# Hurwitz Numbers and Expectation Values

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# Contents

# Introduction

The study of Hurwitz numbers appears in many areas of mathematics, from moduli spaces to representation theory. In this dissertation we aim to give an overview of some of these areas. Hurwitz numbers were first introduced by A.Hurwitz in [11, 12] as counts of ramified coverings of Riemann surfaces. We will focus largely on this classical interpretation of Hurwitz numbers and their relations to the symmetric group, but it is worth bringing to attention the role they have played in the study of other areas of mathematics. Hurwitz numbers did not garner too much attention since their introduction until the latter part of the 20th century, where much interest was taken in simple and double Hurwitz numbers – which count ramified coverings over the Riemann sphere – due to their strong connections between integrable systems and moduli spaces. A broader overview of more recent developments in our understanding of Hurwitz numbers can be found in [18].

Chapter 1 will be dedicated to giving an account of the classical interpretations of Hurwitz numbers. We will define a Hurwitz number as a count of locally complex differentiable maps between Riemann surfaces. As is the case with ordinary complex differentiable functions these maps exhibit extremely strong structure, one of the consequences of this is given by the Riemann-Hurwitz formula. As a result of Riemann's Existence Theorem these maps can be classified by ramified coverings over a fixed genus $g$ surface. With this topological classification of locally complex differentiable maps, we can alternately view a Hurwitz number associated to a genus $g$ surface as an enumeration of ramified coverings satisfy some prescribed data. The remainder of this chapter will then be aimed towards demonstrating the relationship between Hurwitz numbers and the symmetric group, the key to which is given by the monodromy representation of a ramified covering. In short, ramified coverings can be classified algebraically as group homomorphisms from the fundamental group of a fixed base curve into the symmetric group by studying some group action inherent to the class of coverings. In Section 1.2.4 we will demonstrate how powerful this shift in perspective is by computing all Hurwitz numbers of degree 2 and 3 with relative ease. To conclude this chapter, the algebraic interpretation of Hurwitz numbers is expanded upon and encapsulated in Burnside's character formula of Theorem 1.3.10, which gives Hurwitz numbers as products of irreducible characters of the symmetric group.

In Chapter 2 we will give an interpretation of Hurwitz numbers as expectation values of operators on a dense subspace of the fermionic Fock space. This result is simply a reformulation of the character formula mentioned above, but the value of this work comes in the interpretation of this space and its further applications to integrable systems. In particular A.Okounkov in [21], building on the works of [25], shows that the generating function of the double Hurwitz numbers is a solution to the Toda lattice integrable hierarchy of [29] – a family of partial differential equations related to physical models of crystal lattices. This relation extends to the Kadomtsev-Petviashvili (KP) hierarchy [16, p.7], so called as its simplest member is the KP equation which describes nonlinear wave motion. A proof of Witten's conjecture can also be given utilising the fact that the generating function of the simple Hurwitz numbers is a solution to the KP hierarchy [18, p.18]. We will not discuss these topics in detail but instead try to provide a number of references for further reading.

# Chapter 1

# Hurwitz Theory

## 1.1 Hurwitz Numbers

The origin of Hurwitz numbers is rooted in the theory of complex analysis on surfaces, so that is where we will start our discussion. We will begin by revealing the local structure of holomorphic maps and show how this eventually leads to the classification of holomorphic maps in terms of ramified coverings – the key to which is given by Theorem 1.1.12. With this topological understanding of holomorphic maps we define a Hurwitz number associated to a connected Riemann surface.

### 1.1.1 Holomorphic Maps

Holomorphic maps are simply the complex analogue of differentiable functions between real surfaces – real two dimensional compact manifolds without boundary. Throughout this dissertation we will take a **Riemann surface** to be a complex one dimensional compact manifold without boundary. We can consequently be view a Riemann surface as a real surface whose transition functions obey the Cauchy-Riemann equations. Functions that obey the Cauchy-Riemann equations are necessarily orientation preserving, and as a consequence the topological structure of Riemann surfaces are classified by the orientable genus $g$ real surfaces – connected sums of $g$ tori and spheres. In general we take the genus of a compact Riemann surface $X$ with connected components $X_1, \ldots, X_n$, with respective genera $g_1, \ldots, g_n$, to be $g_1 + \ldots g_n + 1 - n$.

**Definition 1.1.1.** *Let $X$ and $Y$ be Riemann surfaces. A continuous map $f : X \to Y$ is said to be* ***holomorphic*** *if $f$ is locally complex differentiable everywhere on $X$.*

A continuous function $f : X \to Y$ is locally complex differentiable everywhere on $X$ if for any $x \in X$ and for any charts $\phi$ and $\psi$ about $x$ and $f(x)$ respectively, $\psi \circ f \circ \phi^{-1}$ is complex differentiable (on any suitable domain). Complex differentiable functions from $\mathbb{C}$ into $\mathbb{C}$ must adhere to very strong properties unlike their real differentiable counterparts. Immediately from the above definition we can see that some of these properties can be extended to holomorphic maps between Riemann surfaces.

**Lemma 1.1.2** (Open Mapping Theorem)**.** *Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces. Then $f$ is an open map.*

*Proof.* Let $U$ be an open subset of $X$, and let $V$ be an open set of $Y$ containing $f(U)$. Take $\phi$ and $\psi$ to be local coordinates on $U$ and $V$ respectively, then $\psi \circ f \circ \phi^{-1}$ is holomorphic and non-constant on $\phi(U)$, where $\phi(U)$ is open (coordinate charts are homeomorphisms). Then by the open mapping theorem of complex analysis $\psi \circ f \circ \phi^{-1}(\phi(U)) = \psi \circ f(U)$ is open. Thus $f(U)$ is an open set in $X$. □

**Theorem 1.1.3** (Liouville's Theorem)**.** *Let $f : X \to Y$ be a holomorphic map of Riemann surfaces, with $Y$ connected. Then either $f$ is a constant function or $f$ is onto.*

*Proof.* If $f$ is non-constant then from Lemma 1.1.2, $f(X)$ is open in $Y$. As $X$ is compact and $f$ continuous by definition, $f(X)$ is compact and thus closed in $Y$. $f(X)$ is clearly non-empty, and so as $Y$ is connected we must have $f(X) = Y$. □

Thanks to the flexibility available to us in the choice of charts when exhibiting a holomorphic map about a point, we can re-phrase the local expression of such a map about any point in as nice a way as one can think of – that being as a power function. The following theorem is a fundamental observation of holomorphic maps, it classifies the local action of holomorphic maps whilst giving us some global information about $f$.

**Theorem 1.1.4.** *Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces with $Y$ connected. Then for all $x \in X$, there is a unique integer $k_x \geq 1$ such that $f$ locally appears as the power function $z \mapsto z^{k_x}$. Moreover, all but finitely many $x \in X$ have $k_x = 1$.*

*Proof.* We will tackle each claim separately.

**Existence:** Choose charts $\phi$ and $\psi$ centred about $x$ and $f(x)$ respectively i.e. such that $\phi(x) = 0 = \psi(f(x))$. Define $F := \psi \circ f \circ \phi^{-1}$ to be the local expression of $f$ with respect to these charts, then $F$ is holomorphic on an open neighbourhood $U$ of $\phi(x) = 0$. Then for $z \in U$ with $|z|$ sufficiently small, $F(z)$ is equal to its Taylor expansion about 0

$$F(z) = a_0 + a_1 z + \cdots + a_k z^k + \ldots$$

Let $k$ be the order of the zero of $F$ at 0 then there is a function $G$ holomorphic on $W \subseteq U$, with $G(0) \neq 0$ and $F(z) = z^k G(z)$ for all $z \in W$. Moreover, as $G(0) \neq 0$, for sufficiently small neighbourhood $\tilde{W} \subseteq W$ about 0, $G$ has a well defined $k$-th root $H$ – i.e. $H$ holomorphic on $\tilde{W}$ with $G(z) = H(z)^k$ for all $z \in \tilde{W}$. Then we can write $F(z) = (zH(z))^k$ for all $z \in \tilde{W}$.

Let $h$ be the holomorphic function on $\tilde{W}$ with $h(z) := zH(z)$. Then $h$ has derivative $h'$, with $h'(z) = H(z) + zH'(z)$, and so $h'(0) \neq 0$ and $h$ is locally invertible about 0. So there is an open neighbourhood $\tilde{U}$ about zero such that $h$ restricts to a biholomorphic map from $\tilde{U}$ to $h(\tilde{U})$. Note then that $h \circ \phi$ gives a chart about $x$

with $h \circ \phi(x) = 0$. Letting $\phi^* = h \circ \phi$, the corresponding local expression of $f$ is given by $F^* = \psi \circ f \circ \phi^{*-1}$, and for any $\tilde{x}$ (sufficiently close to $x$), letting $z = \phi(\tilde{x})$ and $z^* = \phi^*(\tilde{x})$ then $F(z) = (z^*)^k$. Moreover

$$F^*(z^*) = \psi \circ f \circ \phi^{*-1}(\phi^*(\tilde{x})) = \psi \circ f \circ \phi^{-1}(\phi(\tilde{x})) = F(z) = (z^*)^k. \tag{1.1}$$

It is then clear we can choose a chart $\phi^*$ centred about $x$, such that $F^*(z^*) = (z^*)^k$, for some $k \in \mathbb{N}$.

- **Uniqueness:** To show $k$ is unique it suffices to show the order of the zero of $F$ at $\phi(x) = 0$ is independent of the choice of charts centred about $x$ and $f(x)$. Given the transition functions between charts must be holomorphic, if $\tilde{F}$ is another local expression of $f$ arising from charts centred about $x$ and $f(x)$, then $\tilde{F}$ is related to $F$ by $\tilde{F} = g \circ F \circ h$ for some biholomorphic functions $g$ and $h$ with $g(0) = h(0) = 0$. Considering the $n$-th derivatives of $\tilde{F}$ we have

$$\tilde{F}^{(n)}(0) = F^{(n)}(0)g'(0)h'(0) + \text{other terms}\ldots$$

where the 'other terms' involve products of lower derivatives of $F$ at 0. So if $n < k$ then $\tilde{F}^{(n)}(0) = 0$ given $F^{(m)}(0) = 0$ for each $m < k$. Moreover $\tilde{F}^{(k)}(0) = F^{(k)}(0)g'(0)h'(0)$ and as $g$ and $h$ are invertible about 0, they must have non-zero derivatives about zero, and given $F^{(k)}(0) \neq 0$ we have $\tilde{F}^{(k)}(0) \neq 0$ and $\tilde{F}$ has a zero of order $k$ at zero. So $k$ is unique to $x$ and we denote it by $k_x$.

- **Finiteness:** Let $R = \{x \in X : 2 \leq k_x\}$. To show that $R$ is finite we want to show the following property. For any $x \in R$ there is an open neighbourhood $U_x$ about $x$ such that $k_{\tilde{x}} = 1$ for all $\tilde{x} \in U_x \setminus \{x\}$. Let $\phi$ and $\psi$ be charts centred about $x \in X$ and $y = f(x)$ respectively that admits a local expression $F$ with $F(z) = z^{k_x}$ for all $z \in \mathbb{C}$ sufficiently close to 0. $\phi$ is a homomorphism between an open neighbourhood $U_x$ of $x$ and an open neighbourhood of $0 \in \mathbb{C}$, so pick some $\tilde{x} \in U_x \setminus \{x\}$ and let $\tilde{y} = f(\tilde{x})$ – note that $\phi(\tilde{x}) \neq 0$. Let $\tilde{\phi} = \phi - \phi(\tilde{x})$ and $\tilde{\psi} = \psi - \psi(\tilde{y})$, then these are valid charts centred about $\tilde{x}$ and $\tilde{y}$ respectively admitting a local expression

$$\tilde{F}(z) = \tilde{\psi} \circ f \circ \tilde{\phi}^{-1}(z) = F(z + \phi(\tilde{x})) - \psi(\tilde{y})$$

with $\tilde{F}'(z) = F'(z + \phi(\tilde{x}))$. In particular $\tilde{F}'(0) = F'(\phi(\tilde{x})) = k_x\phi(\tilde{x})^{k_x - 1} \neq 0$. So $F$ has a zero of order 1 at 0. Thus by uniqueness we have that $k_{\tilde{x}} = 1$.

  Now $\{U_x\}_{x \in X}$ is an open cover of $X$ with $X$ compact and so there is a subcover $\{U_x\}_{x \in K}$ with $K \subseteq X$ and $|K|$ finite. If $x \in R$ and $x \notin K$ then $x \in U_{x'}$ for some $x \neq x'$ and thus $k_x = 1$ – a contradiction. So $R \subseteq K$ and $R$ is finite.

□

We now have another important structure theorem of holomorphic maps. This starts to hint at the global structure of these maps, saying that non-constant holomorphic maps 'evenly cover' a target space except at those points which have some non-trivial local action of $f$ above them. A proof of Theorem 1.1.5 can be found in [7, p.12] or in most introductory books on the theory of Riemann surfaces and holomorphic maps.

**Theorem 1.1.5.** *Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces with $Y$ connected. Then $|f^{-1}(y)| < \infty$ for any $y \in Y$. Moreover, there is a positive integer $d \geq 1$ such that for all $y \in Y$*

$$\sum_{x \in f^{-1}(y)} k_x = d.$$

*This integer will be called the **degree** of $f$ and denoted* $\deg f$.

It is worth noting that finiteness of the fibre can be attributed to the compactness of $X$, whilst the invariance of the sum can be attributed to the connectedness of $Y$. Also observe that as the set of $x \in X$ with $k_x \geq 2$ is finite, all but finitely many $y \in Y$ have $|f^{-1}(y)| = d$, and the points in $X$ with $k_x \geq 2$ can be thought of as singularities with respect to the action of $f$ – as we will see this intuition is not entirely wrong. Before proceeding, we will give one more important observation of non-constant holomorphic maps that emphasises the 'even covering' nature of these functions.

**Lemma 1.1.6.** *Let $f : X \to Y$ be a non-constant holomorphic map with $Y$ connected and of degree $d$. Recall we take $R := \{x \in X : k_x \geq 2\}$. Then for each $y \in Y \setminus f(R)$ there is a neighbourhood $V$ of $y$ and disjoint open sets $U_i \subset X$ such that*

$$f^{-1}(V) = \bigsqcup_{i=1}^{d} U_i$$

*where $f|_{U_i} : U_i \to V$ is a homeomorphism.*

*Proof.* We can write $f^{-1}(y) = \{x_1, x_2, \ldots x_d\} \subset X$ where all $x_i$'s are distinct. Given $k_{x_i} = 1$ for each $x_i$, $f$ admits a homeomorphism from an open neighbourhood $U_i$ of $x_i$ to an open neighbourhood $V_i$ of $y$. As $X$ is Hausdorff, these $W_i$'s can be taken to be disjoint by appropriately restricting $V_i$ and $W_i$. Taking $V = \bigcap_{i=1}^{d} V_i \neq \emptyset$ and $U_i = W_i \cap f^{-1}(V) \subseteq W_i$, we see that $f(U_i) = V$, and it then remains to check that $f^{-1}(V) = \bigcup_{i=1}^{d} U_i$. $\square$

This structure is very reminiscent of covering spaces which we will discuss in the next section. As we will see before the end of this chapter, holomorphic maps can be classified by ramified covering spaces up to some symmetry.

### 1.1.2 Ramified Coverings

In this section we will introduce covering spaces in an arbitrary topological setting, and of particular interest to us are ramified coverings. The main point of this section is to translate the language and ideas used to talk about general covering spaces into holomorphic maps.

**Definition 1.1.7.** *A continuous surjective function $f : X \to Y$ of topological spaces is a **covering space** of $Y$ if for any $y \in Y$ there is an open neighbourhood $V$ of $y$ such that $f^{-1}(V)$ is a disjoint union of open sets in $X$, where each component is homeomorphic to $V$ under $f$.*

From Lemma 1.1.6 we observe that given a non-constant holomorphic map $f : X \to Y$, with $R = \{x \in X : k_x \geq 2\}$, the restriction of $f$ to $X \setminus f^{-1}(f(R))$ is a covering space of $X \setminus f(R)$. We will develop this correspondence further with Riemann's Existence Theorem in the following sections.

A classic way of visualising a covering space is by describing $f^{-1}(U)$ as a 'stack of pancakes' mapped to $U$ by $f$. Covering spaces exhibit similar properties to those which were discussed in the previous section concerning non-constant holomorphic maps. In particular, if $f : X \to Y$ is a covering space of $Y$, with $X$ compact and $Y$ connected, then there is an integer $d$ with $|f^{-1}(y)| = d$ for all $y \in Y$ – which we will also call the degree of $f$. Again, it is exactly the compactness of $X$ that gives us finiteness and the connectedness of $Y$ the invariance of cardinality. However a more accurate description of holomorphic maps comes by introducing the notion of ramification.

**Definition 1.1.8.** *A continuous surjective function $f : X \to Y$ is said to be a* ***ramified covering*** *of $Y$ if there is a finite set $B \subseteq Y$ such that $f^{-1}(B)$ is finite and $f$ restricted to $X \setminus f^{-1}(B)$ is a covering space of $Y \setminus B$.*

Again from Lemma 1.1.6, given a non-constant holomorphic map $f : X \to Y$ with $Y$ connected we see that $f$ is a ramified covering of $Y$. It is desirable to give the points that do not obey the nice covering structure of a holomorphic map special names, as they characterise a large part of the structure of such a function.

