Delay Differential Equations

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Disclaimer

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DDEs

Definition

A Delay Differential Equation (DDE) is a differential equation where the state variable appears with delayed argument.

This can manifest itself in many ways. Simplest scenario is

Constant Delay DDE

$$\dot{u}(t) = f(t, u(t), u(t-\tau)), \qquad u(t) \in \mathbb{R}^d$$

where delay $\tau > 0$ is constant.

Example: Mackey-Glass Equation

$$\dot{u}(t) = -\gamma u(t) + \beta \frac{u(t-\tau)}{1 + u(t-\tau)^n}, \qquad u(t) \in \mathbb{R}$$

very early model of circulating white blood cell numbers
A scalar equation with chaotic dynamics

Variable Delays

DDEs with Time-Dependent Delay

$$\dot{u}(t) = f(t, u(t), u(t - \tau(t))), \qquad u(t) \in \mathbb{R}^d$$

where delay $\tau(t) \geqslant 0$ is a given function.

Example: Pantograph Equation

$$\dot{u}(t) = au(t) + bu(kt), \qquad u(t) \in \mathbb{R}$$

where a, b and k are parameters with $k \in (0, 1)$.
Here $kt = t - \tau(t)$ with $\tau(t) = (1 - k)t$.
Originates from modelling pantographs!

Relatively complete theory/numerics developed for constant and time-dependent delays.

State-Dependent Delays

DDEs with State-Dependent Delay

$$\dot{u}(t) = f(t, u(t), u(t - \tau(t, u(t)))), \qquad u(t) \in \mathbb{R}^d$$

where delay $\tau(t, u(t)) \geqslant 0$ depends on solution.

Example: Sawtooth Equation

$$\varepsilon \dot{u}(t) = -\gamma u(t) - \kappa u(t - a - cu(t)), \qquad u(t) \in \mathbb{R}$$

with $\varepsilon > 0$, $a > 0$, $c > 0$ and $\gamma + \kappa > 0$.
Model problem introduced by Mallet-Paret and Nussbaum, gets its name from stable period solutions seen in $\varepsilon \to 0$ limit.
$\tau(t, u(t)) = a + cu(t)$ is a delay provided $u(t) \geqslant -a/c$.
Fortunately if $u(t) = -a/c$ then $\dot{u}(t) = -(\gamma + \kappa)u(t) > 0$.

State-Dependent Delays subject of much current research

Delay Equations we won't solve

$\exists$ many other types of delay equations

Neutral Equations

Equation is neutral if derivatives of delay terms appear

$$\dot{u}(t) = f(t, u(t), u(t-\tau_1), \dot{u}(t-\tau_2)), \qquad u(t) \in \mathbb{R}^d$$

These are nasty: ongoing research area

The τ and τ_i above are called *discrete delays*.

Distributed Delay Examples

Finite distributed delay: $\dot{u}(t) = \int_{t-\tau}^{t} f(s, u(s))\, ds$

Infinite delay:

$$\dot{u}(t) = f\left(\int_{-\infty}^{t} u(s) g_a^n(t-s)\ ds\right), \quad g_a^n(x) = \frac{1}{\Gamma(n)} a^n x^{n-1} e^{-ax}$$

Nastier Stuff

Threshold Conditions

Delays can be defined implicity: $\int_{t-\tau}^{t} V(u(s))ds = a$

a is given constant, $V(\cdot)$ a given function; τ must be determined.
[Leibniz can help or hinder here]

Implicit delays also appear in electrodynamics:

Wheeler-Feynman Electrodynamics

proton $p(t)$ and electron $e(t)$ interact through light cones in space-time.

$$\tau_p^{\pm} = \frac{1}{c}\|p(t) - e(t \pm \tau_p^{\pm})\|,$$

Neutral equation, with advanced and retarded terms, and implicit delays that stumped Feynman.

DDE Initial Value Problems (IVPs)

Constant Delay DDE IVP

$$\dot{u}(t) = f(t, u(t), u(t-\tau)), \qquad u(t) \in \mathbb{R}^d, t \geqslant t_0$$

For unique IVP solution for $t \geqslant t_0$

- it is *not* sufficient to specify $u(t_0)$
- To evaluate RHS at t_0 require $u(t_0 - \tau)$
- $\forall s \in [t_0 - \tau, t_0]$ require a value of $u(s)$ to evaluate RHS of DDE at $t = s + \tau \in [t_0, t_0 + \tau]$.

