Classification of Separable Approximately Finite-Dimensional C^*-Algebras

Sean Yang

Department of Mathematics

The University of Auckland

Supervisor: Dr Pedram Hekmati

A Honours thesis of 2025 submitted to fulfill the requirements of the degree BAdvSci(Hons) in Mathematics at the University of Auckland

Abstract

We shall prove the celebrated Elliott's classification theorem [1] of separable approximately finite-dimensional C^*-algebras using the techniques of K-theory, and provide a more categorical interpretation of the main theorem and the results leading up to it. The methodology involved will follow closely to the K-Theory textbook by M. Rørdam [2] alongside further elaboration of techniques involved in certain technical results for the sake of clarity. We shall also generalize the intermediary results involving inductive limits to be inductive limits indexed by directed sets, this includes the continuity of the K_0 functor.

Acknowledgements

I would firstly like to thank my supervisor Pedram Hekmati for always being supportive and understanding of my situation, and always willing to provide guidance and helpful insights for my project. I would also like to thank Dean Miller for helping me prove and understand certain results of the topic. Lastly, I would like to thank my father Xiaoping (Peter) Yang for always looking out for me, and helping the family and me out tirelessly in order to make this thesis much more possible.

Contents

1 | Introduction

The theory of C^*-algebras had never been more fundamental and quintessential in the development of operator theory and related functional analysis topics. Elliott's classification program is one of the ongoing research program in an attempt to classify those C^*-algebras given sufficiently nice conditions. One of the famous, and perhaps the first, classification result of C^*-algebras was provided by G.A. Elliott in his paper in 1976 [1]. Which specifically classifies *separable approximately finite-dimensional algebras* using the tools of K-theory.

The general process goes as follows, for a unital C^*-algebra A, we consider the set of all possible projection matrices (matrices p satisfying $p = p^2 = p^*$) with entries of A, $\mathrm{P}_\infty(A)$, identified by Murray-von Neumann equivalence $\sim$. In the case of $A = \mathbb{C}$, the equivalence relation $\sim$ identifies complex orthogonal projection matrices with their ranks. We consider the set of equivalence classes, $\mathrm{P}_\infty(A)/\sim$, identified by thre relation $\sim$, and endow it with an Abelian semigroup structure by equipping it with the *direct sum* operation $\oplus$. The $\oplus$ operation behaves in a natual way and is compitable with the $\sim$ relation, for example,

$$\begin{pmatrix} 2 & i \\ -1 & 3 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 2 & i & 0 & 0 \\ -1 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The Abelian semigroup $(\mathrm{P}_\infty(A)/\sim, \oplus)$ can be extended further to a more natural structure of Abelian groups by taking the Grothendieck completion, this process introduces inverses from existing elements which results in an Abelian group that is typically considered the smallest Abelian group containing $(\mathrm{P}_\infty(A)/\sim, \oplus)$, or the Abelian group closure of $(\mathrm{P}_\infty(A)/\sim, \oplus)$ if you will. This grants us the K_0-group for A which is an invariant of A, and in fact, we obtain a functor K_0 from the category of unital C^*-algebras $\mathsf{C^*\text{-}Alg_1}$ to the category of Abelian groups Ab. For example, the C^*-algebra of complex numbers $\mathbb{C}$ associates to the Abelian group $\mathbb{Z}$ as one identifies $K_0(\mathbb{C}) \cong \mathbb{Z}$. For the process of a nonunital C^*-algebra A, we initiate the construction from its unitization $\tilde{A}$, then the K_0-group for A with be the kernel of $K_0(\pi)$ where $\pi : \tilde{A} \to \mathbb{C}$ is the natural projection map. Hence one has a functor from the category of C^*-algebras $\mathsf{C^*\text{-}Alg}$ to Ab. This process will be described in Chapter 3.

Unforunately, the invariant K_0 is not enough to distinguish even finite-dimensional C^*-algebras, thus we shall enrich the structure of $K_0(A)$ by giving a 'natural' ordering, making it into an *ordered Abelian group with a distinguished order unit* $(K_0(A), K_0(A^+), [1_A]_0)$. For example, the matrix algebras $\mathcal{M}_2(\mathbb{C})$ and $\mathcal{M}_3(\mathbb{C})$ associates to the triples $(\mathbb{Z}, \mathbb{Z}^+, 2)$, and $(\mathbb{Z}, \mathbb{Z}^+, 3)$, which have different order units, namely 2 and 3 respectively. Hence one has a stronger invariant, and as it turns out, this invariant is sufficient to classify *approximately finite-dimensional algebras*, which is the celebrated Elliott's Theorem 4.5.5 and will be proven using Elliott's Intertwining Argument 5.4.8. The classification theorem shall be the main result of Chapter 4 and this thesis.

The concepts and results introduced will be done as generally as possible despite the main result not requiring the full generality. We will also have extra emphasis on category theory, as we can package the results more neatly in categorical terms. The reader should be familiar with the language of basic category theory; see Appendix 5.1 to 5.3.

2 | Fundamentals on C^*-Algebras

We shall formalize the definitions and fundamental results from C^*-algebras with an extra emphasis on category theory. While most of the results introduced should be well-known, we will also shed light on more specific results that will be used later in the thesis. Most of the well-known results will be cited from the B. Blackadar [3] and M. Takesaki [4] operator theory textbooks.

2.1 C^*-Algebras

Definition 2.1.1. *(C^*-Algebra).* A **C^*-algebra** is a complex Banach space $(A, \|\cdot\|)$ with:

- A multiplication operation $A \times A \to A$ that is an associative bilinear map, and satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.
- An involution operation $A \to A$, typically denoted as *, that satisfies
  - $(a+b)^* = a^* + b^*$;
  - $(\lambda a)^* = \overline{\lambda} a^*$;
  - $(a^*)^* = a$;
  - (C^*-identity). $\|a^* a\| = \|a\|^2$;

  for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

We say A is **unital** if it has a multiplicative identity, which we denote as 1 or 1_A depending on the context, and note that we require that $\|1_A\| = 1$. We say a A is a ***-algebra** if A is not necessarily complete. In general, we denote $\|\cdot\|_A$ to be the norm of A, and the subscripts is omitted if the context clear. We say an element $a \in A$ is:

- **Normal** if $aa^* = a^* a$.
- **Self-adjoint** (or **Hermitian**) if $a = a^*$, and denote $\mathrm{H}(A)$ to be the set of self-adjoints elements of A.
- **Projection** if $a = a^* = a^2$, and denote $\mathrm{P}(A)$ to be the set of projections of A.

  Assume A is unital.
- **Unitary** if $aa^* = a^* a = 1_A$, and denote $\mathcal{U}(A)$ to be the set of unitary elements of A.
- **Invertible in A** if there is a $b \in A$ such that $ab = ba = 1_A$, and we write $a^{-1} = b$, and denote $\mathrm{GL}(A)$ to be the set of all invertible elements of A.

Given a subset $S \subseteq A$, denote $A\langle S \rangle$ to be the smallest not necessarily unital C^*-subalgebra of A that contains S, if $S = \{a_1, ..., a_n\}$, then write $A\langle S \rangle = A\langle a_1, ..., a_n \rangle$. A linear map $\varphi : A \to B$ between C^*-algebra is a **homomorphism** if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$, and we add the prefix * if furthermore $\varphi(a^*) = \varphi(a)^*$ for $a \in A$. We note that *-homomorphisms are automatically Lipschitz continuous with Lipschitz constant 1, i.e. φ is **norm-decreasing**; see [3, II.1.6.6]. If A and B are unital, then we say φ is **unital** if $\varphi(1_A) = 1_B$.

One of the crucial results of C^*-algebra is that if one has a *-homomorphism $\varphi : A \to B$, then φ is an isometry if, and only if, φ is injective; see [3, II.2.2.9]. We say such φ to be ***-embeddings**. We say A is **embedded** in B if there is an injective *-homomorphism from A to B, and we say they are **isomorphic** if the aforementioned *-homomorphism is also surjective, i.e. a ***-isomorphism**. The * prefix will be omitted if the context is clear.

Thus in the category of C^*-algebras, denoted as C*-Alg, the morphisms are *-homomorphisms. We observe that $\mathbb{C}$ is the most trivial nonzero C^*-algebra. We shall also denote C*-Alg_1 to be the category of unital C^*-algebra where the morphisms are unital *-homomorphisms. Here are some common examples of C^*-algebras.

Example 2.1.2. *(Space of Bounded Operators)*. Given Banach spaces X and Y, we denote $\mathcal{B}(X,Y)$ to be the set of all bounded linear operators from X to Y, and $\mathcal{B}(X) = \mathcal{B}(X,X)$. We note that $\mathcal{B}(X)$ is a unital Banach algebra. If furthermore X is a Hilbert space, then $\mathcal{B}(X)$ is a unital C^* -algebra under the Hermitian adjoint. The restriction to compact operators $\mathcal{K}(X)$ is a nonunital C^*-algebra unless X is finite-dimensional. Thus in particular, spaces like $\ell_2(\mathbb{N})$, or $n \times n$ complex matrices $\mathcal{M}_n(\mathbb{C})$ under the conjugate-adjoint are C^*-algebras.

Example 2.1.3. *(Space of Continuous Maps)*. Given topological spaces X and Y, we denote $\mathcal{C}(X,Y)$ to be the space of continuous maps from X to Y. If Y is normed, we denote $\mathcal{C}_0(X,Y)$ to be the space of continuous maps that vanishes at infinity, i.e. $f \in \mathcal{C}_0(X,Y)$ if, and only if, $f : X \to Y$ is continuous and for each $\varepsilon > 0$, there is a compact $K \subseteq X$ such that for all $x \in X \setminus K$, one has $\|f(x)\| \leq \varepsilon$. We note that if Y is normed (resp. a Banach algebra, or C^*-algebra), then $\mathcal{C}_0(X,Y)$ forms a normed space (resp. a Banach algebra, or C^*-algebra), and if X is compact, then $\mathcal{C}(X,Y) = \mathcal{C}_0(X,Y)$ also satisfies those respective properties. In those cases, we can equip those spaces with the supremum norm and pointwise operations. Note that if Y is a commutative C^*-algebra, then X can be assumed to be locally compact Hausdorff (resp. compact Hausdorff) and Y can be assumed to be $\mathbb{C}$ in the case of $\mathcal{C}_0(X,Y)$ (resp. $\mathcal{C}(X,Y)$) by Commutative Gelfand-Naimark 2.4.1. We denote $\mathcal{C}(X) := \mathcal{C}(X,\mathbb{C})$ and $\mathcal{C}_0(X) := \mathcal{C}_0(X,\mathbb{C})$, where the former is a unital C^*-algebra while the latter is nonunital if X is noncompact.

The most natural way of constructing new C^*-algebras from old ones is by taking direct sums.

Example 2.1.4. *(Sum of C^*-algebras)*. Given C^*-algebras A and B, we can define the direct sum $A \oplus B$ of C^*-algebras A and B with elements as (a,b) for $a \in A$ and $b \in B$, and equip $A \oplus B$ with pointwise operations, and the norm $\|(a,b)\| = \max\{\|a\|, \|b\|\}$. Then $A \oplus B$ is indeed a C^*-algebra.

The ideals of C^*-algebras are have elegant properties. Given a two-sided ideal $I \subseteq A$, which means that I is a subspace such that $rI \subseteq I$ for all $r \in A$ (note that we do not require I to be a *-algebra). Then if I is closed, it follows that I is now a *-algebra, hence a C^*-algebra, furthermore, one has a quotient C^*-algebra A/I with a canonical *-homomorphism $\pi : A \to A/I$ where the set and norm of A/I is defined in the context of quotient Banach spaces; see [3, II.5.1.1]. We obtain a following variant of the first isomorphism theorem which will be used in Chapter 4.

Theorem 2.1.5. *(First Isomorphism)*. Let $\varphi : A \to B$ be a *-homomorphism between C^*-algebras A and B, then $\operatorname{im}(\varphi)$ is a C^*-algebra. If $\psi : A \to C$ is a *-homomorphism into a C^*-algebra C, and $\ker(\varphi) \subseteq \ker(\psi)$, then there is a unique *-homomorphism $\mu : \operatorname{im}(\varphi) \to C$ such that $\psi = \mu \circ \varphi$.

Proof. As $I = \ker(\varphi)$ is a closed two-sided ideal of A, then A/I is a quotient C^*-algebra. Define $\nu(a + I) = \varphi(a)$, then it follows that ν is a well-defined map, in particular ν is a *-embedding, so $\operatorname{im}(\varphi) = \operatorname{im}(\nu)$ is a C^*-algebra as ν is an isometry. Now define $\mu : \operatorname{im}(\varphi) \to C$ as $\mu(\varphi(a)) = \psi(a)$, which again is a well-defined map as $\ker(\varphi) \subseteq \ker(\psi)$, and it follows that μ is a *-homomorphism. Thus $\psi = \mu \circ \varphi$ and uniqueness is apparent. ■

It is well-known that all finite-dimensional C^*-algebras are just direct sums of $\mathcal{M}_n(\mathbb{C})$. The result is stated below, and we shall assume this fact.

Theorem 2.1.6. Let A be a finite-dimensional C^*-algebra, then there exists $n_1, ..., n_m \in \mathbb{N}$ for some $m \in \mathbb{N}$ such that

$$A \cong \mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_m}(\mathbb{C}).$$

Proof. See [2, Theorem 7.1.5]. ■

2.2 Unitization

One of the most important concepts is making our algebras unital. This process of *unitization* is extremely relevant in our study of K-theory, so this chapter aims to provide a careful treatment of understanding the structure of unitization and its various functorial properties.

Construction 2.2.1. *(Unitization of C^*-algebras)*. For any C^*-algebra A, we can embed A into a unital C^*-algebra, which we denote as $\tilde{A}$, such that A is a maximal ideal in $\tilde{A}$.
Let $1_{\tilde{A}}$ be some symbol, and consider the direct sum of vector spaces:

$$\tilde{A} = A \oplus \mathbb{C}1_{\tilde{A}} = \{a + \alpha 1_{\tilde{A}} : \alpha \in \mathbb{C}\}.$$

Define mulitiplication in a natural way such that $1_{\tilde{A}}a = a1_{\tilde{A}} = a$ for all $a \in \tilde{A}$, i.e.

$$(a + \alpha 1_{\tilde{A}})(b + \beta 1_{\tilde{A}}) = ab + \alpha b + \beta a + \alpha\beta 1_{\tilde{A}}$$

and also $(a + \alpha 1_{\tilde{A}})^* = a^* + \overline{\alpha} 1_{\tilde{A}}$ for all $a, b \in A$ and $\alpha, \beta \in \mathbb{C}$. Thus it clear that $\sim$ is indeed a *-algebra. The norm and the embedding will be briefly established in the next proposition; refer to [4, p. 3] and [4, Proposition 1.5] for details.

We shall provide a proof sketch on why the unitization is indeed a C^*-algebra, though the norm structure of our unitization will not be relevant to our thesis.

Proposition 2.2.2. The unital algebra $\tilde{A}$ is indeed a C^*-algebra such that A is embedded into $\tilde{A}$ as a maximal ideal.

Proof Sketch. For each $a \in \tilde{A}$, denote $L_a(x) = ax$ for each $x \in A$, and by our definition, L_a is a linear map from A to A. If $a \in A$, it is clear that $\|L_a\| \leq \|a\|$ (given the operator norm), and

$$\|L_a a^*\| = \|a\|^2$$

so $\|L_a\| \geq \|a\|$, hence $\|L_a\| = \|a\|$. Thus $T : a \mapsto L_a$ is an injective *-homomorphism from A to $\mathcal{B}(A)$, hence $T(A)$ is a C^*-algebra. Define $S : \tilde{A} \to T(A) \oplus \mathbb{C} : a \mapsto L_a$, which is a bijective *-homomorphism, and for each $a \in \tilde{A}$, denote $\|a\| = \|S(a)\|$, which makes $\tilde{A}$ a C^*-algebra. The embedding of A as a maximal ideal into $\tilde{A}$ is now clear. ■

We denote $\tilde{A}$ to be a **unitization** A. Though as the construction may suggest, it is not true that $\tilde{A} \cong A \oplus \mathbb{C}$ (as C^*-algebras) in general as the former is unital while the latter may not be. In the case of unital C^*-algebra, we can consider the element $1_{\tilde{A}} - 1_A$ which acts an indicator for elements in A. And as it turns out: A is unital if, and only if, $\tilde{A} \cong A \oplus \mathbb{C}$; see next lemma.