**Definition 1.1.9.** *Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces. For $x \in X$:*

- *we call $k_x$ the* ***ramification index*** *of $x$;*
- *if $k_x \geq 2$ we say $x$ is a* ***ramification point*** *of $f$, and denote the set of all ramification points of $f$ by $R$, and call it the* ***ramification locus*** *of $f$;*
- *let $B = f(R)$. Call $B$ the* ***branch locus*** *of $f$, and the elements of $B$* ***branch points*** *of $f$.*

*Let $y \in Y$, and let $f^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$, such that $k_{x_1} \geq k_{x_2} \geq \cdots \geq k_{x_n}$. The* ***ramification profile*** *of $f$ at $y$ the list of integers $\lambda = (k_{x_1}, k_{x_2}, \ldots, k_{x_n})$. If $Y$ is connected, then from Theorem 1.1.5 $\lambda$ is an integer partition of $d = \deg f$.*

**Example 1.1.10.** *To see how the notions of ramified covering is inherent to complex differentiable functions, consider the function $f : \mathbb{C} \to \mathbb{C} : z \mapsto z^k$ for some fixed $k \geq 1$. $f'(z) = kz^{k-1} = 0$ if and only if $z = 0$, so by the inverse function theorem, at all points $z \neq 0$, $f$ is locally invertible. Now let $w \in \mathbb{C} \setminus \{0\}$, then*

$$f^{-1}(w) = \{\sqrt[k]{|w|}\zeta : \zeta \text{ is a } k\text{-th root of unity}\} \subseteq \mathbb{C} \setminus \{0\}$$

*and so $f$ is locally invertible around each element of $f^{-1}(w)$. Let $V_1, V_2, \ldots, V_k$ be open sets about each distinct root of $w$ (taken to be disjoint without loss of generality), which* *are mapped bijectively by $f$ to open sets $U_1, U_2, \ldots, U_k$ about $w$, respectively (given by the IVT). Taking $U = \bigcap_{i=1}^{k} U_i \ni w$ (which is non-empty and open in $\mathbb{C} \setminus \{0\}$)*

$$f^{-1}(U) = \bigcup_{i=1}^{k} V_i \cap f^{-1}(U),$$

*where $V_i \cap f^{-1}(U)$ is in bijective correspondence with $f(V_i) \cap U = U_i \cap U = U$, under $f$. So $f$ restricted to $\mathbb{C} \setminus \{0\}$, gives a cover of $\mathbb{C} \setminus \{0\}$. Take $B = \{0\}$ as in Definition 1.1.8, where $f^{-1}(B) = \{0\}$, and note that $f|_{\mathbb{C}\setminus\{0\}}$ is continuous, and surjective on $\mathbb{C} \setminus \{0\}$. So indeed $f|_{\mathbb{C}\setminus\{0\}}$ is a covering of $\mathbb{C} \setminus \{0\}$. Moreover $f$ is a ramified covering of $\mathbb{C}$.*

### 1.1.3 Riemann's Existence Theorem

From the previous discussions we have observed that a non-constant holomorphic map of Riemann surfaces with a connected image is a ramified covering. Riemann's Existence Theorem gives a partial converse to this observation by providing a topological classification of holomorphic maps in terms of ramified coverings.

To give this classification we want to establish some notion of when two holomorphic maps are equivalent by identifying and removing some symmetry. For our purposes we do not care about the symmetry of a holomorphic map in its domain.

**Definition 1.1.11.** *Let $f : X \to Y$ and $\tilde{f} : \tilde{X} \to Y$ be holomorphic maps of Riemann surfaces. We say $f$ and $\tilde{f}$ are **isomorphic** if there exists a bijective holomorphic map $\pi : X \to \tilde{X}$, with holomorphic inverse, such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "f"'] & & \tilde{X} \arrow[dl, "\tilde{f}"] \\ & Y & \end{tikzcd}$$

*We consequently call $\pi$ an **isomorphism** of Riemann surfaces.*

**Theorem 1.1.12** (Riemann's Existence Theorem)**.** *Let $Y$ be a connected Riemann surface, and $B \subseteq Y$ a finite set. Let $\tilde{f} : \tilde{X} \to Y \setminus B$ be a topological covering of finite degree. Then there exists a Riemann surface $X$ and holomorphic map $f : X \to Y$ where*

- *$\tilde{X}$ is dense in $X$;*
- *$f|_{\tilde{X}} = \tilde{f}$.*

*Moreover, $f$ is unique up to isomorphism.*

This theorem tells us that up to isomorphism, a holomorphic map of Riemann surfaces is determined by its corresponding ramified covering. Then the position of a set of branch points on a Riemann surface tells you nothing about the covering structure of a holomorphic map with said branch points. This is a very important property of holomorphic maps when it comes to defining a Hurwitz number.

### 1.1.4 Riemann-Hurwitz Formula

Now that we are well aware of the local geometry of holomorphic maps, a fairly straightforward application of this understanding is given by the Riemann-Hurwitz formula, which essentially shows us how the Euler characteristic of a surface is altered by a holomorphic function. This is surprising as we only really have sound understanding of the local description of holomorphic maps yet we can extend this understanding to infer global properties.

**Theorem 1.1.13** (Riemann-Hurwitz Formula)**.** *Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces of degree $d$ with $Y$ connected. Let $g_X$ be the genus of $X$ and $g_Y$ the genus of $Y$. Then*

$$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X}(k_x - 1) \tag{1.2}$$

*Proof.* There are many different ways to prove this theorem. A rather common approach, which can be found in [5, p.56], is as follows. Note the Euler characteristic $\chi$ of a genus $g$ connected surface is given by $2 - 2g$. So when $X$ is connected, we can rephrase equation (1.2) to

$$\chi(X) = d\chi(Y) - \sum_{x \in X}(k_x - 1).$$

To show this consider a graph on $Y$, whose vertices are taken to be the branch points of $f$ and whose edges never cross, and consider the pre-image of such a graph, computing the Euler characteristic in $X$ in terms of the Euler characteristic in $Y$. Using our knowledge of the local description of $f$, we know how the edges and faces about a vertex multiply.

If $X$ is disconnected, consider the restriction of $f$ to each connected component of $X$. The above equation must hold for each restriction and note that the genus of $X$ is $g_1 + g_2 + \cdots + g_n - n + 1$ where $g_1, \ldots, g_n$ are the genera of its connected components. Simply summing the equations obtained from each restriction yields the desired expression. □

Then, if the genus of the base space and the ramification profiles are given, the genus of the covering surface is determined from this equation.

### 1.1.5 Defining Hurwitz numbers

We have now laid all the ground work necessary for establishing the definition of a Hurwitz number associated to a connected Riemann surface.

**Definition 1.1.14.** *Let $g \in \mathbb{Z}$, $d \in \mathbb{N}$ and $\lambda_1, \lambda_2 \ldots, \lambda_n$ partitions of $d$. A* ***Hurwitz cover of type*** $(g, d, \lambda_1, \lambda_2, \ldots, \lambda_n)$ *is a non-constant holomorphic map of Riemann surfaces $f : X \to Y$ such that*

- *$Y$ is of genus $g$,*
- *$f$ is of degree $d$,*

- *$f$ has $n$ branch points, with ramification profiles $\lambda_1, \lambda_2 \ldots, \lambda_n$.*

**Remark 1.1.15.** *We do not need to specify the integer $d$ as this is made implicit from the size of the partitions. However we will include it for ease of reading.*
*The Riemann-Hurwitz formula tells us that the genus $h$ of a covering surface of a Hurwitz cover of type $(g, d, \lambda_1, \ldots, \lambda_n)$ is uniquely determined. Namely, by equation (1.2),*

$$h = d(g-1) + 1 + \frac{1}{2}\left(\sum_{p \in C}(k_p - 1)\right). \tag{1.3}$$

*Two isomorphic non-constant holomorphic maps of compact Riemann surfaces are clearly Hurwitz covers of the same type. But, the data $(g, d, \lambda_1, \ldots, \lambda_n)$ does not determine the isomorphism class of holomorphic maps uniquely, and so not all Hurwitz covers are isomorphic.*

A Hurwitz number is a weighted count of isomorphism classes of Hurwitz covers. All this means is that the symmetry of an isomorphism class of covering space is taken into consideration.

**Definition 1.1.16.** *An* ***automorphism*** *of a non-constant holomorphic map of Riemann surfaces is an isomorphism from $f$ to itself. The collection of automorphisms of $f$ form a group under composition, which we will denote by $Aut(f)$.*

**Remark 1.1.17.** *If $f$ and $f^*$ are isomorphic homomorphic maps, with isomorphism $\pi$ from $f$ to $f^*$, then the mapping $\phi : Aut(f) \to Aut(f^*)$ given by $g \mapsto \pi g \pi^{-1}$ is an isomorphism of groups. In particular $|Aut(f)| = |Aut(f^*)|$ is invariant amongst isomorphism classes of holomorphic maps.*

With this notion of symmetry, we can now define a Hurwitz number.

**Definition 1.1.18** (The Hurwitz Number)**.** *We define the* ***Hurwitz number*** *of type $(g, d, \lambda_1, \ldots, \lambda_n)$ to be the sum*

$$H_{d,g}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{[f]} \frac{1}{|Aut(f)|}, \tag{1.4}$$

*summing over all isomorphism classes $[f]$ of Hurwitz covers of type $(g, d, \lambda_1, \ldots, \lambda_n)$.*

This definition is well defined, by the previous remark. It also offers a good visual interpretation of Hurwitz numbers. Riemann's Existence Theorem tells us an isomorphism class of a Hurwitz cover of type $(g, d, \lambda_1, \ldots, \lambda_2)$ is determined uniquely by its covering structure. So a Hurwitz number can be seen to count distinct ramified coverings of a genus $g$ surface with specified ramification profiles. However this visual interpretation is not all that useful when it comes to actual computation. The automorphism group of a covering $f$ may become increasingly non-trivial when larger sets of data are considered. The problem of computing Hurwitz numbers becomes much simpler by a slight change in perspective. To emphasise the relative difficulty in calculating Hurwitz numbers from the definition we outline the following example.

**Example 1.1.19.** *Let* $Y = \mathbb{CP}^1$*,* $b_1 = 0$*,* $b_2 = \infty$*,* $d \geq 1$ *an integer and* $\lambda_1 = \lambda_2 = (d)$*. We compute*

$$H_{d,0}((d),(d)) = \frac{1}{d}. \tag{1.5}$$

*If* $f : X \to \mathbb{CP}^1$ *is a holomorphic map of degree* $d$ *where* $g_X$ *is the genus of* $X$*, then by the Riemann-Hurwitz formula*

$$2g_X - 2 = -2d + 2(d-1) = -2$$

*and so* $g_X = 0$*. The restriction of a holomorphic map to a connected component of* $X$ *is also a holomorphic map, in particular each connected component of* $X$ *must be of genus* $0$*. So we are forced to conclude that* $X$ *is connected and of genus* $0$*. Then* $X = \mathbb{CP}^1$*.*

*One must now show that the map* $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ *that takes* $z \mapsto z^d$ *– viewing* $\mathbb{CP}^1$ *as the one-point compactification of* $\mathbb{C}$ *– is the unique Hurwitz cover of type* $(0, d, (d), (d))$ *up to isomorphism, and that* $|Aut\, f| = d$*. The latter observation follows by explicitly constructing the automorphisms of* $f$*. In particular, observe that multiplying* $z$ *by a* $d$*-th root of unity* $\zeta$ *yields* $f(\zeta z) = \zeta^d z^d = f(z)$*, it follows that the automorphisms of* $f$ *are exactly the maps* $z \mapsto \zeta^k z$ *with* $1 \leq k \leq d$*. The former can be seen by showing that the bijective holomorphic maps from* $\mathbb{CP}^1$ *to itself are given by Möbius transformations, and that any holomorphic map of type* $(0, d, (d), (d))$ *must be of the form*

$$a\frac{(x-r_1)^d}{(x-r_2)^d} = \left(b\frac{(x-r_1)}{(x-r_2)}\right)^d,$$

*for some* $a \neq 0, r_1, r_2 \in \mathbb{C} \subseteq \mathbb{CP}^1$*, where* $b \in \mathbb{C}$ *with* $b^d = a$*. It is clear from this example, that even the most simple Hurwitz number is difficult to calculate in this manner.*

## 1.2 Monodromy

Monodromy representation gives a representation of the fundamental group of a connected surface, via a group action on the pre-image of a fixed point. In this section we make this notion precise in the context of holomorphic maps, although it may be applicable to ramified coverings of topological spaces in general. Through the study of monodromy representations we can shift the context in which Hurwitz numbers lie from that of analysis to something combinatorial in nature. It should not be understated how powerful this shift in perspective and its consequences are. Throughout this section, unless stated otherwise, results that are given without proof can be found in [6], [3] or any introductory textbook to algebraic topology treating covering spaces.

### 1.2.1 Path Lifting and Monodromy

In this section we build the framework from which the ideas of monodromy representation arise, and towards the end give our main motivation for the following sections.

Let $f : C \to X$ be a non-constant holomorphic map of Riemann surfaces with $X$ connected. Let $B$ be the set of branch points of $f$, $x_0 \in X \setminus B$, and $\gamma : [0,1] \to X \setminus B$ be a curve with $\gamma(0) = \gamma(1) = x_0$.

**Definition 1.2.1.** *We say a curve $\alpha : [0,1] \to C$ is a **lift** of $\gamma$ if the following diagram commutes.*

$$\begin{tikzcd} & C \arrow[d, "f"] \\ {[0,1]} \arrow[ur, "\alpha"] \arrow[r, "\gamma"] & X \end{tikzcd}$$

Observe that for any lift $\alpha$ of $\gamma$, $\alpha(0), \alpha(1) \in f^{-1}(x_0)$. As $x_0$ is not a branch point of $f$ there exist local homeomorphisms about each $p \in f^{-1}(x_0)$ onto a neighbourhood of $x_0$. This gives us some intuition as to why we might expect the following result to hold.

**Lemma 1.2.2.** *For any $p \in f^{-1}(x_0)$ there is a unique lift $\alpha_p$ of $\gamma$ with $\alpha_p(0) = p$, and a unique lift $\tilde{\alpha}_p$ of $\gamma$ with $\tilde{\alpha}_p(1) = p$. Note $\alpha_p$ and $\tilde{\alpha}_p$ need not be distinct.*

With the existence of a unique $\alpha_p$, the following function is well defined.

$$\sigma_\gamma : f^{-1}(x_0) \to f^{-1}(x_0) : p \mapsto \alpha_p(1) \tag{1.6}$$

Moreover, the existence and uniqueness of $\tilde{\alpha}_p$ makes $\sigma_\gamma$ surjective and injective respectively. Thus $\sigma_\gamma$ is a bijection and $\sigma_\gamma \in \mathrm{Sym}(f^{-1}(x_0))$. Hence to any curve based a $x_0$ we can associate an action permuting the pre-images of $x_0$. This is the underlying principle of monodromy representation. We can denote this association by a function $\sigma$ such that

$$\sigma : \mathrm{Cur}(X \setminus B, x_0) \to \mathrm{Sym}(f^{-1}(x_0)) : \gamma \mapsto \sigma_\gamma, \tag{1.7}$$

where $\mathrm{Cur}(X \setminus B, x_0)$ denotes the set of curves in $X \setminus B$ based at $x_0$.

As it is now we do not know enough to extend $\sigma$ to a well defined function on the fundamental group of the punctured surface $\pi_1(X \setminus B, x_0)$. $\pi_1(X \setminus B, x_0)$ consists of homotopy classes of curves in $X \setminus B$, and so we will require $\sigma$ to be invariant with respect to these homotopy classes in order to descend the information carried by $\sigma$ in any meaningful way. It just so happens that a homotopy between paths lifts in a nice enough way to give this desired property.

**Lemma 1.2.3.** *Let $\gamma_1$ and $\gamma_2$ be curves in $X \setminus B$ based at $x_0$. Let $p \in f^{-1}(x_0)$, with $\alpha_{p,1}$ and $\alpha_{p,2}$ the unique lifts of $\gamma_1$ and $\gamma_2$ starting at $p$, respectively. Suppose $\gamma_1$ and $\gamma_2$ are homotopically equivalent. Then $\alpha_{p,1}(1) = \alpha_{p,2}(1)$.*

Hence if $\gamma_1$ and $\gamma_2$ are homotopically equivalent curves based at $x_0$, then $\sigma_{\gamma_1} = \sigma_{\gamma_2}$. Thus the function

$$\Phi_{f,x_0} : \pi_1(X \setminus B, x_0) \to \mathrm{Sym}(f^{-1}(x_0)) : [\gamma] \mapsto \sigma_\gamma \tag{1.8}$$

is indeed well defined – where $[\cdot]$ denotes the corresponding equivalence class of a curve.

It also happens that $\Phi_{f,x_0}$ gives a group homomorphism. The group operation on $\pi_1(X \setminus B, x_0)$ is given by the concatenation of curves, which we will denote by $*$. This observation follows immediately from the following lemma.

**Lemma 1.2.4.** *Let $\gamma_1$ and $\gamma_2$ be curves in $X \setminus B$ based at $x_0$. Then*

$$\sigma_{\gamma_1 * \gamma_2} = \sigma_{\gamma_1} \circ \sigma_{\gamma_2}. \tag{1.9}$$

*Proof.* Choose an arbitrary $p \in f^{-1}(x_0)$, and let $\alpha_p$, $\alpha_{1,p}$, $\alpha_{2,p}$ be the unique lifts of $\gamma_1 * \gamma_2$, $\gamma_1$ and $\gamma_2$ respectively. By uniqueness it follows that $\alpha_p = \alpha_{1,p} * \alpha_{2,p}$ in which case $\sigma_{\gamma_1 * \gamma_2}(p) = \sigma_{\gamma_1}\sigma_{\gamma_2}(p)$. So the above equality holds. □

We will refer to this group homomorphism $\Phi_{f,x_0}$ as the monodromy representation of $f$ at $x_0$.

**Definition 1.2.5.** *Let $f : C \to X$ be a Hurwitz cover of type $(g, d, \lambda_1, \ldots, \lambda_n)$, with branch points $B$, and $x_0 \in X \setminus B$. Then the* ***monodromy representation*** *of $f$ at $x_0$ is a group homomorphism $\Phi_{f,x_0} : \pi_1(X \setminus B, x_0) \to f^{-1}(x_0)$ determined by the above procedure.*

Monodromy representation is an important object in the theory of covering spaces, as it neatly conveys covering structure into an algebraic framework. However the main drawback of this definition is the dependence upon some explicit description of $f$. We will eventually like to remove this dependence and so we introduce the notions associated with labelled monodromy representation.

Observe that $\mathrm{Sym}(f^{-1}(x_0)) \cong S_d$ where $d = |f^{-1}(x_0)|$. This isomorphism can be achieved by labelling the elements of $f^{-1}(x_0)$ from 1 to $d$, we will refer to this bijection $L : f^{-1}(x_0) \to \{1, 2, \ldots, d\}$ as a **labelling**. Define the map

$$\Phi^L_{f,x_0} : \pi_1(X \setminus B) \to S_d : [\gamma] \mapsto L \circ \sigma_\gamma \circ L^{-1}.$$

This is a group homomorphism from $\pi_1(X \setminus B, x_0)$ into $S_d$. Moreover $\Phi^L_{f,x_0}$ obeys the key properties of a monodromy representation i.e. $\Phi^L_{f,x_0}$ is nothing but a re-labelling of $\Phi_{f,x_0}$. For this reason we will refer to $\Phi^L_{f,x_0}$ as a **labelled monodromy representation** of $f$ at $x_0$. It turns out the number of distinct labellings is invariant up to isomorphism of holomorphic maps. Better yet, we can exhibit equality between sets of labelled monodromy representations of isomorphic holomorphic maps.

