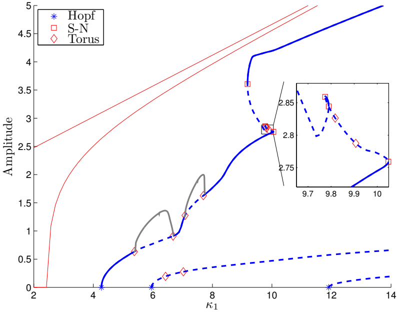
- Lots of Hopf bifurcations
- Bistability explained by singular limit

Stable Torus!

- Parameter intervals with no stable periodic orbits
- Torus stable....compute using *ddesd* as IVP
- Plot projection of torus in $\mathbb{R}^3$: $(u(t), u(t-a_1), u(t-a_2))$
- Can't compute unstable torus from 2nd branch.

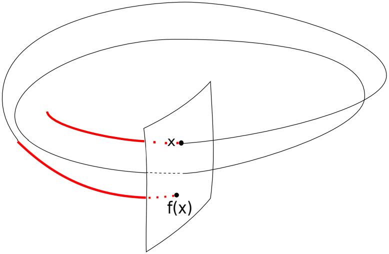
$\gamma = 4.75$, $\varepsilon = c_1 = c_2 = 1$, $\kappa_2 = 2.3$ and $a_2 = 6 \gg a_1 = 1.3 > 0$:





Poincaré Sections

$$\varepsilon \dot{u}(t) = -\gamma u(t) - \kappa_1 u\big(t - a_1 - cu(t)\big) - \kappa_2 u\big(t - a_2 - cu(t)\big), \; u(t) \in \mathbb{R},$$



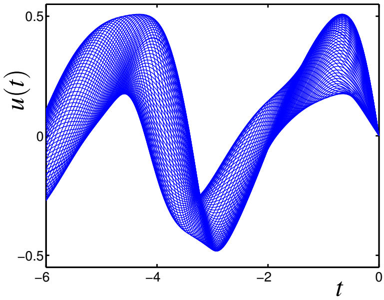
- Used to study persistent dynamics; reduces dimension by 1
- Phase Space is $C = C([-r, 0], \mathbb{R})$ where r is largest delay: $r = a_1\left(1 + \frac{1}{\gamma c}(\kappa_1 + \kappa_2)\right)$
- C is infinite dimensional function space
- Solutions oscillate about $u = 0$ so natural Poincarè section is

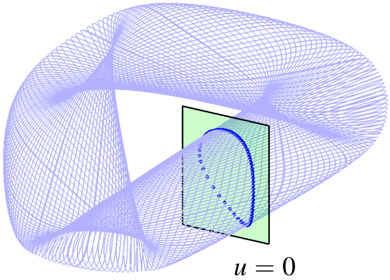
$$\Big\{\varphi \in C : \varphi(0) = 0\Big\}$$

Poincaré Sections

$$\varepsilon\dot{u}(t) = -\gamma u(t) - \kappa_1 u\big(t - a_1 - cu(t)\big) - \kappa_2 u\big(t - a_2 - cu(t)\big), \; u(t) \in \mathbb{R},$$

Poincaré Section reduces dimension by one from ∞ to ∞.

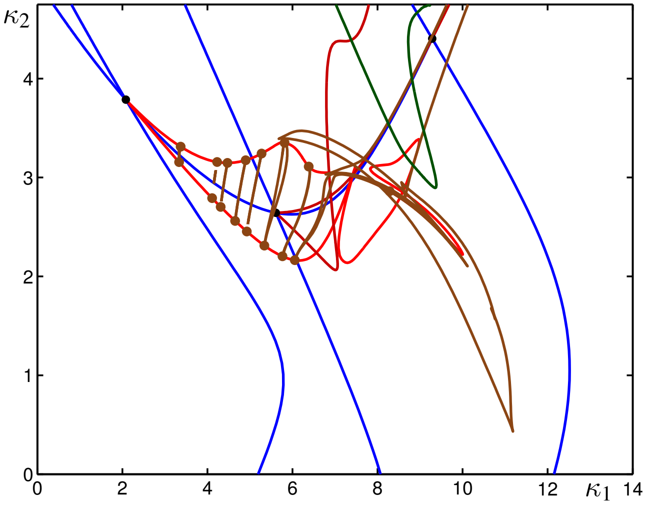




Since solutions oscillate about trivial solution $u = 0$
take $u = 0$ as Poincaré Section and project into $\mathbb{R}^2$ by plotting
$u(t - a_2)$ against $u(t - a_1)$ when $u(t) = 0$ (with $\dot{u}(t) < 0$)