Lemma 2.2.3. Let A be a unital C^*-algebra, and $p = 1_{\tilde{A}} - 1_A$. Then
(i) p is a projection with $ap = pa = 0$ for all $a \in A$.
(ii) $\tilde{A} = A \oplus \mathbb{C}p$ as vector spaces.
(iii) The map $\varphi : A \oplus \mathbb{C} \to \tilde{A}$ given by $\varphi(a, \alpha) = a + \alpha p$ is an isomorphism.
In particular $\tilde{A} \cong A \oplus \mathbb{C}$ if, and only if, A is unital.

Proof. As $1_{\tilde{A}}a = 1_A a = a$ for all $a \in A$, then (i) follows. For each $a + \alpha 1_{\tilde{A}} \in \tilde{A}$, one has

$$a + \alpha 1_{\tilde{A}} = a - \alpha 1_A + \alpha p \in A + \mathbb{C}p,$$

and if $z \in A \cap \mathbb{C}p$, then $z = \alpha 1_{\tilde{A}} - \alpha 1_A = a$ for some $a \in A$ and $\alpha \in \mathbb{C}$. In particular, $a + \alpha 1_A = \alpha 1_{\tilde{A}}$, then by definition of formal sums, one has $\alpha = 0$, and $a + \alpha 1_A = 0$, hence $a = 0$, so $z = 0$. Thus $A \cap \mathbb{C}p = 0$, and the sum is direct; this proves (ii).

It is clear that φ is linear, hence by (ii), φ is bijective. By (i) φ is a *-homomorphism, so this proves (iii). ■

In any case, one induces a **split-exact** sequence[1]

$$0 \longrightarrow A \overset{\iota_A}{\hookrightarrow} \tilde{A} \underset{\lambda_A}{\overset{\pi_A}{\rightleftarrows}} \mathbb{C} \longrightarrow 0 \tag{2.1}$$

where $\iota : A \to \tilde{A} : a \mapsto a$, $\pi : \tilde{A} \to \mathbb{C} : a + \alpha 1_{\tilde{A}} \mapsto \alpha$, and $\lambda : \mathbb{C} \mapsto \tilde{A} : \alpha \mapsto \alpha 1_{\tilde{A}}$. It will be understood that the arrows refer to ι, π, and λ whenever we are presented with the diagram (2.1). The unitization has a nice universal and functorial property, which we can then say that $\tilde{A}$ is *the* unitization of A up to isomorphism by universality, see Appendix 5.2 for a precise definition.

Theorem 2.2.4. *(Universality of Unitization)*. Let φ be a *-homomorphism between C^* algebras A and B, then there is a unique unital *-homomorphism $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ such that $\tilde{\varphi} \circ \iota_A = \iota_B \circ \varphi$. Even more, one has the commutative diagram:

$$\begin{array}{ccccc} A & \overset{\iota_A}{\hookrightarrow} & \tilde{A} & \overset{\pi_A}{\twoheadrightarrow} & \mathbb{C} \\ \varphi \downarrow & & \tilde{\varphi} \downarrow & & \| \\ B & \overset{\iota_B}{\hookrightarrow} & \tilde{B} & \overset{\pi}{\twoheadrightarrow} & \mathbb{C} \end{array} \tag{2.2}$$

Moreover,

(i) φ is injective if, and only if, $\tilde{\varphi}$ is injective.

(ii) φ is surjective if, and only if, $\tilde{\varphi}$ is surjective.

Proof. Define $\tilde{\varphi}(a + \alpha 1_{\tilde{A}}) = \varphi(a) + \alpha 1_{\tilde{B}}$, then it is clear that $\tilde{\varphi}$ is a *-homomorphism such that $\tilde{\varphi} \circ \iota_A = \iota_B \circ \varphi$, and the diagram (2.2) commutes. Suppose $\varphi' : \tilde{A} \to \tilde{B}$ is another unital *-homomorphism such that $\varphi' \circ \iota_A = \iota_B \circ \varphi$, then given $a + \alpha 1_{\tilde{A}} \in \tilde{A}$, one has

$$\varphi'(a + \alpha 1_{\tilde{A}}) = \varphi'(a) + \alpha 1_{\tilde{B}} = \varphi(a) + \alpha 1_{\tilde{B}} = \tilde{\varphi}(a + \alpha 1_{\tilde{A}}),$$

which shows uniqueness. As $\tilde{\varphi} = \varphi \oplus \gamma : A \oplus \mathbb{C}1_{\tilde{A}} \to B \oplus \mathbb{C}1_{\tilde{B}}$ where $\gamma : \mathbb{C}1_{\tilde{A}} \to \mathbb{C}1_{\tilde{B}} : \alpha 1_{\tilde{A}} \to \alpha 1_{\tilde{B}}$ is an isomorphism, then (i) and (ii) follows. ■

Theorem 2.2.5. *(Functoriality of Unitization)*. The unitization operator $\tilde{\cdot}$ defines a faithful functor from C*-Alg to the category of C*-Alg$_1$. This means that, given C^*-algebras A, B, C, and *-homomorphisms $\varphi : A \to B$ and $\psi : B \to C$, one has:

(i) $\widetilde{\psi \circ \varphi} = \tilde{\psi} \circ \tilde{\varphi}$.

(ii) $\widetilde{\mathrm{id}_A} = \mathrm{id}_{\tilde{A}}$.

(iii) (Faithfulness). Suppose $\varphi, \psi : A \to B$ are *-homomorphisms. If $\tilde{\varphi} = \tilde{\psi}$, then $\varphi = \psi$.

Proof. Let $a + \alpha 1_{\tilde{A}} \in \tilde{A}$, one has

$$(\widetilde{\psi \circ \varphi})(a + \alpha 1_{\tilde{A}}) = \psi(\varphi(a)) + \alpha 1_{\tilde{C}} = \tilde{\psi}(\varphi(a) + \alpha 1_{\tilde{B}}) = \tilde{\psi}(\tilde{\varphi}(a + \alpha 1_{\tilde{A}}))$$

so $\widetilde{\psi \circ \varphi} = \tilde{\psi} \circ \tilde{\varphi}$. Also

$$\widetilde{\mathrm{id}_A}(a + \alpha 1_{\tilde{A}}) = \mathrm{id}_A(a) + \alpha 1_{\tilde{A}} = a + \alpha 1_{\tilde{A}}$$

[1] Refer to Definition 5.4.2 for the definition and notation of arrows.

so $\widetilde{\mathrm{id}_A} = \mathrm{id}_{\tilde{A}}$. Hence (i) and (ii) are shown.
In the case of (iii), by Universality of Unitization 2.2.4, one has $\iota_B \circ \varphi = \iota_B \circ \psi$, so $\varphi = \psi$ as ι_B is injective. ■

Example 2.2.6. *(The trivial unitization)*. Note that if 0 is the trivial C^*-algebra, then $\tilde{0} = \mathbb{C}$, and if $0 : A \to B$ is the zero map, then $\tilde{0} = \lambda_B \circ \pi_A$. Which funnily enough, $0 : 0 \to 0$ gives $\tilde{0} = \mathrm{id}_{\mathbb{C}}$.

Example 2.2.7. Given a locally compact Hausdorff space X, the unitization of $\mathcal{C}_0(X)$ is isomorphic to $\mathcal{C}(X')$ where X' is the Alexandroff one-point compacitification of X; refer to [3, II.1.2.2]. In particular, one has the following corresponding: let ∞ be the point at infinity of X', then one has the *-isomorphism $\varphi : \widetilde{\mathcal{C}_0(X)} \to \mathcal{C}(X)$ defined as

$$\varphi\Big(f + \alpha 1_{\widetilde{\mathcal{C}_0(X)}}\Big)(t) = \begin{cases} f(t) & \text{if } t \in X \\ \alpha & \text{if } t = \infty \end{cases}.$$

Hence one has the cooresponding split-exact sequence

$$0 \longrightarrow \mathcal{C}_0(X) \xhookrightarrow{\ \iota\ } \mathcal{C}(X') \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

where $\iota(f) = \varphi(f)$, $\pi(f) = f(\infty)$, and $\lambda(\alpha) = \varphi\Big(0 + \alpha 1_{\widetilde{\mathcal{C}_0(X)}}\Big)$.

2.3 Spectral Theory

The spectral theory of C^*-algebras is elegant and is what makes C^*-algebra standout from other algebraic normed structures as it leads onto *continuous functional calculus* seen in the next chapter. We shall briefly introduce the concept of spectrum and some relevant results following from it. In this chapter, let $A_1 = A$ if A is unital, otherwise, let $A_1 = \tilde{A}$ if A is nonunital and we denote 1 to be the unit in A_1. We denote the **spectrum** of $a \in A$ to be

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } A_1\},$$

which is always a nonempty compact subset of $\mathbb{C}$. In particular, if $a \in A$, then $0 \in \sigma_{\tilde{A}}(a)$; see [5, II.1.4.1 and II.1.4.2]. We may omit the subscript $_A$ if the context clear.

Lemma 2.3.1. Let a, b be elements of some C^*-algebra A. Then:
(i) $\sigma_A(ab) \setminus \{0\} = \sigma_A(ba) \setminus \{0\}$.
(ii) If $f \in \mathbb{C}[z, \bar{z}]$ (f is a complex polynomial), then $\sigma_{\tilde{A}}(f(a)) = f(\sigma_A(a)) = \{f(\lambda) : \lambda \in \sigma_A(a)\}$.
(iii) If B is a C^*-subalgebra of A, then $\sigma_B(a) = \sigma_A(a)$.
(iv) If $\varphi : A \to B$ is a *-homomorphism, then $\sigma_B(\varphi(a)) \subseteq \sigma_A(a)$. If φ is injective, then $\sigma_B(\varphi(a)) = \sigma_A(a)$.

Proof. See [3, II.1.4.2] for (i) to (ii), [4, Proposition 4.8] for (iii) [3, II.1.6.7] for (iv). The rest are easy, with last part of (iv) follows from (iii). ■

Define the **spectral radius** $r(a) = \max_{\lambda \in \sigma(a)} |\lambda|$, and if a is a normal element, then $r(a) = \|a\|$; see [3, II.1.6.3]. So in particular, $\|a\| = \sqrt{r(a^*a)}$, and note that by Lemma 2.3.1 (iii), the spectral radius is independent of the spectrum that the element resides in.

We say an element a is **positive** if a is normal with $\sigma(a) \subseteq [0, \infty)$. It turns out, a is positive if, and only if, $a = x^*x$ for some $x \in A$. We define A^+ to be the set of positive elements of A. Given $a, b \in \mathrm{H}(A)$, define $b \geq a$ if $\sigma_A(b - a) \subseteq [0, \infty)$, thus it follows that $(\mathrm{H}(A), \geq)$ forms an ordered

vector space over $\mathbb{R}$ where the relation $\geq$ is a partial order, i.e. the addition on A^+ and scalar multiplication over $[0, \infty)$ preserves the $\geq$. See [4, Theorem 6.1, p. 23] for the details.
Let $a \in A_1$, and note that $a \geq \lambda$ and $a \in A^+$ for some $\lambda > 0$ implies a is invertible in A_1 as $0 \notin \sigma(a)$. Also given a $a \in \mathrm{H}(A)$, one has $a \leq \|a\| 1_{\tilde{A}}$ as $\sigma(a)$ is bounded by $\|a\|$.

Now given $b \in A$, then $bab^* \geq 0$ if $a \geq 0$, as $a = x^*x$ for some $x \in A$, hence

$$bab^* = bx^*xb^* = (xb^*)^*(xb^*)$$

shows that bab^* is positive. It turns out the product of positive elements are also positive: take $a, b \in A^+$, then following from Lemma 2.3.1 (i) and Example 2.4.7: $\sigma(ab) \cup \{0\} = \sigma\left(ab^{\frac{1}{2}}b^{\frac{1}{2}}\right) \cup \{0\} = \sigma\left(b^{\frac{1}{2}}ab^{\frac{1}{2}}\right) \cup \{0\}$ and as $b^{\frac{1}{2}}ab^{\frac{1}{2}} \geq 0$, thus ab is positive.

The following lemma is used in Chapter 4 and is especially prominent in Lemma 4.1.2.

Lemma 2.3.2. Let p, q be projection elements of a C^*-algebra. Then the following are equivalent:
(i) $q \leq p$.
(ii) $q = qpq$.
(iii) $qp = q$.
(iv) $pq = q$.

Proof. Write $1 = 1_{\tilde{A}}$. (i)$\Longrightarrow$(ii). One has that $q^2 = qqq^* \leq qpq^*$, so $q(1-p)q \leq 0$, hence $q(1-p)q = 0$ as $q(1-p)q^*$ is positive. So $q = qpq$.
(ii)$\Longrightarrow$(iii). As

$$0 = q - qpq = q(1-p)q = (q(1-p))((1-p)q)^*,$$

thus $0 = q(1-p)$ by C^*-identity, so $qp = q$.
(iii)$\Longrightarrow$(iv). As

$$((1-p)q)((1-p)q)^* = (1-p)q(1-p) = (1-p)(q - qp) = 0$$

thus $(1-p)q = 0$ by C^*-identity, so $q = pq$.
(iv)$\Longrightarrow$(i). As $p(1-q) \geq 0$ (product of two projections, hence positive elements), one has $p \geq pq = q$. ■

We say two elements $a, b \in \mathrm{H}(A)$ are **orthogonal** if $ab = 0$, and we write $a \perp b$. Note that $ab = 0$ if, and only if, $ba = 0$ as the elements are self-adjoint. We also say that two *-homomorphisms $\varphi, \psi : A \to B$ between C^*-algebras are **orthogonal** if $\varphi(a)\psi(b) = 0$ for all $a, b \in A$, note that this implies $\psi(a)\varphi(b) = 0$ for all $a, b \in A$ as given $a, b \in A$, one has

$$\psi(a)\varphi(b) = (\varphi(b^*)\psi(a^*))^* = 0^* = 0.$$

The following lemmas are crucial in lifting our morphisms identified by the K_0 functor; these results are exclusively used in Lemma 4.5.3.

Lemma 2.3.3. Let $n \in \mathbb{N}$ and $p_1, \ldots, p_n$ be projections of a C^*-algebra. Then the following are equivalent:
(i) $p_1, \ldots, p_n$ are mutually orthogonal.
(ii) $p_1 + \ldots + p_n$ is a projection.
(iii) $p_1 + \ldots + p_n \leq 1$.

Proof. (i)$\Longrightarrow$(ii). If $p_i p_j = 0$ for distinct $i, j \leq n$, then one has

$$\left(\sum_{i \leq n} p_i\right)^2 = \sum_{i,j \leq n} p_i p_j = \sum_{i \leq n} p_i^2 = \sum_{i \leq n} p_i$$

shows that $p_1 + ... + p_n$ is a projection.
(ii)$\Longrightarrow$(iii). Trivial as $\sigma(p_1 + ... + p_n) \subseteq \{0, 1\}$.
(iii)$\Longrightarrow$(i). Let $i, j \leq n$ be distinct. Then

$$\begin{aligned} & p_i + p_j \leq p_1 + ... + p_n \leq 1 \\ \Longrightarrow\ & p_i(p_i + p_j)p_i^* \leq p_i^* p_i \\ \Longrightarrow\ & p_i + p_i p_j p_i \leq p_i \\ \Longrightarrow\ & p_i p_j p_i \leq 0 \\ \Longrightarrow\ & p_i p_j p_i = 0 \end{aligned}$$

where the last equality holds because $p_i p_j p_i$ is positive. Now

$$\|p_j p_i\|^2 = \left\|(p_j p_i)^* (p_j p_i)\right\| = \|p_i p_j p_j p_i\| = \|p_i p_j p_i\| = 0$$

so $p_j p_i = 0$. This shows (i). ■

We say an element $v \in A$ to be a **partial isometry** if $v^* v$ is a projection. In that case, as $\sigma(vv^*) \cup \{0\} = \sigma(v^* v) \cup \{0\} \subseteq \{0, 1\}$ and vv^* is normal, then vv^* is also a projection by Corollary 2.4.4 and hence v^* is a partial isometry. We shall borrow the relation $vv^* v = v$ in Proposition 3.1.1.

Lemma 2.3.4. Let $v_1, ..., v_n$ be partial isometries of a unital C^*-algebra, and suppose

$$\sum_{i \leq n} v_i^* v_i = 1 = \sum_{i \leq n} v_i v_i^*.$$

Then $\sum_{i=1}^n v_i$ is unitary.

Proof. By preceding lemma, as $v_1 v_1^* + ... + v_n v_n^* = 1$ is a projection, then $v_1 v_1^*, ..., v_n v_n^*$ are mutually orthogonal. Let $i, j \leq n$ be distinct, so

$$v_i^* v_j = v_i^* v_i v_i^* v_j v_j^* v_j = v_i^* 0 v_j = 0.$$

So one has,

$$\left(\sum_{i \leq n} v_i\right)^* \left(\sum_{i \leq n} v_i\right) = \sum_{i,j \leq n} v_i^* v_j = \sum_{i \leq n} v_i^* v_i = 1$$

and similarly $\left(\sum_{i \leq n} v_i\right)\left(\sum_{i \leq n} v_i\right)^* = 1$. Thus $\sum_{i \leq n} v_i$ is unitary. ■

2.4 Continuous Functional Calculus

It turns out every C^*-algebra A has the following identifications.