**Lemma 1.2.6.** *Let $f : C \to X$ and $\tilde{f} : \tilde{C} \to X$ be non-constant holomorphic maps of Riemann surfaces with $X$ connected and branch points $B$. Let $x_0 \in X \setminus B$ and $\Phi_{f,x_0}$ and $\Phi_{\tilde{f},x_0}$ be monodromy representations of $f$ and $\tilde{f}$ at $x_0$, respectively. Then,*

$$|\{\Phi^L_{f,x_0} : L \text{ is a labeling of } f^{-1}(x_0)\}| = \frac{d!}{|Aut(f)|}. \tag{1.10}$$

*Moreover, if $f$ and $\tilde{f}$ are isomorphic,*

$$\left\{\Phi^L_{f,x_0} : L \text{ is a labelling of } f^{-1}(x_0)\right\} = \left\{\Phi^{\tilde{L}}_{\tilde{f},x_0} : \tilde{L} \text{ is a labelling of } \tilde{f}^{-1}(x_0)\right\}. \quad (1.11)$$

*Proof.* To see explicitly the number of distinct labelled monodromy representations we obtain, consider how a labelled monodromy representation $\Phi^L_{f,x_0}$ behaves under symmetries of $f$ compared to $\Phi_{f,x_0}$. An automorphism of $f$ is a bijective holomorphic map $\delta : C \to C$ such that $f \circ \delta = f$. So $\delta$ permutes the elements of $f^{-1}(x_0)$. Considering how lifts of loops in the downstairs behave, let $\rho$ be a loop based a $x_0$, let $p \in f^{-1}(x_0)$, and $\alpha_p$ be the unique lift of $\rho$ starting at $p$ and $\tilde{\alpha}_p$ the unique lift ending at $p$. Note that $\delta \circ \alpha_p$ is also a lift of $\rho$, as $f \circ \delta \circ \alpha_p = f \circ \alpha_p = \rho$. Likewise $\delta \circ \tilde{\alpha}_p$ is a lift. Moreover $\delta \circ \alpha_p$ and $\delta \circ \tilde{\alpha}_p$ are the unique lifts of $\rho$ beginning and ending at $\delta(p)$, respectively. The automorphism $\delta$ sends adjacent lifts to adjacent lifts. This tells us that a cycle of lifts of $\rho$ must be sent to another cycle of lifts of the same size under $\delta$. In particular $\delta$ permutes the elements of $f^{-1}(x_0)$ in a way that keeps them in the same cycle, or completely swaps a cycle with another, with respect to the lifts of $\rho$. Translating this to actions on the cycles of $\sigma_\rho$, they are either left unchanged or swapped with another cycle of the same type, under $\delta$. Either way the permutation is left unchanged i.e. $\sigma_\rho = \delta\sigma_\rho\delta^{-1}$. If $L$ is a labelling of the pre-image points $f^{-1}(x_0)$ then

$$L \circ \sigma_\rho \circ L^{-1} = (L \circ \delta) \circ \sigma_\rho \circ (\delta^{-1} \circ L^{-1}) = (L \circ \delta) \circ \sigma_\rho \circ (L \circ \delta)^{-1}$$

and so $L$ and $L \circ \delta$ are two labellings who under a fixed monodromy representation $\Phi_{f,x_0}$ give rise to the same labelled monodromy representation – $\Phi^L_{f,x_0} = \Phi^{L\circ\delta}_{f,x_0}$. Hence for each distinct labelling $L$ there are $|\mathrm{Aut}(f)| - 1$ other distinct labellings that give the same labelled monodromy representation. There are $d!$ choices of labelling on $f^{-1}(x_0)$, hence

$$|\{\Phi^L_{f,x_0} : L \text{ is a labeling of } f^{-1}(x_0)\}| = \frac{d!}{|\mathrm{Aut(f)}|}.$$

To show the second equation, let $\pi : C \to \tilde{C}$ be an isomorphism of $f$ and $\tilde{f}$, then $\pi$ restricts to a bijection between $f^{-1}(x_0) \to \tilde{f}^{-1}(x_0)$. Moreover the functions $\sigma$ and $\tilde{\sigma}$ obtained by the monodromy procedure are related by $\sigma_\gamma = \pi^{-1}\tilde{\sigma}_\gamma\pi$, for any $\gamma \in \mathrm{Cur}(X \setminus B, x_0)$. Then the monodromy representations $\Phi_{f,x_0}$ and $\Phi_{\tilde{f},x_0}$ are related by conjugation. That is

$$\Phi_{f,x_0}([\gamma]) = \pi^{-1} \circ \Phi_{\tilde{f},x_0}([\gamma]) \circ \pi$$

for all $[\gamma] \in \pi_1(X \setminus B, x_0)$. Let $L$ be a labelling of $f^{-1}(x_0)$, by the equation above we observe that

$$\Phi^L_{f,x_0}([\gamma]) = (L \circ \pi^{-1}) \circ \Phi_{\tilde{f},x_0}([\gamma]) \circ (L \circ \pi^{-1})^{-1} = \Phi^{L\circ\pi^{-1}}_{\tilde{f},x_0}([\gamma])$$

for all $[\gamma] \in \pi_1(X \setminus B, x_0)$ i.e. $\Phi^L_{f,x_0} = \Phi^{L\circ\pi^{-1}}_{\tilde{f},x_0}$. Likewise if $\tilde{L}$ is a labelling of $\tilde{f}^{-1}(x_0)$ we have $\Phi^{\tilde{L}\circ\pi}_{f,x_0} = \Phi^{\tilde{L}}_{\tilde{f},x_0}$. In particular as $\tilde{L} \circ \pi$ and $L \circ \pi^{-1}$ are valid labellings of $f^{-1}(x_0)$ and $\tilde{f}^{-1}(x_0)$ respectively, there is an equality of sets of labelled monodromy representations. ∎

From equation (1.10) and the definition of a Hurwitz number we observe that

$$H_{d,g}(\lambda_1, \ldots, \lambda_n) = \frac{1}{d!} \sum_{[f]} \left| \left\{ \Phi^L_{f,x_0} : L \text{ is a labelling of } f^{-1}(x_0) \right\} \right|, \tag{1.12}$$

where the sum is over all isomorphism classes $[f]$ of Hurwitz covers of type $(g, d, \lambda_1, \ldots, \lambda_n)$.

With equation (1.12), we have made some progress in our efforts to rephrase the definition of a Hurwitz number. However, as things are, we need some explicit description of each isomorphism class of holomorphic map. It turns out we can remove this dependence by introducing the abstract formulation of a monodromy representation.

**Definition 1.2.7.** *Let $X$ be a connected Riemann surface of genus $g$ and $B = \{b_1, \ldots, b_n\}$ a finite subset of $X$ with $x_0 \in X \setminus B$. A* ***labelled monodromy representation of type*** *$(g, d, \lambda_1, \ldots, \lambda_n)$ is a group homomorphism*

$$\Phi : \pi_1(X \setminus B, x_0) \to S_d$$

*such that if $\rho_i$ is a small loop about $b_i$ in $X \setminus B$, $\Phi([\rho_i])$ is of cycle type $\lambda_i$. Also, let $M(g, d, \lambda_1, \ldots, \lambda_n)$ denote the set of all labelled monodromy representations of type $(g, d, \lambda_1, \ldots, \lambda_n)$.*

As to why this is an abstraction of the idea of a monodromy representation as given in Definition 1.2.5, see the discussion at the beginning of Section 1.2.2.

If $f : C \to X$ is a Hurwitz cover of type $(g, d, \lambda_1, \ldots, \lambda_n)$ with $x_0 \in X \setminus B$, and $L$ any labelling of $f^{-1}(x_0)$, then $\Phi^L_{f,x_0}$ is a labelled monodromy representation of type $(g, d, \lambda_1, \ldots, \lambda_n)$. Then it follows that

$$\bigcup_{[f]} \left\{ \Phi^L_{f,x_0} : L \text{ is a labelling of } f^{-1}(x_0) \right\} \subseteq M(g, d, \lambda_1, \ldots, \lambda_n), \tag{1.13}$$

where again the union is taken over all isomorphism classes of Hurwitz covers of type $(g, d, \lambda_1, \ldots, \lambda_n)$.

Our goal for the next section is to show that

$$M(g, d, \lambda_1, \ldots, \lambda_n) \subseteq \bigcup_{[f]} \left\{ \Phi^L_{f,x_0} : L \text{ is a labelling of } f^{-1}(x_0) \right\},$$

so that we obtain equality between all labelled monodromy representations of type $(g, d, \lambda_1, \ldots, \lambda_n)$ and all labelled monodromy representations of Hurwitz covers of type $(g, d, \lambda_1, \ldots, \lambda_n)$. In particular, it remains to show that any labelled monodromy representation $\Phi$ of type $(g, d, \lambda_1, \ldots, \lambda_n)$ can be written as a labelled monodromy representation $\Phi^L_{f,x_0}$ of a Hurwitz cover of type $(g, d, \lambda_1, \ldots, \lambda_n)$. It will turn out that this occurs in a unique manner – that is if $\Phi^L_{f,x_0} = \Phi^{\tilde{L}}_{\tilde{f},x_0} = \Phi$ then $f \cong \tilde{f}$.

### 1.2.2 Construction of Covers

We will show that given a labelled monodromy representation $\Phi : \pi_1(X \setminus B, x_0) \to S_d$ of type $(g, d, \lambda_1, \dots, \lambda_n)$ we can construct a holomorphic map $f : C \to X$ that exhibits $\Phi$ as a labelled monodromy representation, and that such a map is unique up to isomorphism. Note we will not give a rigorous proof, but instead emphasise the main idea behind the construction. The details are then more a matter of book keeping.

This correspondence hinges on whether or not we have enough information to recover a covering space structure of a punctured surface from a group homomorphism of the fundamental group. The key to this idea is that the homomorphism contains information about ramification profiles, that has been translated into cycle types of the generators of the fundamental group. Consider a small loop $\gamma$ winding about a branch point $b$ in the downstairs space $X$ and observe the lifts of such a curve. The local geometry about any point $p \in f^{-1}(b)$ is characterised by the ramification index $k_p$. In particular, by shrinking $\gamma$ to be a sufficiently small loop about $b$, for each $p \in f^{-1}(b)$ we observe there to be $k_p$ lifts of $\gamma$ about $p$, such that when concatenating the distinct $k_p$ lifts about $p$ we obtain a curve that loops about the point $p$ – and only the point $p$ – exactly once. This tells us that the cycle type of $\sigma_\gamma$ is determined by the ramification profile of $b$, and is in fact exactly the ramification profile of $b$. As we showed $\sigma$ is invariant under homotopy, any loop about a single branch point must have cycle type being the ramification profile of that branch point.

One thing to observe at this point is that a description of the ramification profiles is not enough to specify a covering space. In the context of holomorphic maps we see that the data $(g, d, \lambda_1, \lambda_2, \dots, \lambda_n)$ is not enough to specify a unique isomorphism class of holomorphic maps, and hence the underlying covering space structure need not be unique. If this were the case the problem of computing Hurwitz numbers would become almost trivial. To specify a unique covering structure, not only do we need the ramification profiles of branch points, but also information regarding the way in which sheets of the covering space are 'glued' together between ramification points. By fixing a base point $x_0$ – it does not matter which $x_0$ as $X \setminus B$ is always path connected – and observing how the lifts of the generators of $\pi_1(X \setminus B, x_0)$ travel between elements of $f^{-1}(x_0)$, we are able to recover this gluing information.

**Theorem 1.2.8.** *Let $X$ be a connected Riemann surface, and $B$ be a finite set of points in $X$. Fix some base point $x_0 \in X \setminus B$ and let*

$$\Phi : \pi_1(X \setminus B, x_0) \to S_d$$

*be a labelled monodromy representation of type $(g, d, \lambda_1, \dots, \lambda_n)$. Then there exists a Holomorphic map $f : C \to X$ with $\Phi^L_{f,x_0} = \Phi$ for some labelling $L$ of $f^{-1}(x_0)$. Moreover, $f$ is unique up to isomorphism of holomorphic maps.*

*Proof.* In the case that that $\Phi$ gives a transitive group action of $\pi_1(X \setminus B, x_0)$ on $S_d$, a proof of this fact along with a broader introduction to monodromy, can be found in [20, p.84]. Note that this approach is sufficient if one is simply interested in holomorphic maps between connected Riemann surfaces, however we would like to consider a more general approach.

The idea behind the more general proof, which can be found in [5, p.96], is to construct a covering space of $X \setminus B$ using the group homomorphism $\Phi$. As we discussed earlier, $\Phi$ contains enough information to reconstruct a covering space – via ramification profiles and glueing instructions – exhibiting $\Phi$ as monodromy representation up to labelling of points. This procedure is outlined in Example 1.2.9. Applying Riemann's Existence Theorem, this covering space characterises a unique isomorphism class of holomorphic maps, which we will take to be represented by $f$. $f$ is a Hurwitz cover of type $(g, d, \lambda_1, \ldots, \lambda_n)$ and exhibits a labelled monodromy representation $\Phi^L_{f,x_0} = \Phi$, by construction. □

**Example 1.2.9.** *We demonstrate the covering space construction with a specific example. Let $X$ be a complex torus, fix $x_0 \in X$, and let $B = \{b_1, b_2, b_3\} \subseteq X \setminus \{x_0\}$. The fundamental group of the torus with 3 punctures is given by,*

$$\pi_1(X \setminus B, x_0) \cong \langle \alpha, \beta, \rho_1, \rho_2, \rho_3 : \alpha\beta\alpha^{-1}\beta^{-1}\rho_3\rho_2\rho_1 = 1\rangle \tag{1.14}$$

*where $\rho_i$ can be interpreted as a counter-clockwise loop about $b_i$ for $1 \leq i \leq 3$, $\alpha$ is a loop through the hole of the torus, and $\beta$ a loop about the rim. Define a group homomorphism $\Phi : \pi_1(X \setminus B, x_0) \to S_3$ by*

$$\rho_1, \rho_2, \rho_3 \mapsto (1, 2, 3) \quad \textit{and} \quad \alpha, \beta \mapsto e. \tag{1.15}$$

*$\Phi$ is indeed a group homomorphism as*

$$\Phi(\alpha\beta\alpha^{-1}\beta^{-1}\rho_3\rho_2\rho_1) = (1, 2, 3)^3 = e, \tag{1.16}$$

*and so $\Phi$ is a labelled monodromy representation of type $(1, 3, (3), (3), (3))$.*

*Our aim is to construct a covering space of $X \setminus B$ that exhibits the above monodromy representation up to some labelling of the pre-image points. This is achieved by creating 3 identical copies of $X \setminus B$, and glueing them together based on how the lifts travel between the pre-image points of $x$. Note that we expect the covering space to exhibit three points with ramification index 3, so the resulting covering space should extend to a Hurwitz cover of type $(1, 3, (3), (3), (3))$. By the Riemann-Hurwitz formula the resulting covering surface should be of genus 4.*

- *The first step in the procedure is to convert $X \setminus B$ into a suitable identification polygon. Then the gluing procedure simply corresponds to a quotient by an equivalence relation on the edges of the three polygons, based on how lifts of the generators of the fundamental group must travel between them. 'Cutting' along $\alpha$ and $\beta$, $X \setminus B$ is homeomorphic a quotient space of a square with three punctures. Picking a corner of this square, draw curves in $X \setminus B$ from each $b_i$ to to this corner, for each*

$1 \leq i \leq 3$. *Again 'cutting' along theses curves, the resulting space is homeomorphic to a quotient space of a regular decagon – missing some points on the boundary. This procedure is illustrated in Figure 1.1. We will denote this identification polygon $S$ and note that $S \cong X \setminus B$.*

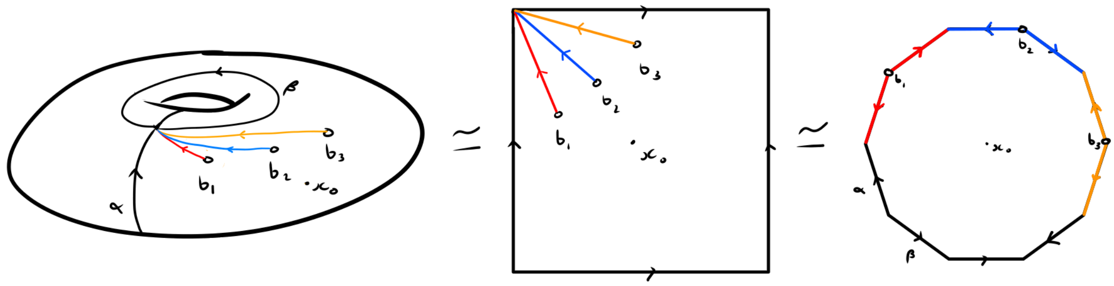



Figure 1.1: A chain of homeomorphisms, converting $X \setminus B$ into an identification polygon.