2 Parameter Bifurcation Diagram

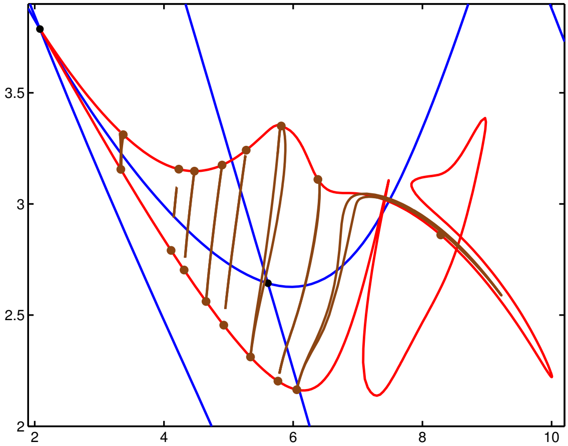
$$\dot{u}(t) = -4.75u(t) - \kappa_1 u\big(t - 1.3 - cu(t)\big) - \kappa_2 u\big(t - 6 - cu(t)\big), \ u(t) \in \mathbb{R},$$



- Hopf bifns (blue)
- 3 Double Hopf Bifns
- Torus Bifns (Red)
- 2 branches of torus bifns originate in each double Hopf
- Folds and Arnold tongues (brown)
- Period-doubling (green)

Arnold tongues

DDEBiftool can continue torus and fold bifurcations in (κ_1, κ_2) for state-dependent DDEs. Use to find Arnold tongues.

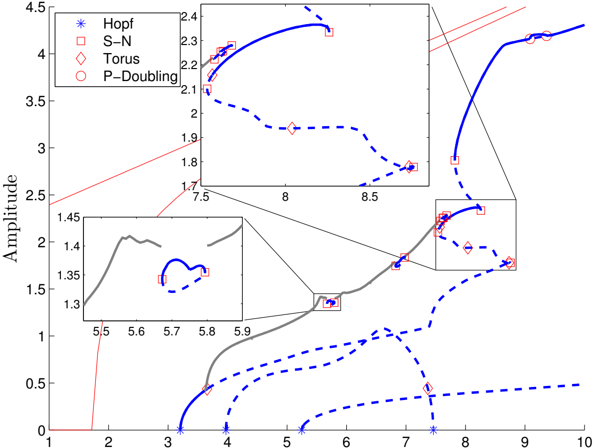


Shown: $p/q = 1/2, 4/9, 3/7, 2/5, 3/8, 1/3, 2/7, 1/4 \to p/(p+q)$.

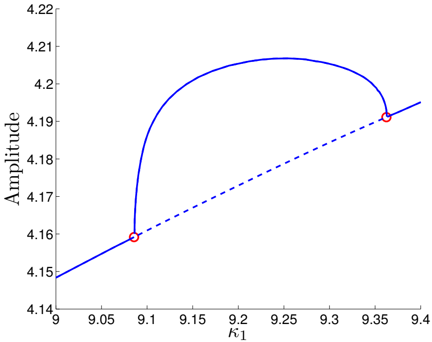
Period Doubling and Torus with Phase Locking

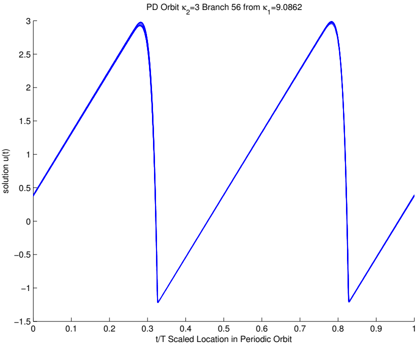
$$\dot{u}(t) = -\gamma u(t) - \kappa_1 u(t - a_1 - u(t)) - \kappa_2 u(t - a_2 - u(t))$$

- $3 = \kappa_2 < \gamma = 4.75$
- New Hopf branch, Period Doubling, Torus with Phase Locking



