Continuation and bifurcation analysis in experiments

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Plan

- how to continue unstable branches of equilibria and periodic orbits in experiments
  Lecture 1
  - motivating examples
  - short intro to concepts for feedback control
- **Lecture 2** detailed examples (simple mechanical experiments)

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- **Lecture 3** Continuation for delay-differential equations (mostly done by Tony): looking under the hood

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  - short intro to concepts for feedback control
- **Lecture 2** detailed examples (simple mechanical experiments)
- **Lecture 3** Continuation for delay-differential equations (mostly done by Tony): looking under the hood
- **Afternoons & hands-on workshop** help with DDE-Biftool (for delay equations), AUTO, coco
- (for myself) learn more coco

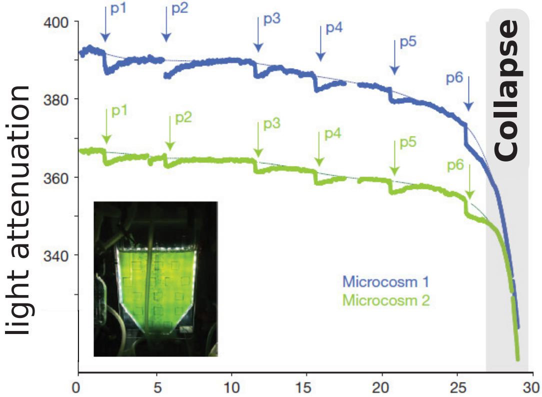
Material

DDE-Biftool (for delay)	⇒	sourceforge
slides	⇒	pdf’s
exercises (for pen&paper, computer)	⇒	pdf’s
papers & references	⇒	pdf’s
computer (matlab/octave) demos etc	⇒	source files

Motivating example I

Lab experiment on cyanobacteria
collapse of population

Veraart *et al*
Nature 2012



time (days)
(gradual increase in light stress)

Motivating example II

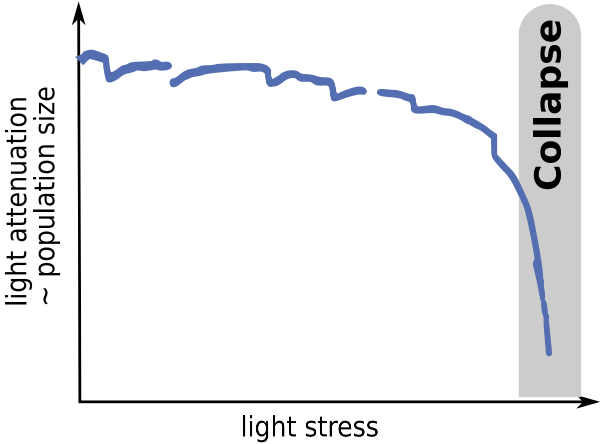
Mechanism:

- bacteria susceptible to light stress
- bacteria shade each other
- if light stress too high
  $\Rightarrow$reproduction goes down (death rate up)
  $\Rightarrow$shading reduced
  $\Rightarrow$light stress increases
  (**positive feedback loop**)
  $\Rightarrow$fold/saddle-node bifurcation

Motivating example III

Lab experiment

Veraart *et al*
Nature 2012



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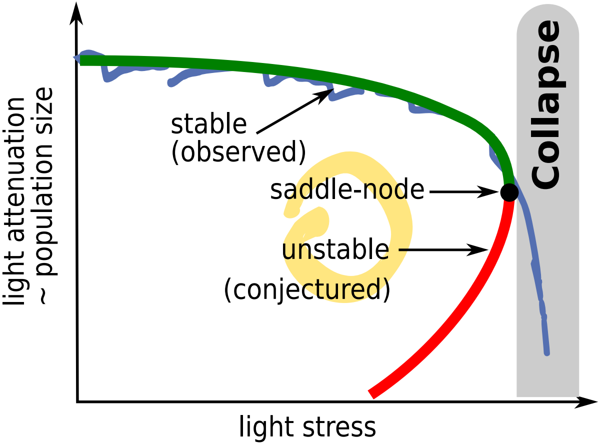
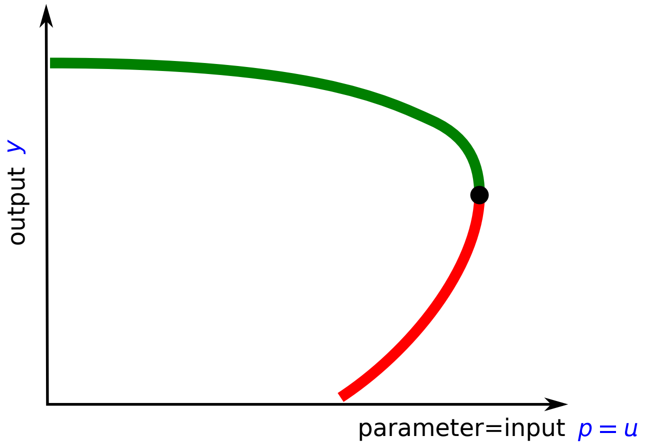