- *Let $S_1, S_2, S_3$ be three disjoint copies of the space $S$, with functions $f_i : S_i \to S$ being identifications between the two spaces. Let $\tilde{C} = S_1 \sqcup S_2 \sqcup S_3$, and define the map $\tilde{f} : \tilde{C} \to S$ by*

$$\tilde{f}(x) = \begin{cases} f_1(x) & \textit{if } x \in S_1; \\ f_2(x) & \textit{if } x \in S_2; \\ f_3(x) & \textit{if } x \in S_3. \end{cases} \tag{1.17}$$

  *Next we want to find a suitable equivalence relation $\sim$ on the edges of the polygons in $\tilde{C}$, such that the quotient space $\tilde{C}/\sim$ together with the descendant of $\tilde{f}$, gives a suitable covering space of $S$. Let $x_i = f_i^{-1}(x_0)$ for each $i \in \{1, 2, 3\}$ and let our labelling be such that $L : x_i \mapsto i$.*

- *Consider the curve $\rho_1$ about the puncture $b_1$, depicted in Figure 1.2. As $\rho_1 \mapsto (1, 2, 3)$, we want $\rho_1$ to exhibit three distinct lifts under the map $f : \tilde{C}/\sim \to S$: traversing from $x_1$ to $x_2$, $x_2$ to $x_3$, and from $x_3$ to $x_1$.*

  *$\rho_1$ can be thought to be split into two parts, one travelling away from the center of $S$ and one traversing towards the center. Observe the pre-image of the image of $\rho_1$ in $S$ under $\tilde{f}$. This will simply look like three copies of the right hand side of Figure 1.2, one for each $S_i$. We can then, for example, rearrange the polygons $S_1$ and $S_2$ so that the part of the preimage traversing from $x_1$ to some edge in $S_1$ and the part travelling from some edge to $x_2$ in $S_2$ align. We take the resulting edges that align to be equivalent under our relation $\sim$. This provides a lift of $\rho_1$ under $f$ traversing from $x_1$ to $x_2$. We repeat this procedure so that four other edges are made equivalent, yielding the remaining two required lifts of $\rho_1$ under $f$. We have colour coded the lifts and edges made equivalent in Figure 1.3. If you have ever played a game of Tantrix, it might be a helpful analogy for visualising the process.*

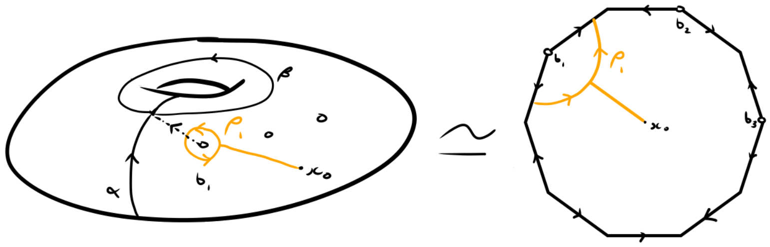



Figure 1.2: This illustrates how $\rho_1$ appears in the identification polygon $S$.

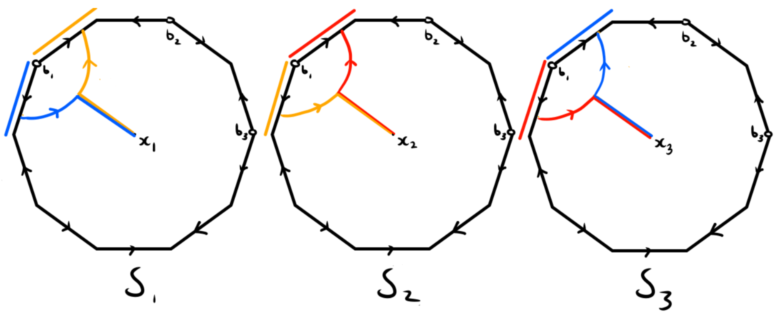



Figure 1.3: Each colour represents a different lift of $\rho_1$, and which edges are to be identified.

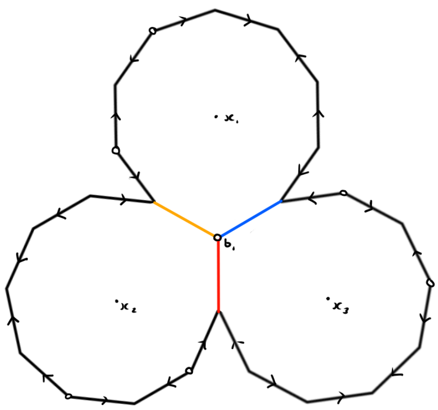



Figure 1.4: The resulting space from gluing together $S_1, S_2, S_3$ along edges specified by the lifts of $\rho_1$. Note the puncture at $b_1$ remains.

*Quotienting out the edges made equivalent under $\sim$ by $\rho_1$ we obtain the space in Figure 1.4. This space exhibits local geometry expected about the point $b_1$ i.e. it has ramification index 3 under $f$. The space also clearly remains a covering space of $S$ if we were to spontaneously re-glue the remaining edges.*

- *Continuing with this procedure for $\rho_2$, $\rho_3$, $\alpha$ and $\beta$ we eventually identify each edge of the polygons $S_1, S_2, S_3$ with some other edge to obtain a closed surface. Note that $\alpha$ and $\beta$ are mapped to the identity under $\Phi$, and so under this procedure, for each polygon $S_1, S_2, S_3$ we simply glue the two sides corresponding to $\alpha$ back together, and likewise for the sides corresponding to $\beta$ (i.e. no two distinct polygons are glued together along these edges). $f$ then gives a covering of $X \setminus B$.*

*For a simpler example, consider how one might use this procedure to construct the covering space corresponding to the map $f : \mathbb{CP}^1 \to \mathbb{CP}^1 : z \mapsto z^2$.*

### 1.2.3 Hurwitz Numbers and Monodromy Representations

Equipped with the main theorem of the previous section the desired correspondence between Hurwitz numbers and Monodromy representations is then almost immediate.

Theorem 1.2.8 tells us that

$$M(g, d, \lambda_1, \ldots, \lambda_n) \subseteq \bigcup_{[f]} \left\{\Phi^L_{f,x_0} : L \text{ is a labelling of } f^{-1}(x_0)\right\},$$

and that

$$M(g, d, \lambda_1, \ldots, \lambda_n) \cap \left\{\Phi^L_{f,x_0} : L \text{ is a labelling of } f^{-1}(x_0)\right\} \cap \left\{\Phi^{\tilde{L}}_{\tilde{f},x_0} : \tilde{L} \text{ is a labelling of } \tilde{f}^{-1}(x_0)\right\} = \emptyset$$

when $f \ncong \tilde{f}$. In conjunction with equation (1.13) we have that

$$M(g, d, \lambda_1, \ldots, \lambda_n) = \bigsqcup_{[f]} \left\{\Phi^L_{f,x_0} : L \text{ is a labelling of } f^{-1}(x_0)\right\}. \tag{1.18}$$

Then the theorem below follows.

**Theorem 1.2.10.** *The Hurwitz number can be written as*

$$H_{d,g}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{d!} |M(g, d, \lambda_1, \ldots, \lambda_n)|. \tag{1.19}$$

*Proof.* It follows from equation (1.18) that

$$|M(g, d, \lambda_1, \ldots, \lambda_n)| = \sum_{[f]} \left|\left\{\Phi^L_{f,x_0} : L \text{ is a labelling of } f^{-1}(x_0)\right\}\right|.$$

Then by equation (1.12) the result follows. □

This result is incredibly powerful as it translates the problem of computing Hurwitz numbers into something combinatorial. This result says that a Hurwitz number is a count of certain group homomorphism from the fundamental group of a punctured genus $g$ surface into the symmetric group. This theorem is what will ultimately bridge the gap between Hurwitz numbers and the symmetric group – specifically giving Hurwitz numbers as the product of characters of the symmetric group in the form of Burnside's character formula. Before proceeding we will spend some time playing with examples of how we might compute Hurwitz numbers using this monodromy interpretation.

### 1.2.4 Computations

We will now display the powerful computing power of this shift in interpretation with the following examples. Computations that might have seemed impossible in the setting of curve counting, become relatively easy. In Example 1.2.11 we will highlight this disparity by recalling Example 1.1.19. In Examples 1.2.12 and 1.2.13 we will compute all Hurwitz numbers of degree 2 and 3 respectively. Note throughout these examples let $\chi = 2 - 2g$ be the Euler characteristic of a connected genus $g$ surface.

**Example 1.2.11.** *Recall our calculation of $H_{d,0}((d),(d))$. The Riemann sphere with two punctures is homeomorphic to the circle $S^1$. The resulting fundamental group is then the free group generated by one element, isomorphic to $\mathbb{Z}$. We then want to count group homomorphisms $\Phi : \mathbb{Z} \to S_d$, such that $\Phi(1)$ is of cycle type $(d)$. There are $d!/d = (d-1)!$ such options, and so by equation (1.19) we have*

$$H_{d,0}((d),(d)) = \frac{(d-1)!}{d!} = \frac{1}{d}. \tag{1.20}$$

**Example 1.2.12.** *It is now suiting to note some properties of Hurwitz numbers. Given the right hand side of equation (1.19) is counting homomorphisms between finite groups, we are guaranteed that Hurwitz numbers are indeed finite. Moreover, just as we might not always expect a monodromy representation to exist under certain conditions we do not expect Hurwitz numbers to always be non-zero. In this example we will attempt to calculate all Hurwitz numbers of degree 2,*

$$H_{2,g}((2)^n) \tag{1.21}$$

*where $(2)^n$ denotes $(2),(2),\dots,(2)$ $n$ times. The Riemann-Hurwitz formula imposes a restriction on the genus of the covering space $h$, in particular, $2h = 4g - n - 2$. We immediately see that $n$ cannot be odd (we cannot have half genus), so we are forced to conclude*

$$H_{2,g}((2)^n) = 0 \tag{1.22}$$

*when $n$ is odd.*

*Now let $n$ be even. The fundamental group of a genus $g$ surface with $n$ punctures has a finite presentation*

$$G = \langle \alpha_1, \beta_1, \dots \alpha_g, \beta_g, \rho_1, \dots, \rho_n : \left( \prod_{i=1}^{g} [\alpha_i, \beta_i] \right) \rho_1 \dots \rho_n = e \rangle. \tag{1.23}$$

*A relevant monodromy representation is a group homomorphisms from $\Phi : G \to S_2$ such that each $\rho_i$ has cycle type $(2)$. Note that we must have $\Phi(\rho_i) = (1,2)$ for each $\rho_i \in G$, as $S_2 = \{e, (1,2)\}$. Then our choices of $\Phi(\rho_i)$ are fully determined, and so monodromy representations may only be distinguished by choices of each $\alpha_i, \beta_i$. As $S_2$ is Abelian and $n$ is even, the relation*

$$e = \Phi\left( \left( \prod_{i=1}^{g} [\alpha_i, \beta_i] \right) \rho_1 \dots \rho_n \right) = \Phi\left( \prod_{i=1}^{g} [\alpha_i, \beta_i] \right) (1,2)^n = \prod_{i=1}^{g} [\Phi(\alpha_i), \Phi(\beta_i)] \tag{1.24}$$

*is satisfied regardless of the choice of* $\alpha_i$ *of* $\beta_i$*. So there are* $2^{2g}$ *distinct monodromy representations determined by the choice of each* $\alpha_i$ *and* $\beta_i$*. Hence by equation (1.19)*

$$H_{2,g}((2)^n) = \frac{1}{2!}2^{2g} = 2^{2g-1} = 2^{1-\chi}. \tag{1.25}$$

**Example 1.2.13.** *For a final example consider Hurwitz numbers of the form*

$$H_{3,g}((3)^m, (2,1)^{2n}) \tag{1.26}$$

*for* $m, n \geq 1$*. Let us first focus on the case when* $g = 0$*. The fundamental group is given by*

$$G = \langle \rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_{2n} : \rho_1 \ldots \rho_m \sigma_1 \ldots \sigma_{2n} = e \rangle. \tag{1.27}$$

*We count group homomorphisms sending* $\rho_i$*'s to elements of* $S_3$ *of cycle type* $(3)$ *and* $\sigma_j$*'s to elements of cycle type* $(2,1)$ *(i.e. transpositions). The defining relation of the group* $G$ *must be satisfied, and so letting* $\Phi$ *denote an appropriate homomorphism*

$$\Phi(\rho_1 \cdots \rho_m \sigma_1 \cdots \sigma_{2n-1}) = \Phi(\sigma_{2n}). \tag{1.28}$$

*Note that regardless of choice of homomorphism* $\Phi$*,* $\Phi(\rho_i)$ *is an even permutation, and* $\Phi(\sigma_j)$ *is odd, and thus the product* $\Phi(\rho_1 \cdots \rho_m \sigma_1 \cdots \sigma_{2n-1})$ *is an odd element of* $S_3$*, and so is always a transposition i.e. a valid choice of* $\Phi(\sigma_{2n})$*. Hence we can choose* $\Phi(\rho_1), \ldots, \Phi(\rho_m), \Phi(\sigma_1), \ldots, \Phi(\sigma_{2n-1})$ *freely, of which there are* $2^m \cdot 3^{2n-1}$ *choices, and set* $\Phi(\sigma_{2n})$ *as the corresponding product. Clearly we cannot make any other choices as a group homomorphism is determined by the image of the generators, so*

$$H_{3,0}((3)^m, (2,1)^{2n}) = \frac{1}{3!}2^m 3^{2n-1} = 2^{m-1}3^{2(n-1)}. \tag{1.29}$$

*Now consider* $g = 1$*. The fundamental group of a torus with* $2n + m$ *punctures is given by*

$$G = \langle \alpha, \beta, \rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_{2n} : [\alpha, \beta]\rho_1 \ldots \rho_m \sigma_1 \ldots \sigma_{2n} = e \rangle \tag{1.30}$$

*Focusing on* $\alpha$ *and* $\beta$*, we observe that* $\Phi([\alpha, \beta]) \in S_3' = \{e, (1,2,3), (1,3,2)\}$*, so regardless of the choice of* $\alpha$ *and* $\beta$*,* $\Phi([\alpha, \beta])$ *is even. Thus, freely choosing the image of* $\alpha, \beta, \rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_{2n-1}$ *(with appropriate restriction to cycle type) we observe that*

$$\Phi([\alpha, \beta]\rho_1 \cdots \rho_m \sigma_1 \cdots \sigma_{2n-1}) = \Phi(\sigma_{2n}) \tag{1.31}$$

*is an odd permutation in* $S_3$*, and thus* $\Phi(\sigma_{2n})$ *is necessarily a transposition and* $\Phi$ *a valid choice of monodromy representation. There are* $6 \cdot 6 = 2^2 \cdot 3^2$ *choices of the images of* $\alpha$ *and* $\beta$ *(as there are no cycle type restrictions), and* $2^m \cdot 3^{2n-1}$ *choices for the remaining generators (excluding* $\sigma_{2n}$ *which is fully determined by these choices). Hence*

$$H_{3,1}((3)^m, (2,1)^{2n}) = \frac{1}{3!}2^2 3^2 2^m 3^{2n-1} = 2^{m+1}3^{2n}. \tag{1.32}$$

*Finally let $g$ be an arbitrary positive integer. Then the arguments above generalise very easily to this case. The fundamental group of a genus $g$ surface with $2n+m$ punctures is given by*

$$G = \langle \alpha_i, \beta_i, \rho_j, \sigma_k : \prod_{i=1}^{g} [\alpha_i, \beta_i] \prod_{j=1}^{m} \rho_j \prod_{k=1}^{2n} \sigma_k = e \rangle \tag{1.33}$$

*Freely choosing the image of the generators of type $\alpha_i, \beta_i, \rho_j$ and $\sigma_k$ for $1 \leq k \leq 2n-1$ we have that their product is necessarily an odd element of $S_3$ and thus a transposition, and so the corresponding homomorphism is a valid monodromy representation (assuming the generators are chosen with cycle type restrictions). So there are $(6 \times 6)^g$ choices of the image of the generators of type $\alpha_i, \beta_i$, and $2^m 3^{2n-1}$ of the remaining generators. Hence*

$$H_{3,g}((3)^m, (2,1)^{2n}) = \frac{1}{3!} 2^{2g} 3^{2g} 2^m 3^{2n-1} = 2^{(m-1)+2g} 3^{2(n-1)+2g}. \tag{1.34}$$

*Moreover, if we let $a, b \in \mathbb{N}$ then*

$$H_{3,g}((3)^a, (2,1)^b) = 2^{a-\chi} 3^{b-\chi} \cdot (1 + (-1)^b). \tag{1.35}$$

*Note this example was made significantly easier by the fact that the only odd elements of $S_3$ are transpositions. We could not use the same argument for calculating Hurwitz numbers of degree 4 for example, as $S_4$ has three odd cycle types, and more cases would need to be considered.*

## 1.3 Burnside's Character Formula

This section will provide one of the most powerful results in the study of Hurwitz numbers – that being Burnside's character formula. Making use of Theorem 1.2.10 one obtains a readily computable formula for all Hurwitz numbers in terms of irreducible characters of the symmetric group. First we will give a brief overview of the representation theory of the symmetric group and then proceed with the proof of the formula.

### 1.3.1 Representations of the Symmetric Group

Representation theory allows us to study the structure of a group using linear algebra. Often we can tell more about the group structure by considering the ways in which we can represent a group rather than studying its abstract presentation. By a representation of a group $G$ we mean a complex vector space $V$ endowed with some linear group action on $V$. That is, we identify elements of $G$ with elements of $GL(V)$ via some group homomorphism. Of particular interest are representations where the group action of $G$ cannot be isolated to a proper subspace of $V$, wherein these representations are said to be irreducible. For a finite group $G$ there are as many distinct irreducible representations as there are conjugacy classes of $G$. Moreover, any representation of $G$ on a finite dimensional vector space $V$ can be decomposed into direct sums of irreducible representations. So the irreducible representations of a finite group serve as building blocks for all other finite dimensional representations of $G$.

In the case of the symmetric group $S_n$, the conjugacy classes are indexed by partitions of $n$ corresponding to the distinct cycle types in $S_n$. The irreducible representations of $S_n$ are given by the Specht modules $S^\lambda$. The construction of these representations would take us on a diversion from the main focus of this section, a good outline of the construction can be found in [26], and [14]. If $V$ is a finite dimensional representation of $S_n$, it can be written as

$$V \cong \bigoplus_{\lambda \vdash n} S^{\lambda^{\oplus d_\lambda}},$$

where $d_\lambda = \dim(\mathrm{Hom}_{S_n}(S^\lambda, V))$. A rather important representation of $S_n$ – and in fact any finite group – is given by the group algebra, which encodes all irreducible representations of $S_n$.

**Definition 1.3.1.** *The **group algebra** of $S_n$ is a finite dimensional complex algebra $\mathbb{C}S_n$, with vector space basis $S_n$ and where multiplication is inherited from $S_n$. That is $\sigma \cdot \pi = \sigma\pi$ for any $\sigma, \pi \in S_n$.*

The group algebra $\mathbb{C}S_n$ carries a natural representation of $S_n$ given simply by left multiplication. So $\mathbb{C}S_n$ is an $n!$ dimensional representation of $S_n$, often called the regular representation of $S_n$. We can decompose $\mathbb{C}S_n$ in terms of Specht modules as

$$\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} S^{\lambda^{\oplus \dim(S_\lambda)}}.$$

In particular we can find a basis of $\mathbb{C}S_n$ such that for any $\sigma \in S_n$ the corresponding matrix under the regular representation is an $n! \times n!$ matrix in block diagonal form – where each block corresponds to the restriction of the action of $\sigma$ onto an irreducible sub-representation of $\mathbb{C}S_n$.