For uniqueness of IVP solution need an initial function

$$u(t) = \varphi(t), \quad \forall t \in [t_0 - \tau, t_0]$$

Provided φ is Lipschitz and $f = f(t, u, v)$ is Lipschitz in its arguments this is sufficient for local existence and uniqueness even in state-dependent case.

Breaking Points and Smoothing

$$\dot{u}(t) = f(t, u(t), u(t-\tau)), \quad t \geqslant t_0$$
$$u(t) = \varphi(t), \qquad t \in [t_0 - \tau, t_0]$$

Breaking Point at t_0

Usually $\quad \dot{\varphi}(t_0) \neq f(t_0, \varphi(t_0), \varphi(t_0 - \tau))$
so $\dot{u}(t_0^-) \neq \dot{u}(t_0^+)$. This is a *breaking point*.

Breaking Points at $t_0 + k\tau$

$$\ddot{u}(t) = f_t(t, u(t), u(t-\tau)) + \dot{u}(t) f_u(t, u(t), u(t-\tau)) + \dot{u}(t-\tau) f_v(t, u(t), u(t-\tau)).$$

So $\ddot{u}$ generically discontinuous at $t_0 + \tau$ and similarly,
$u^{(k+1)}(t)$ discontinuous at $t = t_0 + k\tau$ for $k \geqslant 0$.

- Smoothing: $u(t) \in C^{k+1}$ for $t \geqslant t_0 + k\tau$
- *No* such smoothing for neutral problems

DDEs as Dynamical Systems

Phase space of DS is set of (initial) states of system:

$$\{u_t : u_t(\theta) = u(t+\theta),\ \theta \in [-\tau, 0]\}$$

But for $t \in (t_0, t_0+\tau)$ $\exists\, \theta \in (-\tau, 0)$ s.t. $t+\theta = t_0$.
$u_t(\theta)$ is not differentiable at this θ.

Phase Space of continuous functions

$$\{\varphi : \varphi \in C([-\tau, 0], \mathbb{R}^d)\}$$

Phase space is infinite dimensional even for scalar $d = 1$ problems

Retarded Functional Differential Equations

$$\dot{u}(t) = F(t, u_t), \qquad F : \mathbb{R} \times C \to \mathbb{R}^d$$

- Lack of differentiability is a serious hindrance to theory

Linearization for Autonomous Constant Delay DDEs

Scalar Example

Suppose $f(u,v)$ satisfies $f(0,0)=0$ so $u=0$ is a steady state then

$$\dot{u}(t)=f(u(t),u(t-\tau))=f_u(0,0)u(t)+f_v(0,0)u(t-\tau)+h.o.t$$

and linearization is

$$\dot{u}(t)=f_u(0,0)u(t)+f_v(0,0)u(t-\tau)=\mu u(t)+\sigma u(t-\tau)$$

Positing $u(t)=e^{\lambda t}$ gives transcendental *characteristic equation*

$$\lambda-\mu-\sigma e^{-\tau\lambda}=0.$$

Let $\lambda=x+iy$ and take real and imaginary parts:

$$x-\mu-\sigma e^{-\tau x}\cos(y\tau)=y+\sigma e^{-\tau x}\sin(y\tau)=0$$

Infinitely many roots, all lie on curve $y=\pm\sqrt{\sigma^2 e^{-2\tau x}-(x-\mu)^2}$

- Laplace transforms show all solutions are exponentials
- Finitely many roots to right of any vertical line in $\mathbb{C}$;
- All characteristic roots satisfy $x<|\mu|+|\sigma|$
- Stable manifolds is infinite dimensional

Linerization for DDEs in $\mathbb{R}^d$

$$\dot{u}(t) = f(u(t), u(t-\tau_1), \ldots, u(t-\tau_m))$$

Let $f(u, v_1, \ldots, v_m) : \mathbb{R}^d \times \mathbb{R}^{md} \to \mathbb{R}^d$ satisfy $f(0, 0, \ldots, 0) = 0$, so $u = 0$ is a steady state.
Linearization is variational equation

$$\dot{u}(t) = A_0 u(t) + \sum_{j=1}^m A_j u(t-\tau_j),$$

where $A_0 = f_u$ and $A_j = f_{v_j}$ are $d \times d$ matrices evaluated at the steady-state (essentially a Jacobian matrix for each 'delay').
There is nontrivial solution $u(t) = e^{\lambda t}\underline{v} \in \mathbb{R}^d$ with $\Delta(\lambda)\underline{v} = 0$ if

$$0 = \det(\Delta(\lambda)), \quad \Delta(\lambda) = \lambda I_d - A_0 - \sum_{j=1}^m A_j e^{-\lambda\tau_j}.$$