Theorem 2.4.1. *(Commutative Gelfand-Naimark).* Let A be a commutative C^*-algebra, then there is a locally compact Hausdorff space X such that A is isomorphic to $\mathcal{C}_0(X)$. If A is unital, then X can be chosen as compact Hausdorff, i.e. $A \cong \mathcal{C}(X)$.

Proof. See [3, II.2.2.4]. ■

Theorem 2.4.2. *(Gelfand-Naimark-Segal).* Let A be a C^*-algebra, then A is embedded in $\mathcal{B}(H)$ for some Hilbert space H. If A is separable, then H can be chosen to be separable.

Proof. See [3, II.6.4.10]. ■

Following from Commutative Gelfand-Naimark 2.4.1, we achieve the following result to obtain an effective way of constructing C^*-algebra elements; this is the strength and elegance of C^*-algebras.

Theorem 2.4.3. *(Continuous Functional Calculus).* Let a be a normal element of a C^*-algebra A. Then one has a *-isomorphism:

$$\mathcal{C}(\sigma_A(a)) \to \tilde{A}\langle a, 1_{\tilde{A}}\rangle : f \mapsto f(a)$$

where if $p(z) = \sum_{i,j\leq n} c_{i,j} z^i \overline{z}^j \in \mathbb{C}[z, \overline{z}]$, then $p(a) = \sum_{i,j\leq n} c_{i,j} a^i (a^*)^j$, and if $(p_n)_{n\in\mathbb{N}}$ is a sequence of polynomials that converges uniformly to a $f \in \mathcal{C}(\sigma(a))$, then $f(a) = \lim_{n\to\infty} p_n(a)$. Furthermore, given $a \in A$ and $f \in \mathcal{C}(\sigma(a))$:

(i) If $f(0) = 0$, then $f(a) \in A\langle a\rangle$.
(ii) One has $\mathrm{im}(f) = f(\sigma(a)) = \sigma(f(a))$.
(iii) If B is a C^*-algebra and $\varphi : A \to B$ is a *-homomorphism, then $\varphi(f(a)) = f(\varphi(a))$.

Proof. See [3, II.2.3.1 and II.2.3.2]. ∎

For example, this gives us a convenient way of generating unitary elements out of self-adjoint elements. Let $a \in A$ be a self-adjoint element, then $\sigma(a) \subseteq \mathbb{R}$. Define $f(t) = e^{it}$ on $\sigma(a)$, then f is unitary and $f \in \mathcal{C}(\sigma(a))$, thus $f(a) = e^{ia}$ is a unitary element. In fact, one has an immediate corollary of identifying self-adjoint, unitary, and projection elements.

Corollary 2.4.4. Let $a \in A$ be normal. Then the following holds:

(i) a is a self-adjoint if, and only if, $\sigma_A(a) \subseteq \mathbb{R}$.
(ii) a is a projection if, and only if, $\sigma_A(a) \subseteq \{0, 1\}$.
(iii) If A is unital, then a is unitary if, and only if, $\sigma_A(a) \subseteq \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$.

Proof. See [3, II.2.3.4]. ∎

This next lemma gives a sufficient condition on when the map $a \mapsto f(a)$ is continuous given normal elements $a \in A$.

Lemma 2.4.5. Let $K \subseteq \mathbb{C}$ be a nonempty compact set, and denote Ω to be the set of all normal elements with spectrum contained in K from a C^*-algebra A. Given $f \in \mathcal{C}(K)$, then the induced map $a \mapsto f(a)$ from Ω, is continuous.

Proof. Let $\varepsilon > 0$ and by Stone-Weierstrass theorem, there is a polynomial $p \in \mathbb{C}[z, \overline{z}]$ such that $\|f - p\|_\infty < \varepsilon$. It is clear that p is continuous from Ω to A by $a \mapsto p(a)$, so given $a \in A$, there is a $\delta > 0$ such that for all $b \in A$ with $\|a - b\| < \delta$, one has $\|p(a) - p(b)\| < \varepsilon$. In particular,

$$\|f(a) - f(b)\| \leq \|f(a) - p(a)\| + \|p(a) - p(b)\| + \|p(b) - f(b)\| < 3\varepsilon,$$

hence f is continuous from Ω. ∎

Using the theory of positive elements and continuous functional calculus, we also have a lemma regarding the invertibility of elements and the unitalizability of our C^*-subalgebras. This lemma will in Chapter 4.

Lemma 2.4.6. Let B be a C^*-subalgebra of a unital C^*-algebra A. Let $a \in B$, then one has

(i) Suppose B is unital. Then a is left-invertible in B if, and only if, a^*a is invertible in B.
(ii) If a is invertible in A, then B is unital and a is invertible in B.

Proof.

(i) Suppose a is left-invertible. Let b be the left-inverse of a and note that $\|b\| \neq 0$. As $b^*b \leq \|b\|^2 1$, then $1 = (ba)^*(ba) = a^*b^*ba \leq a^*\|b\|^2 1a = \|b\|^2 a^*a$, so $0 < \|b\|^{-2} \leq a^*a$. Thus a^*a is invertible. If a^*a is invertible, then $(a^*a)^{-1}a^*$ is a left-inverse of a.
(ii) As a is invertible, then so are a^*a and aa^*. As $0 \notin K = \sigma_A(aa^*) \cup \sigma_A(a^*a)$, and as K is compact, then we can define a continuous function $f \in \mathcal{C}(\mathbb{C})$ such that

$$f(z) = \begin{cases} 0 & \text{if } z \notin K \\ \frac{1}{z} & \text{if } z \in K \end{cases}.$$

By the Continuous Functional Calculus 2.4.3 (i), as $f(0) = 0$, one has that $f(aa^*) = (aa^*)^{-1} \in A\langle aa^* \rangle$ and $f(a^*a) = (a^*a)^{-1} \in A\langle a^*a \rangle$. In particular, $(a^*a)^{-1}, (aa^*)^{-1} \in A\langle a \rangle \subseteq B$, now a has a left-inverse $(a^*a)^{-1}a^*$ in B and a right-inverse $a^*(aa^*)^{-1}$ in B, so a is invertible in B. In particular $1 = a^{-1}a \in B$, so B is unital. ■

Here are some common constructions of new elements via continuous functional calculus, which are analogous to the modulus and argument of complex numbers.

Example 2.4.7. *(Absolute Value)*. Let $a \in A$ be an element, then a^*a is positive, so $\sigma(a^*a) \subseteq [0, \infty)$, thus the map $f(t) = t^{\frac{1}{n}}$ is in $C(\sigma(a^*a))$. Hence we can define $f(a^*a) = (a^*a)^{\frac{1}{n}} \in A\langle a^*a \rangle$ as $f(0) = 0$ such that $f(a^*a)^n = a^*a$. Define the **absolute value** of a to be $|a| := (a^*a)^{\frac{1}{2}}$, so it follows that $|a|^2 = a^*a$, and $|a|$ is invertible if a is, with $|a|^{-1} = |a^{-1}|$.
Now for each bounded set $B \subseteq A^+$, then $R = \sup\{\|a\| : a \in B\}$ is finite, in particular, $B \subseteq \{a \in A : \sigma(a) \subseteq [0, R]\}$ as the norm and spectral radius are equivalent for normal elements, hence by Lemma 2.4.5, the map $x \mapsto x^{\frac{1}{2}}$ on B is continuous. Thus the square root map on A^+ is continuous as it is continuous on each bounded subset, thus the absolute value map on A is continuous as a composition of continuous maps.

Example 2.4.8. *(Polarization)*. Let a be an invertible element of a unital C^*-algebra A. Then one can define $\omega(a) := a|a|^{-1}$, and note that

$$\omega(a)^*\omega(a) = |a|^{-1}a^*a|a|^{-1} = |a|^{-1}|a|^2|a|^{-1} = 1$$

and

$$\omega(a)\omega(a)^* = a|a|^{-1}|a|^{-1}a^* = a|a^{-1}|^2a^* = a(a^*a)^{-1}a^* = 1$$

hence $\omega(a)$ is unitary, and we say $\omega(a)$ is the **polarization** of a. Thus ω is a continuous map from $\mathrm{GL}(A)$ to $\mathcal{U}(A)$ as a composition of continuous maps. Furthermore, $\omega|_{\mathcal{U}(A)} = \mathrm{id}_{\mathcal{U}(\mathcal{A})}$.

2.5 Matrix Algebras

Given a C^*-algebra A, and $m, n.p, q \in \mathbb{N}$, we can construct matrices with entries from A equipped with the natural matrix operations. That is, denote $\mathcal{M}_{m,n}(A)$ to be the set of all tuples $(a_{ij})_{\substack{i \leq n \\ j \leq m}}$ where $a_{ij} \in A$ for all $i \leq m$ and $j \leq n$, we shall omit the indexing subscripts if the context is clear. Define $(a_{ij})^* = (a_{ji}{}^*) \in \mathcal{M}_{n,m}(A)$. Denote the space of square matrices to be $\mathcal{M}_n(A) = \mathcal{M}_{n,n}(A)$, which is called a **matrix algebra of A**.

We denote $0_{m,n}$ or 0_n to be the zero matrices of $\mathcal{M}_{m,n}(A)$ and $\mathcal{M}_n(A)$ respectively. Similarly, let 1_n be the identity matrix of $\mathcal{M}_n(A)$. In general, given $a \in \mathcal{M}_{m,n}(A)$ and $b \in \mathcal{M}_{p,q}(A)$, define

$$a \oplus b = \mathrm{diag}(a, b) = \begin{pmatrix} a & 0_{m,q} \\ 0_{p,n} & b \end{pmatrix} \in \mathcal{M}_{m+p,n+q}(A).$$

Construction 2.5.1. *(Induced map between matrix algebras)*. Given a map φ between C^*-algebras A to B, we denote the induced map $\varphi_n : \mathcal{M}_n(A) \to \mathcal{M}_n(B)$ defined as $\varphi_n((a_{ij})) = (\varphi(a_{ij}))$. In any case, we will just write φ instead of φ_n if the context is clear, and note that if φ is a *-homomorphism, then so is φ_n.

Let $a = (a_{ij}) \in \mathcal{M}_n(A)$, define the norm $\|a\| = \|\varphi_n(a)\|$, where φ is an *-embedding $\varphi : A \to \mathcal{B}(H)$ for some Hilbert space H, which exists by Gelfand-Naimark-Segal 2.4.2, and we have the induced map $\varphi_n : \mathcal{M}_n(A) \to \mathcal{B}(H^n)$ given by

$$\varphi_n(a)h = \begin{pmatrix} \varphi(a_{11}) & \cdots & \varphi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \varphi(a_{n1}) & \cdots & \varphi(a_{nn}) \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \varphi(a_{11})h_1 + \cdots + \varphi(a_{n1})h_n \\ \vdots \\ \varphi(a_{n1})h_1 + \cdots + \varphi(a_{nn})h_n \end{pmatrix}.$$

Thus $\mathcal{M}_n(A)$ also forms a C^*-algebra under the usual matrix operations; see [2, 1.3] for more details. Given $a \in \mathcal{M}_n(A)$, one must have $\|a\| = \sqrt{r(a^*a)}$, thus the choice of the norm is independent of the embedding. Note that $\mathcal{M}_n$ has a natural functorial property shown in the next lemma.

Lemma 2.5.2. *(Functoriality of Matrix Algebras)*. Let $n \in \mathbb{N}$, then $\mathcal{M}_n$ defines a covariant faithful exact functor that preserves zero. That is, give C^*-algebras A, B, C, and *-homomorphisms $\varphi : A \to B$, $\psi : B \to C$, one has the following:

(i) $(\psi \circ \varphi)_n = \psi_n \circ \varphi_n$.
(ii) $(\mathrm{id}_A)_n = \mathrm{id}_{\mathcal{M}_n(A)}$.
(iii) If 0 is the trivial C^*-algebra, then so is $\mathcal{M}_n(0)$.
(iv) If $0 : A \to B$ is the trivial map, then so is 0_n.
(v) If one has an exact sequence:

$$0 \longrightarrow A \overset{\varphi}{\hookrightarrow} B \overset{\psi}{\twoheadrightarrow} C \longrightarrow 0$$

then one induces an exact sequence:

$$0 \longrightarrow \mathcal{M}_n(A) \overset{\varphi_n}{\hookrightarrow} \mathcal{M}_n(B) \overset{\psi_n}{\twoheadrightarrow} \mathcal{M}_n(C) \longrightarrow 0$$

(vi) Suppose $\varphi, \psi : A \to B$ are *-homomorphisms. If $\mathcal{M}_n(\varphi) = \mathcal{M}_n(\psi)$, then $\varphi = \psi$.

Proof. Statements (i) to (iv), and (vi) should be obvious. For (v), claim that $\mathrm{im}(\varphi_n) = \ker(\psi_n)$. Given $a \in \mathcal{M}_n(A)$, then $\varphi(a_{ij}) \in \ker(\psi)$, so $\psi(\varphi(a_{ij})) = 0$ for all $i, j \leq n$, hence $\psi(\varphi(a)) = 0$, thus $\mathrm{im}(\varphi_n) \subseteq \ker(\psi_n)$. Given $b \in \ker(\psi_n)$, then $\psi(b) = 0$ implies $b_{ij} \in \ker(\psi)$, so there is a $a_{ij} \in A$ such that $\varphi(a_{ij}) = b_{ij}$ for each $i, j \leq n$. Define $a = (a_{ij}) \in \mathcal{M}_n(A)$, then $\varphi(a) = b$ is obvious, so $\mathrm{im}(\varphi_n) = \ker(\psi_n)$. As the claim does not rely on the fact that φ is injective and ψ is surjective, then following from the claim, one has $\ker(\varphi_n) = 0$ and $\mathrm{im}(\psi_n) = \mathcal{M}_n(C)$, as required. ■

The next lemma provides a natural bound on the norm on the of matrices. In particular, we have convergence in matrices if, and only if, we have convergence between matrix elements.

Lemma 2.5.3. Let A be a C^*-algebra and $n \in \mathbb{N}$. Then given $a = (a_{ij}) \in \mathcal{M}_n(A)$, one has

$$\max_{i,j \leq n} \|a_{ij}\| \leq \|a\| \leq \sum_{i,j \leq n} \|a_{ij}\|.$$

In particular, for each $k \in \mathbb{N}$ and $a^{(k)} = \left(a_{ij}^{(k)}\right) \in \mathcal{M}_n(A)$, the sequence $(a^{(k)})_{k \in \mathbb{N}}$ converges to $a = (a_{ij}) \in \mathcal{M}_n(A)$ if, and only if, $\lim_{k \to \infty} a_{ij}^{(k)} = a_{ij}$ for each $i, j \leq n$.

Proof. Fix $i, j \leq n$ and let $e^{(ij)}$ be a matrix such that its (i,j)-entry is a_{ij} and zero everywhere else. Then for any $h \in H^n$, one has

$$\|\varphi_n(e^{(ij)})h\| = \|\varphi(a_{ij})h_j\| \leq \|\varphi(a_{ij})\|\|h_j\| \leq \|\varphi(a_{ij})\|\|h\|,$$

so taking the supremum over $\|h\| = 1$ one obtains, $\left\|\varphi_n(e^{(ij)})\right\| \leq \left\|\varphi(a_{ij})\right\|$. Similarly, for each $h \in H$, one can define $h' = (0, \ldots, h, \ldots, 0) \in H^n$ where h is in the jth entry, so

$$\|\varphi(a_{ij})h\| = \|\varphi_n(e)h'\| \leq \|\varphi_n(e)\|\|h'\| = \|\varphi_n(e)\|\|h\|$$

and taking the supremum over $\|h\| = 1$, one obtains $\|\varphi(a_{ij})\| \leq \|\varphi_n(e^{(ij)})\|$. Hence

$$\|e^{(ij)}\| = \|\varphi_n(e^{(ij)})\| = \|\varphi(a_{ij})\| = \|a_{ij}\|.$$

Thus

$$\|a\| = \left\| \sum_{i,j\leq n} e^{(ij)} \right\| \leq \sum_{i,j\leq n} \|e^{(ij)}\| = \sum_{i,j\leq n} \|a_{ij}\|.$$

Given a $h \in H^n$ and $i, j \leq n$, let $h' = (0, ..., h_j, ..., 0) \in H^n$ where h_j is in the jth entry, so one has

$$\|\varphi_n(e^{(ij)})h\| = \|\varphi(a_{ij})h_j\| \leq \sqrt{\sum_{i\leq n} \|\varphi(a_{ij})h_j\|^2} = \|\varphi_n(a)h'\| \leq \|\varphi_n(a)\|\|h\|$$

so taking the supremum over $\|h\| = 1$, one obtains $\|a_{ij}\| \leq \|a\|$, hence $\max_{i,j\leq n}\|a_{ij}\| \leq \|a\|$. The rest follows. ■

We shall see how matrix algebras pair with common C^*-algebras and their natural constructions.

