Looking under the hood of DDE-Biftool

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Plan

- how to continue unstable branches of equilibria and periodic orbits in experiments
  Lecture 1
  - motivating examples
  - short intro to concepts for feedback control
- **Lecture 2** detailed examples (simple mechanical experiments)
  + periodic orbits close to Hopf bifurcation
- ⇒**Lecture 3** Continuation for delay-differential equations (mostly by Tony): looking under the hood
- **Afternoons & hands-on workshop** help with DDE-Biftool (for delay equations), AUTO, coco
- (for myself) learn more coco

Purpose

- point out where methods are different from ODE methods
- often less reliable than ODE methods

⇒ good to know why things fail

Alternative

knut (developed and maintained by Robert Szalai (Univ. Bristol)

- stand-alone C++ program with graphical user interface
- can also perform analysis of neutral delay-differential equations

$\Longrightarrow$review article Dirk Roose & Robert Szalai in red book

History

- developed by K. Engelborghs (KU Leuven, Belgium), as a PhD project under supervision of Dirk Roose
- later additions from Leuven (version 2.x):
  - connecting orbits between saddle equilibria (Giovanni Samaey)
  - alternative heuristics for linear stability (Koen Verheyden)

  was octave compatible
- version 3.x interface and internal changes (JS),
- extensions:
  - local bifurcations of periodic orbits (JS)
  - systems with rotation symmetry (JS)
  - normal form computations for equilibria (B. Wage, Y. Kuznetsov from Utrecht, Netherlands)

Type of DDEs

- DDE has form

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_k), p)$$

- Either τ_j are parameters, or functions

$$\begin{aligned} \dot{x}(t) &= f(x_0, \ldots, x_k, p) \\ x_0 &= x(t) \\ x_j &= x(t - \tau_j(x_0, \ldots, x_{j-1}), p)) \quad (j = 1 \ldots k) \end{aligned}$$

depending on previously defined delayed states.

Example

$$\dot{x}(t) = \mu - x(t - x(t - x(t - x(t))))$$

```
ntau=3;
funcs=set_funcs(...
    'sys_rhs',  @(x,p)      p-x(1,ntau+1,:),...
    'sys_ntau', @()         ntau,...
    'sys_tau',  @(nr,x,p)   x(1,nr,:),...
    'x_vectorized',true);
```

- equilibrium $x = \mu$ has Hopf bifurcation at $\mu = \pi/2$
- periodic orbits have fold
- arbitrary levels of nesting needed for bifurcations of periodic orbits

Linear stability of equilibria

$$0 = f(x, \ldots, x, p)$$

- Consider $\tilde{x} \in C([-\tau_{\max}, 0]; \mathbb{R}^n)$, $h = \tau_{\max}/N$.
- discretise: single time step size h of integration method $x^{(k)} = \tilde{x}(-kh)$, $k = 0 \ldots N$
- linear large eigenvalue problem

$$\mu x^{(0)} = x^{(0)} + h[A_0 x_0 + \ldots + A_k x_k] \quad \text{(Euler)}$$
$$\mu x^{(k)} = x^{(k-1)} \quad k = 1 \ldots N$$

  where

$$x_j = \text{interpolation of } \tilde{x} \text{ at } -\tau_j, \qquad A_j = \partial_j f(x, \ldots, x, p)$$

- Then $\mu = \exp(h\lambda)$.

Linear stability of equilibria

- use higher order multi-step method instead of Euler
- **How to choose** h?
  - large h $\Rightarrow$poor accuracy
  - small h $\Rightarrow$large matrix
- heuristics: h such that λ with $\mathrm{Re}\lambda > -r_{\min}$ accurate
- requires a-priori estimates where $\lambda \in \mathbb{C}$ lie
  - $\det[\lambda I - A_0 - A_1 e^{-\lambda\tau_1} - \ldots - A_k e^{-\lambda\tau_k}] = 0$
  - $|A_j e^{-\lambda\tau_j}| \le |A_j e^{r_{\min}\tau_j}|$

$\Rightarrow$ Engelborghs, Roose, Luzyanina, Breda

Linear stability of equilibria

- heuristics fails often
- $r_{\min}$ not specified by user $\Rightarrow r_{\min} = 1/\tau_{\max}$
- mean examples:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau)$$

  where τ large, A_1 small (lasers)
- sometimes also failure for $0 \leq \tau \ll 1$