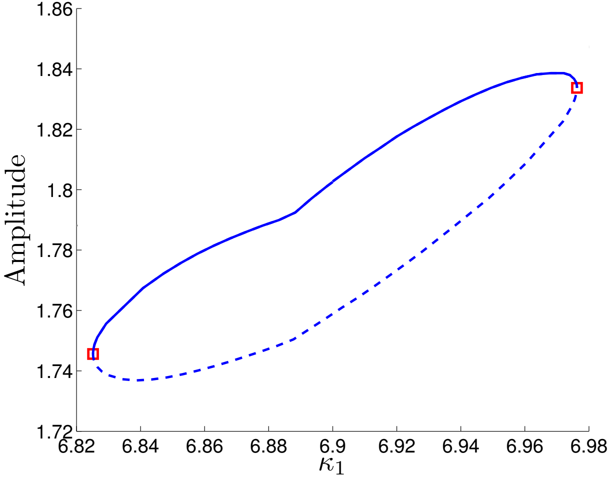
Period Doubled Orbits





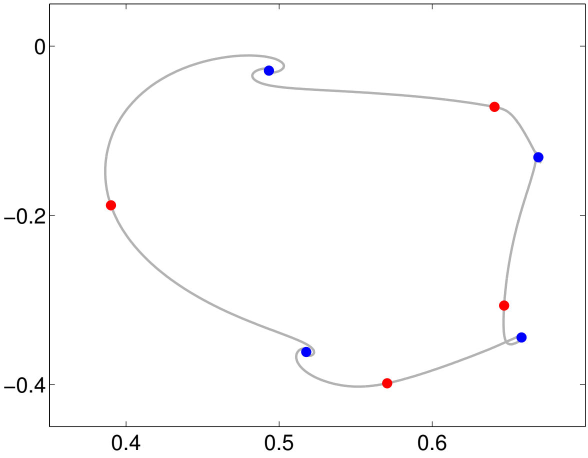
- Small parameter interval where periodic orbit loses stability to a period doubled solution
- With D. Barton extended ddebiftool to compute branch switching to period doubled orbit
- Further period doublings observed for larger values of κ_2.

Phase Locked 'Torus' for $\kappa_1 \in [6.8252, 6.9763]$



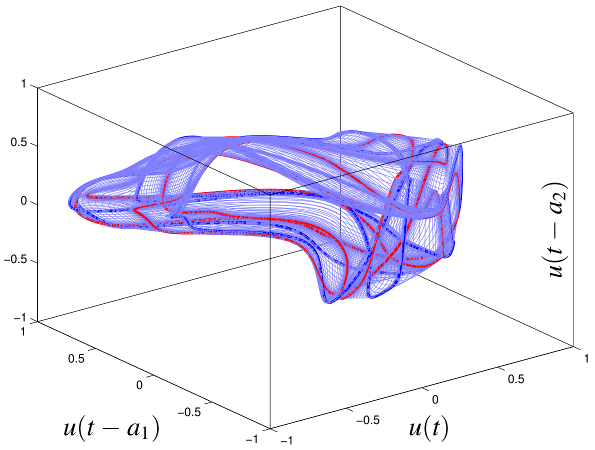
- *ddesd* solution $\to$ stable phase locked orbit
- use *ddebiftool* to compute isola of periodic orbits
- unstable orbit of saddle type with one positive Floquet multiplier

Phase Locked 'Torus' for $\kappa_1 \in [6.8252, 6.9763]$



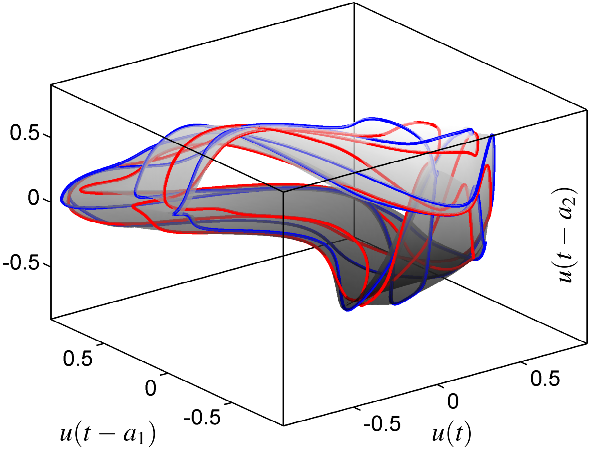
- Perturb unstable periodic orbit with Eigenfunction
- Hence compute orbit in unstable manifold with *ddesd*
- Stable orbit has complex dominant Floquet multiplier: not a torus!