Letting $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the partitions of $n$, we have

$$\sigma \mapsto \begin{bmatrix} \begin{bmatrix} B_{\lambda_1} & & \\ & \ddots & \\ & & B_{\lambda_1} \end{bmatrix} & & & \\ & \begin{bmatrix} B_{\lambda_2} & & \\ & \ddots & \\ & & B_{\lambda_2} \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} B_{\lambda_p} & & \\ & \ddots & \\ & & B_{\lambda_p} \end{bmatrix} \end{bmatrix}, \tag{1.36}$$

where $B_{\lambda_i}$ is a $\dim(S_{\lambda_i}) \times \dim(S_{\lambda_i})$ square matrix appearing $\dim(S_{\lambda_i})$ times in the matrix representation on $\sigma$.

To any representation of $S_n$ we can associate a function $\chi : S_n \to \mathbb{C}$ given by taking the trace of the corresponding matrix representation of an element.

**Definition 1.3.2.** *Let $\rho : S_n \to GL(V)$ be a finite dimensional representation of $S_n$. The* ***character*** *of $\rho$ is a function $\chi^\rho : S_n \to \mathbb{C}$ given by*

$$\chi^\rho(\sigma) = trace(\rho(\sigma)).$$

*A character is said to be* ***irreducible*** *if $\rho$ is.*

By fixing a basis of $V$, the trace can be viewed as the sum of the diagonal entries of the matrix representation of $\rho(g)$ – which is independent of the basis chosen. Importantly, the character of an irreducible representation is unique up to isomorphism, allowing one to transition the study of irreducible representations of a group into irreducible characters. Characters are constant over conjugacy classes as $\rho$ is a homomorphism and the trace is invariant under conjugation, and in particular the collection of all irreducible characters are a basis for the set of class functions – functions $f : S_n \to \mathbb{C}$ that are constant over conjugacy classes of $S_n$. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the partitions of $n$ and $\chi^{\lambda_i}$ be the irreducible character associated to $S^{\lambda_i}$. We can associate to $S_n$ a $p \times p$ invertible matrix $\Phi$ whose entries are $\Phi_{ij} = \chi^{\lambda_i}(\lambda_j) := \chi^{\lambda_i}(g)$ – where $g$ is of cycle type $\lambda_j$ – we call $\Phi$ the **character table** of $S_n$.
Importantly we have,

$$\chi^\lambda((1^n)) = \dim S^\lambda$$

for any $\lambda \vdash n$.

**Example 1.3.3.** *Here we give the character table of the $S_4$. The partitions of 4 are given by $(1^4), (4), (2, 1^2), (3, 1), (2, 2)$, so $\Phi$ is a $5 \times 5$ matrix. The entries can be readily calculated by means we will discuss in later sections – in particular by using the Murnaghan-Nakayama rule.*

| $S_4$ | $(1^4)$ | $(4)$ | $(2,2)$ | $(3,1)$ | $(2,1^2)$ |
|---|---|---|---|---|---|
| $\chi^{(1^4)}$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\chi^{(4)}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi^{(2,2)}$ | 2 | 0 | 2 | $-1$ | 2 |
| $\chi^{(3,1)}$ | 3 | $-1$ | $-1$ | 0 | $-1$ |
| $\chi^{(2,1^2)}$ | 3 | 1 | $-1$ | 0 | $-1$ |

The set of class functions forms a complex vector space under pointwise addition and scalar multiplication, and can be endowed with an inner product,

$$\langle f, h \rangle = \frac{1}{|S_n|} \sum_{g \in S_n} f(g)\overline{h(g)} = \sum_{i=1}^{p} \frac{f(\lambda_i)\overline{h(\lambda_i)}}{|\xi(\lambda_i)|}, \tag{1.37}$$

where $\lambda_1, \ldots, \lambda_n$ are the partitions of $S_n$ and $f(\lambda_i) = f(g)$ where $g$ is of cycle type $\lambda_i$ – here $|\xi(\lambda)|$ denotes the size of the centraliser of an element of cycle type $\lambda$. The set of irreducible characters $\{\chi^{\lambda_1}, \chi^{\lambda_2}, \ldots, \chi^{\lambda_p}\}$ of $S_n$ are an orthonormal basis for the space of class functions,

$$\langle \chi^{\lambda_i}, \chi^{\lambda_j} \rangle = \delta_{ij}. \tag{1.38}$$

Moreover it can be shown that

$$\sum_{\mu \vdash n} \chi^{\mu}(\lambda_i)\overline{\chi^{\mu}(\lambda_j)} = |\xi(\lambda_i)|\delta_{ij}. \tag{1.39}$$

Equations (1.38) and (1.39) are referred to as the **orthogonality relations** of the character table $\Phi$, corresponding to products of the rows and columns of the matrix. It is also worth noting that all character values of a $S_n$ are real.

All of what we have discussed can be found in any introductory text book to representation theory – for example [14] or [8]

### 1.3.2 The Center $\mathcal{Z}(\mathbb{C}S_n)$

Schur's lemma is an extremely useful tool in representation theory. It tells us how group actions are allowed to translate between finite dimensional irreducible representations. One of its consequences is of particular importance to us.

**Lemma 1.3.4** (Schur)**.** *Let $V$ be an irreducible representation of $S_n$. Let $\tau \in GL(V)$ such that*

$$\sigma(\tau v) = \tau(\sigma v)$$

*for all $v \in V$ for any $\sigma \in S_n$. Then $\tau$ is a scalar multiple of the identity.*

As one might guess form this theorem we will be particularly interested in a commutative subalgebra of $\mathbb{C}S_n$.

**Definition 1.3.5.** *The* ***center*** *of the group algebra $\mathcal{Z}(\mathbb{C}S_n)$ is the subalgebra of $\mathbb{C}S_n$ consisting of all central elements. That is*

$$\mathcal{Z}(\mathbb{C}S_n) = \{\sigma \in \mathbb{C}S_n : \sigma\delta = \delta\sigma \text{ for all } \delta \in \mathbb{C}S_n\}.$$

**Remark 1.3.6.** *Let $\sigma \in \mathbb{C}S_n$ then the map $\tau : v \mapsto \sigma v$ on $\mathbb{C}S_n$ gives an element of $GL(\mathbb{C}S_n)$. Moreover $\tau|_{S^\lambda}$ is an element of $GL(S^\lambda)$. If $\sigma \in \mathcal{Z}(\mathbb{C}S_n)$ then $\tau|_{S^\lambda}$ commutes with all elements of $GL(V)$ of the form $v \mapsto \delta v$ where $\delta \in S_n$. Hence by Schur's lemma above $\tau|_{S^\lambda}$ is simply a scalar multiple of the identity automorphism on $S^\lambda$. By fixing a basis as described earlier we have that $\tau$ is a block diagonal matrix of the form in equation (1.36), where each component is of the form*

$$B_{\lambda_i} = b_{\lambda_i} I_{\dim S^{\lambda_i} \times \dim S^{\lambda_i}}$$

*for some $b_{\lambda_i} \in \mathbb{C}$. In particular*

$$\tau \mapsto \begin{bmatrix} b_{\lambda_1} I_1 & & & \\ & b_{\lambda_2} I_2 & & \\ & & \ddots & \\ & & & b_{\lambda_p} I_p \end{bmatrix},$$

*where $I_i$ is the $(\dim S^{\lambda_i})^2 \times (\dim S^{\lambda_i})^2$ identity matrix.*

With this observation, we have the motivation behind the following lemma. A precise run down of this result can be found in [28, p. 109], although in a more general approach than we have taken.

**Lemma 1.3.7.** *Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the partitions of $n$. Then $\mathcal{Z}(\mathbb{C}S_n)$ has a basis $\{e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_p}\}$ indexed by partitions of $n$, where*

$$e_{\lambda_i} e_{\lambda_j} = \begin{cases} e_{\lambda_i} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

In particular, there is a basis of $\mathbb{C}S_n$ such that the matrix representation of the action of $\mathcal{Z}(\mathbb{C}S_n)$ on $\mathbb{C}S_n$ is specified by

$$e_{\lambda_i} \mapsto \begin{bmatrix} 0_1 & & & & & & \\ & \ddots & & & & & \\ & & 0_{i-1} & & & & \\ & & & I_i & & & \\ & & & & 0_{i+1} & & \\ & & & & & \ddots & \\ & & & & & & 0_p \end{bmatrix}$$

where $0_j$ and $I_j$ are the $(\dim S^{\lambda_j})^2 \times (\dim S^{\lambda_j})^2$ zero and identity matrices. From this we observe $\mathcal{Z}(\mathbb{C}S_n)$ is a $p$-dimensional subalgebra of $\mathbb{C}S_n$.

$\mathcal{Z}(\mathbb{C}S_n)$ has another basis indexed by partitions of $S_n$.

**Definition 1.3.8.** *Let $C_\lambda \subseteq S_n$ denote the conjugacy class of elements with cycle type $\lambda$. We denote by $c_\lambda$ the sum of all elements of cycle type $\lambda$ in $S_n$, or formally*

$$c_\lambda := \sum_{\sigma \in C_\lambda} \sigma \in \mathbb{C}S_d. \tag{1.40}$$

Let $\lambda \vdash n$ and $g \in S_n$, then observe that

$$g^{-1} c_\lambda g = c_\lambda,$$

as conjugation permutes the entries of a conjugacy class. It follows that $c_\lambda$ is a central element. So if $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the partitions of $n$, then $\{c_{\lambda_1}, c_{\lambda_2}, \ldots, c_{\lambda_p}\}$ are $p$ linearly independent central elements, and forms a basis for $\mathcal{Z}(\mathbb{C}S_n)$.

We have established that there are two basis of $\mathcal{Z}(\mathbb{C}S_n)$ indexed by partitions, $\{e_\lambda\}_{\lambda\vdash n}$ and $\{c_\lambda\}_{\lambda\vdash n}$. Knowing how to change between these two basis is critical to the character formula. Ordering the partitions of $n$ as $\lambda_1, \lambda_2, \ldots, \lambda_p$, the change of bases matrix from $\{e_\lambda\}_{\lambda\vdash n}$ to $\{c_\lambda\}_{\lambda\vdash n}$, is given by

$$\frac{1}{d!}\begin{bmatrix} \dim S^{\lambda_1}\chi^{\lambda_1}(\lambda_1) & \dim S^{\lambda_2}\chi^{\lambda_2}(\lambda_1) & \ldots & \dim S^{\lambda_p}\chi^{\lambda_p}(\lambda_1) \\ \dim S^{\lambda_1}\chi^{\lambda_1}(\lambda_2) & \dim S^{\lambda_2}\chi^{\lambda_2}(\lambda_2) & \ldots & \dim S^{\lambda_p}\chi^{\lambda_p}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ \dim S^{\lambda_1}\chi^{\lambda_1}(\lambda_p) & \dim S^{\lambda_2}\chi^{\lambda_2}(\lambda_p) & \ldots & \dim S^{\lambda_p}\chi^{\lambda_p}(\lambda_p) \end{bmatrix}$$

or in other terms

$$e_\lambda = \frac{\dim S^\lambda}{d!} \sum_{\mu\vdash n} \chi^\lambda(\mu) c_\mu \quad \text{and,} \quad c_\lambda = |C^\lambda| \sum_{\mu\vdash n} \frac{\chi^\mu(\lambda)}{\dim S^\mu} e_\mu. \tag{1.41}$$

The expression for $c_\lambda$ can be verified using the orthogonality relations of the character table.

This will be all the machinery we will need for the remainder of this chapter.

### 1.3.3 Counting Cycle Types and The Character Formula

In this section we will prove the character theorem using what we have discussed so far. The idea behind the proof is to express counts of monodromy representations as the coefficient in a product of elements in $\mathcal{Z}(\mathbb{C}S_n)$. Changing basis between $\{e_\lambda\}_{\lambda\vdash n}$ and $\{c_\lambda\}_{\lambda\vdash n}$ we can manipulate the expression to give a computable formula for these counts. So first let us consider how counts of monodromy representations are related to taking products in the group algebra in the following lemma.

Let $\sigma \in \mathbb{C}S_d$ and $g \in S_d$ we denote by $[g]\sigma$ the coefficient of $g$ in $\sigma$.

**Lemma 1.3.9.** *Let $d \geq 1$ and $g$ be integers, and $\lambda_1, \ldots, \lambda_n$ be partitions of $d$. Then*

$$H_{d,g}(\lambda_1, \ldots, \lambda_n) = \frac{1}{d!}[e]\left(\sum_{\lambda \vdash d} |\xi(\lambda)| c_\lambda^2\right)^g c_{\lambda_n} c_{\lambda_{n-1}} \cdots c_{\lambda_1} \tag{1.42}$$

*Proof.* As mentioned, consider the number of monodromy representations $|M(g, d, \lambda_1, \lambda_2, \ldots, \lambda_n)|$. Then from equation (1.19) it follows that we must prove

$$|M(g, d, \lambda_1, \lambda_2, \ldots, \lambda_n)| = [e]\left(\sum_{\lambda \vdash d} |\xi(\lambda)| c_\lambda^2\right)^g c_{\lambda_n} c_{\lambda_{n-1}} \cdots c_{\lambda_1}, \tag{1.43}$$

and we are done.

Consider the zero genus case. $[e]c_\lambda c_{\lambda_{n-1}} \cdots c_{\lambda_1}$ is exactly the number of products of elements of $S_d$, such that $\sigma_n \sigma_{n-1} \cdots \sigma_1 = e$, where $\sigma_i$ has cycle type $\lambda_i$. This is precisely the number of monodromy representations of type $(0, d, \lambda_1, \ldots, \lambda_n)$, and so

$$|M(0, d, \lambda_1, \lambda_2, \ldots, \lambda_n)| = [e]c_\lambda c_{\lambda_{n-1}} \cdots c_{\lambda_1} \tag{1.44}$$

So it remains to see how we can encode the information regarding the genus in the product above.

$|M(g, d, \lambda_1, \lambda_2, \ldots, \lambda_n)|$ is the number of choices of $\mu_1, \mu_2, \ldots, \mu_g, \eta_1, \ldots, \eta_g, \sigma_1, \ldots, \sigma_n \in S_d$ such that

$$\prod_{i=1}^{g} [\mu_i, \eta_i] \sigma_n \sigma_{n-1} \ldots \sigma_1 = e. \tag{1.45}$$

and $\sigma_j$ is of cycle type $\lambda_j$. For each commutator $[\mu_i, \eta_i] = (\eta_i^{-1})^{\mu_i} \eta_i$, $\tilde{\eta}_i := (\eta_i^{-1})^{\mu_i}$ and $\eta_i$ are in the same conjugacy class. Taking any two choices of $\tilde{\eta}_i$, $\eta_i \in C_\lambda$ (where $\eta_i$ is of cycle type $\lambda$) there are $|\xi(\lambda)|$ choices corresponding of $\mu_i$ in $S_d$ giving $(\eta_i^{-1})^{\mu_i} = \tilde{\eta}_i$. With this correspondence, for a fixed $1 \leq i \leq g$, we can encode all possible choices of $\eta_i$ and $\mu_i$ in terms of choices of $\eta_i$ and $\tilde{\eta}_i$ (up to multiplicity), as elements of the product

$$\sum_{\lambda \vdash d} |\xi(\lambda)| c_\lambda c_\lambda. \tag{1.46}$$

Thus all possible products, as in the left hand side of equation (1.45) (up to multiplicity) can be represented as elements of the product

$$\left(\sum_{\lambda \vdash d} |\xi(\lambda)| c_\lambda^2\right)^g c_{\lambda_n} c_{\lambda_{n-1}} \ldots c_{\lambda_1}. \tag{1.47}$$

Moreover, those products satisfying equation (1.45) will evaluate to the identity. So the number of such products may be taken to be the coefficient of $e$ in the expression above. Namely,

$$|M(g,d,\lambda_1,\lambda_2,\ldots,\lambda_n)| = [e]\left(\sum_{\lambda\vdash d}|\xi(\lambda)|c_\lambda^2\right)^g c_{\lambda_n}c_{\lambda_{n-1}}\ldots c_{\lambda_1} \tag{1.48}$$

□

The character formula then arises as a simple calculation using the change of basis formulas in equation (1.41).