Local bifurcations of equilibria

- Hopf bifurcation:

$$\begin{aligned} 0 &= f(x, \ldots, x, p) && \text{(equilibrium)} \\ i\omega v &= [A_0 + A_1 \mathrm{e}^{-i\omega\tau_1} + \ldots + A_k \mathrm{e}^{-i\omega\tau_k}]v && \text{(Hopf)} \\ 1 &= v_{\mathrm{ref}}^H v && \text{(fix } v\text{)} \end{aligned}$$

- $A_j = \partial_j f(x, \ldots, x, p)$
- variables $x \in \mathbb{R}^n, v \in \mathbb{C}^n, \omega \in \mathbb{R}$; $n + 2n + 2$ equations

$\Rightarrow$ $p \in \mathbb{R}$ for Newton iteration
$p \in \mathbb{R}^2$ for branch continuation

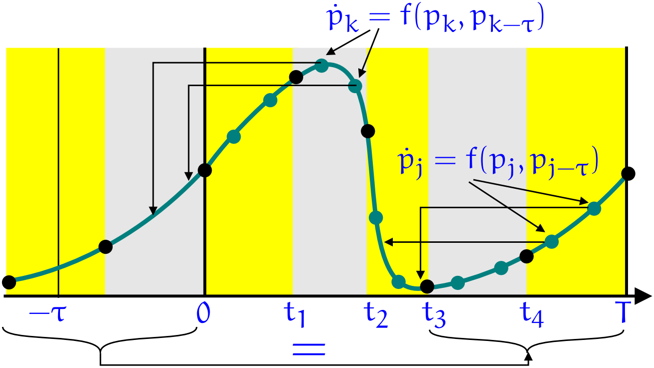
- fold identical to ODE case

Periodic orbits

periodic boundary-value problem:

$$\dot{x} = Tf\left(x, x(t-\tau/T)_{\mathrm{mod}[0,1]}\right), \quad x(0) = x(1)$$

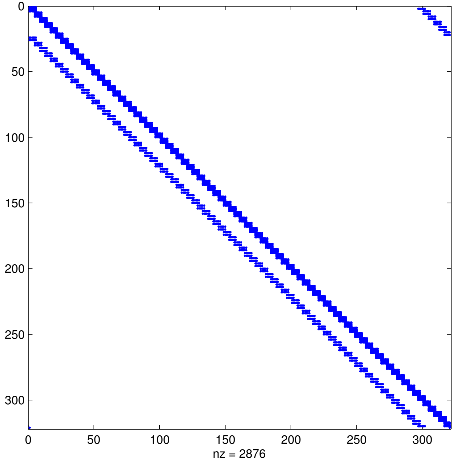
collocation on $[0,1]$: N polynomials p_k, degree d
⇒1d problem



Periodic orbits

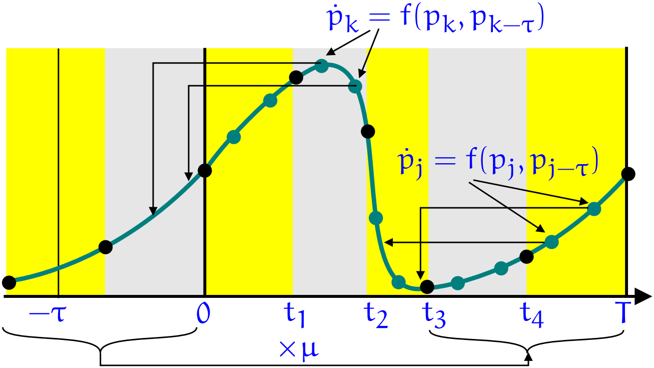
collocation on $[0,1]$: N polynomials p_k, degree d
(no super-convergence)
$\Longrightarrow$nonlinear system for $n\,N\,d$ variables