Phase Locked ‘Torus’ for $\kappa_1 \in [6.8252, 6.9763]$



- Returning to 3D projection to $(u(t), u(t-a_1), u(t-a_2))$,
- plot stable (blue), unstable (red) orbit on attractor, along with orbits (blue-grey) in unstable manifold of the red orbit
- $\kappa_1 = 6.93$
- More spaghetti than torus......

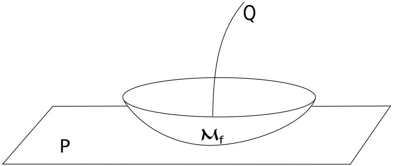
Phase Locked 'Torus' for $\kappa_1 \in [6.8252, 6.9763]$



- ... it is better to render unstable manifold of unstable orbit on not-a-torus as a surface
- $\kappa_1 = 6.93$

Double Hopf Bifurcation Normal Form

- Centre Manifold reduction to 4-dim ODE
  (two pairs imaginary characteristic values at double Hopf).
- Let $C = P \oplus Q$ where P is centre eigenspace.
  Then centre manifold M_f given as a graph in C over space P



- Follow [Belair & Campbell 94, Guo & Wu 2013] approach for constant delay DDEs.
- But our delays are not constant. Also how do we expand the nonlinearity??

Double Hopf Bifurcation Normal Form

$\dot{u}(t) = -\gamma u(t) - \kappa_1 u\big(t - a_1 - c_1 u(t)\big) - \kappa_2 u\big(t - a_2 - c_2 u(t)\big),$
DDE is state-dependent without nonlinearity

Expand delays about their steady state $u = 0$ values

$$u(t - a_i - cu(t)) = u(t - a_i) + \dot{u}(t - a_i)(-cu(t)) + \frac{1}{2}\ddot{u}(t - a_i)(-cu(t))^2 + \ldots$$

Use original DDE to remove $\dot{u}$, $\ddot{u}$ terms etc

$$\begin{aligned}\dot{u}(t) &= -\gamma u(t) - \kappa_1 u(t - a_1 - cu(t)) - \kappa_2 u(t - a_2 - cu(t)) \\ &= -\gamma u(t) - \kappa_1 u(t - a_1) - \kappa_2 u(t - a_2) + h.o.t.\end{aligned}$$

Hence

$$\dot{u}(t - a_i) = -\gamma u(t - a_i) - \kappa_1 u(t - a_1 - a_i) - \kappa_2 u(t - a_2 - a_i) + h.o.t.$$

We obtain....

Double Hopf Bifurcation Normal Form

$$\dot{u}(t) = -\gamma u(t) - \kappa_1 u\big(t - a_1 - c_1 u(t)\big) - \kappa_2 u\big(t - a_2 - c_2 u(t)\big),$$

DDE is state-dependent without nonlinearity

Expansion:

$$\dot{u}(t) = -\gamma u(t) - \sum_{i=1}^{2} \kappa_i u(t - a_i) \quad \text{(linear)}$$

$$- \textstyle\sum_{i=1}^{2} \kappa_i c u(t) \Big[\gamma u(t - a_i) + \sum_{j=1}^{2} \kappa_j u(t - a_i - a_j)\Big] \quad \text{(quadratic)}$$

$$- \textstyle\sum_{i,j=1}^{2} \kappa_i \kappa_j c^2 u(t) u(t-a_i) \Big[\gamma u(t-a_i-a_j) + \sum_{m=1}^{2} \kappa_m u(t-a_i-a_j-a_m)\Big]$$

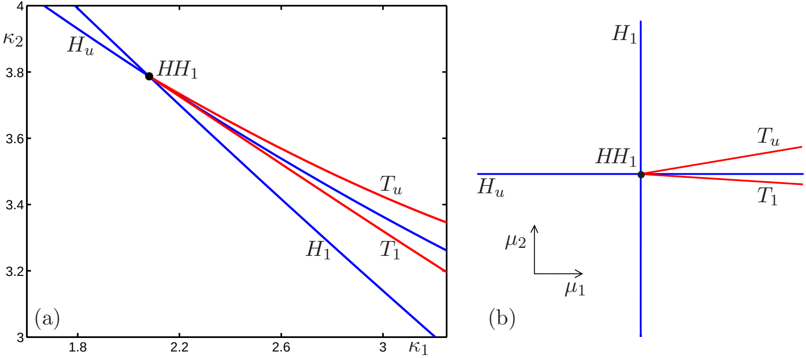
$$- \tfrac{1}{2}(cu(t))^2 \textstyle\sum_{i=1}^{2} \kappa_i \Big[\gamma^2 u(t-a_i) + 2\gamma \sum_{j=1}^{2} \kappa_j u(t-a_i-a_j) + \sum_{j,m=1}^{2} \kappa_j \kappa_m u(t-a_i-a_j-a_m)\Big]$$