Motivating example III

How to find unstable branch? Assume $\dot{y}(t) = f(y(t), p) +$ “noise”



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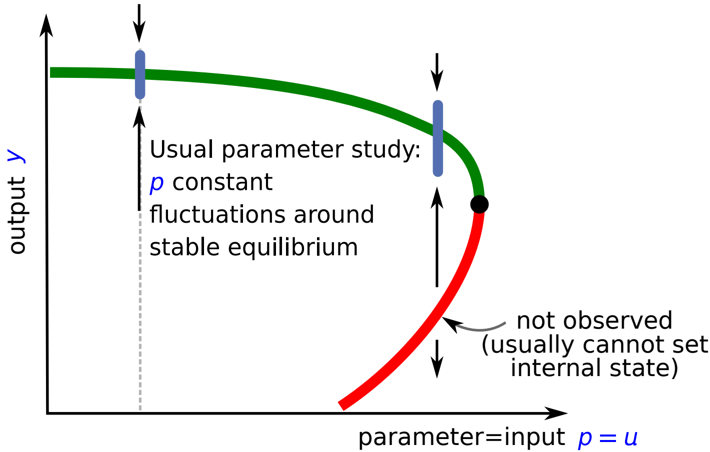

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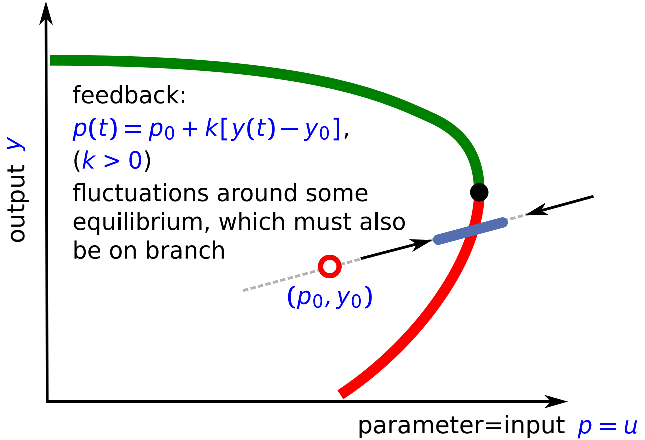

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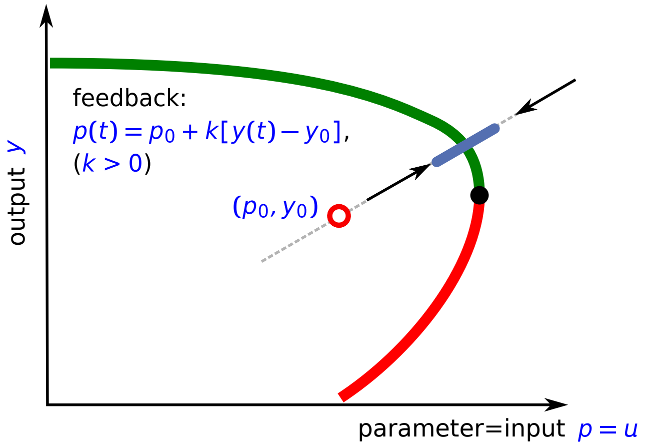

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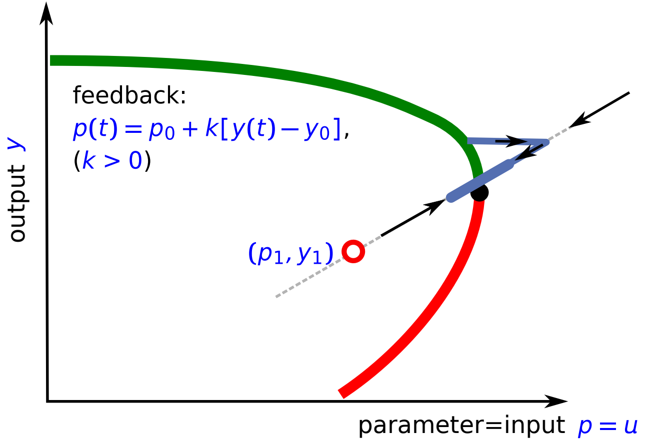

