Delay Differential Equations

Tony Humphries

McGill

tony.humphries@mcgill.ca

at

2016 NZMRI Summer School
Continuation Methods in Dynamical Systems
Raglan, New Zealand

12th, 14th & 15th January 2016

Table of contents

Mackey-Glass Equation

Mackey-Glass Equation

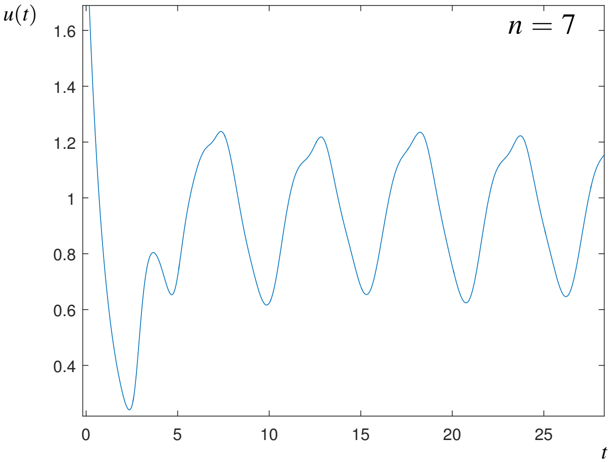
$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$

- [Mackey & Glass 1977] (see also scholarpedia article)
- Early toy model of circulating white blood cell numbers
- A scalar equation with chaotic dynamics
- $u(t) \in \mathbb{R}$, but phase-space $C = C([-\tau, 0], \mathbb{R})$ is ∞-dimensional. How should we represent solutions??
- Mackey and Glass project into $\mathbb{R}^2$ by plotting $u(t-\tau)$ vs $u(t)$.
- Equivalently we have $\mathcal{P} : C \to \mathbb{R}^2$ defined by

$$\mathcal{P}u_t = (u_t(0), u_t(-\tau))$$

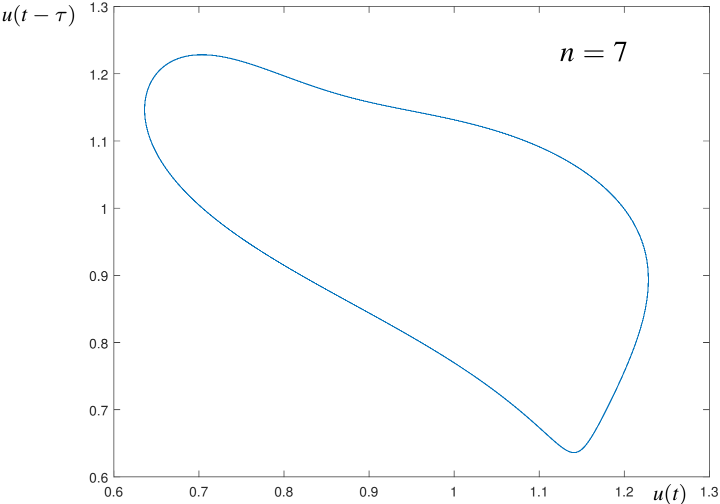
Mackey-Glass Equation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad \tau = \beta = 2,\ \gamma = 1$$



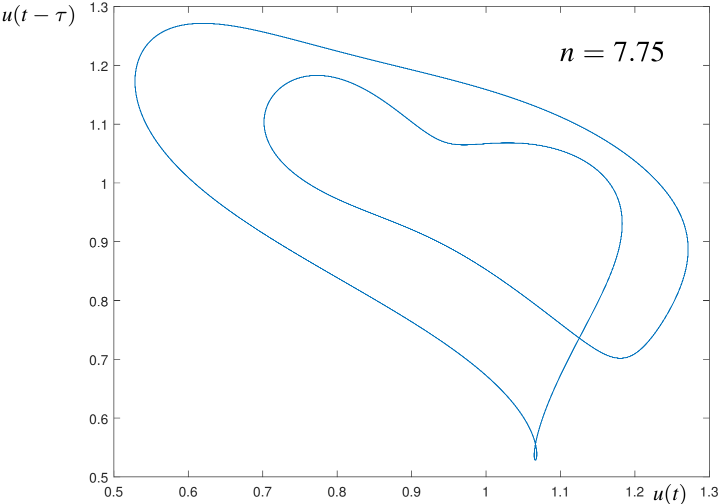
Mackey-Glass Equation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad \tau = \beta = 2,\ \gamma = 1$$