**Theorem 1.3.10** (Burnside's Character Formula)**.** *Let $g$ and $d \geq 1$ be integers, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ partitions of $d$. Then*

$$H_{d,g}(\lambda_1,\lambda_2,\ldots,\lambda_n) = \sum_{\lambda\vdash d}\left(\frac{\dim S^\lambda}{d!}\right)^{2-2g}\prod_{i=1}^n \frac{|C_{\lambda_i}|\chi^\lambda(\lambda_i)}{\dim S^\lambda}. \tag{1.49}$$

*Proof.* From Lemma 1.3.9 we have the equality,

$$H_{d,g}(\lambda_1,\lambda_2,\ldots,\lambda_n) = \frac{1}{d!}[e]\left(\sum_{\lambda\vdash d}|\xi(\lambda)|c_\lambda^2\right)^g c_{\lambda_n}c_{\lambda_{n-1}}\ldots c_{\lambda_1}. \tag{1.50}$$

Using the equations of (1.41), and the multiplicative properties of the basis $\{e_\lambda\}_{\lambda\vdash d}$.

$$c_{\lambda_n}c_{\lambda_{n-1}}\ldots c_{\lambda_1} = \prod_{i=1}^n\left(\sum_{\mu\vdash d}\frac{|C_{\lambda_i}|\chi^\mu(\lambda_i)}{\dim S^\mu}e_\mu\right) = \sum_{\mu\vdash d}\left(\prod_{i=1}^n \frac{|C_{\lambda_i}|\chi^\mu(\lambda_i)}{\dim S^\mu}\right)e_\mu.$$

And likewise

$$\begin{aligned}
\sum_{\lambda\vdash d}|\xi(\lambda)|c_\lambda^2 = \sum_{\lambda\vdash d}|\xi(\lambda)|\left(\sum_{\mu\vdash d}\frac{|C_\lambda|\chi^\mu(\lambda)}{\dim S^\mu}e_\mu\right)^2 &= \sum_{\lambda\vdash d}|\xi(\lambda)|\sum_{\mu\vdash d}\left(\frac{|C_\lambda|\chi^\mu(\lambda)}{\dim S^\mu}\right)^2 e_\mu \\
&= \sum_{\mu\vdash d}\left(\sum_{\lambda\vdash d}|\xi(\lambda)||C_\lambda|^2\left(\frac{\chi^\mu(\lambda)}{\dim S^\mu}\right)^2\right)e_\mu \\
&= \sum_{\mu\vdash d}\frac{d!}{(\dim S^\mu)^2}\left(\sum_{\lambda\vdash d}|C_\lambda|(\chi^\mu(\lambda))^2\right)e_\mu \\
&= \sum_{\mu\vdash d}\left(\frac{d!}{\dim S^\mu}\right)^2\left(\sum_{\lambda\vdash d}\frac{(\chi^\mu(\lambda))^2}{|\xi(\lambda)|}\right)e_\mu \\
&= \sum_{\mu\vdash d}\left(\frac{d!}{\dim S^\mu}\right)^2\langle\chi^\mu,\chi^\mu\rangle e_\mu \\
&= \sum_{\mu\vdash d}\left(\frac{d!}{\dim S^\mu}\right)^2 e_\mu,
\end{aligned}$$

So

$$\left(\sum_{\lambda\vdash d}|\xi(\lambda)|c_\lambda^2\right)^g = \sum_{\mu\vdash d}\left(\frac{d!}{\dim S^\mu}\right)^{2g} e_\mu$$

and

$$\begin{aligned}
\left(\sum_{\lambda\vdash d}|\xi(\lambda)|c_\lambda^2\right)^g c_{\lambda_n}c_{\lambda_{n-1}}\dots c_{\lambda_1} &= \sum_{\mu\vdash d}\left(\frac{d!}{\dim S^\mu}\right)^{2g}\left(\prod_{i=1}^n \frac{|C_{\lambda_i}|\chi^\mu(\lambda_i)}{\dim S^\mu}\right)e_\mu \\
&= \sum_{\mu\vdash d}\left(\frac{d!}{\dim S^\mu}\right)^{2g}\left(\prod_{i=1}^n \frac{|C_{\lambda_i}|\chi^\mu(\lambda_i)}{\dim S^\mu}\right)\left(\frac{\dim S^\mu}{d!}\sum_{\lambda\vdash d}\chi^\mu(\lambda)c_\lambda\right) \\
&= \sum_{\lambda\vdash d}\left[\sum_{\mu\vdash d}\left(\frac{d!}{\dim S^\mu}\right)^{2g-1}\chi^\mu(\lambda)\left(\prod_{i=1}^n \frac{|C_{\lambda_i}|\chi^\mu(\lambda_i)}{\dim S^\mu}\right)\right]c_\lambda
\end{aligned}$$

Observing that $e = c_{(1^d)}$ and that $\chi^\mu((1^6)) = \dim S^\mu$, one has

$$\begin{aligned}
\frac{1}{d!}[e]\left(\sum_{\lambda\vdash d}|\xi(\lambda)|c_\lambda^2\right)^g c_{\lambda_n}c_{\lambda_{n-1}}\dots c_{\lambda_1} &= \sum_{\mu\vdash d}\left(\frac{d!}{\dim S^\mu}\right)^{2g-1}\frac{\chi^\mu((1^6))}{d!}\left(\prod_{i=1}^n \frac{|C_{\lambda_i}|\chi^\mu(\lambda_i)}{\dim S^\mu}\right) \\
&= \sum_{\mu\vdash d}\left(\frac{d!}{\dim S^\mu}\right)^{2g-2}\left(\prod_{i=1}^n \frac{|C_{\lambda_i}|\chi^\mu(\lambda_i)}{\dim S^\mu}\right)
\end{aligned}$$

giving us the character formula. □

**Example 1.3.11.** *Using the character formula we calculate $H_{3,g}((3)^a,(2,1)^b)$ for $a,b \geq 1$, as we did in Example 1.2.13. The partitions of 3 are $(1^3),(2,1)$ and $(3)$, and the character table of $S_3$ is given below.*

| | $(1^3)$ | $(3)$ | $(2,1)$ |
|---|---|---|---|
| $\chi^{(1^3)}$ | 1 | 1 | −1 |
| $\chi^{(3)}$ | 1 | 1 | 1 |
| $\chi^{(2,1)}$ | 2 | 1 | 0 |

*Using equation (1.50), we have*

$$\begin{aligned}
H_{3,g}((3)^a,(2,1)^b) &= \sum_{\lambda\vdash 3}\left(\frac{\dim S^\lambda}{3!}\right)^{2-2g}\left(\frac{|C_{(3)}|\chi^\lambda((3))}{\dim S^\lambda}\right)^a\left(\frac{|C_{(2,1)}|\chi^\lambda((2,1))}{\dim S^\lambda}\right)^b \\
&= \frac{2^a3^b}{(3!)^{2-2g}}\sum_{\lambda\vdash 3}\left(\dim S^\lambda\right)^{2-2g-a-b}\chi^\lambda((3))^a\chi^\lambda((2,1))^b \\
&= 2^{a-(2-2g)}3^{b-(2-2g)}\left(1+(-1)^b\right) \\
&= 2^{a-\chi}3^{b-\chi}\left(1+(-1)^b\right)
\end{aligned}$$

There are a few cases of Hurwitz numbers that behave nicely with the character formula.

**Definition 1.3.12.** *Let $\mu, \nu \vdash d$ and $r$ a positive integer, we define the **double Hurwitz number** by*

$$H_d^r(\mu, \nu) := H_{d,0}(\mu, \nu, (2, 1^{d-1})^r). \tag{1.51}$$

*From the equation (1.49),*

$$H_d^r(\mu, \nu) = \frac{1}{|\xi(\mu)||\xi(\nu)|} \sum_{\lambda \vdash d} \chi^\lambda(\mu)\chi^\lambda(\nu) \left( \frac{|C_{(2,1^{d-1})}|\chi^\lambda((2, 1^{d-1}))}{\dim S^\lambda} \right)^r \tag{1.52}$$

A double Hurwitz number counts coverings of the Riemann sphere with two special branch points – and some arbitrary number of simple ramifications. In the case that $r = 0$, from the orthogonality relations of the character table of $S_d$,

$$H_d^0(\mu, \nu) = \frac{1}{|\xi(\mu)||\xi(\nu)|} \sum_{\lambda \vdash d} \chi^\lambda(\mu)\chi^\lambda(\nu) = \frac{1}{|\xi(\mu)|}\delta_{\mu,\nu}.$$

As one might expect, by considering monodromy representations of the fundamental group, $\mu$ and $\nu$ have to be identical, and $\frac{1}{|\xi(\mu)|} = |C_\mu|/d!$.

It becomes convenient to shorten the expression of a double Hurwitz number by introducing the **weighted character**, in particular, for $\lambda, \mu \vdash d$

$$f^\lambda(\mu) = \frac{|C_\mu|\chi^\lambda(\mu)}{\dim S^\lambda}. \tag{1.53}$$

Also observe that we can rewrite Burnside's character formula in terms of weighted characters in the obvious way, which appears in some texts. For our purposes shorten $f_2^\lambda := f^\lambda((2, 1^{d-1}))$, where $d$ will be made clear from context. Then equation (1.52) becomes

$$H_d^r(\mu, \nu) = \frac{1}{|\xi(\mu)||\xi(\nu)|} \sum_{\lambda \vdash d} \chi^\lambda(\mu)\chi^\lambda(\nu) \left( f_2^\lambda \right)^r. \tag{1.54}$$

**Definition 1.3.13.** *Let $r \geq 0, d \geq 1$ and $g$ be integers, and $\mu \vdash d$. A **simple Hurwitz number** is defined by*

$$H_d^r(\mu) := H_{d,0}(\mu, (2, 1^{d-1})^r) \tag{1.55}$$

Using the character formula,

$$H_d^r(\mu) = \frac{1}{|\xi(\mu)|} \sum_{\lambda \vdash d} \left( \frac{\dim S^\lambda}{d!} \right)^{1-2g} \chi^\lambda(\mu)(f_2^\lambda)^r \tag{1.56}$$

Here we exchange freedom in the choice of ramification for freedom in the choice of the genus. In the case $r = g = 0$, we have,

$$H_d^0(\mu) = \frac{1}{d!|\xi(\mu)|} \sum_{\lambda \vdash d} \chi^\lambda((1^6))\chi^\lambda(\mu) = \frac{1}{d!}\delta_{\mu,(1^d)}, \tag{1.57}$$

corresponding to the covers of an unramified Riemann sphere.

# Chapter 2

# Expectation Values

In this chapter we discuss how Hurwitz numbers arise as expectation values of operators on an infinite dimensional vector space. The framework we will introduce in order to achieve this is but a small piece in a much larger effort conducted by a group of physicists know as the Kyoto school in the late 20th century, which laid the ground work for the establishment of the connections between certain generating functions of Hurwitz numbers and solutions to infinite families of partial differential equations. A shorter overview of this construction can be found in [10] along with some further applications that we have not mentioned. The conclusion of this section will not contribute much on its own to this pursuit however, but we will provide an outline of a remarkable connection between simple Hurwitz numbers and the KP hierarchy. For the interested reader, we have given some references that explore the connections mentioned above in greater detail towards the end of this section.

## 2.1 Fermionic Fock space

### 2.1.1 Construction of $\mathcal{F}$

In this section we will construct an infinite dimensional Hilbert space which has a very natural interpretation as a sort of 'state space' of electrons in a Dirac sea, which we encapsulate in terms of Maya diagrams.

Let $\mathbb{Z}_{\frac{1}{2}}$ denote the set of half integers i.e. $\mathbb{Z}_{\frac{1}{2}} = \{\dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots\}$. Let $V$ be a complex vector space with basis indexed by half integers, namely

$$V := \bigoplus_{i \in \mathbb{Z}_{\frac{1}{2}}} \mathbb{C} v_i, \tag{2.1}$$

for some fixed basis $\{v_i : i \in \mathbb{Z}_{\frac{1}{2}}\}$.

**Definition 2.1.1.** *For a sequence $X = \{i_1, i_2, i_3, \dots\}$ of half integers, denote*

$$\mathbf{v}_X := v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots \tag{2.2}$$

*If $X$ satisfies the following conditions:*

- $i_1 > i_2 > i_3 > \dots$
- *there is some $N \in \mathbb{N}$ with $i_n = i_N + (n - N)$ for all $n \geq N$.*

*then we call $\mathbf{v}_X$ a **semi-infinite wedge** (or abbreviated to **wedge**).*

If $\mathbf{v}_X$ and $\mathbf{v}_Z$ are semi-infinite wedges, and if $X$ and $Z$ are equal as sets, then by the alternating property of the wedge product $\mathbf{v}_X = \pm\mathbf{v}_Z$. Otherwise $\mathbf{v}_X$ and $\mathbf{v}_Z$ are linearly independent by construction. We also observe that there are countably many of these semi-infinite wedges. In particular, although each semi-infinite wedge is an object in an infinite dimensional vector space, we only require a finite amount of information to describe one i.e. the vectors $v_{i_1}, v_{i_2}, \dots v_{i_N}$.

A useful tool for visualising semi-infinite wedges is through the use of Maya diagrams. These diagrams are to be thought of as black beads on a string, placed at positions indexed by $\mathbb{Z}_{\frac{1}{2}}$. Given a wedge $\mathbf{v}_X$, the corresponding Maya diagram is obtained by placing a black bead at the position indexed by $i$ if $i \in X$, leaving the positions indexed by $i \notin X$ blank. As mentioned before, $v_X$ can be described by a finite number of indices. This is what allows us to draw Maya diagrams, where all place holders not drawn to the left of the diagram are empty, and all place holders to the right of the diagram are taken to be filled by a black bead.

**Example 2.1.2.** *The Maya diagram of the wedges $v_{\frac{5}{2}} \wedge v_{\frac{3}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{7}{2}} \wedge \cdots$ and $v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{5}{2}} \wedge \cdots$ are given by*

$$\cdots \circ \circ \bullet \circ \bullet \underset{0}{|} \circ \bullet \circ \bullet \bullet \cdots \quad \textit{and} \quad \cdots \circ \circ \circ \circ \circ \underset{0}{|} \bullet \bullet \bullet \bullet \bullet \cdots$$

*respectively. Note that the Maya diagram uniquely determines a semi-infinite wedge, so we will tend to talk about wedges and Maya diagrams interchangeably.*

It may be relevant to think of these diagrams as systems of fermions occupying energy levels in some Dirac sea. In fact the origin of the fermionic Fock space was precisely motivated to describe systems of an unspecified number of fermions, and in turn has a related space called the bosonic Fock space construed from vector spaces of polynomials in many variables.

**Definition 2.1.3.** *Let $E := \{e_1, e_2, e_3, \dots\}$ be the set of all semi-infinite wedges. We can endow the vector space $F :=$ span $W$ with an inner product $\langle\cdot,\cdot\rangle$ such that $W$ is taken to be an orthonormal set and extended by sesquilinearity. The **fermionic Fock space** $\mathcal{F}$ is taken to be the Hilbert space completion of $F$, that is*

$$\mathcal{F} := \overline{span}\, E. \tag{2.3}$$

Alternatively one can view $\mathcal{F}$ as the Hilbert space completion of the free complex vector space generated by all Maya diagrams, with a similarly defined inner product. It is standard theory of Banach spaces that any normed linear space can be embedded as a dense subspace of a Banach space, so our definition of $\mathcal{F}$ as the completion of $\operatorname{span} E$ is well defined. Given $\mathcal{F}$ has a countable orthonormal basis $E$ it follows that $\mathcal{F}$ is separable and isometrically isomorphic to the space $\ell^2(\mathbb{N})$ as Hilbert spaces, as is the case for most Hilbert spaces used in physics. Moreover for any $x \in \mathcal{F}$ we can write,

$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i := \lim_{n\to\infty} \sum_{i=1}^{n} \langle x, e_i \rangle e_i \tag{2.4}$$

### 2.1.2 Operators on $\mathcal{F}$

With a sound description of the Hilbert space $\mathcal{F}$ we can define a number of fairly natural linear operators on this space. By equation (2.4) and Hahn-Banach Extension Theorem, it is enough to define a continuous linear operator $A$ on $\operatorname{span} E$ and extend uniquely to a continuous linear operator $\hat{A}$ on $\mathcal{F}$, with $\hat{A}|_{\operatorname{span} E} = A$ and operator norms $\|A\| = \|\hat{A}\|$, specified uniquely by

$$\hat{A}x = \sum_{i=1}^{\infty} \langle x, e_i \rangle A e_i. \tag{2.5}$$

Again this does not appear to differ from the usual intuition of how an operator acts on a vector space, and so we will often refer to the operators $A$ and $\hat{A}$ interchangeably.

Within the context of Maya diagrams it is natural to define operators that correspond to adding or removing a bead. These are what we will refer to as the wedging and contracting operators, whose names become clear in the following definition.

**Definition 2.1.4.** *For each $n \in \mathbb{Z}_{\frac{1}{2}}$, we define the linear operators $\psi_n$ such that for each semi-infinite wedge $\mathbf{v}_X \in W$, $X = (i_k)_{k\in\mathbb{N}}$,*

$$\psi_n \mathbf{v}_X = \begin{cases} 0 & \text{if } n \in X, \\ (-1)^j v_{i_1} \wedge \cdots \wedge v_{i_j} \wedge v_n \wedge v_{i_{j+1}} \cdots & \text{if } n \notin X, \text{ where } i_j < n < i_{j+1}. \end{cases} \tag{2.6}$$

*and extended by linearity to $\operatorname{span} E$. Likewise we define $\psi_n^\dagger$ on $\operatorname{span} E$ to be such that*

$$\psi_n^\dagger \mathbf{v}_X = \begin{cases} (-1)^{j-1} v_{i_1} \wedge \cdots \wedge v_{i_{j-1}} \wedge v_{i_{j+1}} \cdots & \text{if } n \in X, \text{ where } n = i_j, \\ 0 & \text{if } n \notin X. \end{cases} \tag{2.7}$$

*We will often call $\psi_n$ and $\psi_n^\dagger$ the **wedging** and **contracting** operators respectively.*

From the definitions of $\psi_n$ and $\psi_n^\dagger$, for any semi-infinite wedge $e_i \in E$ we have $\psi_n e_i, \psi_n^\dagger e_i \in -E \cup \{0\} \cup E$. In particular, $\|\psi_n e_i\|, \|\psi_n^\dagger e_i\| \leq 1$ and it follows that the wedging and contracting operators are bounded linear operators on $\operatorname{span} E$ and thus continuous on $\operatorname{span} E$. We also observe that $\|\psi_n x\|, \|\psi_n^\dagger x\| \leq \|x\|$ for each $x \in \operatorname{span} E$ and have operator norms $\|\psi_n\| = \|\psi_n^\dagger\| = 1$. Given that $\psi_n$ and $\psi_n^\dagger$ are continuous linear operators on $\operatorname{span} E$ we can extend them to operators on $\mathcal{F}$ as we discussed above, and by $\psi_n$ and $\psi_n^\dagger$ we will mean the operators $\hat{\psi}_n$ and $\hat{\psi}_n^\dagger$ on $\mathcal{F}$ unless specified otherwise.

It is worth mentioning how these operators interact with one another.