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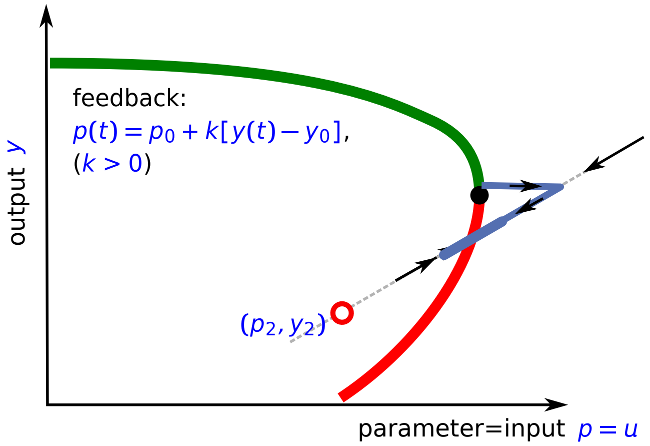


Motivating example IV

Check for saddle-node normal form:

$$\dot{y} = -p - y^2$$

⇒Equilibrium $y_s = \sqrt{-p}$ stable, $y_u = -\sqrt{-p}$ unstable.

$$p(t) = p_0 + k[y(t) - y_0]$$

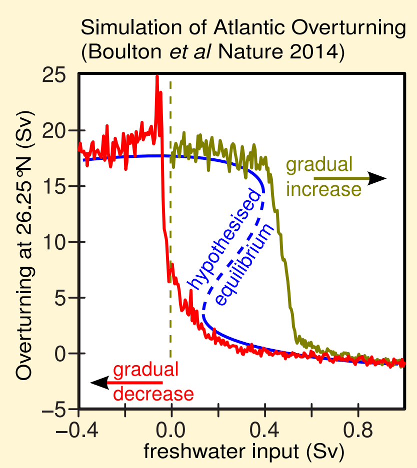
Equilibria satisfy

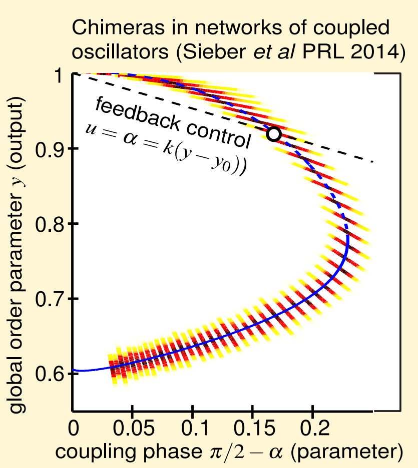
$$\begin{aligned} 0 &= -p_{\text{eq}} - y_{\text{eq}}^2 \\ &= -\{p_0 + k[y_{\text{eq}} - y_0]\} - y_{\text{eq}}^2 \end{aligned}$$

Stability: $-k - 2y_{\text{eq}}$ ⇒ stable for $y_{\text{eq}} \geq -k/2$.

Other examples

Computer simulations as experiments





General terms and conditions

To do (in experiments)

- Continue equilibria & periodic orbits that are either
  - dynamically unstable or
  - depend sensitively on system parameters

Constraints

- no setting of internal state possible
- accuracy independent of model
- good estimate for error
- avoid system identification
- no real-time computations

Feedback control

- Assume that experiment is dynamical system with input and output governed by

$$\dot{x}(t) = f(x(t), p, u(t))$$
$$y(t) = g(x(t))$$

- $x(t) \in \mathbb{R}^n$ $\Leftarrow$ internal state
  $y(t) \in \mathbb{R}^k$ (often $k = 1$) $\Leftarrow$ output
  $u(t) \in \mathbb{R}^\ell$ (often $\ell = 1$) $\Leftarrow$ input

- **feedback control**:
  $u(t)$ is permitted to depend on $y(t)$ (and its past)

- Assume that system has, for

$$u = 0,$$

  equilibrium x_*,
  or periodic orbit $x_*(t)$.

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- **feedback control**:
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  equilibrium x_*, $\Rightarrow$ **look at this first, linearize**
  or periodic orbit $x_*(t)$.

