Continuation and bifurcation analysis in experiments II

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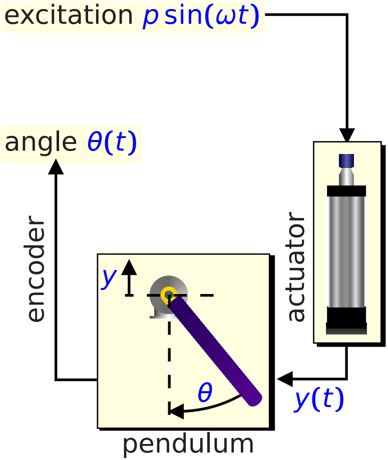
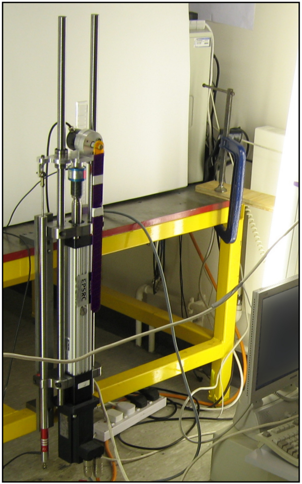
Plan

- how to continue unstable branches of equilibria and periodic orbits in experiments
  Lecture 1
  - motivating examples
  - short intro to concepts for feedback control
- ⇒**Lecture 2** detailed examples (simple mechanical experiments)
  + periodic orbits close to Hopf bifurcation
- **Lecture 3** Continuation for delay-differential equations (mostly done by Tony): looking under the hood
- **Afternoons & hands-on workshop** help with DDE-Biftool (for delay equations), AUTO, coco
- (for myself) learn more coco

Example: rotation in pendulum

Schema





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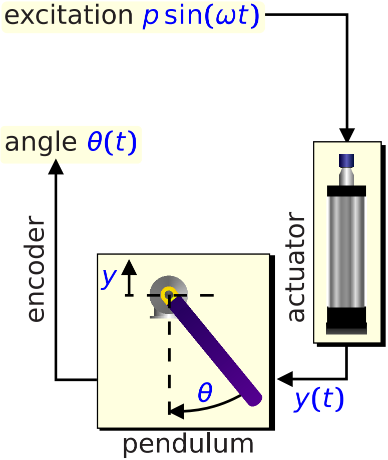
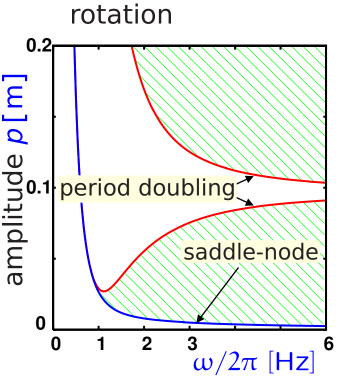


Numerics



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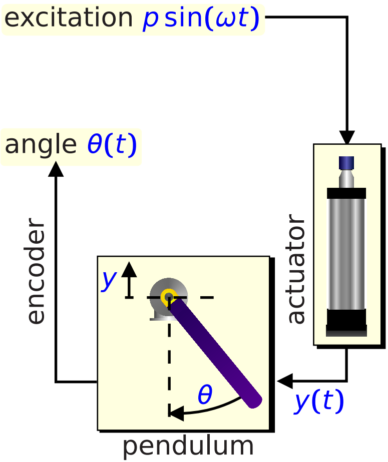

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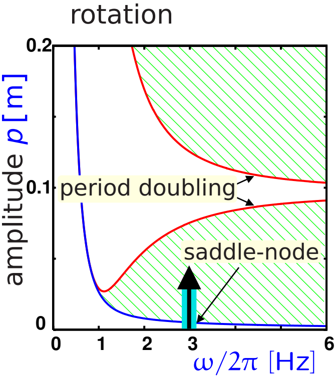