$$+ \mathcal{O}(4)$$

Benefit: Expansion of nonlinearity with constant delays

Cost: Have $n(n+3)/2$ delays at nth order$(= 2, 5, 9, 14, 20, \ldots)$

Double Hopf Bifurcation Normal Form

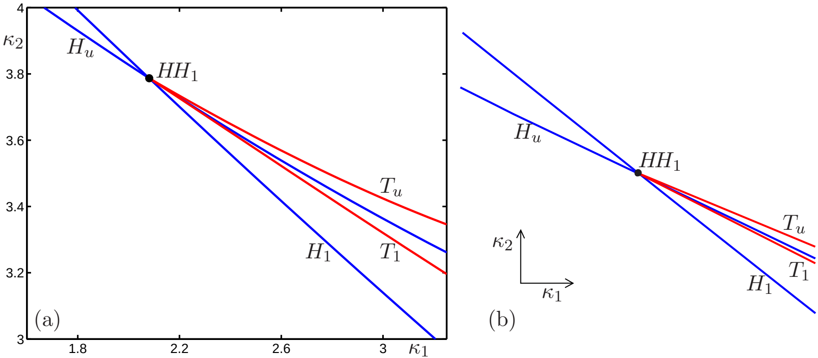


(a) Bifurcations from full state-dependent DDE

(b) Double Hopf Normal Form for 9 constant delays expansion

- Normal form analysis yields two branches of torus bifurcations emerging from first double Hopf point, as did numerical investigation
- Look different because normal form uses real parts of eigenvalues as parameters

Double Hopf Bifurcation Normal Form



(a) Bifurcations from full state-dependent DDE

(b) Double Hopf Normal Form for 9 constant delays expansion

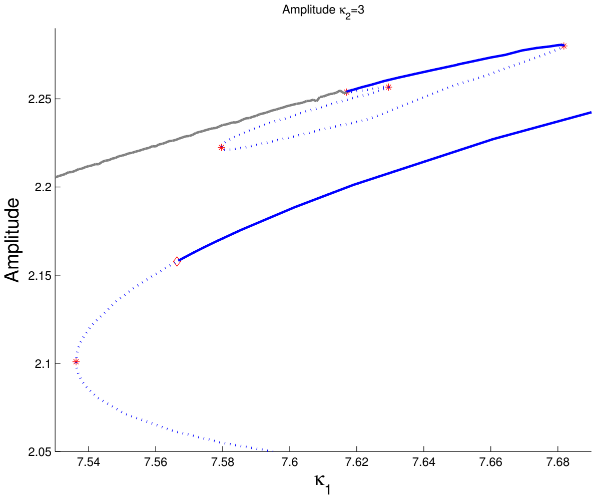
- Normal form analysis yields two branches of torus bifurcations emerging from first double Hopf point
- Remap normal form unfolding to (κ_1, κ_2): Eureka!
- Note: Right figure determined *only* from single point HH_1

Normal Form Coefficients

	Computed Normal Form	DDE-BIFTOOL			
		H_1 High	H_1 Low	H_u High	H_u Low
κ_1	2.080920227069893	2.080905301795540		2.080662320398254	
κ_2	3.786800923405767	3.786811738802836		3.786929718494380	
ω_1	2.487102830659818	2.487103286770640		1.582142631415513	
ω_2	1.582152129599611	1.582151566193548		2.487110459273053	
θ	5.291049995477214	5.2909997813	5.2909980111	-0.0222756426	-0.0222756534
δ	-0.022289571330146	-0.0222816360	-0.0222817195	5.2909133110	5.2909132195