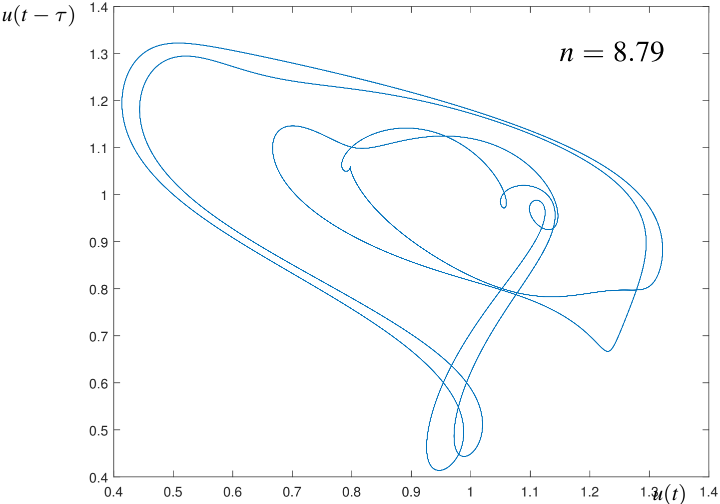
Mackey-Glass Equation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad \tau = \beta = 2,\ \gamma = 1$$



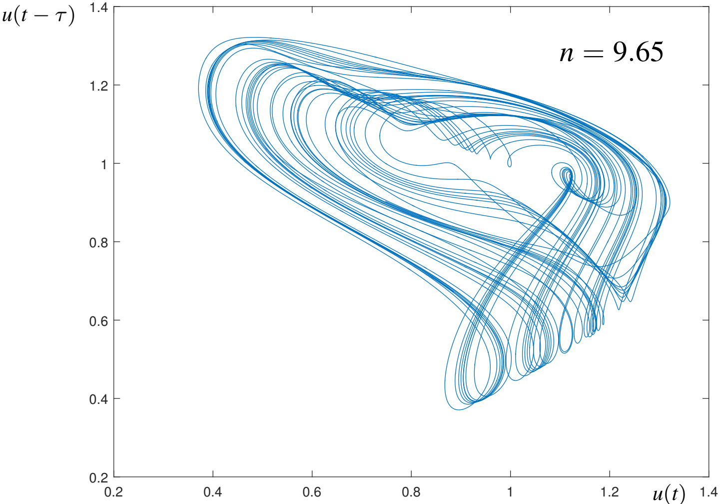
Mackey-Glass Equation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad \tau = \beta = 2,\ \gamma = 1$$



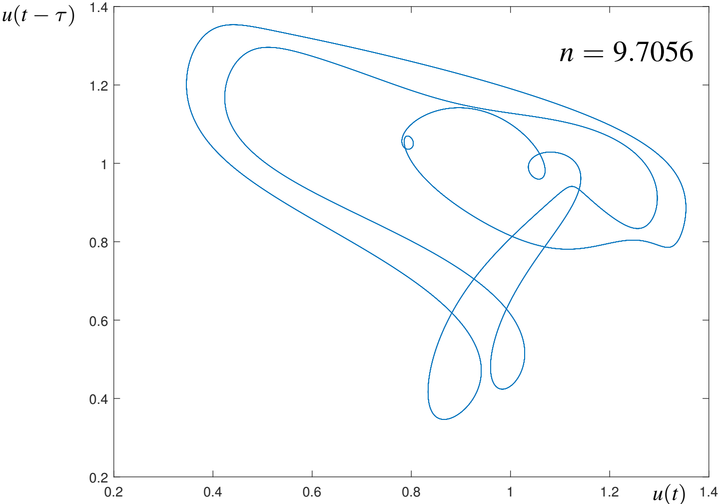
Mackey-Glass Equation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad \tau = \beta = 2,\ \gamma = 1$$