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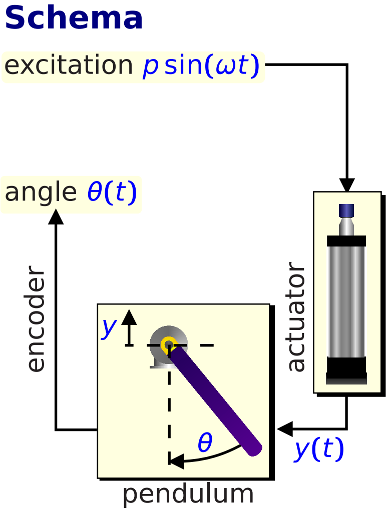

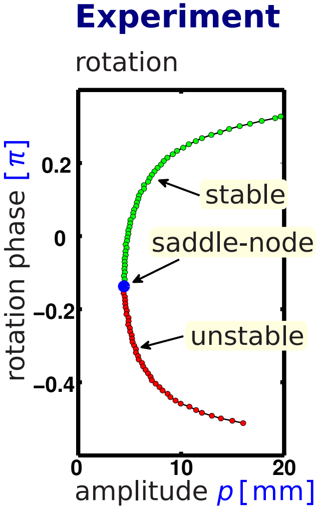

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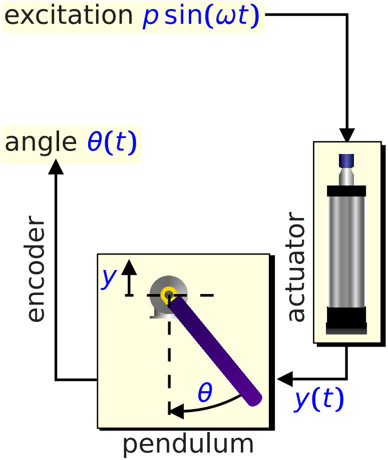
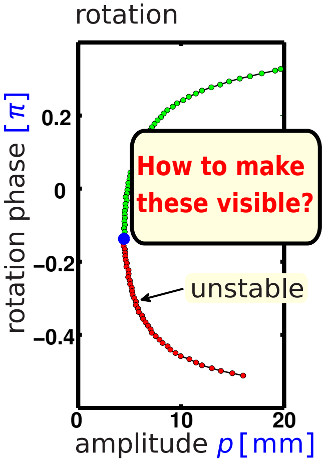


Experiment



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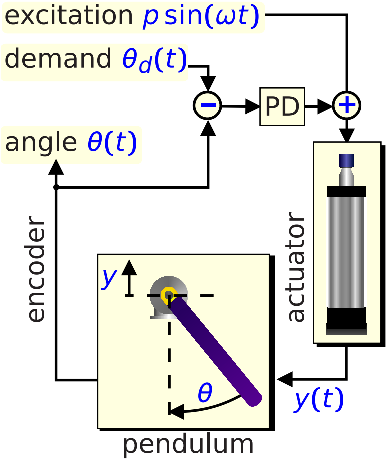


Control

- feed back $\theta - \theta_d$

⇒ always stable

⇒ output not natural

- How to choose θ_d?

⇒ find θ_d
such that $\theta_d - \theta = 0$

Computational side

Unknowns: θ_d (Fourier modes),
p (parameter)

"Equation": $\theta_d - \theta[\theta_d, p] = 0$

$\Rightarrow$ **use Newton iteration**

Newton iteration requires function $\mathrm{eqs}(\theta_d, p)$:

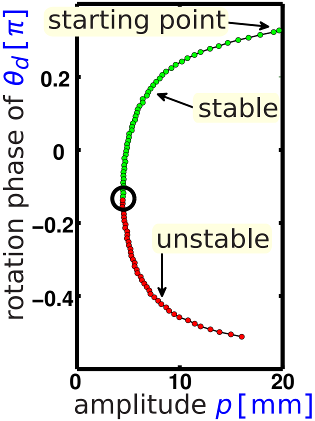
- set control demand to θ_d, excitation amplitude p
- wait until output θ is periodic again
- return Fourier modes of $\theta - \theta_d$