**Lemma 2.1.5.** *For each $n, m \in \mathbb{Z}_{\frac{1}{2}}$, the wedging and contracting operators obey the relations*

$$\begin{aligned} \psi_n \psi_m + \psi_m \psi_n &= 0, \\ \psi_n^\dagger \psi_m^\dagger + \psi_m^\dagger \psi_n^\dagger &= 0, \\ \psi_n \psi_m^\dagger + \psi_m^\dagger \psi_n &= \delta_{nm}. \end{aligned}$$

*Moreover, $\psi_n$ and $\psi_n^\dagger$ are indeed adjoint.*

*Proof.* These relations can be easily shown considering the action of these operators on $E$ in terms of their Maya diagrams. We show the most interesting example. The operators of the form $\psi_n \psi_m^\dagger$ and $\psi_m^\dagger \psi_n$ have a nice interpretation in terms of Maya diagrams: by attempting to add a black bead at position $n$ and then attempting to remove a black bead at position $m$, multiplying the result by the number of black beads one needs to jump across between positions $n$ and $m$. Consider the Maya diagram of an arbitrary semi-infinite-wedge $\mathbf{v}$. If $n \neq m$, there are four cases to consider, corresponding to whether or not there is a black bead at positions $n$ or $m$, and it is simply a matter of checking that indeed $(\psi_n \psi_m^\dagger + \psi_m^\dagger \psi_n)\mathbf{v} = 0$ for each case. If $n = m$ then, $\psi_n \psi_m^\dagger \mathbf{v} = 0$ and $\psi_m^\dagger \psi_n \mathbf{v} = \mathbf{v}$ if there is not a black bead at position $n$, or $\psi_n \psi_m^\dagger \mathbf{v} = \mathbf{v}$ and $\psi_m^\dagger \psi_n \mathbf{v} = 0$ if there is a black bead at position $n$. This shows the last expression given above.

For any $x, y \in \mathcal{F}$, we have

$$\begin{aligned} \langle \psi_n x, y \rangle &= \left\langle \sum_{i=1}^{\infty} \langle x, e_i \rangle \psi_n e_i, \sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \right\rangle = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle x, e_i \rangle \langle y, e_j \rangle \langle \psi_n e_i, e_j \rangle, \\ \langle x, \psi_n^\dagger y \rangle &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle x, e_i \rangle \langle y, e_j \rangle \langle e_i, \psi_n^\dagger e_j \rangle. \end{aligned}$$

To show that $\psi_n^\dagger$ is the adjoint of $\psi_n$ it is enough to show that $\langle \psi_n e_i, e_j \rangle = \langle e_i, \psi_n^\dagger e_j \rangle$ for any two semi-infinite wedges $e_i$ and $e_j$. Note $\psi_n e_i = \pm e_j$ if and only if $\pm e_i = \psi_n^\dagger e_j$, and the above equality holds. So if $\psi_n e_i \neq \pm e_j$ then $\pm e_i \neq \psi_n^\dagger e_j$ and we have both $\langle \psi_n e_i, e_j \rangle = 0 = \langle e_i, \psi_n^\dagger e_j \rangle$. So regardless the equality holds and $\psi_n^\dagger$ and $\psi_n$ are adjoint. $\square$

A family of operators of particular interest to us are the shift operators $\alpha_n$ which count all the valid ways a black bead can be picked up from a Maya diagram and shifted

$n$ places to the left. Although one has to be careful to avoid unaccounted infinite sums occurring, and for this reason we introduce the normal ordering of an operator of the form $\psi_n\psi_m^\dagger$.

**Definition 2.1.6.** *For each $n, m \in \mathbb{Z}_{\frac{1}{2}}$ we define the* ***normal ordering*** *of $\psi_n\psi_m^\dagger$ to be the operator given by*

$$E_{n,m} := \begin{cases} \psi_m\psi_n^\dagger & \text{if } 0 < n, \\ -\psi_n^\dagger\psi_m & \text{if } n < 0. \end{cases} \tag{2.8}$$

*For each $n \in \mathbb{Z}$ define the* ***shift operator***

$$\alpha_n := \sum_{i \in \mathbb{Z}_{\frac{1}{2}}} E_{i,i+n} \tag{2.9}$$

The normal ordering of $\psi_m\psi_n^\dagger$ is more traditionally denoted by $:\psi_m\psi_n^\dagger:$, however we opted for a slightly more compact notation. Using the anti-commutator relations established in Lemma 2.1.5 we observe that when $n \neq m$, $E_{n,m} = \psi_m\psi_n^\dagger$. However if $n = m$ then $E_{n,n} = \psi_n\psi_n^\dagger$ if $0 < n$ and $E_{n,n} = \psi_n\psi_n^\dagger - 1$ if $n < 0$, obtaining an operator distinct from $\psi_n\psi_n^\dagger$. The one rule that one should remember is that $E_{n,n} \neq \psi_n\psi_n^\dagger$.

The introduction of this normal ordering may seem somewhat unnecessary, however this construction makes the operator $\alpha_0$ well defined, as well as allowing us to always write $\alpha_n$ as a finite sum of bounded linear operators when acting on an element of span $E$. We also give $\alpha_0$ a special name, often called the **charge** operator. Given an arbitrary semi-infinite wedge $\mathbf{v}_X$ one can check that

$$\alpha_0\mathbf{v}_X = \sum_{n \in \mathbb{Z}_{\frac{1}{2}}} E_{n,n}\mathbf{v}_X = \sum_{n \in \mathbb{Z}_{\frac{1}{2}}, 0<n} E_{n,n}\mathbf{v}_X + \sum_{n \in \mathbb{Z}_{\frac{1}{2}}, n<0} E_{n,n}\mathbf{v}_X = \left(|X^+| - |X^-|\right)\mathbf{v}_X,$$

where $X^+ = X \cap \{n \in \mathbb{Z}_{\frac{1}{2}} : 0 < n\}$ and $X^- = (\mathbb{Z}_{\frac{1}{2}} \setminus X) \cap \{n \in \mathbb{Z}_{\frac{1}{2}} : n < 0\}$. So $\mathbf{v}_X$ is an eigenvector of $\alpha_0$ with eigenvalue $q = (|X^+| - |X^-|)$ called the **charge** of $\mathbf{v}_X$. So $\mathcal{F}$ has an orthonormal basis of eigenvectors of $\alpha_0$, these being the semi-infinite wedges.

**Example 2.1.7.** *It may not be immediately clear from the expressions above, but the charge of a semi-infinite wedge is simply the difference in the number of black beads to the left of the zero mark, and the number of empty place holders to the right of the zero mark. Below are a few examples.*

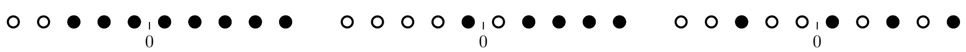



Figure 2.1: Maya diagrams of semi-infinite wedges of charge $q = 3, 0, -1$ (left to right).

*We can think of semi-infinite wedges whose Maya diagrams resemble that of the far left diagram above i.e. with all black beads squished to the right, as being representative* *of all wedges with some particular charge. In particular, moving beads respects the charge and we will soon show that indeed any wedge can be obtained by re-arranging beads on a Maya diagram of this form.*

Unfortunately, as one might expect, the shift operators $\alpha_n$ for $n \in \mathbb{N} \setminus \{0\}$ are not bounded on $\operatorname{span} E$, as we can always construct a semi-infinite wedge $\mathbf{v}$, so that $\|\alpha_n \mathbf{v}\| = M$ for any $M \in \mathbb{N}$. So there is no immediate way to extend $\alpha_n$ to an operator on $\mathcal{F}$, however we will not need to make use of this for now and we think of $\alpha_n$ as acting on a dense subspace $\operatorname{span} E$ of $\mathcal{F}$.

We now show some key characteristics of the shift operators $\alpha_n$ – when $n \neq 0$.

**Lemma 2.1.8.** *For any $n, m \in \mathbb{Z} \setminus \{0\}$,*

$$[\alpha_n, \alpha_m] = -n\delta_{n,-m}. \tag{2.10}$$

*Moreover we have that*

$$\alpha_n^\dagger = \alpha_{-n},$$

*on $\operatorname{span} E$.*

*Proof.* Noting that we can write $[AB, C] = A(BC + CB) - (AC + CA)B$, and using the relations of Lemma 2.1.5 one can write

$$[E_{i,j}, \psi_k] = \delta_{i,k}\psi_j \quad \text{and} \quad [E_{i,j}, \psi_k^\dagger] = -\delta_{j,k}\psi_i^\dagger,$$

for $i \neq j$ given we can write $E_{i,j} = \psi_j \psi_i^\dagger$. Then one can check that

$$[\alpha_n, \psi_k] = \psi_{k+n} \quad \text{and} \quad [\alpha_n, \psi_k^\dagger] = -\psi_{k-n}^\dagger.$$

Then, using the fact that $[A, BC] = [A, B]C + B[A, C]$ we calculate

$$\begin{aligned}
[\alpha_n, \alpha_m] &= \sum_{i \in \mathbb{Z}_{\frac{1}{2}}} [\alpha_n, E_{i,i+m}] \\
&= \sum_{i \in \mathbb{Z}_{\frac{1}{2}}} \left( [\alpha_n, \psi_{i+m}]\psi_i^\dagger + \psi_{i+m}[\alpha_n, \psi_i^\dagger] \right) \\
&= \sum_{i \in \mathbb{Z}_{\frac{1}{2}}} \left( \psi_{i+n+m}\psi_i^\dagger - \psi_{i+m}\psi_{i-n}^\dagger \right).
\end{aligned}$$

If $n \neq -m$ then we can split the sum above, and change indices on the right hand sum from $i \mapsto i + n$ to get $[\alpha_n, \alpha_m] = 0$. If $n = -m$ then

$$[\alpha_n, \alpha_m] = \sum_{i \in \mathbb{Z}_{\frac{1}{2}}} \left( \psi_i \psi_i^\dagger - \psi_{i-n}\psi_{i-n}^\dagger \right) = \sum_{i \in \mathbb{Z}_{\frac{1}{2}}} \left( \psi_i \psi_i^\dagger + \psi_{i-n}^\dagger \psi_{i-n} - 1 \right) = -n.$$

To show that $\alpha_n^\dagger = \alpha_{-n}$ it remains to show that $\langle \alpha_n e_i, e_j \rangle = \langle e_i, \alpha_{-n} e_j \rangle$ for any $e_i, e_j \in E$, as was the case when we considered the wedging and contracting operators.

Note that $|\langle \alpha_n e_i, e_j\rangle| \in \{0,1\}$, as there is no way to move two distinct beads to yield the same Maya diagram. If $\langle \alpha_n e_i, e_j\rangle = 0$ then there is no way to move a bead in the Maya diagram of $e_i$ to obtain $e_j$, then clearly there is no way to move a bead in $e_j$ to get to $e_i$ – else one could simply reverse the action to get $\langle \alpha_n e_i, e_j\rangle \neq 0$. So $\langle \alpha_n e_i, e_j\rangle = 0 = \langle e_i, \alpha_{-n} e_j\rangle$. If $|\langle \alpha_n e_i, e_j\rangle| = 1$ then there is a way to alter the Maya diagram from $e_i$ to get to $e_j$, reversing the direction of the alteration we have $|\langle e_i, \alpha_{-n} e_j\rangle| = 1 = |\langle \alpha_n e_i, e_j\rangle|$. Given the number of black beads jumped across is the same we can remove the absolute values and we are done. □

### 2.1.3 Partitions and Characters

Similar to how we can assign a charge to a semi-infinite wedge we can assign a partition. Moreover, the charge together with a partition uniquely determines the wedge. There are many ways of identifying a partition corresponding to a wedge, so bear this in mind during the procedure we will outline.

Let $\mathbf{v}_X$ be a semi-infinite wedge and consider the corresponding Maya diagram. The aim is to construct a Young diagram by drawing its boundary above the Maya diagram. Above each empty place holder draw a line segment traversing down and to the left, and above each black bead draw a line segment traversing up and to the right. Draw these line segments so that they form one continuous line above the Maya diagram. Excluding the two infinitely long straight lines that are formed in this procedure, we obtain the desired boundary. As a line can never traverse down and to the left or up and to the left we can guarantee that this will indeed form the outline of a rotated Young diagram. The columns traversing up and to the left we take to be the components of the corresponding partition i.e. such that the resulting Young diagram has be rotated $\frac{3\pi}{4}$ radians counter-clockwise.

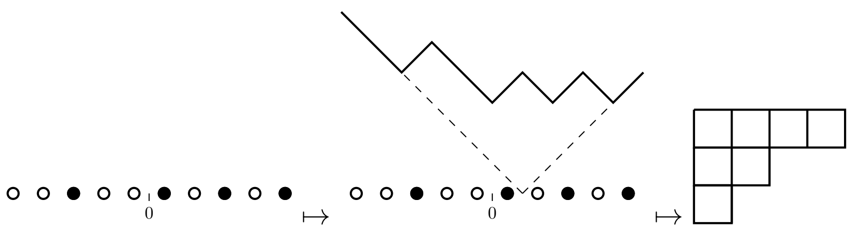



Figure 2.2: How a Young diagram is obtained from a Maya diagram.

One may then ask where the corner of the Young diagram is situated with respect to the Maya diagram, as in the above diagram. Observe the point on the continuous line above the 0 index on the Maya diagram, and consider the rectangle between this point and where the corner of the Young diagram lies above the Maya diagram. Then one can see the index of the corner of the Young diagram is given by the difference in the length of the sides of the rectangle, as depicted in figure 2.3. This is simply the difference between the number of black beads to the left of 0 and the number of empty place holders to the right of 0: precisely the charge corresponding to $\mathbf{v}_X$

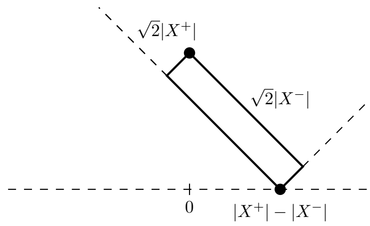



Figure 2.3: Determining where the corner of the Young diagram rests on the Maya diagram.

With these observations, it is enough to see that a Maya diagram – and thus a semi-infinite wedge – is uniquely determined by a tuple $(q, \lambda)$ consisting of an integer $q$ and partition $\lambda$. We will be interested in how operators behave with these partitions and so it is convenient to index semi-infinite wedges in this way, using ket notation we define $|q, \lambda\rangle := \mathbf{v}_X$ where $\mathbf{v}_X$ is a semi-infinite wedge whose Maya diagram can be identified by $(q, \lambda)$.

As mentioned before, the basis elements $|q, 0\rangle$ are 'distinguished' members of the set of all wedges of charge $q$. That is all semi-infinite wedges $|q, \lambda\rangle$ with same $q$ are obtained in orbit of $|q, 0\rangle$ under the operators $E_{i,j} = \psi_j \psi_i^\dagger$, $j \neq i$. The operators $E_{i,j}$ correspond to attempting to pick up a black bead at position $i$ and place it at position $j$ in regards to Maya diagrams, and as a result preserve the charge, yet alter the corresponding partition. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, then we can write

$$\begin{aligned}
\psi_{q-\frac{1}{2}+\lambda_1} \psi^\dagger_{q-\frac{1}{2}} |0, q\rangle &= |q, (\lambda_1)\rangle, \\
\psi_{(q-1)-\frac{1}{2}+\lambda_2} \psi^\dagger_{(q-1)-\frac{1}{2}} |(\lambda_1), q\rangle &= |q, (\lambda_1, \lambda_2)\rangle \\
&\vdots \\
\psi_{(q-(n-1))-\frac{1}{2}+\lambda_n} \psi^\dagger_{(q-(n-1))-\frac{1}{2}} |q, (\lambda_1, \ldots, \lambda_{n-1})\rangle &= |q, \lambda\rangle
\end{aligned}$$

**Example 2.1.9.** *Constructing the wedge $|0, (4, 2)\rangle$ from $|0, 0\rangle$:*

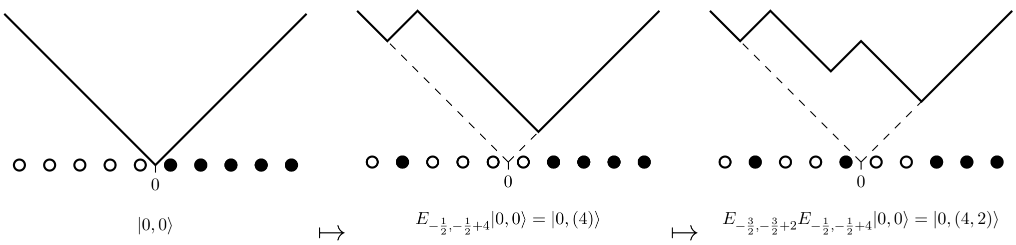


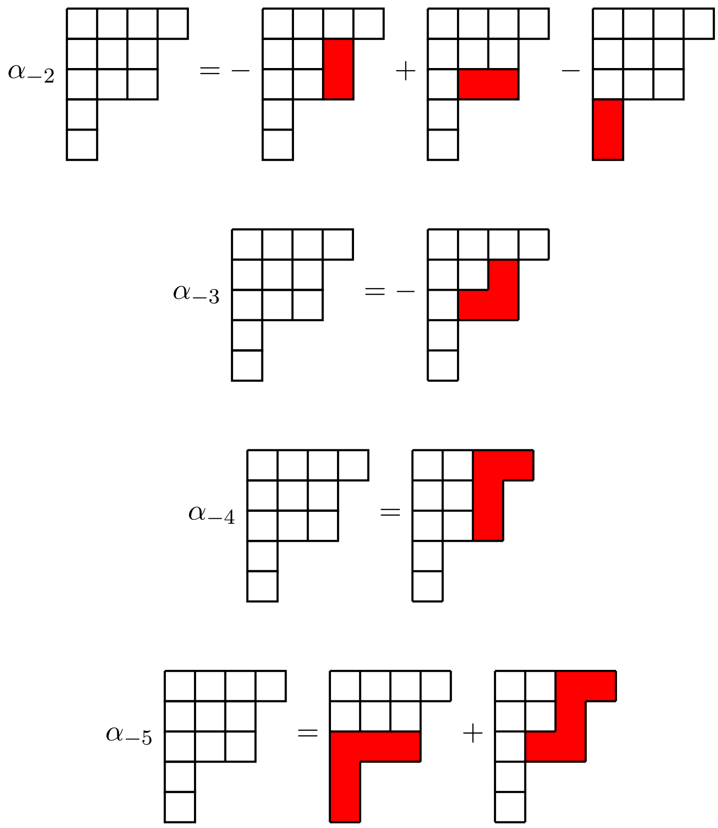

Figure 2.4

More generally the action of $E_{i,j}$ with $i \neq j$ can be seen as an attempt to add or remove a connected component along the boundary of the corresponding partitions – i.e. a **border strip** or **rim hook** of the corresponding partition – where sign of the resulting partition is dependent upon the number of black beads between $i$ and $j$, given by the number of rows the border strip traverses in the corresponding Young diagram. Then, for any $n \in \mathbb{N}$, the operator $\alpha_n$ counts all possible ways in which a border strip with $n$ squares – the length of the border strip – can be added to a partition, and $\alpha_{-n}$ counts all possible ways in which a border strip of length $n$ can be removed from a partition (up to sign).