Example 2.5.4. *(Some common matrix algebras)*. Let A be a C^*-algebra, X be a locally compact Hausdorff space, H be a Hilbert space, and $n, m \in \mathbb{N}$. Then:

(i) One has the obvious identification: $\mathcal{M}_n(\mathcal{M}_m(A)) \cong \mathcal{M}_{nm}(A)$.

(ii) One has the obvious identification $\mathcal{M}_n(\mathcal{C}_0(X, A)) \cong \mathcal{C}_0(X, \mathcal{M}_n(A))$. This is given by $\Phi : \mathcal{M}_n(\mathcal{C}_0(X, A)) \to \mathcal{C}_0(X, \mathcal{M}_n(A))$ where given $f = (f_{ij}) \in \mathcal{M}_n(\mathcal{C}_0(X, A))$, define $(\Phi(f))_{ij} \in \mathcal{C}_0(X, A)$ as $[(\Phi(f))_{ij}](x) = f_{ij}(x)$ for all $x \in X$ and $i, j \leq n$. It follows that Φ is a well-defined *-isomorphism by Lemma 2.5.3.

(iii) One has the identification $\mathcal{M}_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$. Let $i \leq n$, define $\pi_i : H^n \to H$ as the ith coordinate projection map, and $\lambda_i : H \to H^n$ as the ith coordinate inclusion map, so one defines

$$\Phi : \mathcal{B}(H^n) \to \mathcal{M}_n(\mathcal{B}(H)) : T \mapsto (\pi_i \circ T \circ \lambda_j)_{i,j\leq n}.$$

This map is clearly linear, and as $\lambda_i \circ \pi_j = \delta_{ij}$ where $\delta_{ij} = \mathrm{id}_{H^n}$ whenever $i = j$ and 0 otherwise, then given $S, T \in \mathcal{B}(H^n)$, one has

$$\Phi(S)\Phi(T) = (\pi_i \circ S \circ \lambda_j)(\pi_i \circ T \circ \lambda_j) = \Phi(ST)$$

by direct computation. Given $x \in H$ and $y \in H^n$, one has $\langle \lambda_i x, y\rangle = \langle x, y_i\rangle = \langle x, \pi_i y\rangle$, so $\lambda_i^* = \pi_i$, thus $\Phi(T^*) = (\pi_j \circ T^* \circ \lambda_i) = \Phi(T)^*$. So Φ is a *-homomorphism.

If $\Phi(T) = 0$, then given $x \in H^n$, one has $(T\lambda_i x_i)_j = 0$ for each $i, j \leq n$, thus $T\lambda_i x_i = 0$, and as $x = \sum_{i\leq n} \lambda_i x_i$, one has $Tx = 0$. Hence $T = 0$, and thus Φ is injective, hence an isomorphism.

Example 2.5.5. *(Unitization of matrix algebras)*. Let A be a C^*-algebra, then it is not true in general that $\widetilde{\mathcal{M}_n(A)} \cong \mathcal{M}_n(\tilde{A})$. Indeed, referring to Example 2.2.6, one has

$$\widetilde{\mathcal{M}_n(0)} \cong \tilde{0} \cong \mathbb{C} \not\cong \mathcal{M}_n(\mathbb{C}) \cong \mathcal{M}_n(\tilde{0})$$

unless $n = 1$. It is clear that $\mathcal{M}_n(\tilde{A})$ is usually much larger than $\widetilde{\mathcal{M}_n(A)}$, e.g. $\dim(\mathcal{M}_2(\tilde{\mathbb{C}})) = 8$ and $\dim(\widetilde{\mathcal{M}_2(\mathbb{C})}) = 5$.[1]

Let I be the unit of $\widetilde{\mathcal{M}_n(A)}$ and $1 = \mathrm{diag}(\underbrace{1_{\tilde{A}}, ..., 1_{\tilde{A}}}_{n \text{ times}})$. Let $M_n = \mathcal{M}_n(A) + \mathbb{C}1$, this is a closed subspace as a sum of a closed subspace and a finite-dimensional subspace, and hence it is clear that this is a unital C^*-algebra. As $\mathcal{M}_n(A) \cap \mathbb{C}1 = 0$, then we can view the sum as a vector space direct sum, then one has the *-isomorphism

[1] By Lemma 2.2.3, $\tilde{\mathbb{C}} \cong \mathbb{C} \oplus \mathbb{C}$ and $\widetilde{\mathcal{M}_2(\mathbb{C})} \cong \mathcal{M}_2(\mathbb{C}) \oplus \mathbb{C}$ as they are unital algebras.

$$\Phi : M_n \to \widetilde{\mathcal{M}_n(A)} : a + \alpha 1 \mapsto a + \alpha I.$$

Thus one can identify $\widetilde{\mathcal{M}_n(A)}$ as a C^*-subalgebra of $\mathcal{M}_n(\tilde{A})$ given by M_n. Define $\Psi : \widetilde{\mathcal{M}_n(A)} \to \mathcal{M}_n(A)$ given by $\Psi(a) = \Phi^{-1}(a)$, so one has a commutative diagram with exact rows (by Functoriality of Matrix Algebras 2.5.2),

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}_n(A) & \xhookrightarrow{\iota_A} & \mathcal{M}_n(\tilde{A}) & \xrightarrow{\pi_A} & \mathcal{M}_n(\mathbb{C}) & \longrightarrow & 0 \\
 & & \Vert & & \uparrow \Psi & & \uparrow j & & \\
0 & \longrightarrow & \mathcal{M}_n(A) & \xhookrightarrow{\iota_{\mathcal{M}_n(A)}} & \widetilde{\mathcal{M}_n(A)} & \xrightarrow{\pi_{\mathcal{M}_n(\mathbb{C})}} & \mathbb{C} & \longrightarrow & 0
\end{array}$$

where $j : \mathbb{C} \to \mathcal{M}_n(\mathbb{C}) : \alpha \to \alpha I$. Hence $\mathcal{M}_n(\tilde{A}) \cong \widetilde{\mathcal{M}_n(A)}$ if, and only if, j is an isomorphism by Five Lemma 5.4.5, which only holds if, and only if, $n = 1$.

Example 2.5.6. *(Matrix algebra of direct sums)*. Let A and B be C^*-algebras and $n \in \mathbb{N}$, then $\mathcal{M}_n(A \oplus B) \cong \mathcal{M}_n(A) \oplus \mathcal{M}_n(B)$. Note that given $A_1 = \left(a_{ij}^{(1)}\right), A_2 = \left(a_{ij}^{(2)}\right) \in \mathcal{M}_n(A)$ and $B_1 = \left(b_{ij}^{(1)}\right), B_2 = \left(b_{ij}^{(2)}\right) \in \mathcal{M}_n(B)$, then (using Einstein summation notation)

$$\mathcal{M}_n(A \oplus B) \ni \left(\left(a_{ij}^{(1)}, b_{ij}^{(1)}\right)\right)\left(\left(a_{ij}^{(2)}, b_{ij}^{(2)}\right)\right) = \left(\left(a_{kj}^{(1)} a_{ik}^{(2)}, b_{kj}^{(1)} b_{ik}^{(2)}\right)\right)$$

$$\leftrightarrow \left(\left(a_{kj}^{(1)} a_{ik}^{(2)}\right), \left(b_{kj}^{(1)} b_{ik}^{(2)}\right)\right) = (A_1 A_2, B_1 B_2) = (A_1, B_1)(A_2, B_2) \in \mathcal{M}_n(A) \oplus \mathcal{M}_n(B)$$

shows our equivalence. We shall make use of this identification going forward.

2.6 The Unitary Group of C^*-Algebras

We shall dedicate this chapter to discuss the structure of $\mathcal{U}(A)$, which will be relevant in proving the split-exactness of the K_0 functor; see Lemma 3.4.5. Note that $\mathcal{U}(A)$ forms a group under the usual multiplication. Let

$$\mathcal{U}_n(A) = \mathcal{U}(\mathcal{M}_n(A))$$

for each $n \in \mathbb{N}$. Given $u \in \mathcal{U}(A)$, define $\mathrm{ad}u : A \to A$ as $(\mathrm{ad}u)(a) = u^* a u$, then it follows that $\mathrm{ad}u$ is a *-isomorphism. Indeed, it should go without saying that $\mathrm{ad}u$ is a *-homomorphism, so note that:

$$u^* a u = u^* b u \implies a = b$$
$$(\mathrm{ad}u)(u a u^*) = a$$

shows injectivity and surjectivity respectively. Note that for $u \in \mathcal{U}(A)$, one has $\|u\|^2 = \|u^* u\| = \|1\| = 1$, hence $\|u\| = 1$.

Given $a, b \in A$, and a subset $B \subseteq A$, we write $\boldsymbol{a \sim_h b}$ **in** $\boldsymbol{B}$ if there is a continuous map, called a **path**, $\gamma : [0, 1] \to B$ (B has the subspace topology) such that $\gamma(0) = a$ and $\gamma(1) = b$. It is clear that $\sim_h$ defines an equivalence relation, and we denote

$$\mathcal{U}^0(A) := \left\{u \in \mathcal{U}(A) : u \underset{h}{\sim} 1 \text{ in } \mathcal{U}(A)\right\},$$
$$\mathcal{U}_n^0(A) := \mathcal{U}^0(\mathcal{M}_n(A)).$$

We first prove a nice structural identification of $\mathcal{U}^0(A)$.

Lemma 2.6.1. Let u, v be unitary elements of a unital C^*-algebra A. Then:

(i) If $\sigma(u) \subsetneq \mathbb{S}^1$, then $u \in \mathcal{U}^0(A)$. In particular, $u = e^{ih}$ for some self-adjoint $h \in A$.

(ii) $\|u - v\| \leq 2$ always, and if $\|u - v\| < 2$, then $u \sim_h v$ in $\mathcal{U}(A)$.

Proof.

(i) If $\sigma(u) \subsetneq \mathbb{S}^1$, then there is a $\theta \in (-\pi, \pi]$ such that $e^{i\theta} \notin \sigma(u)$. Consider the argument map $\arg : \mathbb{S}^1 \setminus \{e^{i\theta}\} \to (\theta, \theta + 2\pi)$, which is continuous, and observe that $e^{i\arg(u)} = u$ and $\arg(u)$ is self-adjoint. Thus $t\arg(u)$ is self-adjoint for all $t \in [0, 1]$, and the map $t \mapsto e^{it\arg(u)}$ shows that $1_A \sim_h e^{i\arg(u)} = u$. Hence $u \in \mathcal{U}^0(A)$.

(ii) Clearly $\|u - v\| \le \|u\| + \|v\| = 2$. Observe that $\|u^*v - 1\| = \|u^*(v - u)\| \le \|v - u\| < 2$, so $-2 \notin \sigma(u^*v - 1)$ so $-1 \notin \sigma(u^*v)$. As u^*v is unitary, then by part (i), $u^*v \sim_h 1$, so $u \sim_h v$ by left mulitiplication of u. ■

Proposition 2.6.2. *(Structure of $\mathcal{U}^0(A)$).* Let A be a unital C^*-algebra. Then

(i) $\mathcal{U}^0(A)$ is a normal subgroup of $\mathcal{U}(A)$.

(ii) $\mathcal{U}^0(A)$ is the path-connected and a clopen subspace of $\mathcal{U}(A)$.

(iii) An element a is in $\mathcal{U}^0(A)$ if, and only if,

$$a = \exp(ih_1) \cdots \exp(ih_k)$$

for some self-adjoint elements $h_1, ..., h_k \in A$ with $k \in \mathbb{N}$.

Proof. Note that $\mathcal{U}^0(A)$ is path-connected from definition. Define

$$G = \{\exp(ih_1) \cdots \exp(ih_k) : h_1, ..., h_k \in \mathrm{H}(A) \text{ for } k \in \mathbb{N}\}.$$

Note that if h is self-adjoint, then $\exp(ih)$ is unitary; see discussion in Chapter 2.4. In particular, note that given $t \in [0, 1]$, th is self-adjoint, and hence $\exp(ith) \in \mathcal{U}(A)$. By Lemma 2.4.5 the map $f : \Omega \to A$ defined as $f(x) = \exp(ix)$ is continuous where Ω is from the lemma defined by $K = [-\|h\|, \|h\|]$. As $\sigma(th) \subseteq K$ for each $t \in [0, 1]$, then $t \mapsto \exp(ith)$ is continuous as a composition of continuous map of f and $t \mapsto th$. Thus one has $\exp(ih) \sim_h \exp(i0) = 1$ in $\mathcal{U}(A)$, so $\exp(ih) \in \mathcal{U}^0(A)$. As $\mathcal{U}(A)$ is closed under multiplication, and multiplication preserves $\sim_h$, then it follows that $G \subseteq \mathcal{U}^0(A)$.

It is clear that G is a subgroup of $\mathcal{U}(A)$ by observing that $\exp(-ih) = \exp(ih)^{-1}$.

Let $u \in \mathcal{U}(A)$ and $v \in G$ with $\|u - v\| < 2$, then $\|1 - uv^*\| = \|(v - u)v^*\| \le \|v - u\| < 2$, then following from the proof of Lemma 2.6.1 (ii) and the conclusion of Lemma 2.6.1 (i), we observe that $uv^* = e^{ih}$ for some self-adjoint $h \in A$, hence $u = e^{ih}v \in G$, so G is open in $\mathcal{U}(A)$.

As G is an open subgroup, we observe that G is also closed in $\mathcal{U}(A)$, as $\mathcal{U}(A) \setminus G$ is a union of the cosets of G, which are homeomorphic to G.

As G is a nonempty clopen set in $\mathcal{U}(A)$ and is a subset of a connected set $\mathcal{U}^0(A)$, then $G = \mathcal{U}^0(A)$. So it suffices to show that $\mathcal{U}^0(A)$ is normal. Indeed, given $u \in \mathcal{U}^0(A)$ and $v \in \mathcal{U}(A)$, note that one has a continuous map $t \mapsto u_t$ in $\mathcal{U}(A)$ with $u_0 = u$ and $u_1 = 1$, so $t \mapsto v^*u_tv$ is a continuous map then in $\mathcal{U}(A)$ with $v^*uv = v^*u_1v \sim_h v^*u_0v = 1$, hence $v^*uv \in \mathcal{U}^0(A)$, as required. ■

The following lemmas gives sufficient conditions on when unitary elements in B can be identified as lifted unitary elements from A. They are used for showing Half Exactness of K_0 3.4.7 in Chapter 3.

Lemma 2.6.3. *(Whitehead).* Let u, v be unitary elements of a unital C^*-algebra A. then

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \underset{h}{\sim} \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \underset{h}{\sim} \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \underset{h}{\sim} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \quad \text{in } \mathcal{U}_2(A).$$

In particular, $u \oplus u^* \sim_h 1_2$.

Proof. Let $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. As $-1 \notin \sigma(w)$, and the matrix is unitary, then $w \sim_h 1_2$ by preceding lemma. Hence as

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} w$$

by replacing the second instance of w with 1_2 to get

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \underset{h}{\sim} \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

Replacing both instances of w with 1_2 to get

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \underset{h}{\sim} \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}.$$

The rest follows. ■

Lemma 2.6.4. Let A and B be unital C^*-algebras and let $\varphi : A \to B$ be a surjective *-homomorphism.

(i) $\varphi(\mathcal{U}^0(A)) = \mathcal{U}^0(B)$.

(ii) If $u \in \mathcal{U}(B)$ and suppose there is a $v \in A$ such that $u \sim_h \varphi(v)$ in $\mathcal{U}(B)$, then $u \in \varphi(\mathcal{U}(A))$.

(iii) For $u \in \mathcal{U}(B)$, there is a $v \in \mathcal{U}_2^0(A)$ such that $\varphi(v) = u \oplus u^*$ where $\varphi : \mathcal{M}_2(A) \to \mathcal{M}_2(B)$ is the induced matrix map.

Proof.

(i) As φ is surjective, then φ is unital, hence by continuity, one has $\varphi(\mathcal{U}^0(A)) \subseteq \mathcal{U}^0(B)$. Note that by Structure of $\mathcal{U}^0(A)$ 2.6.2, elements of $\mathcal{U}^0(B)$ is a $\exp(ih)$ where h is self-adjoint in B, and note that by Continuous Functional Calculus 2.4.3 (iii), one has $\varphi(\exp(ix)) = \exp(i\varphi(x))$ for $x \in A$. With that in mind, let $h \in B$ be self-adjoint, then there is a $x \in A$ such that $\varphi(x) = h$. Now $k = (x + x^*)/2$ is self-adjoint in A and $\varphi(k) = h$, then $\exp(ik) \in \mathcal{U}^0(A)$, and $\varphi(\exp(ik)) = \exp(ih)$, hence it follows that $\varphi(\mathcal{U}^0(A)) = \mathcal{U}^0(B)$.