\+ **pseudo-arclength condition**

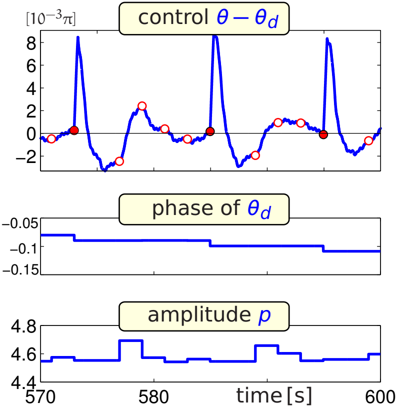
Experimental results

Experiment

bifurcation diagram



time profile of continuation



Experimental results

video

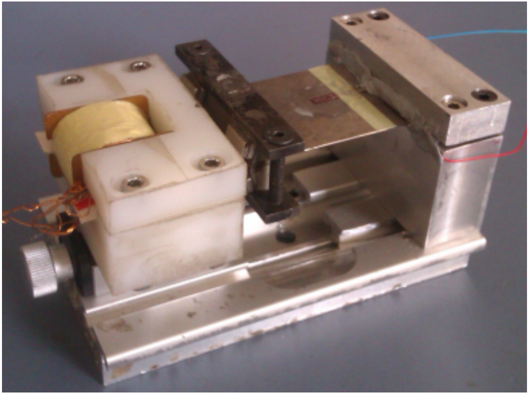
Comments

- model accurate except for damping, but damping has large influence on bifurcation diagram
- true dimension of nonlinear system for Newton iteration: 1 (phase shift ϕ)

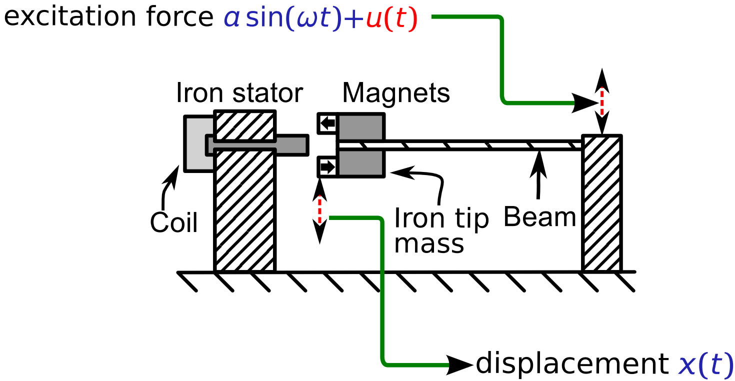
  other harmonics found with "time-delayed feedback"/Picard iteration

David Barton's results (Bristol)

Nonlinear Energy Harvester



David Barton's results (Bristol)



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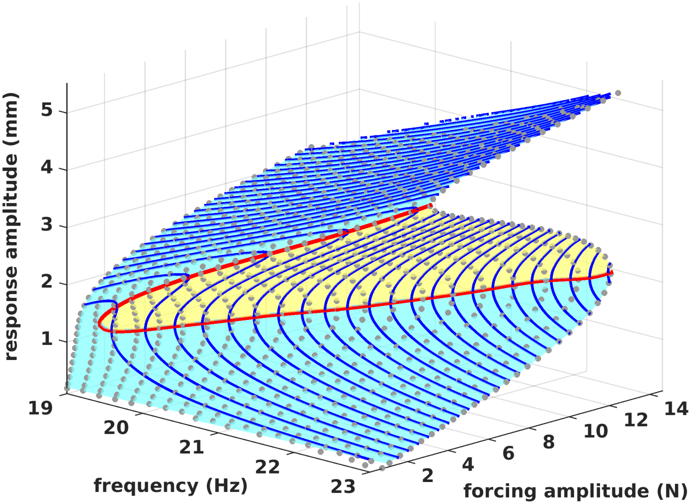