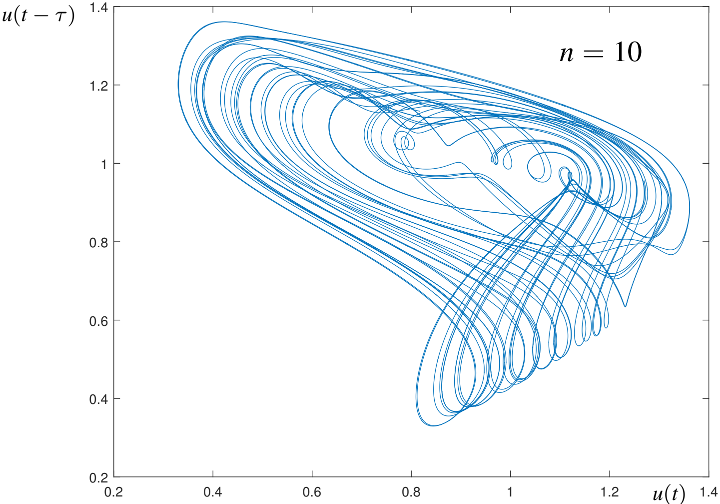
Mackey-Glass Equation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad \tau = \beta = 2,\ \gamma = 1$$



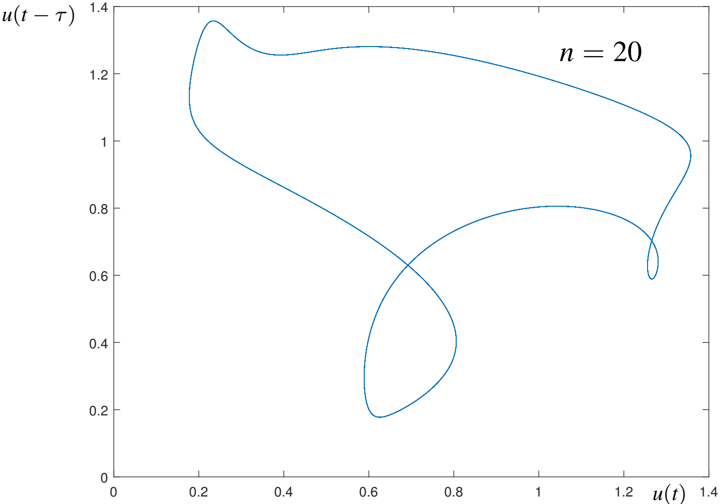
Mackey-Glass Equation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad \tau = \beta = 2, \ \gamma = 1$$



Mackey-Glass Equation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad \tau = \beta = 2,\ \gamma = 1$$



Continuation

- Previous Mackey-Glass simulations performed using `dde23`; the matlab fixed delay adaptive step-size dde solver
- It only reveals stable dynamics
- Plots show last 200 time units of a 1000 time unit integration to show asymptotic behaviour
- 20 plot points per time unit (use `deval` to evaluate solution); never use mesh points.
- Reduced tolerances: AbsTol$= 1e-8$; RelTol$= 1e-6$.

Preaching to the Converted:

We need continuation to study the bifurcations that lead to these dynamics

Steady states

Consider autonomous DDEs

Steady States

A steady-state of

$$\dot{u}(t) = f(u(t), u(t-\tau))$$

is a constant function $\varphi \in C = C([-\tau, 0, \mathbb{R}^d)$ st

$$\varphi(\theta) = u^* \ \forall \theta \in [-\tau, 0], \quad \text{s.t.} \quad f(u^*, u^*) = 0, \ \forall t \geqslant t_0$$

- Solving $f(u^*, u^*) = 0$ is equivalent to setting $\tau = 0$ and solving for fixed points of resulting ODE; just algebra.
- Stability of steady-state of DDE in C given by characteristic equation; not directly related to fixed point of ODE in $\mathbb{R}^d$ unless delay τ is small.