**Example 2.1.10.** *We will not explicitly prove the assertion above but will provide a number of examples to convince the reader. Another overview of this process can be found in [15]. Consider the partition $|0,(4,3,3,1,1)\rangle$. One can verify that*

$$
\begin{aligned}
\alpha_{-2}|0,(4,3,3,1,1)\rangle &= -|0,(4,2,2,1,1)\rangle + |0,(4,3,1,1,1)\rangle - |0,(4,3,3)\rangle, \\
\alpha_{-3}|0,(4,3,3,1,1)\rangle &= -|0,(4,2,1,1,1)\rangle, \\
\alpha_{-4}|0,(4,3,3,1,1)\rangle &= |0,(2,2,2,1,1)\rangle, \\
\alpha_{-5}|0,(4,3,3,1,1)\rangle &= |0,(4,3)\rangle + |0,(2,2,1,1,1)\rangle,
\end{aligned}
$$

*where in terms of Young diagrams*



*The parts of the partition that have been highlighted are border strips of the original partition that can be thought of as being 'removed' by the shift operator.*

This interpretation of the action of the shift operators is incredibly reminiscent of the recursive procedure specified by Murnaghan-Nakayama rule which calculates irreducible characters of the symmetric group by counting all the ways one can remove border strips from a partition, where the size of the strips is determined by another partition. A proof of this result can be found in [17].

**Lemma 2.1.11** (Murnaghan-Nakayama Rule)**.** *Let $\lambda, \mu \vdash d$ where $\mu = (\mu_1, \mu_2, \dots, \mu_n)$*

$$\chi^\lambda(\mu) = \sum_{|\beta|=|\mu_1|} (-1)^{height(\beta)} \chi^{\lambda-\beta}(\mu - \mu_1) \tag{2.11}$$

*where the sum is over border strips $\beta$ of $\lambda$ of length $|\mu_1|$, $\lambda - \beta$ is the partition resulting from removing $\beta$, and $\mu - \mu_1$ is the partition $(\mu_2, \mu_3, \dots, \mu_n)$. Here $height(\beta)$ is the number of rows $\beta$ traverses minus 1.*

One can recursively apply this formula to calculate $\chi^\lambda(\mu)$ as simply a sum of 1's and $-1$'s. To encode the irreducible characters of the symmetric in terms of coefficients of elements in $\mathcal{F}$, we need operators that exhibit the recursive nature of this rule.

**Definition 2.1.12.** *Given a partition $\mu \vdash d \in \mathbb{N}$, with $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ define the operator*

$$\alpha_\mu = \prod_{i=1}^{n} \alpha_{\mu_i} \quad \textit{and} \quad \alpha_{-\mu} = \prod_{i=1}^{n} \alpha_{-\mu_i} \tag{2.12}$$

*By the commutation relations given in Lemma 2.1.8, the order of the products above does not matter, and we have $\alpha_\mu^\dagger = \alpha_{-\mu}$.*

Then as one might expect, we can almost directly apply the above operators to obtain irreducible characters of the symmetric group.

**Lemma 2.1.13.** *Let $\mu \vdash d \in \mathbb{N}$. Then*

$$\alpha_{-\mu}|q, \lambda\rangle = \chi^\lambda(\mu)|q, 0\rangle \quad \textit{and} \quad \alpha_\mu|q, 0\rangle = \sum_{\lambda \vdash d} \chi^\lambda(\mu)|q, \lambda\rangle \tag{2.13}$$

*for any fixed $\lambda \vdash d$.*

*Proof.* The first formula is more or less a straight forward application of the Murnaghan-Nakayama rule, given the interpretation of the action of $\alpha_n$ on a partition.

Given the first formula holds, for any $\lambda \vdash d$ the inner product evaluates to

$$\langle q, 0|\alpha_{-\mu}|q, \lambda\rangle = \chi^\lambda(\mu).$$

So given the irreducible characters of the symmetric group are all real valued, we have that

$$\langle q, \lambda|\alpha_{-\mu}^\dagger|q, 0\rangle = \langle q, \lambda|\alpha_\mu|q, 0\rangle = \overline{\chi^\lambda(\mu)} = \chi^\lambda(\mu).$$

So we can write

$$\alpha_\mu|q,0\rangle = \sum_{\lambda\vdash d} \chi^\lambda(\mu)|q,\lambda\rangle,$$

given that $\alpha_\mu|q,0\rangle$ must be a sum of semi-infinite wedges, corresponding to partitions of size $d$ and charge $q$.

□

This is a remarkable formula to emerge so easily from simple considerations of how the basis elements behave under certain operations. Given partitions $\mu$ and $\nu$ of $d$, we can calculate the product of the columns of the character table of $S_d$. That is

$$\langle q,0|\alpha_{-\mu}\alpha_\nu|q,0\rangle = \sum_{\lambda\vdash d} \chi^\lambda(\nu)\langle q,0|\alpha_{-\mu}|q,\lambda\rangle = \sum_{\lambda\vdash d} \chi^\lambda(\nu)\chi^\lambda(\mu) \tag{2.14}$$

### 2.1.4 Matrix elements

Given the close relationship between Hurwitz numbers and the symmetric group, and the relationship displayed above between fermionic Fock space and the symmetric group, a natural approach is to then bridge the gap. In particular, using the character equation of 1.3.10 we can encode certain Hurwitz numbers as expectation values of certain operators on fermionic Fock space. Before proceeding we want a way of encoding information regarding the simple ramifications of a Hurwitz number into the expectation value of an operator.

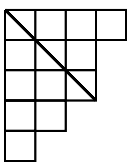
**Definition 2.1.14.** *Define the operator $F_2$ on span $E$ by*

$$F_2 := \sum_{i\in\mathbb{Z}_{\frac{1}{2}}} \frac{1}{2} i^2 E_{i,i}.$$

This is well defined on span $E$ in same way that $\alpha_0$ is well defined on span $E$ – only a finite number of terms act non-trivially given a semi-infinite wedge. Again this operator fails to be bounded, yet this still does not concern us.

To realise the action of $F_2$ on a semi-infinite wedge we require a new way of viewing a partition, and with it a useful result. To a partition $\lambda$ we can associate two decreasing sequences of half integers $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$. These are constructed by splitting the Young diagram of $\lambda$ along the diagonal sloping down and to the right, and letting $a_i$ and $b_i$ be the number of boxes – including the half boxes – to the right and below of the $i$th diagonal box, respectively.

**Example 2.1.15.** *Consider the partition $\lambda = (4, 3, 3, 2, 1)$,*



*We then have* $(a_1, a_2, a_3) = (3 + \frac{1}{2}, 1 + \frac{1}{2}, \frac{1}{2})$ *and* $(b_1, b_2, b_3) = (4 + \frac{1}{2}, 2 + \frac{1}{2}, \frac{1}{2})$.

Likewise given any two decreasing sequences of half integers $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$, we can construct a partition. These tuples are referred to as the modified Frobenius notation of a partition. There is then a nice result between this notation and the weighted character $f_2(\lambda)$ which we will make use of.

**Lemma 2.1.16** (Frobenius)**.** *Let* $\lambda$ *be a partition with modified Frobenius notation* $(a_1, a_2, \dots, a_n)$ *and* $(b_1, b_2, \dots, b_n)$*. Then the weighted character* $f_2^\lambda$ *is given by*

$$f_2^\lambda = \frac{1}{2} \sum_{i=1}^{n} \left( a_i^2 - b_i^2 \right). \tag{2.15}$$

With a slight change in notation, this result can be found as an exercise in [8, p.52]. Note that many other weighted characters can be expressed as polynomials in a similar way, derivations of these can be found in [27].

Given a wedge $|0, \lambda\rangle$, let $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ be the modified Frobenius notation of the partition $\lambda$. Observe that $(a_1, a_2, \dots, a_n)$ describes the positions of black beads to the left of the 0 mark on the Maya diagram of $|0, \lambda\rangle$, and $(b_1, b_2, \dots, b_n)$ the empty positions to the right of the 0 mark on the Maya diagram.

**Example 2.1.17.** *Consider the partition* $\lambda = (4, 3, 3, 2, 1)$.

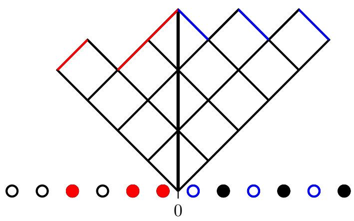



Figure 2.5: Maya diagram of $|0, (4, 3, 3, 2, 1)\rangle$ with corresponding Young diagram.

*We see that the Maya diagram corresponding to* $\lambda$ *has empty slots to the right of* 0 *(coloured blue) at positions* $-(4 + \frac{1}{2})$, $-(2 + \frac{1}{2})$, $-\frac{1}{2}$, *and black beads placed to the right of* 0 *(coloured red) at positions* $3 + \frac{1}{2}$, $1 + \frac{1}{2}$ *and* $\frac{1}{2}$. *We see that the black beads are placed at positions* $a_1, a_2, a_3$ *and empty slots at* $-b_1, -b_2, -b_3$.

Define the operator $F_2$ on $\operatorname{span} E$ by

$$F_2 := \sum_{i \in \mathbb{Z}} \frac{1}{2} i^2 E_{i,i}.$$

This is indeed well defined as the action of $F_2$ on a semi-infinite wedge can always be written in a finite number of terms. Then let $\lambda$ be a partition with modified Frobenius notation $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. Then by Lemma 2.1.16 and the observations above

$$\begin{aligned} F_2|0,\lambda\rangle &= \frac{1}{2}\left(\sum_{i\in\mathbb{Z},i<0} i^2\psi_i\psi_i^\dagger - \sum_{i\in\mathbb{Z},i<0} i^2\psi_i^\dagger\psi_i\right)|0,\lambda\rangle \\ &= \frac{1}{2}\left(\sum_{i=1}^{n}(a_i)^2 - \sum_{i=1}^{n}(b_i)^2\right)|0,\lambda\rangle \\ &= f_2^\lambda|0,\lambda\rangle \end{aligned}$$

We have successfully encoded the weighted character $f_2^\lambda$ as an eigenvalue of an operator on span $E$.

**Theorem 2.1.18.** *Let $\mu, \nu \vdash d$ and $r$ a positive integer, then*

$$H_d^r(\mu,\nu) = \frac{1}{|\xi(\mu)||\xi(\nu)|}\langle 0,0|\alpha_\mu F_2^r \alpha_{-\nu}|0,0\rangle \tag{2.16}$$

*Proof.* This is more or less a straight forward application of what we have discussed so far in this chapter. By above and Lemma 2.1.13

$$\begin{aligned} \langle 0,0|\alpha_{-\mu}F_2^r\alpha_\nu|0,0\rangle &= \sum_{\lambda\vdash d}\chi^\lambda(\nu)\langle 0,0|\alpha_\mu F_2^r|0,\lambda\rangle \\ &= \sum_{\lambda\vdash d}\chi^\lambda(\nu)(f_2^\lambda)^r\langle 0,0|\alpha_\mu|0,\lambda\rangle \\ &= \sum_{\lambda\vdash d}\chi^\lambda(\nu)(f_2^\lambda)^r\chi^\lambda(\mu) \end{aligned}$$

Then using Burnside's character formula we obtain equation (1.52), and

$$\langle 0,0|\alpha_{-\mu}F_2^r\alpha_\nu|0,0\rangle = |\xi(\mu)||\xi(\nu)|H_d^r(\mu,\nu).$$

□

As it is now we have done nothing but rewrite equation (1.54) in terms of operators on $\mathcal{F}$, relying on the mutual connection between Hurwitz numbers, Fock space and the symmetric group. The power of this approach comes from the context in which we have set Hurwitz numbers.

### 2.1.5 Further Applications

To end, we will give a brief outline of how one can show the generating function of the simple Hurwitz numbers is a solution to the KP hierarchy. We will follow the approach given in [4]. Another approach can be found in [25], and a similar result is given for double Hurwitz numbers in [21]. The process we will outline arises fairly naturally from the context in which we have placed Hurwitz numbers above. Note the fermionic Fock space has a symmetric counterpart $\mathcal{B}$ dubbed the bosonic Fock space, consisting of polynomials in an infinite number of variables. In particular,

$$\mathcal{B} := \bigoplus_{q\in\mathbb{Z}} \mathbb{C}[x_1, x_2, \dots]z^q.$$

The space $\mathbb{C}[x_1, x_2, \dots]$ has a natural basis given by Schur polynomials $s_\lambda$, so $\mathcal{B}$ has a basis $\{s_\lambda z^q\}$ indexed by partitions and integers. $\mathcal{F}$ and $\mathcal{B}$ are isomorphic as vector spaces under the identification $\sigma : |q, \lambda\rangle \mapsto s_\lambda z^q$. This isomorphism preserves actions of the infinite dimensional Clifford and Heisenberg algebras on $\mathcal{F}$ and $\mathcal{B}$, this observation is often called the boson-fermion correspondence.

The infinite dimensional Lie group $GL_\infty$ is defined by

$$GL_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} : a_{ij} = \delta_{ij} \text{ for all but finitely many } a_{ij}\}.$$

Endowing $\mathcal{F}$ with a particular representation of the closure $\overline{GL_\infty}$, it follows that the solutions of the KP hierarchy are exactly those in the image of the orbit of the vacuum vector $|0,0\rangle$ under the action of $\overline{GL_\infty}$ i.e. $\sigma\left(\overline{GL_\infty}|0,0\rangle\right)$ are exactly the solutions to the $KP$ hierarchy.

The generating function of the simple Hurwitz numbers $H(z, x_1, x_2, \dots)$ can be written as

$$H(z, x_1, x_2, \dots) = \sum_{d=1}^{\infty}\sum_{r=1}^{\infty}\sum_{\mu\vdash d} H_d^r(\mu) x_{\mu_1} x_{\mu_2} \dots x_{\mu_n} \frac{z^r}{r!},$$

where $\mu_1, \dots, \mu_n$ are the components of each partition $\mu$. The 'cut-and-join' operator $J$ is defined on $\mathcal{B}$ by

$$J = \frac{1}{2}\left(\sum_{n\in\mathbb{N}}\sum_{i+j=n} (i+j)x_i x_j \frac{\partial}{\partial x_{i+j}} + ij x_{i+j}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right),$$

and it can be show that $H(z, x_1, x_2, \dots)$ is a solution to the equation

$$\frac{\partial}{\partial z} H(z, x_1, x_2, \dots) = JH(z, x_1, x_2, \dots),$$

a proof of which can be found in [9]. Utilising equation (1.57) we observe that

$$H(0, x_1, x_2, \dots) = e^{x_1}$$

and so we can write

$$H(z, x_1, x_2, \dots) = e^{zJ}e^{x_1}.$$

One can then construct an operator $\tilde{H} \in \overline{GL_\infty}$, written as exponentials of shift operators ($\alpha_n$'s) such that under the boson-fermion correspondence

$$\sigma\left(\tilde{H}|0,0\rangle\right) = e^{zJ}e^{x_1} = H(z, x_1, x_2, \dots).$$

It follows that $H(z, x_1, x_2, \dots)$ is a solution to the $KP$ hierarchy.

For those that are interested in Gromov-Witten theory, a collection of works by A.Okounkov and R.Pandharipande [23, 22, 24] utilises this understanding of Hurwitz numbers to prove a handful of remarkable results in the subject.

# Conclusion

In summary, a Hurwitz number is a topological invariant of a genus $g$ surface, given by an enumeration of ramified coverings admitting some specified collection of ramification profiles. These invariants are ultimately combinatorial in nature, as emphasised by the correspondence between monodromy representation and ramified coverings, allowing us to exhibit Hurwitz numbers as counts of group homomorphism between finite groups. With some effort, these counts can be represented as products of irreducible characters of the symmetric group with Burnside's character formula. Furthermore, by considering certain operators on fermionic Fock space, we can represent products of irreducible characters of the symmetric group as expectation values. Many other results concerning the nature of Hurwitz numbers can be shown working within this formalism. The expressions obtained for the double Hurwitz numbers from each of these considerations are given below.

$$
\begin{aligned}
H_d^r(\mu,\nu) &= \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|} \\
&= \frac{1}{d!}\left|\left\{(\rho,\sigma,\tau_1,\ldots,\tau_r)\in (S_d)^{\times(2+r)} : \rho\sigma\tau_1\cdots\tau_r = e, \tau_i \in C_{(2,1^{d-2})}, \rho\in C_\mu, \sigma\in C_\nu\right\}\right| \\
&= \frac{1}{|\xi(\mu)||\xi(\nu)|}\sum_{\lambda\vdash d}\chi^\lambda(\mu)\chi^\lambda(\nu)\left(f_2^\lambda\right)^r \\
&= \frac{1}{|\xi(\mu)||\xi(\nu)|}\langle 0,0|\alpha_\mu F_2^r \alpha_{-\nu}|0,0\rangle.
\end{aligned}
$$

An interesting theme to note throughout the first chapter is the exhibition of a 'conservation of complexity'. That is, by changing the complexity of the language in which we talk about Hurwitz numbers, the difficulty in calculation changes accordingly. For example, it is fairly simple to give the original formulation of Hurwitz numbers in terms of counts of holomorphic maps, yet this definition is cumbersome to use and much effort is needed to carry out any calculations using it. With the classification of holomorphic maps in terms of ramified coverings and by introducing monodromy representation, we were able to bring holomorphic maps into a realm of further abstraction where the machinery we developed was able to focus on identifiable characteristics of isomorphism classes of holomorphic maps, rather than explicit descriptions. As a result Hurwitz numbers can be calculated with relative ease using this perspective, as much of the work is already done for us through this abstraction. In the case of Burnside's character formula, simplicity in the formulaic description is sacrificed in favour of ease of calculation: in knowing all the irreducible characters, and conjugacy classes, of the symmetric group $S_n$, all Hurwitz numbers of degree $n$ can be calculated directly using this formula. This observation is not unique to our discussion, yet it seems indicative on any sound mathematical theory.

As a final note, we should acknowledged the rich theory behind Riemann surfaces which underpins our discussion on holomorphic maps. For example, the Riemann-Hurwitz formula is an immediate consequence of the famous Riemann-Roch theorem. Much of Chapter 1 can alternately be found in [20], which gives a broad introduction to Riemann surfaces.

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