(ii) As $1 \sim_h \varphi(v)^*u$, then $u\varphi(v)^* \in \mathcal{U}^0(B)$, then by part (i), there is a $w \in \mathcal{U}^0(A)$ such that $\varphi(w) = u\varphi(v)^*$, in particular, $\varphi(wv) = u$. Hence $u \in \varphi(\mathcal{U}(A))$.

(iii) By Whitehead 2.6.3, one has $u \oplus u^* \sim_h 1_2$ in $\mathcal{U}_2(A)$, and as the induced map $\varphi : \mathcal{M}_2(A) \to \mathcal{M}_2(B)$ is a surjective *-homomorphism, then by the rest follows from part (ii). ■

3 | The K_0-Theory for C^*-algebras

We proceed to construct our first K_0-group for C^*-algebras. The general idea is that given a C^*-algebra A, we identify projection matrices of A by the Murray-von Neumann relation, which is compatible with the direct sum operation. Hence we obtain an Abelian semigroup structure by quotienting out the relation, and take the Grothendieck completion to obtain the associated Abelian groups called the K_0-groups.

The first chapter is to discover the relationship between the various relations on our projection matrices. The two main relations we are interested in are the Murray-von Neumann relation–where we identify projections by partial isometries–and the homotopic relation $\sim_h$. It will be shown that the Murray-von Neumann relation is equivalent to identifying bounded operators on Hilbert spaces by the same rank, the Murray-von Neumann relation provides an algebraic relation between projections, which allows algebraic approach to compute the K_0-groups. The homotopic relation allows a more topological approach, and in fact identifications on the K_0-group level.

The second chapter will be a brief construction of Grothendieck groups, where we essentially 'generate' Abelian groups from Abelian semigroups by introducing inverses. As it will be shown later that we obtain an Abelian semigroup, in fact a monoid, through quotienting out the Murray-von Neumann relation on the space of projection matrices.

The third chapter will bring the first two chapters together in order to finally establish the K_0 -group for C^*-algebras. We discuss the immediate consequences, and the functoriality property of such constructions.

3.1 Equivalence Relations on Projections

Let $n \in \mathbb{N}$. Denote $\mathrm{P}_n(A)$ be the set of all $n \times n$ projection matrices $a \in \mathcal{M}_n(A)$, and let $\mathrm{P}_\infty(A) = \bigcup_{k \in \mathbb{N}} \mathrm{P}_k(A)$.

Assuming A is unital.

Let $\mathcal{U}_\infty(A) = \bigcup_{k \in \mathbb{N}} \mathcal{U}_k(A)$.

Denote $\mathrm{GL}_n(A)$ to be the set of all $n \times n$ invertible matrices $a \in \mathcal{M}_n(A)$.

Given $a, b \in A$, we say they are:

- **Similar** if there is a $g \in \mathrm{GL}(A)$ such that $a = g^{-1}bg$, and we write $a \sim_s b$.
- **Unitarily equivalent** if there is a $u \in \mathcal{U}\big(\tilde{A}\big)$ such that $a = ubu^*$, and we write $a \sim_u b$. If $u \in \mathcal{U}(A)$, we say $a \sim_u b$ in $\mathcal{U}(A)$.
- **Murray-von Neumann equivalent** if there is a $v \in A$ such that $a = v^*v$, and $b = v^*v$, and we write $a \sim_0 b$.

We denote $\mathrm{GL}^0_k(A) := \{g \in \mathrm{GL}_k(A) : g \sim_h 1_k \text{ in } \mathrm{GL}(A)\}$ with $\mathrm{GL}^0(A) := \mathrm{GL}^0_1(A)$.

Given two continuous maps $f : A \to B$ and $g : A \to B$ between topological spaces A and B, then they are **pointwise homotopic in B** if there is a map $F : [0,1] \times A \to B$ such that $F(0,a) = f(a)$ and $F(1,a) = g(a)$ for all $a \in A$, and $t \mapsto F(t,a)$ is continuous for each $a \in A$, and we write $f \sim_h g$, in particular, one has $f(a) \sim_h g(a)$ for each $a \in A$. We omit the prefix pointwise if F is continuous, which automatically implies pointwise homotopy.

It is clear that $\sim_h$, $\sim_u$, and $\sim_s$ are equivalent relations, and it turns out $\sim_0$ is an equivalent relation on $\mathrm{P}(A)$. We will show that in a more general setting: given $p \in \mathrm{P}_n(A)$ and $q \in \mathrm{P}_m(A)$, then we write $p \sim q$ is there is a $v \in \mathcal{M}_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$, and note that v are partial isometries.[1]

Proposition 3.1.1. The relation $\sim$ on $\mathrm{P}_\infty(A)$ is an equivalence relation. Furthemore:
(i) If $v \in \mathcal{M}_{m,n}(A)$ is a partial isometry, then $vv^*v = v$.
(ii) If $\varphi : A \to B$ is a *-homomorphism to a C^*-algebra B, then $p \sim q$ implies $\varphi(p) \sim \varphi(q)$ for all $p, q \in \mathrm{P}_\infty(A)$.

Proof. Let $p \in \mathrm{P}_k(A)$, $q \in \mathrm{P}_m(A)$, and $r \in \mathrm{P}_n(A)$. It is clear that $p = p^*p$ and $p = pp^*$, so $\sim$ is reflexive.
Given $v \in \mathcal{M}_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$, then replacing v with v^* to get $q \sim p$, so $\sim$ is symmetric. Assume $m \geq n$, and let $z = (v - vv^*v) \oplus 0_{1,m-n+1} \in \mathcal{M}_{m+1,m+1}(A)$, so

$$\begin{aligned} z^*z &= \big((v^* - v^*vv^*) \oplus 0_{m-n+1,1}\big)\big((v - vv^*v) \oplus 0_{1,m-n+1}\big) \\ &= \Big(v^*v - (v^*v)^2 - (v^*v)^2 + (v^*v)^3\Big) \oplus 0_{m-n+1,m-n+1} = 0, \end{aligned}$$

hence $\|z\|^2 = \|z^*z\| = 0$, thus $z = 0$, i.e. $v = vv^*v$. Thus $v^* = v^*vv^*$, in particular, $vv^* = vv^*vv^* = (vv^*)^2$, and as vv^* is self-adjoint, then v^* is also a partial isometry. This shows (i).
Suppose $p \sim q$ and $q \sim r$, so there is a $v \in \mathcal{M}_{k,m}(A)$, and $w \in \mathcal{M}_{m,n}(A)$ such that $p = v^*v$, $q = vv^* = w^*w$, and $r = ww^*$, then

$$(wv)^*(wv) = v^*w^*wv = v^*qv = (v^*v)^2 = p,$$

and

$$(wv)(wv)^* = wvv^*w^* = wqw^* = (ww^*)^2 = r,$$

so $p \sim r$. Hence $\sim$ is transitive, thus $\sim$ is an equivalence relation.
Let $p, q \in \mathrm{P}_\infty(A)$ with $p \sim q$, then $p = vv^*$ and $q = v^*v$ for some $v \in \mathcal{M}_{m,n}(A)$ and $m, n \in \mathbb{N}$, thus $\varphi(p) = \varphi(v)\varphi(v)^*$ and $\varphi(q) = \varphi(v)^*\varphi(v)$, so $\varphi(p) \sim \varphi(q)$. This shows (ii). ■

The next propositions tells us how the relations are related to each other.

Proposition 3.1.2. Let p, q be projections of a unital C^*-algebra A. Then the following are equivalent:
(i) $p \sim_u q$ in $\mathcal{U}\big(\tilde{A}\big)$;
(ii) $p \sim_u q$ in $\mathcal{U}(A)$;
(iii) $p \sim q$ and $1_A - p \sim 1_A - q$.
Note that if A is nonunital, then $p \sim q$ is now implied by $p \sim_u q$.

Proof. Let $p = 1_{\tilde{A}} - 1_A$, so $\tilde{A} = A + \mathbb{C}f$. Note that $af = fa = 0$ for all $a \in A$.
(i)$\Longrightarrow$(ii). If $p \sim_u q$, then there is a $z \in \mathcal{U}\big(\tilde{A}\big)$ and $\lambda \in \mathbb{C}$ such that $p = zqz^*$. Now there is a $u \in A$ such that $z = u + \lambda f$, now observe that

$$1_{\tilde{A}} = z^*z = u^*u + |\lambda|^2 f = u^*u - |\lambda|^2 1_A + |\lambda|^2 1_{\tilde{A}}$$

so $|\lambda|^2 = 1$ by comparing $1_{\tilde{A}}$, and $u^*u - |\lambda|^2 1_A = 0$ by comparing elements in A, hence $u^*u = 1_A$. Similarly, $uu^* = 1_A$ by looking at $zz^* = 1_{\tilde{A}}$, thus $u \in \mathcal{U}(A)$. Now observe that

$$q = z^*pz = \big(u^* + \overline{\lambda}f\big)p(u + \lambda f) = u^*pu,$$

as required.
(ii)$\Longrightarrow$(iii). Suppose $q = upu^*$ for some $u \in \mathcal{U}(A)$, and let $v = up$ and $w = u(1_A - p)$. Now one has

[1] Recall that v is a partial isometry if v^*v is a projection; see discussion after Lemma 2.3.3.

$$\begin{aligned} v^*v = pu^*up^* = p \quad \text{and} \quad vv^* = upp^*u^* = q, \\ \text{similarly } w^*w = 1_A - p \quad \text{and} \quad ww^* = 1_A - q. \end{aligned} \tag{3.1}$$

(iii)$\Longrightarrow$(i). Suppose there are $v, w \in A$ such that (3.1) is satisfied. Let $z = v + w + f$, then

$$\begin{aligned} z^*z &= (v^* + w^* + f)(v + w + f) = p + v^*w + w^*v + (1_A - p) + f \\ &= 1_{\tilde{A}} + (v^*w + w^*v) \end{aligned}$$

and similarly

$$zz^* = 1_{\tilde{A}} + (vw^* + wv^*).$$

Now

$$v^*w = (v^*q)((1_A - q)w) = v^*(q(1_A - q))w = 0$$

similarly $w^*v = vw^* = wv^* = 0$, hence $z \in \mathcal{U}\left(\tilde{A}\right)$. Finally,

$$\begin{aligned} zpz^* &= (v + w + f)p(v^* + w^* + f) = vpv^* + wpw^* + vpw^* + wpv^* \\ &= q^2 + 0 + 0 + 0 = q. \end{aligned}$$

■

Lemma 3.1.3. Let a, b be elements of a unital C^*-algebra A. If $a \in \mathrm{GL}(A)$ and $\|a - b\| \leq \left\|a^{-1}\right\|^{-1}$, then $b \in \mathrm{GL}(A)$ and $a \sim_h b$ in $\mathrm{GL}(A)$. In particular, if $\|a\| < \lambda$ for some $\lambda > 0$, then $\pm\lambda \notin \sigma(a)$.

Proof. The inverse of b is given by the series

$$\frac{1}{b} = \frac{1}{a - (a - b)} = \frac{a^{-1}}{1 - a^{-1}(a - b)} = a^{-1} \sum_{n \geq 0} \left(a^{-1}(a - b)\right)^n$$

which exists as $\|a - b\| \leq \left\|a^{-1}\right\|^{-1}$. Define $c_t = tb + (1 - t)a$ for $t \in [0, 1]$, thus

$$\|a - c_t\| = \|a - tb - a + ta\| \leq t\|a - b\| \leq \left\|a^{-1}\right\|^{-1},$$

hence $c_t \in \mathrm{GL}(A)$, thus $a = c_0 \sim_h c_1 = b$ as $t \mapsto c_t$ is continuous.
If $\|a\| < \lambda$ for some $\lambda > 0$, then

$$\frac{1}{a \pm \lambda 1} = \frac{\pm\lambda^{-1}}{1 \pm \lambda^{-1}a} = \pm\lambda^{-1} \sum_{n \geq 0} \left(\mp\lambda^{-1}a\right)^n$$

which exists as $\|a/\lambda\| < 1$. Hence $a \pm \lambda 1$ is invertible, so $\pm\lambda \notin \sigma(a)$. ■

Lemma 3.1.4. Let a, b be self-adjoint elements in a unital C^*-algebra A such that $b = zaz^{-1}$ for some $z \in \mathrm{GL}(A)$, then $b = \omega(z)a\omega(z)^*$. Hence, $a \sim_s b$ implies $a \sim_u b$ in $\mathcal{U}(A)$.

Proof. Note that $bz = za$ implies $z^*b = az^*$, hence

$$|z|^2 a = (z^*z)a = z^*(bz) = (az^*)z = a|z|^2.$$

Thus a commutes with $|z|^2$, hence everything in $A\langle 1, |z|^2\rangle$. By considering the map $f : t \mapsto t^{-\frac{1}{2}}$ on $\sigma\left(|z|^2\right)$, one has $|z| = f\left(|z|^2\right) \in A\langle 1, |z|^2\rangle$, so a commutes with $|z|$. Now

$$\omega(z)a\omega(z)^* = z|z|^{-1}a|z|^{-1}z^* = za\left(z^*z\right)^{-1}z^* = b.$$

■

Proposition 3.1.5. Let p, q be projections of a C^*-algebra A. Then $p \sim_h q$ in $\mathrm{P}(A)$ if, and only if, there is a $u \in \mathcal{U}^0\left(\tilde{A}\right)$ such that $q = upu^*$.

Proof. "$\Longleftarrow$". Suppose $q = upu^*$ for some $u \in \mathcal{U}^0\left(\tilde{A}\right)$, then $u \sim_h 1_{\tilde{A}}$, in particular $q \sim_h 1_{\tilde{A}}p1_{\tilde{A}} = p$.
"$\Longrightarrow$". Suppose $p \sim_h q$ in $\mathrm{P}(A)$, then by compactness, there is a $n \in \mathbb{N}$ and $p_1, ..., p_n \in \mathrm{P}(A)$

such that $p = p_1 \sim_h p_2 \sim_h \dots \sim_h p_n = q$ with $\|p_i - p_{i+1}\| < 1/2$ for each $1 \le i \le n-1$. So by transitivity of $\sim_u$, we may assume $\|p - q\| < 1/2$. Write $1 \underset{\tilde{A}}{=}$, and define

$$z = pq + (1-p)(1-q) \in \tilde{A}$$

now one has

$$\begin{aligned}\|z - 1\| &= \|pq + 1 - p - q + pq - 1\| \\ &= \|pq - p^2 + pq - q^2\| \\ &= \|p(q-p) + (p-q)q\| \\ &= (\|p\| + \|q\|)\|p - q\| \\ &\le 2\|p - q\| < 1,\end{aligned}$$

hence $z \in \mathrm{GL}(A)$ and $z \sim_h 1$ in $\mathrm{GL}(A)$ by Lemma 3.1.3.
Thus $u := \omega(z) \sim_h \omega(1) = 1$ in $\mathcal{U}(A)$ (see Example 2.4.8), so $u \in \mathcal{U}^0(\tilde{A})$. As $pz = pq = zq$, hence $q = z^{-1}pz$, hence by Lemma 3.1.4, one has $q = u^*pu$, as required. ■

Lemma 3.1.6. Let p be a projection in a C^*-algebra A, and $a \in A$ be self-adjoint. Let $\delta = \|a - p\|$, then $\sigma(a) \subseteq [-\delta, \delta] \cup [1-\delta, 1+\delta]$.

Proof. Let $t \in \mathbb{R} \setminus ([-\delta, \delta] \cup [1-\delta, 1+\delta])$ and $d > \delta$ be the distance of t to $\{0, 1\}$. Now $p - t1$ is invertible in $\tilde{A}$, so

$$\|(p - t1)^{-1}\| = r((p - t1)^{-1}) = \max\{|-t|^{-1}, |1-t|^{-1}\} = d^{-1}$$

as $\sigma((p - t1)^{-1}) = \{-t^{-1}, (1-t)^{-1}\}$ by Continuous Functional Calculus 2.4.3 (ii). Thus

$$\begin{aligned}\|1 - (p - t1)^{-1}(a - t1)\| &= \|(p - t1)^{-1}(p - t1 - a + t1)\| \\ &\le \|(p - t1)^{-1}\|\|p - a\| < d^{-1}\delta < 1,\end{aligned}$$

hence $(p - t1)^{-1}(a - t1)$ is invertible, hence so is $a - t1$, thus $1 \notin \sigma(a)$, as required. ■

Proposition 3.1.7. Let p, q be projections on a C^*-algebra A. Then $\|p - q\| \le 1$ always, and if $\|p - q\| < 1$, then $p \sim_h q$ in $\mathrm{P}(A)$.