Mackey-Glass Steady-States and Bifurcations

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$

Setting $\dot{u}(t) = 0$ and $u(t) = u(t-\tau) = u$ gives $\gamma u(1+u^n) = \beta u$ so

Steady-states are $u = 0$ and $u = (\beta/\gamma - 1)^{1/n} > 0$ when $\beta > \gamma > 0$.
In physiology parameters and variables are non-negative.

Linearization at $u = 0$ is

$$\dot{u}(t) = -\gamma u(t) + \beta u(t-\tau),$$

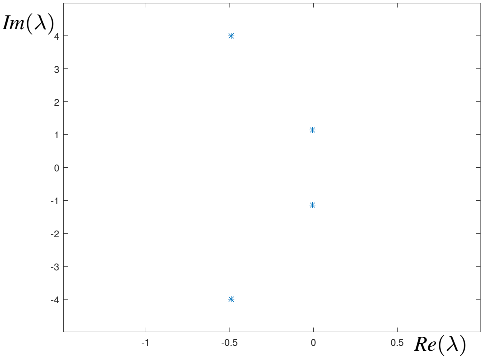
and $u = 0$ is stable for $\beta < \gamma$ ($\lambda = 0$ is a char value when $\beta = \gamma$).

Transcritical bifurcation at $u = 0$, $\beta = \gamma$.

Mackey-Glass Steady-States and Bifurcations

At non-trivial steady-state characteristic equation is

$$\lambda/\gamma + 1 - (1 - n(1 - \gamma/\beta))e^{-\lambda\tau} = 0$$



- For $\tau = \beta = 2$, $\gamma = 1$ and varying n
- Close to $n = 5$ a pair of characteristic values cross imaginary axis left to right
- Steady-state loses stability; a Hopf bifurcation occurs
- Characteristic values computed numerically using DDEBiftool.

DDEBiftool

DDEBiftool

A matlab/octave package for bifurcation analysis of DDEs

- Maintained by Jan Sieber
- Downloadable from sourceforge
- Manual also at arXiv:1406.7144
- Learn by following the demos

Getting Started

1. Download. Add to matlab path.
2. Define problem
3. Initialize a branch
4. Continue it
5. GetStability & Branch switch

Manual is not always exhaustive

DDEBiftool Set Up

Mackey-Glass with $\gamma = 1$, $\beta = 2$

```
gamma=1.0; beta_ind=1; n_ind=2; tau_ind=3;
f=@(x,xtau,beta,n)beta*xtau./(1+xtau.^n)-gamma*x;
funcs=set_funcs('sys_rhs',@(xx,p)f(xx(1,1,:)...
        ,xx(1,2,:),p(1),p(2)),...
        'sys_tau',@()tau_ind,'x_vectorized',true);
beta0=2; n0=4; tau0=2;
```

Nontrivial Steady-state branch

```
x0=(beta0-1)^(1/n0); contpar=n_ind;
nontriv_eqs=SetupStst(funcs,'x',x0,'parameter',...
 [beta0,n0,tau0],'step',0.1,'contpar',contpar,...
 'max_step',[contpar,0.1],'max_bound',[contpar,21]);
nontriv_eqs=br_contn(funcs,nontriv_eqs,1000);
nontriv_eqs=br_stabl(funcs,nontriv_eqs,0,1);
nunst_eqs=GetStability(nontriv_eqs);
ind_hopf=find(nunst_eqs<2,1,'last');
```

Hopf and Periodic Branches

Verify Bifurcation

```
evals=nontriv_eqs.point(ind_hopf).stability.l0;
plot(real(evals),imag(evals),'*')
```