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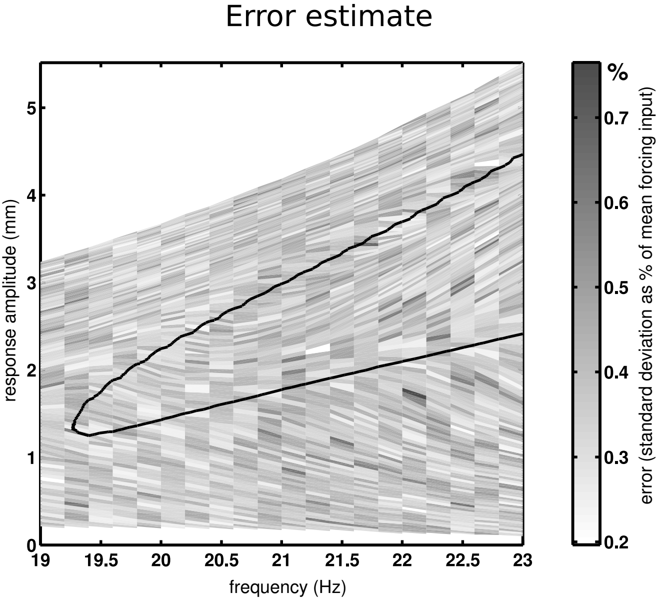


Computational example for Hopf bifurcation

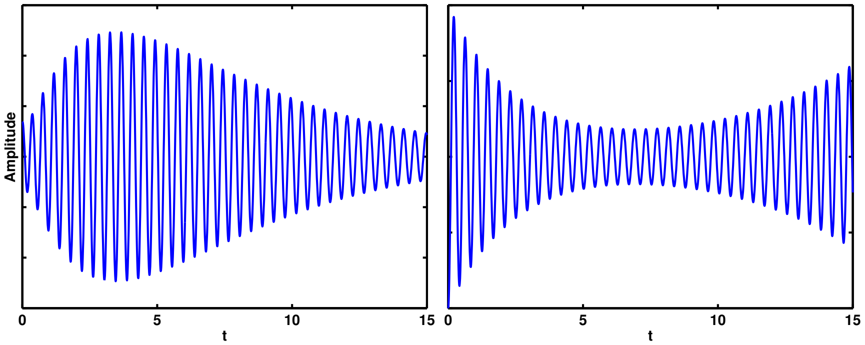
- Haken-Kelso-Bunz (HKB) oscillator:

$$\ddot{x} + \dot{x}(\alpha x^2 + \beta \dot{x}^2 - \gamma) + \omega^2 x = u + \text{“noise”}$$

- nonlinearity:
  mixture of nonlinear damping terms: $\dot{x}(\alpha x^2 + \beta \dot{x}^2)$
- Equilibrium at $x = 0$, bifurcation parameter γ
  - stable for $\gamma < 0$,
  - unstable for $\gamma > 0$,
  - Hopf bifurcation at $\gamma = 0$
- fix $\alpha = 1$, $\omega = 2$,
- $\beta = -0.1$ (small periodic orbits unstable), or
  $\beta = \quad 0.2$ (small periodic orbits stable)

General problems for parameter scan

- Do transient oscillations decay or grow (slowly)?
- Non-normality: below are two linear systems



Effect of noise/disturbance in parameter scan

- stable Hopf bifurcation:

  noise-induced fluctuations gradually grow in amplitude and become more coherent

- unstable Hopf bifurcation:

  same, but at some point before Hopf bifurcation escape occurs.