Proof. Let $u = 1 - 2p \in \tilde{A}$, then $u^* = u$, and $u^*u = (1-2p)^2 = 1$, so $u \in \mathcal{U}(\tilde{A})$, hence $\|u\| = 1$. In particular, $\|p - q\| \le \|\frac{1}{2} \cdot 1 - p\| + \|\frac{1}{2} \cdot 1 - q\| \le 1$. Now suppose $\|p - q\| < 1$.
Let $a_t = (1-t)p + tq$, so $t \mapsto a_t$ is continuous on $t \in [0, 1]$ and $p = a_0 \sim_h a_1 = q$ in A. Let $t \in [0, 1]$, and it suffices to show $a_t \in P(A)$. Now one has

$$\min\{\|a_t - p\|, \|a_t - q\|\} = \min\{t\|p - q\|, (1-t)\|p - q\|\} \le \frac{1}{2}\|p - q\| < \frac{1}{2}.$$

Let $\delta = \frac{\|p-q\|}{2}$ and $\Omega = \{a \in A : \sigma(a) \subseteq [-\delta, \delta] \cup [1-\delta, 1+\delta]\}$, thus by Lemma 3.1.6, one has $a_t \in \Omega$. As $\delta < \frac{1}{2}$, then define $f \in \mathcal{C}([-\delta, \delta] \cup [1-\delta, 1+\delta])$ by $f|_{[-\delta,\delta]} = 0$ and $f|_{[1-\delta,1+\delta]} = 1$, thus $f(a_t) \in \mathrm{P}(A)$.
As a_t is self-adjoint, then by Lemma 2.4.5, the map $t \mapsto f(a_t)$ is continuous, so

$$p = \mathrm{id}_{\sigma(p)}(p) = f(p) = f(a_0) \underset{h}{\sim} f(a_1) = q \text{ in } \mathrm{P}(A).$$

■

It turns out, these relations are rather 'equivalent' when lifted into matrices.

Proposition 3.1.8. Let p, q be projections in a C^*-algebra A. Then
(i) If $p \sim q$, then $p \oplus 0_1 \sim_u q \oplus 0_1$ in $\mathcal{M}_2(A)$.
(ii) If $p \sim_u q$, then $p \oplus 0_1 \sim_h q \oplus 0_1$ in $\mathcal{M}_2(A)$.

Proof.

(i) Suppose there is a $v \in A$ such that $p = v^*v$ and $q = vv^*$. Note $1 = 1_{\tilde{A}}$ here. Define:

$$u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}.$$

Using the fact that $1-p$ is also a projection, and $vp = v = qv$, one can conclude that $u, w \in \mathcal{U}_2(\tilde{A})$. Now

$$wu\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}u^*w^* = w\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}w^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$wu = \begin{pmatrix} qv + (1-q)(1-p) & (1-q)v^* \\ (1-q)v + q(1-p) & (1-q) + qv^* \end{pmatrix} = \begin{pmatrix} v-p-q+qp & v^* - qv^* \\ q - qp & qv^* - q \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{U}_2(\tilde{A}).$$

As $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the identity in $\widetilde{\mathcal{M}_2(A)}$, then $wu \in \mathcal{U}\left(\widetilde{M_2(A)}\right)$ by the identification made in Example 2.5.5. So $p \oplus 0_1 \sim_u q \oplus 0_1$.

(ii) Suppose there is a $u \in \mathcal{U}(\tilde{A})$ such that $q = upu^*$. By Whitehead 2.6.3, there is a path $t \mapsto w_t$ in $\mathcal{U}_2(\tilde{A})$ on $t \in [0,1]$ such that $w_0 = 1_2$ and $w_1 = u \oplus u^*$. Thus it is clear that $t \mapsto w_t(p \oplus 0_1)w_t^*$ is a path on $[0,1]$ from $p \oplus 0_1$ to $q \oplus 0_1$. ■

Thus one has current the roadmap of relations as shown in Diagram 1.

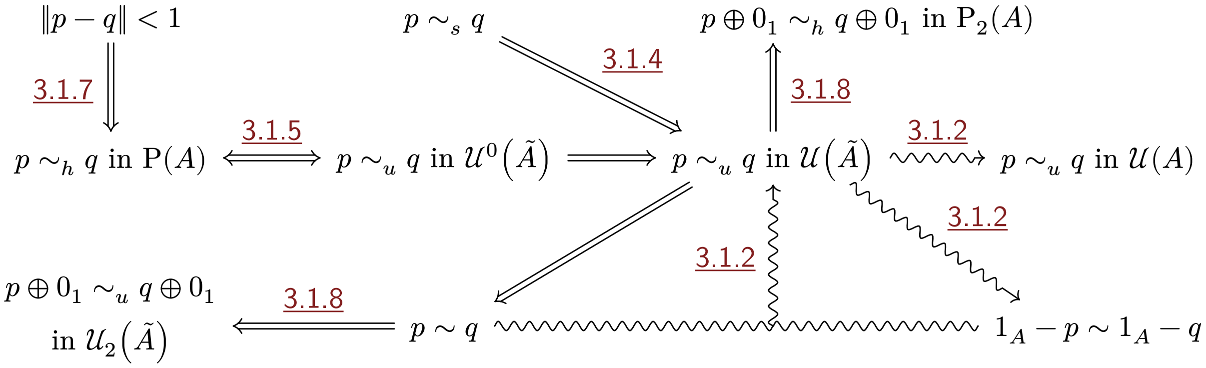


Diagram 1: The roadmap of relations given projection elements p and q in a C^*-algebra A, where $\rightsquigarrow$ means that the underlying C^*-algebra is unital.

Here is the promised result where the $\sim$ identifies bounded operators on Hilbert spaces by its rank. Furthermore, in the finite-dimensional case, one has that $\sim$ and $\sim_u$ are equivalent. Note that given a bounded operator T on a Hilbert space H, we define $\operatorname{rank}(T) = \dim(\operatorname{im}(T))$ where the dimension is the cardinality of Schauder (orthonormal) basis of the underlying Hilbert space.

Proposition 3.1.9. Let H be a Hilbert space, and $p, q \in P_\infty(\mathcal{B}(H))$. Then

(i) One has $p \sim q$ if, and only if, $\operatorname{rank}(p) = \operatorname{rank}(q)$.

(ii) One has $p \sim_u q$ if, and only if, $\operatorname{rank}(p) = \operatorname{rank}(q)$ and $\dim(\ker(p)) = \dim(\ker(q))$.

(iii) One has $\operatorname{rank}(p \oplus q) = \operatorname{rank}(p) + \operatorname{rank}(q)$ as cardinal numbers (or infinities).

Assume H is finite-dimensional.

(iii) One has $p \sim q$ if, and only if, $p \sim_u q$.

(iv) One has $\operatorname{tr}(p) = \operatorname{rank}(p)$, where p is identified as a matrix. In particular, if $\operatorname{rank}(p) = k$, then $p \sim I_k$.

Proof. Assume $p \in P_n(\mathcal{B}(H))$ and $q \in P_m(\mathcal{B}(H))$ for some $n, m \in \mathbb{N}$, and we make the identification $\mathcal{M}_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$, see Example 2.5.4.

(i) Suppose $p \sim q$, then then there is a $v \in \mathcal{B}(H^n, H^m)$ such that $p = v^*v$ and $q = vv^*$. By Proposition 3.1.1, one has $v = qv = vp$, so we can consider v as a map from $\mathrm{im}(p)$ to $\mathrm{im}(q)$, which are Hilbert spaces as $\mathrm{im}(p) = \ker(\mathrm{id}_H - p)$ is closed and similarly for $\mathrm{im}(q)$. Let $x \in H^m$, as $q = qq = vpv^*$, then $v(pv^*x) = qx$ shows that v is surjective. Let $x, y \in H^n$, so as
$$\langle vpx, vpy\rangle_H = \langle px, v^*vpy\rangle_H = \langle px, ppy\rangle_H = \langle px, py\rangle_H,$$
so v is an isometry. Thus $\mathrm{im}(p) \cong \mathrm{im}(q)$ as Hilbert spaces, hence $\mathrm{rank}(p) = \mathrm{rank}(q)$. If $\mathrm{rank}(p) = \mathrm{rank}(q)$, then there is a surjective linear isometry $v : \mathrm{im}(p) \to \mathrm{im}(q)$. Let $x, y \in H^n$, one has that
$$\langle v^*vpx, py\rangle_{\mathrm{im}(p)} = \langle vpx, vpy\rangle_{\mathrm{im}(q)} = \langle px, py\rangle_{\mathrm{im}(p)} = \langle ppx, py\rangle_{\mathrm{im}(p)},$$
thus the maps v^*v and p agree on the inner product of $\mathrm{im}(p)$, hence $v^*v = p|_{\mathrm{im}(p)}$. Similarly, $v^* : \mathrm{im}(q) \to \mathrm{im}(p)$ defines a linear isometry, and by similar argument, one has $vv^* = q|_{\mathrm{im}(q)}$, so $v = vp|_{\mathrm{im}(p)} = qv$. Define $w : H^n \to H^m$ as $wx = vpx$ for $x \in H^n$, so $w^*w = p^*v^*vp = ppp = p$ and $ww^* = vpp^*v^* = vpv^* = qvv^* = qq = q$, so $p \sim q$, as required.

(ii) By Diagram 1, part (i), and the fact that $\ker(p) = \mathrm{im}(1_A - p)$, one has
$$\begin{aligned} p \underset{u}{\sim} q &\iff p \sim q \text{ and } 1_A - p \sim 1_A - q \\ &\iff \mathrm{rank}(p) = \mathrm{rank}(q) \text{ and } \dim(\ker(p)) = \dim(\ker(q)). \end{aligned}$$

(iii) Note that one has the map $p \oplus q : H^n \oplus H^m \to H^n \oplus H^m$, hence $\mathrm{im}(p \oplus q) \cong \mathrm{im}(p) \oplus \mathrm{im}(q)$, then the statement follows.

(iv) If H if finite-dimensional, then by rank-nullity theorem, one has $\mathrm{rank}(p) = \mathrm{rank}(q)$ if, and only if, $\dim(\ker(p)) = \dim(\ker(q))$, the rest follows from (ii).

(v) Choose a basis B for H, and denote $[p]_B$ to be the matrix of p with rest to the basis B. Note that by the cyclic property of trace, $\mathrm{tr}(p) = \mathrm{tr}([p]_B)$ is independent of the basis B. Since $\mathrm{tr}(p)$ is the sum of eigenvalues of p, and the only eigenvalues of p are 0 and 1, then it follows that $\mathrm{tr}(p) = \dim(\ker(p - \mathrm{id}_H)) = \dim(\mathrm{im}(p)) = \mathrm{rank}(p)$, as required. ■

3.2 Grothendieck Groups

The purpose of this section is to provide a careful treatment of extending Abelian semigroups into full on Abelian groups, which will be the last ingredient we need to construct our K_0-groups.

Recall that a **semigroup** $(S, +)$ is a nonempty set S equipped with a binary operation $+ : S \times S \to S$ that is only needed to be associative. We say the semigroup $(S, +)$ is Abelian if $+$ is furthermore commutative, and we say $(S, +)$ is a **monoid** if there exists an additive identity 0 such that $s + 0 = 0 + s = s$ for all $s \in S$. We will be mainly interested in Abelian semigroups, so assume $(S, +)$ as such. We say S has the **cancellation property** if for each $x, y, z \in S$, $x + z = y + z$ implies $x = y$.

Define a relation $\sim$ on $S \times S$ as such:
$$(x_1, y_1) \sim (x_2, y_2) \text{ if, and only if, there is a } z \in S \text{ such that } x_1 + y_2 + z = x_2 + y_1 + z.$$
Note that this relation is clearly reflexive as $S \neq \emptyset$ and symmetric, and further work can show that this relation is transitive. Define $\mathbb{G}(S) = S \times S/\sim$, and define $\langle x, y\rangle$ to be an equivalence class of $(x, y) \in S \times S$ under $\sim$. Define the operation $+$ on $G(S)$ by
$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle.$$
It follows that $(\mathbb{G}(S), +)$ is an Abelian group with additive identity given by $0 = \langle x, x\rangle$, and additive inverses given by $-\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in S$; for details, refer to [6, p. 39]. We say $\mathbb{G}(S)$ is

the **Grothendieck group** or **Grothendieck completion** of S, and we also have a *canonical map* or *Grothendieck map* $\gamma_S : S \to \mathbb{G}(S)$ given by $\gamma_S(x) = \langle x+y, y\rangle$ for a fixed $y \in S$, note that γ_S is well-defined and is independent of the choice of $y \in S$.

We have some nice properties of Grothendieck groups in general, in particularly, (iv) in the next proposition tells us that Grothendieck completions of Abelian semigroups with the cancellation property are precisely the smallest Abelian group that is generated by the semigroup.

Proposition 3.2.1. *(Structure of Grothendieck Groups)*. Let S be an Abelian semigroup.

(i) $\mathbb{G}(S) = \gamma_S(S) - \gamma_S(S)$.
(ii) Given $x, y \in S$, one has $\gamma_S(x) = \gamma_S(y)$ if, and only if, $x + z = y + z$ for some $z \in S$.
(iii) γ_S is injective if, and only if, S has the cancellation property.
(iv) Let H be an Abelian group, and suppose S is an Abelian semisubgroup of H. Then S has the cancellation property and $\mathbb{G}(S) \cong \langle S\rangle = S - S$.

Proof.

(i) Clearly $\gamma_S(S) - \gamma_S(S) \subseteq \mathbb{G}(S)$. Now given $\langle x, y\rangle \in \mathbb{G}(S)$, one has
$$\langle x, y\rangle = \langle x, y\rangle + 2\langle y, y\rangle = \langle x+2y, 3y\rangle = \langle x+y, y\rangle - \langle y+y, y\rangle = \gamma_S(x) - \gamma_S(y),$$
so $\mathbb{G}(S) = \gamma_S(S) - \gamma_S(S)$.

(ii) Suppose $\gamma_S(x) = \gamma_S(y)$, so $\langle x+y, y\rangle = \langle y+x, x\rangle$. Then there is a $w \in S$ such that
$$x + y + x + w = y + y + x + w,$$
choose $z = y + x + w$, and we are done. Conversely, suppose $x + z = y + z$ for some $z \in S$, then
$$\gamma_S(x) = \langle x+z, z\rangle = \langle y+z, z\rangle = \gamma_S(y).$$

(iii) This follows from (ii).

(iv) It is clear that $S - S \subseteq \langle S\rangle$ (the smallest subgroup of H containing S), and $S - S$ is a subgroup of H, so $\langle S\rangle \subseteq S - S$. Hence $\langle S\rangle = S - S$. It is clear that S has the cancellation property, thus γ_S is injective, so it follows that $\mathbb{G}(S) = \gamma_S(S) - \gamma_S(S) \cong S - S = \langle S\rangle$, as required. ■

Thus by part (iv) of the preceding proposition, given an Abelian group H, we may assume that $\mathbb{G}(H)$ is H. Immediately, we have the universality and functoriality properties of the Grothendieck completion $\mathbb{G}$.

Theorem 3.2.2. *(Universality of Grothendieck Completion)*. If $\varphi : S \to H$ is an additive map between an Abelian semigroup S and an Abelian group H. Then there is a unique homomorphism $\psi : \mathbb{G}(S) \to H$ such that $\psi \circ \gamma_S = \varphi$.

Proof. Define $\psi : \mathbb{G}(S) \to H : \langle x, y\rangle \mapsto \varphi(x) - \varphi(y)$. Suppose $\langle x, y\rangle = \langle x', y'\rangle$ for $x, x', y, y' \in S$, then there is a $z \in S$ such that $x + y' + z = x' + y + z$, and one has
$$\psi(\langle x, y\rangle) - \psi(\langle x', y'\rangle) = \varphi(x) - \varphi(y) - \varphi(x') + \varphi(y') = \varphi(x + y' + z) - \varphi(x' + y + z) = 0,$$
so ψ is well-defined. It is clear that $\psi \circ \gamma_S = \varphi$ and ψ is a homomorphism. Thus existence is shown. Suppose there is another homomorphism $\psi' : \mathbb{G}(S) \to H$ such that $\psi' \circ \gamma_S = \varphi$. Given $\langle x, y\rangle \in \mathbb{G}(S)$, by Structure of Grothendieck Groups 3.2.1 (i), there are $x', y' \in S$ such that $\langle x, y\rangle = \gamma_S(x') - \gamma_S(y')$, thus one has
$$\psi'(\langle x, y\rangle) = \psi'(\gamma_S(x') - \gamma_S(y')) = (\psi' \circ \gamma_S)(x') - (\psi' \circ \gamma_S)(y') = \varphi(x) - \varphi(y) = \psi(\langle x, y\rangle),$$
shows $\psi = \psi'$, as required. ■

Theorem 3.2.3. *(Functoriality of Grothendieck Completion).* If $\varphi : S \to T$ is an additive map between Abelian semigroups S and T. Then there is a unique homomorphism $\mathbb{G}(\varphi) : \mathbb{G}(S) \to \mathbb{G}(T)$ such that $\gamma_T \circ \varphi = \mathbb{G}(\varphi) \circ \gamma_S$.