Hopf and Periodic Branches

2-Follow Hopf

```
[hbranch,suc]=SetupHopf(funcs,nontriv_eqs,ind_hopf,...
 'contpar',[n_ind,tau_ind],'max_bound',...
 [n_ind,21;tau_ind,4],'dir',n_ind,'step',1e-3);
hbranch=br_contn(funcs,hbranch,200);
hbranch=br_rvers(hbranch);
hbranch=br_contn(funcs,hbranch,200);
```

Hopf and Periodic Branches

2-Follow Hopf

```
[hbranch,suc]=SetupHopf(funcs,nontriv_eqs,ind_hopf,...
 'contpar',[n_ind,tau_ind],'max_bound',...
 [n_ind,21;tau_ind,4],'dir',n_ind,'step',1e-3);
hbranch=br_contn(funcs,hbranch,200);
hbranch=br_rvers(hbranch);
hbranch=br_contn(funcs,hbranch,200);
```

1-Compute Periodic Orbit

```
[per_orb,suc]=SetupPsol(funcs,nontriv_eqs,ind_hopf,...
 'print_residual_info',1,'intervals',20,'degree',4,...
 'max_bound',[contpar,21],'max_step',[contpar,0.1]);
per_orb=br_contn(funcs,per_orb,200);
per_orb=br_stabl(funcs,per_orb,0,1);
[nunst_per,dom]=GetStability(per_orb,...
 'exclude_trivial',true);
ind_pd=find(diff(nunst_per)==1);
```

Period Doubling

2-Follow Branch of Period Doublings

```
[pdfuncs,pdbranch1,suc]=SetupPeriodDoubling(funcs,...
  per_orb,ind_pd(1),'contpar',[n_ind,tau_ind],...
  'max_bound',[n_ind,21;tau_ind,4],'dir',n_ind,...
  'step',1e-3);
pdbranch1=br_contn(pdfuncs,pdbranch1,200);
pdbranch=br_rvers(pdbranch1);
pdbranch=br_contn(pdfuncs,pdbranch,200);
```

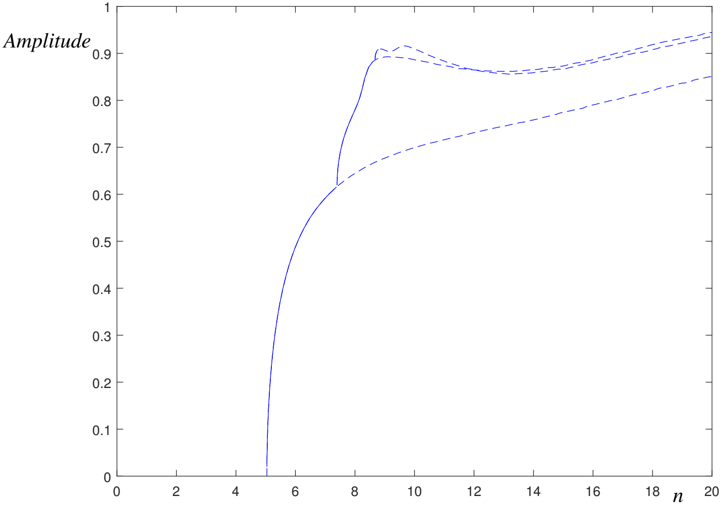
1-Continue Period-Doubled Solution

```
[per2,suc]=DoublePsol(funcs,per_orb,ind_pd(1));
per2=br_contn(funcs,per2,200);
per2=br_stabl(funcs,per2,0,1);
[nunst_per2,dom,triv_defect]=GetStability(per2,...
  'exclude_trivial',true)
```