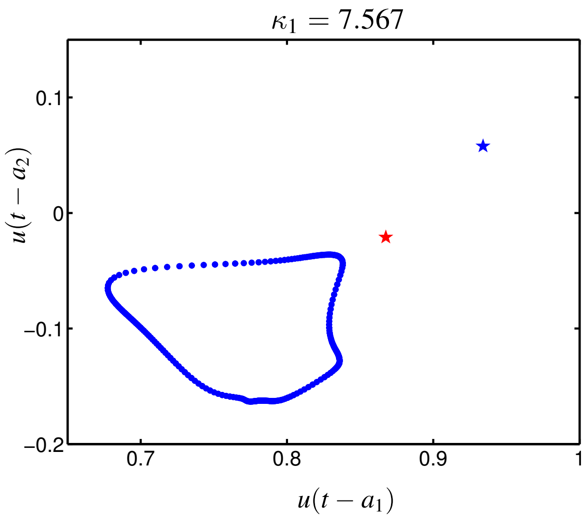
- We implemented double-Hopf normal form computations using characteristic equation and symbolic differentiation.
- The `nmfm` DDEBiftool extension also computes normal forms
- It finds bifurcations by searching along computed branch
- Methods agree to several digits of accuracy

Torus Break Up for $\kappa_1 \in [7.5796, 7.6818]$



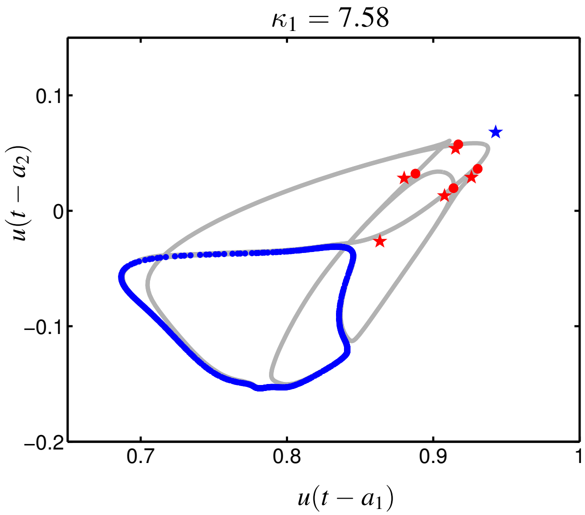
- Study locking region $\kappa_1 \in [7.5796, 7.6818]$
- Very different than previous case
- Torus is destroyed in a complex sequence of bifurcations

Torus Break Up for $\kappa_1 \in [7.5796, 7.6818]$



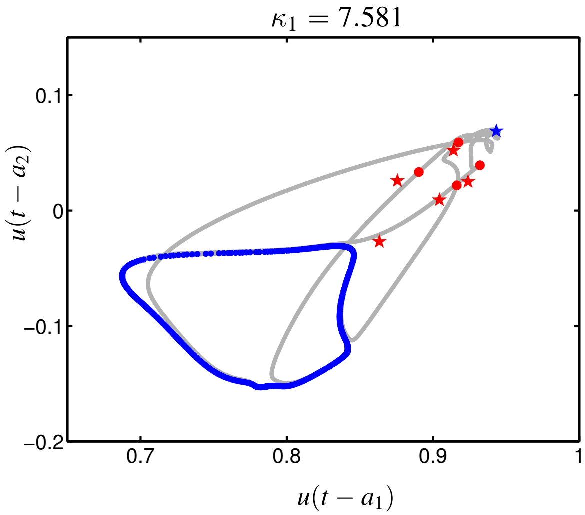
- At $\kappa_1 = 7.5363$ two unstable periodic orbits created in SN bifn on principal branch of periodic orbits
- At $\kappa_1 = 7.5664$ one of these orbits gains stability in a close to 1:4 resonant torus bifn. Stable QP torus coexists with stable periodic orbit
- Shown is quasi-periodic torus and stable and unstable periodic orbits

Torus Break Up for $\kappa_1 \in [7.5796, 7.6818]$



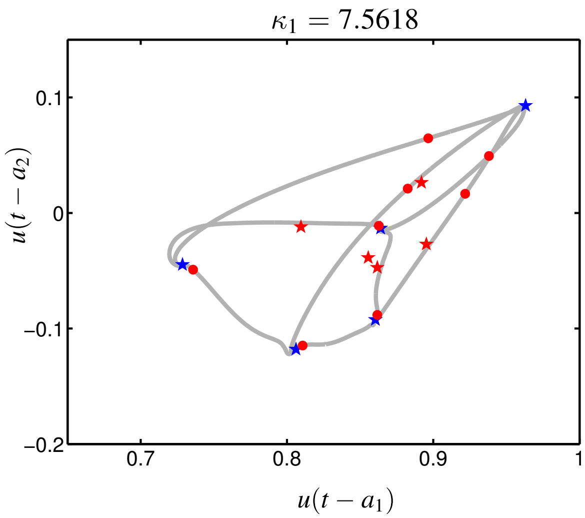
- At $\kappa_1 = 7.5796$ two Period-“4” orbits created in SN bifn.
- One orbit has 1d unstable manifold connecting on both sides to stable QP-torus
- Other orbit has 2d unstable manifold
- Shown is $\kappa_1 = 7.58$