Proof. Define $\mathbb{G}(\varphi) : \mathbb{G}(S) \to \mathbb{G}(T) : \langle x, y\rangle \mapsto \langle \varphi(x), \varphi(y)\rangle$. Suppose $\langle x, y\rangle = \langle x', y'\rangle$ for $x, x', y, y' \in S$, then there is a $z \in S$ such that $x + y' + z = x' + y + z$, and one has

$$\begin{aligned}\mathbb{G}(\varphi)(\langle x, y\rangle) - \mathbb{G}(\varphi)(\langle x', y'\rangle) &= \langle \varphi(x), \varphi(y)\rangle + \langle \varphi(y'), \varphi(x')\rangle \\ &= \langle \varphi(x + y'), \varphi(y + x')\rangle + \langle \varphi(z), \varphi(z)\rangle \\ &= \langle \varphi(x + y' + z), \varphi(y + x' + z)\rangle \\ &= 0,\end{aligned}$$

so $\mathbb{G}(\varphi)$ is well-defined. It is clear that $\mathbb{G}(\varphi) \circ \gamma_S = \gamma_T \circ \varphi$, and $\mathbb{G}(\varphi)$ is a homomorphism. Thus existence is shown.

Suppose there is another homomorphism $\varphi' : \mathbb{G}(S) \to \mathbb{G}(T)$ such that $\varphi' \circ \gamma_S = \gamma_T \circ \varphi$. Given $\langle x, y\rangle \in \mathbb{G}(S)$, by Structure of Grothendieck Groups 3.2.1 (i), there are $x', y' \in S$ such that $\langle x, y\rangle = \gamma_S(x') - \gamma_S(y')$, thus one has

$$\varphi'(\langle x, y\rangle) = (\varphi' \circ \gamma_S)(x') - (\varphi' \circ \gamma_S)(y') = (\mathbb{G}(\varphi) \circ \gamma_S)(x') - (\mathbb{G}(\varphi) \circ \gamma_S)(y') = \mathbb{G}(\varphi)(\langle x, y\rangle),$$

so $\varphi' = \mathbb{G}(\varphi)$, as required. ■

Here are some immediate examples of Grothendieck completions. Note that the second example is rather nondegenerate despite having seemingly more structure than the first.

Example 3.2.4.

- We note that $\mathbb{N} = \{1, 2, ...\}$ under usual addition $+$ forms an Abelian semigroup with the cancellation property. As $\mathbb{N}$ is identified as a subgroup of $\mathbb{Z}$, in particular, $\mathbb{N} - \mathbb{N} = \mathbb{Z}$, then $\mathbb{G}(\mathbb{N}) \cong \mathbb{Z}$.
- If consider $\mathbb{N} \cup \{\infty\}$ where $\infty + x = \infty$ for all $x \in \mathbb{N}$, then $(\mathbb{N} \cup \{\infty\}, +)$ is an Abelian semigroup with no cancellation property, in particular, one observes that $\mathbb{G}(\mathbb{N} \cup \{\infty\}) = 0$ as $\langle x, y\rangle = \langle \infty, \infty\rangle$ for all $x, y \in \mathbb{N} \cup \{\infty\}$.

3.3 The K_{00}-Group Construction

In this chapter, we aim to build our first K_{00}-group, which will serve as our foundation for the K_0-groups. In the preceding section, we have established the Murray-von Neumann relation $\sim$ on $\mathrm{P}_\infty(A)$ is an equivalence relation, so we denote the equivalence classes of $\sim$ by $[\cdot]$. We now establish that $\sim$ is a relation that is compatible with the $\oplus$ operation. With this, one can construct an Abelian semigroup $(\mathrm{P}_\infty(A)/\sim, +)$.

Proposition 3.3.1. Let A be a C^*-algebra, and $p, q, r, s \in \mathrm{P}_\infty(A)$. Then:

(i) $p \sim p \oplus 0_n$ for each $n \in \mathbb{N}$.
(ii) If $p \sim q$ and $r \sim s$, then $p \oplus r \sim q \oplus s$.
(iii) $p \oplus q \sim q \oplus p$.
(iv) Let $n \in \mathbb{N}$, if $p, q \in \mathrm{P}_n(A)$ and $p \perp q$, then $p + q \sim p \oplus q$.
(v) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$

Proof. Let $m, n \in \mathbb{N}$.

(i) Suppose $p \in \mathrm{P}_m(A)$, and let $v = \begin{pmatrix} p \\ 0_{n,m}\end{pmatrix} \in M_{m+n,n}(A)$, so $p = v^*v \sim vv^* = p \oplus 0_n$.

(ii) There are matrices v and w such that $p = v^*v$, $q = vv^*$, $r = w^*w$, and $s = ww^*$. Define $u = v \oplus w$, then $u^*u = p \oplus r \sim uu^* = q \oplus s$.
(iii) Let $u = \begin{pmatrix} 0_{m,n} & q \\ p & 0_{n,m} \end{pmatrix} \in \mathcal{M}_{n+m}(A)$, then $u^*u = p \oplus q \sim uu^* = q \oplus p$.
(iv) Suppose $pq = 0$, then $qp = 0$. Define $u = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathcal{M}_{2n,n}(A)$, then $u^*u = p + q \sim uu^* = p \oplus q$.
(v) Trivial. ■

We define a binary operation $+$ on $\mathrm{P}_\infty(A)/\sim$ by
$$[p] + [q] = [p \oplus q].$$
By Proposition 3.3.1 (ii) and (v), $+$ is well-defined and associative, by (i), $0 = [0_n]$ is the additive identity for each $n \in \mathbb{N}$, and finally by (iii), $+$ is commutative. Thus $(\mathrm{P}_\infty(A)/\sim, +)$ is an Abelian monoid.

Given a C^*-algebra A, by Grothendieck completion, one has an Abelian group, called the **K_{00}-group for A** defined as
$$K_{00}(A) := \mathbb{G}(\mathrm{P}_\infty(A)/\sim, +).$$
We denote the classes of $K_{00}(A)$ to be $[p]_0$ where $[\cdot]_0 : \mathrm{P}_\infty(A) \to K_{00}(A)$ is a composition of the class map $[\cdot]$ under the relation $\sim$, and the Grothendieck map $\gamma_A : \mathrm{P}_\infty(A)/\sim \to K_{00}(A)$. Thus we immediately note that $K_{00}(A) = \{[p]_0 - [q]_0 : p, q \in \mathrm{P}_\infty(A)\}$. We say that $K_{00}(A)$ has the **cancellation property** if the underlying Abelian semigroup $\mathrm{P}_\infty(A)/\sim$ has the cancellation property, which is equivalent to $[p]_0 = [q]_0$ if, and only if, $p \sim q$ for all $p, q \in \mathrm{P}_\infty(A)$ by the Structure of Grothendieck Groups 3.2.1 (iii). We shall consider another relation called the *stable equivalence* which allows us to identify when $[p]_0 = [q]_0$.

Construction 3.3.2. *(Stable Equivalence)*. We define the **stable equivalence relation $\sim_s$** on $\mathrm{P}_\infty(A)$ where A is any C^*-algebra such that for any $p, q \in \mathrm{P}_n(A)$ for some $n \in \mathbb{N}$, we write $p \sim_s q$ if there is a $r \in \mathrm{P}_n(A)$ such that $p \oplus r \sim q \oplus r$. It can be easily verified that this relation is reflexive and symmetric, to show transitivity, observe that $p \oplus a \sim q \oplus a$ and $q \oplus b \sim r \oplus b$, then as $a \oplus b \sim b \oplus a$, one has
$$p \oplus (a \oplus b) \sim q \oplus (a \oplus b) \sim r \oplus (a \oplus b)$$
shows $p \sim_s r$, if $p \sim_s q$ and $q \sim_s r$. The reason we introduce this relation is that it can be used to show that it is equivalent to the $[\cdot]_0$ relation for K_{00}-groups. If A is furthermore unital, then it can be shown that $p \sim_s q$ if, and only if, $p \oplus 1_n \sim q \oplus 1_n$ for some $n \in \mathbb{N}$. Indeed, given any $r \in \mathrm{P}_n(A)$, let $v = \begin{pmatrix} r & 1_n - r \end{pmatrix}$, one has $v^*v = r \oplus (1_n - r)$ and $vv^* = 1_n$, so $r \oplus (1_n - r) \sim 1_n$, hence
$$p \oplus 1_n \sim p \oplus r \oplus (1_n - r).$$
So if $p \sim_s q$, then $p \oplus 1_n \sim q \oplus 1_n$ for all $n \in \mathbb{N}$, and if $p \oplus 1_n \sim q \oplus 1_n$ for some $n \in \mathbb{N}$, then $p \sim_s q$.

Proposition 3.3.3. *(Structure of K_{00})*. Let A be a unital C^*-algebra, then
(i) $[p \oplus q]_0 = [p]_0 + [q]_0$ for all $p, q \in \mathrm{P}_\infty(A)$.
(ii) $[0_n]_0 = 0$ for all $n \in \mathbb{N}$.
(iii) If $p, q \in \mathrm{P}_n(A)$ and $p \sim_h q$, then $[p]_0 = [q]_0$.
(iv) If $p, q \in \mathrm{P}_n(A)$ and $p \perp q$, then $[p + q]_0 = [p]_0 + [q]_0$.
(v) One has $[p]_0 = [q]_0$ if, and only if, $p \sim_s q$ for all $p, q \in \mathrm{P}_\infty(A)$.
(vi) $K_{00}(A) = \{[p]_0 - [q]_0 : p, q \in \mathrm{P}_n(A), n \in \mathbb{N}\}$.

Proof. For (i), observe that
$$[p \oplus q]_0 = \gamma_A([p \oplus q]) = \gamma_A([p] + [q]) = \gamma_A([p]) + \gamma_A([q]) = [p]_0 + [q]_0,$$

so $[p]_0 + [0_n]_0 = \gamma_A([p \oplus 0_n]) = \gamma_A([p]) = [p]_0$, thus $[0_n]_0 = 0$ hence (ii) is proven. For (iii), note that $p \sim_h q$ implies $p \sim q$, so one has $[p] = [q]$, hence $[p]_0 = [q]_0$. For (iv), if $p \perp q$, then by Proposition 3.3.1 (iv):

$$[p+q]_0 = \gamma_A([p] + [q]) = [p]_0 + [q]_0.$$

For (v), $[p]_0 = [q]_0$ means that there is a $r \in \mathrm{P}_n(A)$ such that $[p] + [r] = [q] + [r]$, in particular, $p \oplus r \sim q \oplus r$, so $p \sim_s q$. Similarly, if $p \sim_s q$, then there is a $r \in \mathrm{P}_n(A)$ such that $p \oplus r \sim q \oplus r$, hence $[p] + [r] = [q] + [r]$, thus $[p]_0 = [q]_0$.

For (vi), it is clear that $\{[p]_0 - [q]_0 : p, q \in \mathrm{P}_n(A), n \in \mathbb{N}\} \subseteq K_{00}(A)$. Now given $p \in \mathrm{P}_n(A)$ and $q \in \mathrm{P}_m(A)$, and suppose $m \geq n$, then $p \oplus 0_{m-n} \in \mathrm{P}_m(A)$ and $[p \oplus 0_{m-n}]_{00} - [q]_0 \in K_{00}(A)$ by (ii), so it follows that $K_{00}(A) = \{[p]_0 - [q]_0 : p, q \in \mathrm{P}_n(A), n \in \mathbb{N}\}$. ■

Theorem 3.3.4. *(Universal Property of K_{00}-Groups).* Let A be a C^*-algebra, let S be an Abelian semigroup, and suppose that $\mu : \mathrm{P}_\infty(A) \to S$ is a map such that

(i) $\mu(p \oplus q) = \mu(p) + \mu(q)$ for all $p, q \in \mathrm{P}_\infty(A)$.

(ii) If $p, q \in \mathrm{P}_\infty(A)$ satisifies $p \sim q$, then $\mu(p) = \mu(q)$.

Then there is a unique homomorphism $\nu : K_{00}(A) \to \mathbb{G}(S)$ such that $\nu \circ [\cdot]_0 = \gamma_S \circ \mu$. Furthermore:

(a) If $\hat{\mu}$ is injective and S has cancellation property, then ν is injective.

(b) If μ is surjective, then ν is surjective.

Proof. Note that by (ii) one has a map $\hat{\mu} : \mathrm{P}_\infty(A)/\sim \to S$ such that $\mu = \hat{\mu} \circ [\cdot]$, and $\hat{\mu}$ is an additive map by (i). Hence we can define $\nu' : \mathrm{P}_\infty(A)/\sim \to \mathbb{G}(S)$ as $\nu([p]) = \gamma_S(\hat{\mu}([p]))$, which is an additive map, thus by the Universality of Grothendieck Completion 3.2.2, one has a homomorphism $\nu : K_{00}(A) \to \mathbb{G}(S)$ such that $\nu \circ [\cdot]_0 = \nu'$, the uniqueness ν follows.

For (a), if $\hat{\mu}$ is injective and S has the the cancellation property, then γ_S has the cancellation property by theStructure of Grothendieck Groups 3.2.1 (iii), γ_S is injective, hence ν is injective as a composition of injective maps.

For (b), if μ is surjective, then $\hat{\mu}$ is surjective as $[\cdot]$ is surjective. Let $z \in \mathbb{G}(S)$, then by the Structure of Grothendieck Groups 3.2.1 (i), one has $z = \gamma_S(x) - \gamma_S(y)$ for $x, y \in S$, hence there are $x', y' \in \mathrm{P}_\infty(A)/\sim$ such that $\hat{\mu}([x']) = x$ and $\hat{\mu}([y']) = y$. Thus $\nu\big([x']_0 - [y']_0\big) = z$, shows that ν is surjective. ■

Given a *-homomorphism $\varphi : A \to B$ between C^*-algebras, this induces a natural *-homomorphism betweem between matrix algebras $\mathcal{M}_n(A)$ to $\mathcal{M}_n(B)$ by Construction 2.5.1. As *-homomorphisms preserves projections, then we can induce a map $\varphi : \mathrm{P}_\infty(A) \to \mathrm{P}_\infty(B)$. Consider the following composition $\mu = [\cdot]_0 \circ \varphi : \mathrm{P}_\infty(A) \to K_{00}(B)$, then as φ preserves the $\sim$ relation, then μ is invariant under the $\sim$ relation, i.e. it satisfies (iii) of the Universal Property of K_{00}-Groups 3.3.4. As (i) and (ii) of the Universal Property of K_{00}-Groups 3.3.4 is also satisfied, then there is a unique homomorphism $K_{00}(\varphi) : K_{00}(A) \to K_{00}(B)$ such that $[\cdot]_0 \circ K_{00}(\varphi) = \mu$, in particular, one has the commutative diagram:

$$\begin{array}{ccc} \mathrm{P}_\infty(A) & \xrightarrow{\varphi} & \mathrm{P}_\infty(B) \\ {\scriptstyle [\cdot]_0}\downarrow & & \downarrow{\scriptstyle [\cdot]_0} \\ K_{00}(A) & \xrightarrow[K_{00}(\varphi)]{} & K_{00}(B) \end{array}$$

Given this construction from *-homomorphisms between C^*-algebras to group homormophisms between K_{00}-groups, one has that K_{00} defines a functor that preserves the zero objects.

Theorem 3.3.5. *(Functoriality of K_{00}).* Let A, B, C be C^*-algebras and $\varphi : A \to B$, $\psi : B \to C$ be *-homomorphisms. One has:

(i) $K_{00}(\mathrm{id}_A) = \mathrm{id}_{K_{00}(A)}$.
(ii) $K_{00}(\psi \circ \varphi) = K_{00}(\psi) \circ K_{00}(\varphi)$.
(iii) The K_{00} preserves the zero map $0 : A \to B$, so $K_{00}(0) : K_{00}(A) \to K_{00}(B)$ is the trivial homomorphism.
(iv) If 0 is the trivial C^*-algebra, then $K_{00}(0)$ is the trivial group.