And onwards to more period doublings

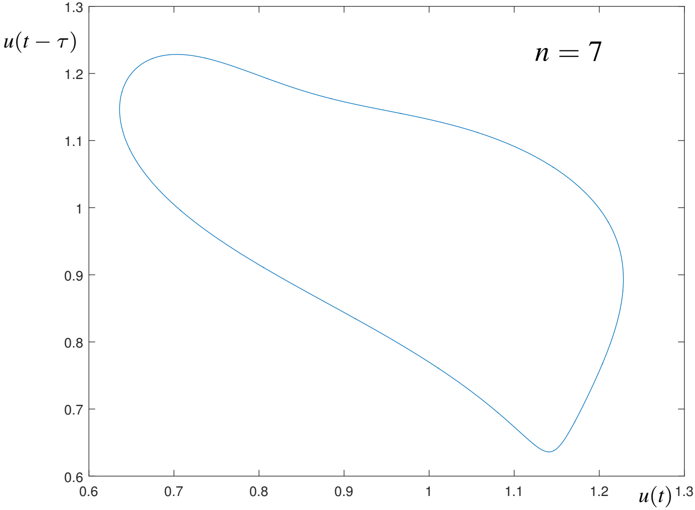
Mackey-Glass Period-Doublings

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$



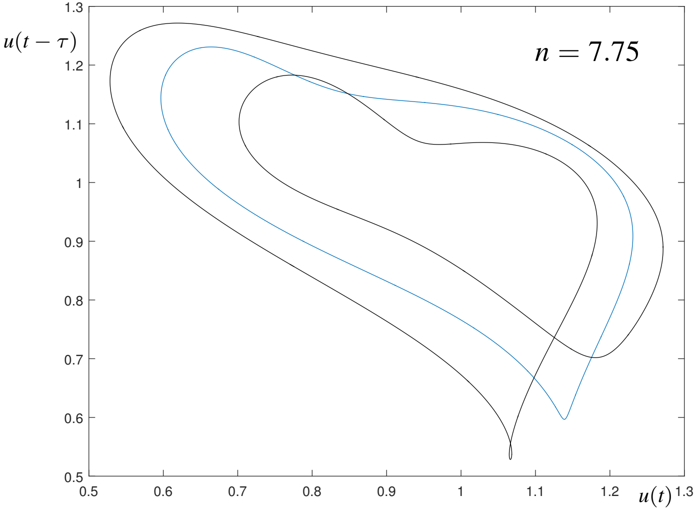
Mackey-Glass Period-Doublings

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$



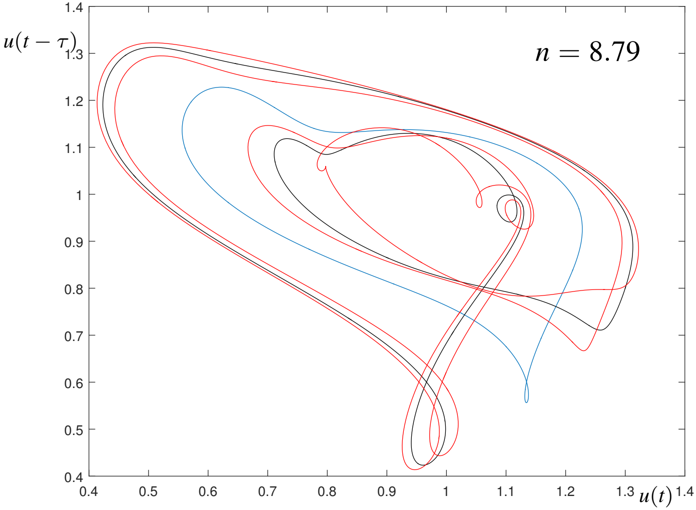
Mackey-Glass Period-Doublings

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$



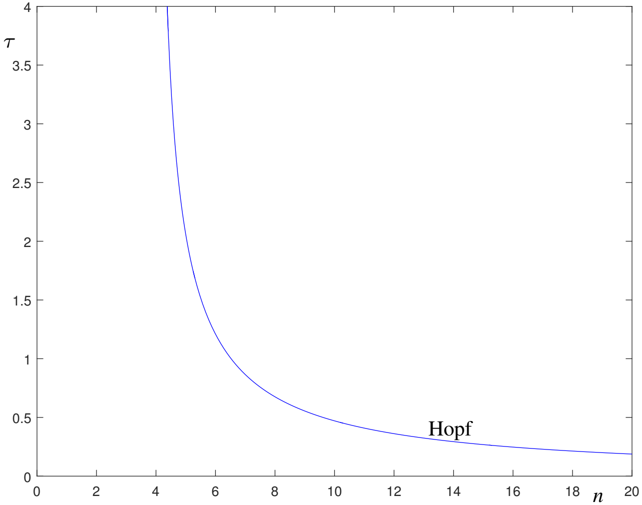
Mackey-Glass Period-Doublings

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$



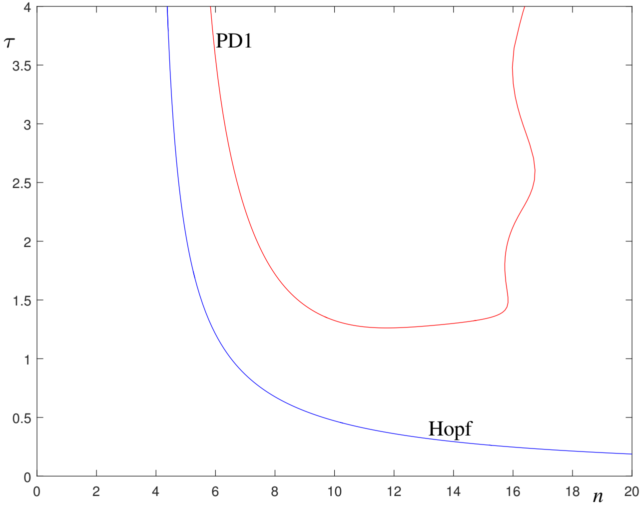