Torus Break Up for $\kappa_1 \in [7.5796, 7.6818]$



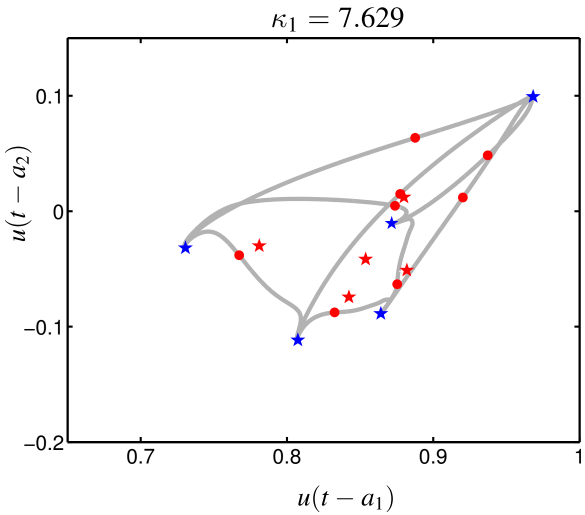
- for $7.58 < \kappa_1 < 7.581$ there is a 'square'-bifurcation characteristic of 1-4 resonance
- The period-"4" orbit with 1d unstable manifold now connects to stable QP-torus on one side and to stable period-"1" orbit on principal branch on the other side

Torus Break Up for $\kappa_1 \in [7.5796, 7.6818]$



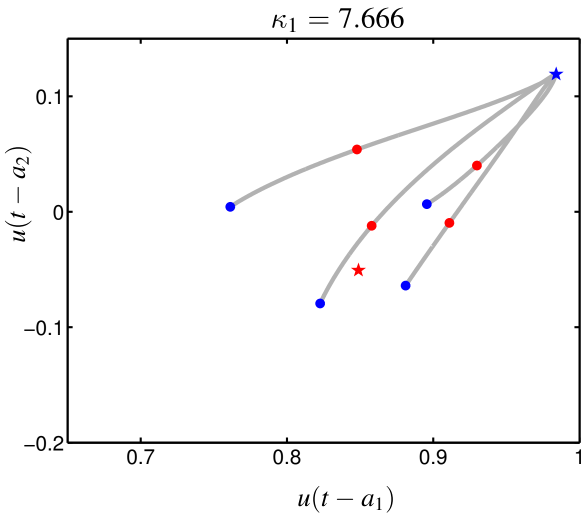
- At $\kappa_1 = 7.617$ two period-"4" orbits created on torus in SN bifn.
- Phase Locked Dynamics on torus for $\kappa_1 > 7.617$
- Unstable manifold of unstable period-"4" orbit fills stable torus.
- Stable orbit has complex dominant Floquet multiplier: not a torus!
- One Period-"4" orbit is approaching unstable orbit on 'torus'

Torus Break Up for $\kappa_1 \in [7.5796, 7.6818]$



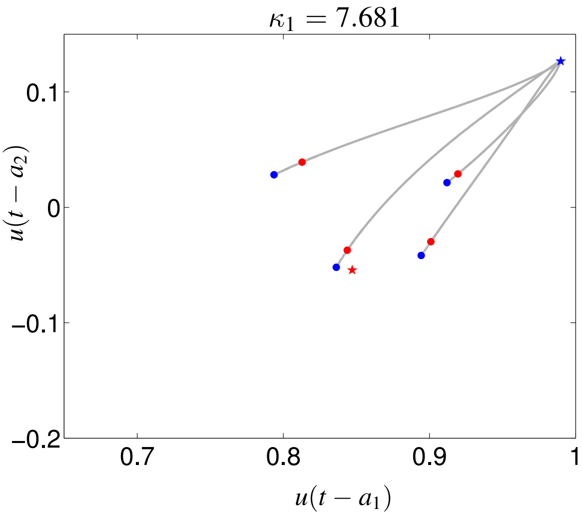
- At $\kappa_1 = 7.6295$ 'torus' is destroyed
- Unstable period-"4" orbit disappears in SN bifn with approaching period-"4" orbit.