Linear feedback control — state feedback

- Single-input system:

$$\dot{x} = Ax - bu, \quad A \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^n, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}$$

- **state feedback** means u is linear combination of x
$\Rightarrow u = k^T x$ ($k \in \mathbb{R}^n$, called **gains**)

Eigenvalues of r.h.s. matrix in $\dot{x} = Ax - bk^T x$ can be freely assigned by choosing gains k:

Theorem

1. Let $p(\lambda)$ arbitrary polynomial of degree n.
2. Let $M_c = [b, Ab, \dots, A^{n-1}b]$ regular ($\det M_c \neq 0$).

$\Rightarrow$ There exist gains $k \in \mathbb{R}^n$ such that p is characteristic polynomial of $A - bk^T$.

Condition 2: **controllability**

Linear feedback control — output feedback

- Single-input single-output (SISO) system:

$$\dot{x} = Ax - bu, \quad A \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^n, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}$$
$$y = c^T x, \qquad c \in \mathbb{R}^n, \qquad y \in \mathbb{R}.$$

- **output feedback**: u depends on y and its past.

Linear feedback control — output feedback

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- **output feedback**: u depends on y and its past.
- Let k, $\tilde{k} \in \mathbb{R}^n$ be some gains.
- Construct observer: $\dot{\tilde{x}} = \left[A - bk^T\right]\tilde{x} - \tilde{k}\left[c^T\tilde{x} - y\right]$.
- Set $u = k^T\tilde{x}$.

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$\Rightarrow$ Error $e = x - \tilde{x}$ satisfies $\quad \dot{e} = \left[A - \tilde{k}c^T\right]e$

$\Rightarrow$ x satisfies $\quad \dot{x} = [A - bk^T]x + bk^T e$.

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⇒ x satisfies $\quad \dot{x} = [A - bk^T]x + bk^Te$.

- If $M_o = [c, A^Tc, \ldots, (A^T)^{n-1}c]$ regular ($\det M_o \neq 0$) (**observability**)

⇒ apply Eigenvalue Assignment Theorem to $A^T - c\tilde{k}^T$.

Back to equilibrium of nonlinear system

- equilibrium x_* of $\dot{x} = f(x, p, 0)$,
- assuming state feedback controllability, there exist gains $k \in \mathbb{R}^n$ such that in

  $$\dot{x}(t) = f(x(t), p, u(t)), \qquad u(t) = k^T[x_* - x(t)]$$

  x_* is a stable equilibrium
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  1. partial derivatives $\partial_1 f$, $\partial_3 f$ in $(x_*, p, 0)$
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  2. location of x_* (appears in u)

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- This assumes that we know two things:
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  2. location of x_* (appears in u)
- Small errors in $\partial_1 f$, $\partial_3 f$ $\Rightarrow$gains k still stabilizing
- error in x_*: $x_{\mathrm{ref}} \neq x_* \quad \Rightarrow \quad \dot{x} = f(x, p, k^T(x_{\mathrm{ref}} - x))$
  has stable equilibrium $\lim_{t \to \infty} x(t) =: x_{\mathrm{eq}} \neq x_*$.

Controlled experiment = fixed point map

- assume experiment with controllable equilibrium x_* (location unknown), and stabilising feedback

$$u(t) = k^T[x_{\text{ref}} - x(t)]$$

⇒this defines **nonlinear fixed point problem**

- $F : (p, x_{\text{ref}}) \mapsto \lim_{t\to\infty} x(t)$
- One evaluation of F:
  1. set system parameters to p,
  2. set input to feedback law to $u(t) = k^T[x_{\text{ref}} - x(t)]$ (x is state/output)
  3. Wait until transients have settled: $F(p, x_{\text{ref}}) := \lim_{t\to\infty} x(t)$
- x_{ref} is equilibrium of uncontrolled experiment if and only if

$$F(p, x_{\text{ref}}) = x_{\text{ref}} \quad \text{(which implies} \quad \lim_{t\to\infty} u(t) = 0\text{)}$$

Same for periodic orbit

- Assume we have experiment and feedback control

$$u(t) = k^T[x_{\text{ref}}(t) - x(t)]$$

  that stabilises periodic orbit $x_*(t)$ (autonomous or forced) locally.

- if x_{ref} has period T, $x_{\text{ref}} \approx x_*$ and $x(0) \approx x_*(0)$

⇒ $x(t)$ converges to T-periodic output $x_\infty(t)$:

$$x(t) - x_\infty(t) \to 0$$

- x_∞ depends locally uniquely & smoothly on x_{ref}

  ⇒(for autonomous periodic orbits) map

$$F : (p, x_{\text{ref}}(\cdot T), T) \mapsto x_\infty(\cdot T)$$

  in space of periodic functions on $[0, 1]$.

- x_* is fixed point of F: $F(p, x_*, T_*) = x_*$.