structure of Jacobian J

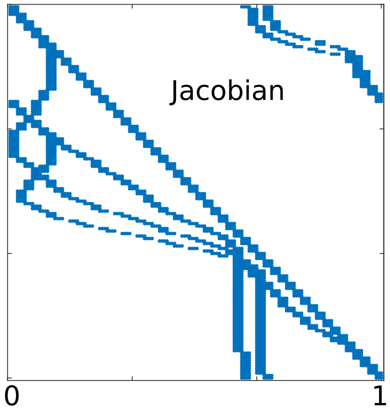


Stability of periodic orbits

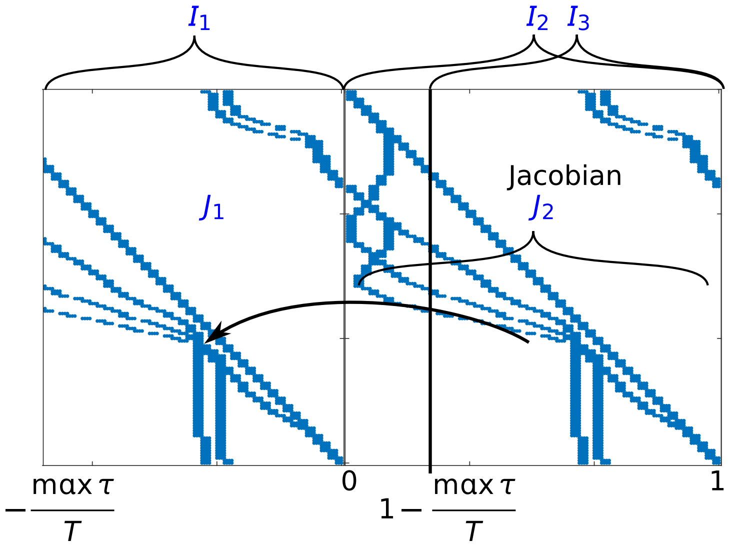
Determined by Floquet multipliers of linearised periodic problem.
Jacobian J from Newton iteration can be re-used:



Stability of periodic orbits



Stability of periodic orbits



$$0 = J_1 y(I_1) + J_2 y(I_2)$$

$$0 = \mu y(I_1) - y(I_3)$$

Local bifurcations of periodic orbits

Set up as extended periodic boundary-value problems, e.g., torus bifurcation (for single delay)

$$\dot{x} = f(x(t), x(t-\tau), p)$$

$$\dot{z} = \frac{\mathrm{i}\pi\omega}{T} z + A_0(t) z + A_1(t) \mathrm{e}^{-i\pi\omega\tau} z(t-\tau)$$

where

$$A_j(t) = \partial_{j+1} f(x(t), x(t-\tau), p)$$

extended by

$$1 = \int_0^T z(t)^H z(t) \mathrm{d}t, \quad 0 = \int_0^T \mathrm{Re} z(t)^T \mathrm{Im} z(t) \mathrm{d}t$$

variables $x \in \mathbb{R}^n$, $z \in \mathbb{C}^n$, $\omega \in \mathbb{R}$; $n + 2n + 2$ equations
$\Rightarrow p \in \mathbb{R}$ for Newton iteration
$\Rightarrow p \in \mathbb{R}^2$ for branch continuation

Normal forms for equilibrium bifurcations

(by Y. Kuznetsov & B. Wage)

- Hopf bifurcation: sign of Lyapunov coefficient ℓ_1 determines stability

⇒ DDE near Hopf bifurcation on center manifold reduced to

$$\dot{z} = (p + i\omega)z + \ell_1 |z|^2 z \qquad (z \in \mathbb{C})$$

- codimension-two bifurcations come in various types, classified by their normal forms,
- require expansion of $f(x_0, x_1, \ldots, x_k, p)$ to order $3 \ldots 5$ in x_j.

Bifurcations with rotational symmetry

- one big application: lasers
- structure: $A = -A^T \in \mathbb{R}^{n\times n}$

$$\dot{x} = f(x(t), x(t-\tau), p),$$
$$e^{A\phi} f(x, y, p) = f(e^{A\phi} x, e^{A\phi} y, p)$$

solution types:

$x(t) = e^{A\omega t} x_0,$ ⇒rotations (relative equilibria)

$x(t) = e^{A\omega t} x_p(t),$

x_p T-periodic ⇒relative periodic orbits

⇒ ω is always additional variable to be solved for

⇒ one more phase condition

Future & support

- more normal form support by Y. Kuznetsov & co-workers
- coco port to enable
  - multi-dimensional atlas continuation
  - coupling between DDEs and other problems
  - benefit from nonlinear solvers
  - cleanup
- support of DDE-Biftool (trouble-shooting and bug fixing) will likely continue
- possible extensions (demand and feasibility)
  - support for neutral equations, delay-differential-algebraic equations
  - basic bifurcations of symmetric systems