Matlab demo

Hopf bifurcation as regular root

- assume that we have entire state as output (x, $\dot{x}$)
- set (PD control, stabilising for k_1, k_2 large)

$$u = -k_1(x - x_{\mathrm{ref}}(t)) - k_2(\dot{x} - \dot{x}_{\mathrm{ref}}(t)) \quad \text{where}$$

$$x_{\mathrm{ref}}(t) = \delta \sin(2\pi t/T) \qquad (\delta \text{ small}).$$

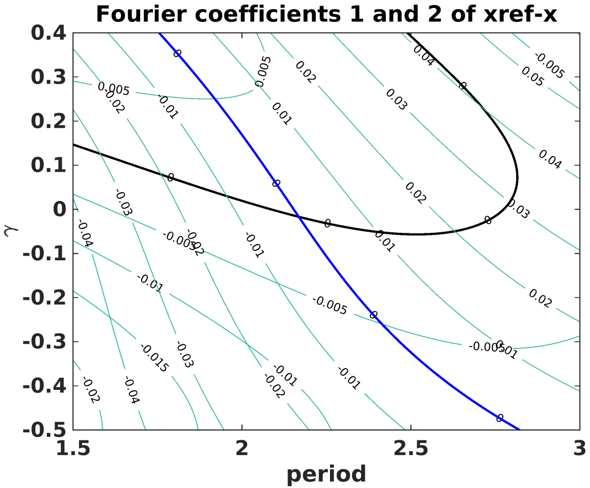
- Check for $x(t)$ after transients have settled:

$$f_1(T, \gamma) = \int_0^T \sin(2\pi t/T)[x(t) - x_{\mathrm{ref}}(t)]\mathrm{d}t$$

$$f_2(T, \gamma) = \int_0^T \cos(2\pi t/T)[x(t) - x_{\mathrm{ref}}(t)]\mathrm{d}t$$

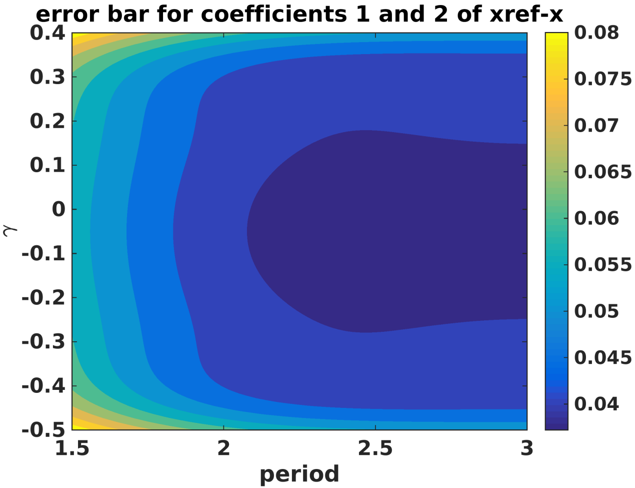
- If $f_{1,2}(T, \gamma)$ has regular root ⇒uncontrolled system has periodic orbit of amplitude δ, period T at parameter γ.

Hopf bifurcation as regular root



⇒small-amplitude periodic orbit present

Hopf bifurcation as regular root



⇒small-amplitude periodic orbit present

Hopf bifurcation as regular root

matlab picture

Comments:

- transients always finite
- result depends on gains
  becomes independent of gains as noise level $\rightarrow 0$.
- Hopf bifurcation defined in limit noise level $\rightarrow 0$.

Further remarks about contination in experiments

Compared to numerical contination

1. evaluation of $F : (x_{\text{ref}}, p) \to x$ is slow

⇒ restriced to low dimension of control inputs and small number of (eg) Fourier modes

2. low accuracy of F (relative error $\approx 10^{-2}$ in very clean experiments)

⇒ restricted to well-conditioned problems

⇒ F. Schilder's `coco` toolbox `continex`

3. limiting factor: ability to provide stabilising (!) real-time feedback

⇒ hardest part is problem specific

⇒new algorithms needed