Torus Break Up for $\kappa_1 \in [7.5796, 7.6818]$



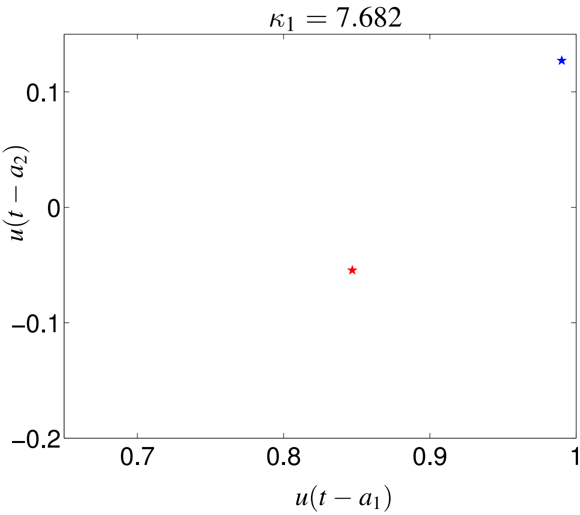
- At $\kappa_1 = 7.6295$ 'torus' is destroyed
- Unstable period-"4" orbit disappears in SN bifn with approaching period-"4" orbit.
- Stable periodic orbit from torus persists & coexists with stable period-"1" orbit.
- Stable orbits are connected by unstable manifold of remaining unstable period-"4" orbit.

Torus Break Up for $\kappa_1 \in [7.5796, 7.6818]$



- At $\kappa_1 = 7.6818$ last vestige of torus disappears.
- Stable and unstable period-“4” orbits destroyed in SN bifn

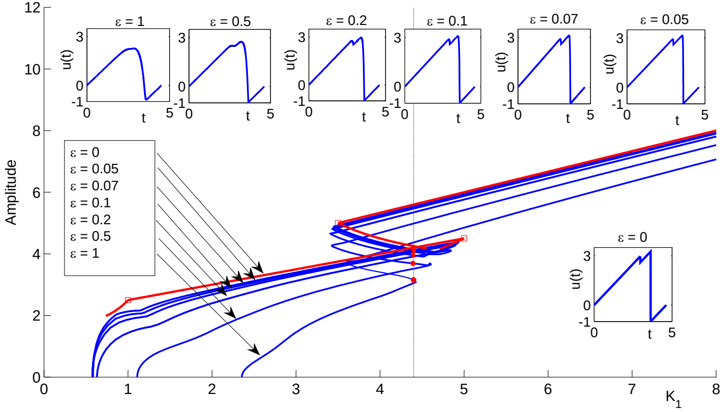
Torus Break Up for $\kappa_1 \in [7.5796, 7.6818]$



- At $\kappa_1 = 7.6818$ last vestige of torus disappears.
- Stable and unstable period-“4” orbits destroyed in SN bifn

Singularly Perturbed State-Dependent DDE $\varepsilon \to 0$

$$\varepsilon \dot{u}(t) = -u(t) - \kappa_1 u(t-1-cu(t)) - \tfrac{1}{2} u\big(t-6-cu(t)\big), \; u(t) \in \mathbb{R},$$



- DDEBiftool branches 'converge' as $\varepsilon \to 0$
- Have a singular solution theory (red curves) for limiting solutions and branches; finds folds and cusps

Summary & Conclusions

References

[H,DeMasi,Magpantay,Upham,DCDS 2012]: Initial Results
[H,Bernucci,Calleja,Homayounfar,Snarski,JDDE 2015/2016]: Singularly Perturbed Results
[Magpantay,H arxiv 2015]: Razumikhin stability
[Calleja,H,Krauskopf, *in prep*]: Torus, Double Hopf, Two-Parameter

Conclusions

- In absence of any other nonlinearity, State-Dependency of delays is enough to drive very interesting dynamics including full gamut of bifurcations associated with tori.
- Approximating state-dependent delays by constant delays could suppress interesting/important dynamics.
- Dynamics can be investigated using a combination of techniques, including continuation, that give consistent results.