- Characteristic equation has infinitely many roots
- Variational equation soln: $u(t) = \sum_i \alpha_i e^{\lambda_i t}\underline{v}_i$
- Finitely many λ_i with $Re(\lambda_i) > \beta$ for any $\beta \in \mathbb{R}$.
- State-dependent DDEs are linearized by freezing the delays

Simple Numerical Methods

Method of steps

For $t \in [t_0, t_0+\tau]$: $\dot{u}(t) = f(t, u(t), u(t-\tau)) = f(t, u(t), \varphi(t-\tau))$
Solve as nonautonomous ODE for $t \in [t_0, t_0+\tau]$. Repeat.

- Tedious if there is a small delay, and fails if a delay vanishes (recall pantograph equation)

Sub-multiple step-sizes

Most methods for ODEs generate sequence u_n where $u_n \approx u(t_n)$ and $t_n = t_0 + n\Delta t$ or with variable step-size $t_{n+1} = t_n + \Delta t_{n+1}$.
For constant delay DDE with single delay τ could try favourite numerical method with constant step-size $\Delta t = \tau/m$ for some integer m. Then if $u_n \approx u(t_n)$ we have $u_{n-m} \approx u(t_n - \tau)$

- Fails if delays are variable or state-dependent, if variable step-size is desired for accuracy, if there are multiple non-commensurate constant delays.

Continuous Runge-Kutta Methods

RK Methods for ODEs

Let $u_n \approx u(t_n)$ where $\dot{u}(t) = f(t, u(t)), \quad u(t_0) = u_0,$

Standard RK method defines s intermediate stages per step

$$Y_i = u_n + h \sum_{j=1}^{s} a_{ij} K_j, \quad K_i = f(t_n + c_i h, Y_i), \quad i = 1, \ldots, s$$

$$u_{n+1} = u_n + h \sum_{i=1}^{s} b_i K_i,$$

(think of $Y_i \approx u(t_n + c_i h)$, typically $c_i \in [0, 1]$)

Continuous RK Methods

Define functions $b_i(\theta)$ & let

$$\eta(t_n + \theta h) = u_n + h \sum_{i=1}^{s} b_i(\theta) K_i, \quad \theta \in [0, 1]$$

defines continuous extension of solution for $t \in [t_n, t_{n+1}]$.
Proceed step by step.

- If $a_{ij} = 0$ for $j \geqslant i$ method is explicit
- For (implicit) collocation RK methods $b_i(\theta)$ defined naturally

Continuous Runge-Kutta Methods (CRKs) for DDEs

$$\dot{u}(t) = f(t, u(t), u(t-\tau)), \qquad u(t) = \varphi(t), \ t \in [t_0 - \tau, t_0].$$

Continuous Runge-Kutta (CRK) method is defined by

$$Y_i^{(n)} = u_n + h \sum_{j=1}^{s} a_{ij}\, f(t_n + c_j h, Y_j^{(n)}, \tilde{Y}_j^{(n)}), \qquad \tilde{Y}_j^{(n)} = \eta(t_n + c_j h - \tau),$$
$$u_{n+1} = u_n + h \sum_{j=1}^{s} b_j(1)\, f(t_n + c_j h, Y_j^{(n)}, \tilde{Y}_j^{(n)})$$

$\tilde{Y}_j^{(n)}$ called *spurious stages*, $\eta(t)$ is the *continuous extension* of numerical solution:

$$\eta(t_m + \theta h) = u_m + h \sum_{j=1}^{s} b_j(\theta) f(t_n + c_j h, Y_j^{(m)}, \tilde{Y}_j^{(m)}), \quad m \leqslant n, \ \theta \in [0, 1]$$

- defines numerical solution as a continuous function for $t \in [t_0, t_0 + T]$.
- if $a_{ij} = 0$ for $j \geqslant i$ & $h \leqslant \tau$ method is explicit
- $b_i(\theta)$ polynomial (of degree equal to order of method)
- $h > \tau$ called *overlapping*. Generalised methods can be explicit
- Spurious stages ruin super-convergence of quadrature methods