Mackey-Glass 2-Parameter Continuation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$



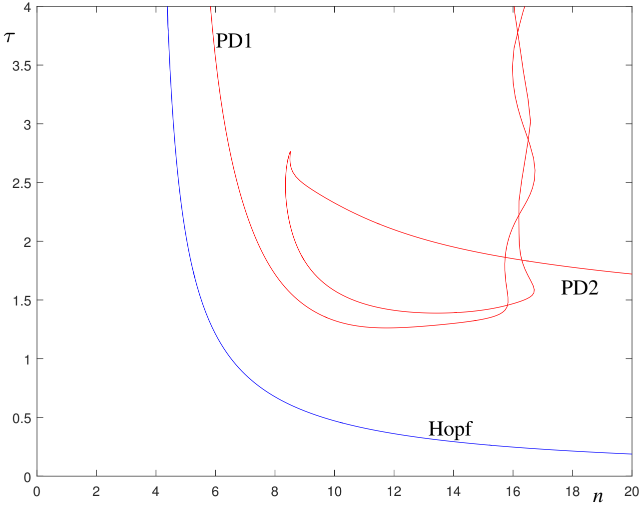
Mackey-Glass 2-Parameter Continuation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$



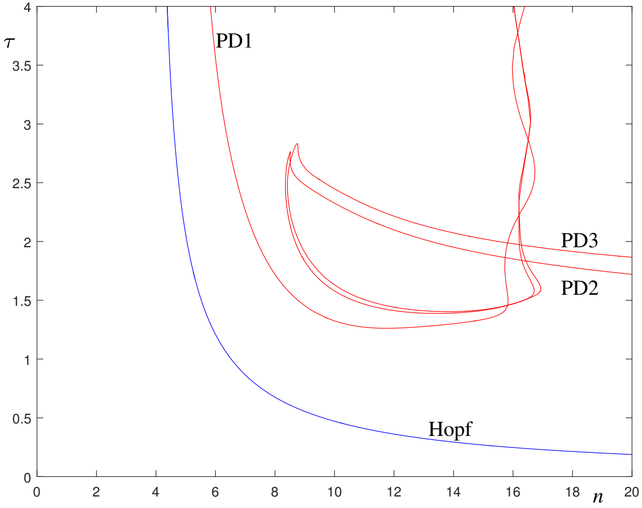
Mackey-Glass 2-Parameter Continuation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$



Mackey-Glass 2-Parameter Continuation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$



Mackey-Glass 2-Parameter Continuation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1+u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$