Proof. Trivial. ■

Lemma 3.3.6. If A and B are C^*-algebras and $\varphi, \psi : A \to B$ are orthogonal *-homomorphisms, then $\varphi + \psi : A \to B$ is a *-homomorphism, and $K_{00}(\varphi + \psi) = K_{00}(\varphi) + K_{00}(\psi)$.

Proof. Note $\varphi + \psi$ is a *-homomorphism is trivially proven. For all $p \in \mathrm{P}_\infty(A)$, one has $\varphi(p) \perp \psi(p)$, hence by the Structure of K_{00} 3.3.3 (iv), one has

$$K_{00}(\varphi + \psi)[p]_0 = [(\varphi + \psi)(p)]_0 = [\varphi(p) + \psi(p)]_0 = [\varphi(p)]_0 + [\psi(p)]_0 = K_{00}(\varphi)[p]_0 + K_{00}(\psi)[p]_0,$$

so $K_{00}(\varphi + \psi) = K_{00}(\varphi) + K_{00}(\psi)$. ■

We say two C^*-algebras A and B are **homotopic** if there are *-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi \circ \psi \sim_h \mathrm{id}_B$ (pointwise homotopy) and $\psi \circ \varphi \sim_h \mathrm{id}_A$.

Theorem 3.3.7. *(Homotopy Invariance of K_{00}).* Let A and B be C^*-algebras. If A and B are homotopic, then $K_{00}(A) \cong K_{00}(B)$.

Proof. Let $\varphi : A \to B$ and $\psi : B \to A$ be the associated *-homomorphisms. So for each $t \in [0, 1]$, there is a *-homomorphism $h_t : A \to A$ with $h_0 = \psi \circ \varphi$ and $h_1 = \mathrm{id}_A$, such that for each $a \in A$, the map $t \mapsto h_t(a)$ is continuous. So given $p \in \mathrm{P}_\infty(A)$, one has that $(\psi \circ \varphi)(p) \sim_h p$, thus

$$[p]_0 = [(\psi \circ \varphi)(p)]_0 = K_{00}(\psi \circ \varphi)[p]_0 = (K_{00}(\psi) \circ K_{00}(\varphi))[p]_0$$

so $K_{00}(\psi) \circ K_{00}(\varphi) = \mathrm{id}_{K_{00}(A)}$. Similarly, $K_{00}(\varphi) \circ K_{00}(\psi) = \mathrm{id}_{K_{00}(B)}$. Thus $K_{00}(A) \cong K_{00}(B)$.■

Proposition 3.3.8. Let A be a unital C^*-algebra, then the split-exact sequence

$$0 \longrightarrow A \overset{\iota}{\hookrightarrow} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

induces a split-exact sequence of groups

$$0 \longrightarrow K_{00}(A) \overset{K_{00}(\iota)}{\hookrightarrow} K_{00}\big(\tilde{A}\big) \underset{K_{00}(\lambda)}{\overset{K_{00}(\pi)}{\rightleftarrows}} K_{00}(\mathbb{C}) \longrightarrow 0 \tag{3.2}$$

Proof. Let $q = 1_{\tilde{A}} - 1_A$, and by Lemma 2.2.3, $\tilde{A} = A \oplus \mathbb{C}q$, and define

$$\mu : \tilde{A} \to A : a + \alpha q \mapsto a \quad \text{and} \quad \nu : \mathbb{C} \to \tilde{A} : \alpha \mapsto aq.$$

Note that

$$\mathrm{id}_A = \mu \circ \iota, \quad \mathrm{id}_{\tilde{A}} = \iota \circ \mu + \nu \circ \pi, \quad \pi \circ \iota = 0, \quad \pi \circ \lambda = \mathrm{id}_{\mathbb{C}},$$

thus by functoriality of K_{00} and Lemma 3.3.6, one has

$$\mathrm{id}_{K_{00}(A)} = K_{00}(\mu) \circ K_{00}(\iota)$$
$$K_{00}(\pi) \circ K_{00}(\lambda) = \mathrm{id}_{K_{00}(\mathbb{C})}$$
$$K_{00}(\pi) \circ K_{00}(\iota) = 0$$

$$\mathrm{id}_{K_{00}(\tilde{A})} = K_{00}(\iota) \circ K_{00}(\mu) + K_{00}(\nu) \circ K_{00}(\pi).$$

It suffices to show $\ker(K_{00}(\iota)) = 0$, $\operatorname{im}(K_{00}(\pi)) = K_{00}(\mathbb{C})$, $\operatorname{im}(K_{00}(\iota)) = \ker(K_{00}(\pi))$, and $K_{00}(\pi) \circ K_{00}(\lambda) = \mathrm{id}_{K_{00}(\mathbb{C})}$, which the last identity is already shown.
By first identity: $\ker(K_{00}(\iota)) \subseteq \ker(K_{00}(\mu) \circ K_{00}(\iota)) = \ker\left(\mathrm{id}_{K_{00}(A)}\right) = 0$.
By second identity: $K_{00}(\mathbb{C}) = \operatorname{im}\left(\mathrm{id}_{K_{00}(\mathbb{C})}\right) = \operatorname{im}(K_{00}(\pi) \circ K_{00}(\lambda)) \subseteq \operatorname{im}(K_{00}(\pi))$.
By third identity: $\operatorname{im}(K_{00}(\iota)) \subseteq \ker(K_{00}(\pi))$. Now given $[p]_0 \in \ker(K_{00}(\pi))$, one has $[p]_0 = K_{00}(\iota)(K_{00}(\mu)[p]_0)$ by the fourth identity, so $\ker(K_{00}(\pi)) \subseteq \operatorname{im}(K_{00}(\iota))$. ■

Proposition 3.3.9. *(Structure of $K_{00}(\mathcal{B}(H))$).* Let H be a Hilbert space, then the rank map $\mathrm{rank} : \mathrm{P}_\infty(\mathcal{B}(H)) \to \mathbb{N}_0 \cup \{\infty\}$ satisfies the conditions of Universal Property of K_{00}-Groups 3.3.4. Furthermore:
(i) If H is infinite-dimensional, then $K_{00}(\mathcal{B}(H)) = 0$.
(ii) If H is finite-dimensional, then $K_0(\mathrm{tr}) : K_{00}(\mathcal{B}(H)) \to \mathbb{Z}$ defined as $K_0(\mathrm{tr})([p]_0) = \mathrm{tr}(p)$ defines an isomorphism, and $K_{00}(\mathcal{B}(H))$ has the cancellation property.

Proof. Note that by Proposition 3.1.9, the map rank satisfies the conditions of Universal Property of K_{00}-Groups 3.3.4, and rank is injective on $\mathrm{P}_\infty(\mathcal{B}(H))/\sim$.
(i) If H is infinite-dimensional, then $\mathrm{rank}(\mathrm{id}_H) = \infty$, then $\mathrm{rank} : \mathrm{P}_\infty(\mathcal{B}(H)) \to \mathbb{N}_0 \cup \{\infty\}$ is surjective, thus by Universal Property of K_{00}-Groups 3.3.4, there is an isomorphism from $K_{00}(\mathcal{B}(H))$ to $\mathbb{G}(\mathbb{N}_0 \cup \{0\})$, which we note $\mathbb{G}(\mathbb{N}_0 \cup \{\infty\}) = 0$ by Example 3.2.4.
(ii) If H is finite-dimensional, then rank surjects onto $\mathbb{N}_0$, thus $\mathrm{P}_\infty(A)/\sim$ has the cancellation property as it embeds into $\mathbb{N}_0$ as Abelian semigroups. By Proposition 3.1.9, one has $\mathrm{rank} = \mathrm{tr}$, thus by Universal Property of K_{00}-Groups 3.3.4, the $K_0(\mathrm{tr}) : K_{00}(\mathcal{B}(H)) \to \mathbb{Z}$ (we identify $\mathbb{G}(\mathbb{N}_0) \cong \mathbb{Z}$) defined as $K_0(\mathrm{tr})([p]_0) = \mathrm{tr}(p)$ is an isomorphism. ■

3.4 The K_0-Group Construction

Unfortunately, the K_{00} functor does not have nice properties such as preserving some variations of exactness or being additive, thus we introduce a slightly more complicated structure than K_{00}, which does indeed have those nice properties. Hence the aim of this section is to properly introduce the K_0-group for C^*-algebras, and prove the nice functorial properties of K_0 which K_{00} lacks. Given a C^*-algebra A, we define the **K_0-group for A** to be

$$K_0(A) := \ker(K_{00}(\pi))$$

where $\pi : \tilde{A} \to \mathbb{C}$ is the natural projection map. Note that we see K_0 immediately generalizes K_{00} in the unital case.

Proposition 3.4.1. If A is a unital C^*-algebra, then $K_0(A) \cong K_{00}(A)$.

Proof. By Proposition 3.3.8, we see that $K_0(A) = \operatorname{im}(K_{00}(\iota)) \cong K_{00}(A)$ if A is unital as $K_{00}(\iota)$ is now injective. ■

We may identify $K_0(A)$ as $K_{00}(A)$ if A is unital. Note that for $[p]_0 \in K_0(A)$, then $[\pi(p)]_0 = 0$, which means that $\pi(p)$ is a zero matrix, so $p \in \mathrm{P}_\infty(A)$. The functoriality of K_0 actually carries over pretty well. Given a *-homomorphism $\varphi : A \to B$ between C^*-algebras A and B, by the Functoriality of Unitization 2.2.5, one induces a commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\iota} & \tilde{A} & \xrightarrow{\pi_A} & \mathbb{C} \\ \varphi \big\downarrow & & \tilde{\varphi} \big\downarrow & & \big\| \\ B & \xrightarrow{\iota} & \tilde{B} & \xrightarrow{\pi_B} & \mathbb{C} \end{array}$$

Then funtoriality of K_{00}, one induces a commutative diagram

$$\begin{array}{ccccc} K_{00}(A) & \xrightarrow{K_{00}(\iota)} & K_{00}\big(\tilde{A}\big) & \xrightarrow{K_{00}(\pi_A)} & K_{00}(\mathbb{C}) \\ K_{00}(\varphi) \big\downarrow & & K_{00}(\tilde{\varphi}) \big\downarrow & & \big\| \\ K_{00}(B) & \xrightarrow[K_{00}(\iota)]{} & K_{00}\big(\tilde{B}\big) & \xrightarrow[K_{00}(\pi_B)]{} & K_{00}(\mathbb{C}) \end{array}$$

Where $K_{00}(\varphi)$ and $K_{00}(\tilde{\varphi})$ are uniquely determined, and define $K_0(\varphi) : K_0(A) \to K_0(B)$ as $K_{00}(\tilde{\varphi})|_{K_0(A)}$. We first need to verify that $K_0(\varphi)$ actually maps into $K_0(B)$, so given $[p]_0 \in K_0(A)$, one has

$$K_{00}(\pi_B)(K_0(\varphi)[p]_0) = K_{00}(\pi_A)[p]_0 = 0$$

so $K_0(\varphi)$ is a well-defined map. Finally, just like K_{00}, we see that K_0 shares many similar properties.

Theorem 3.4.2. *(Functoriality of K_0).* Let A, B, C be C^*-algebras, and $\varphi : A \to B$ and $\psi : B \to C$ be *-homomorphisms. Then

(i) $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$.
(ii) $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$.
(iii) If 0 is the trivial C^*-algebra, then $K_0(0)$ is the trivial group.
(iv) If $0 : A \to B$ is the zero map, then $K_0(0) : K_0(A) \to K_0(B)$ is the trivial homomorphism.

Proof.

(i) Note that $\widetilde{\mathrm{id}_A} = \mathrm{id}_{\tilde{A}}$, thus given $[p]_0 \in K_0(A)$, one has

$$K_0(\mathrm{id}_A)[p]_0 = K_{00}\big(\widetilde{\mathrm{id}_A}\big)[p]_0 = \big[\widetilde{\mathrm{id}_A}(p)\big]_0 = [p]_0$$

so $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$.

(ii) Note that $\widetilde{\psi \circ \varphi} = \tilde{\psi} \circ \tilde{\varphi}$, so one has

$$K_0(\psi \circ \varphi) = K_{00}\big(\widetilde{\psi \circ \varphi}\big)|_{K_0(A)} = K_{00}\big(\tilde{\psi}\big) \circ K_{00}(\tilde{\varphi})|_{K_0(A)} = K_0(\psi) \circ K_0(\varphi).$$

(iii) Note that $\tilde{0} = \mathbb{C}$, so as the projection map $\pi : \tilde{0} \to \mathbb{C}$ is the identity map, then $K_0(0) = \ker(\pi) = 0$.

(iv) The zero map is a composition of the sequence

$$A \to 0 \to B$$

so then by (ii) and (iii), one the $K_0(0)$ is the composition of the sequence

$$K_0(A) \to 0 \to K_0(B)$$

which is the trivial homomorphism. ∎

Proposition 3.4.3. *(Homotopy Invariance of K_0).* Let A, B be C^*-algebras and $\varphi, \psi : A \to B$ be *-homomorphisms. Then

(i) If $\varphi \sim_h \psi$, then $K_0(\varphi) = K_0(\psi)$.
(ii) If A and B are homotopic with maps $\varphi : A \to B$ and $\psi : B \to A$, then $K_0(A) \cong K_0(B)$ where $K_0(\varphi) = K_0(\psi)^{-1}$.

Proof. If $\varphi \sim_h \psi$, then so is $\tilde{\varphi} \sim_h \tilde{\psi}$, then $K_0(\varphi) = K_0(\psi)$ from the Homotopy Invariance of K_{00} 3.3.7, which adapting its proof and using (i), (ii) follows. ∎

To further investigate the property of $K_0(A)$ groups, we define the **scalar map** $s_A : \lambda_A \circ \pi_A : \tilde{A} \to \tilde{A}$, so $s_A(a + \alpha 1_{\tilde{A}}) = \alpha 1_{\tilde{A}}$. Note that $\pi_A \circ s_A = \pi_A$ and $\mathrm{id}_{\tilde{A}} - s_A$ maps into A. Let $s_n : \mathcal{M}_n(\tilde{A}) \to \mathcal{M}_n(\tilde{A})$ be the induced map via Construction 2.5.1, and we say $a \in \mathcal{M}_n(\tilde{A}) \cup \tilde{A}$ is **scalar** if $s(a) = a$. The scalar mapping has the following natural property: given a *-homomorphism $\varphi : A \to B$ between C^*-algebras A and B, then one has the commutative diagram:

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{\varphi}} & \tilde{B} \\ {\scriptstyle s_A}\downarrow & & \downarrow{\scriptstyle s_B} \\ \tilde{A} & \xrightarrow{\tilde{\varphi}} & \tilde{B} \end{array} \tag{3.3}$$

Proposition 3.4.4. *(Structure of K_0).* Let A be a C^*-algebra, then:

(i) $K_0(A) = \left\{[p]_0 - [s(p)]_0 : p \in \mathrm{P}_n(\tilde{A}) \text{ for } n \in \mathbb{N}\right\}$.

(ii) For each $p, q \in \mathrm{P}_\infty(\tilde{A})$, the following are equivalent:
  (a) $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$.
  (b) There are $r_1, r_2 \in \mathrm{P}_\infty(\mathbb{C})$ such that $p \oplus r_1 \sim q \oplus r_2$.
  (c) There are $k, l \in \mathbb{N}$ such that $p \oplus 1_k \sim q \oplus 1_l$ in $\mathrm{P}_\infty(\tilde{A})$.

(iii) If $p \in \mathrm{P}_\infty(\tilde{A})$ satisfies $[p]_0 = [s(p)]_0$, then $p \oplus 1_k \sim s(p) \oplus 1_k$ in $\mathrm{P}_\infty(\tilde{A})$ for some $k \in \mathbb{N}$.

(iv) If $\varphi : A \to B$ is a *-homomorphism, then
$$K_0(\varphi)([p]_0 - [s(p)]_0) = [\tilde{\varphi}(p)]_0 - [s(\tilde{\varphi}(p))]_0$$
for each $p \in \mathrm{P}_\infty(\tilde{A})$.

Proof.

(i) Let $p \in \mathrm{P}_\infty(\tilde{A})$, then as $\pi \circ s = \pi$, so
$$K_{00}(\pi)([p]_0 - [s(p)]_0) = [\pi(p)]_0 - [\pi(s(p))]_0 = 0$$
hence $[p]_0 - [s(p)]_0 \in K_0(A)$. Let $g \in K_0(A)$, then there is a $n \in \mathbb{N}$ and $u, v \in \mathrm{P}_n(\tilde{A})$ such that $g = [u]_0 - [v]_0$ by the Structure of K_{00} 3.3.3 (vi), and let
$$p = u \oplus 1_n - v \quad \text{and} \quad q = 0_n \oplus 1_n$$
so $[p]_0 - [q]_0 = [u]_0 - [v]_0 = g$. As $s(q) = q$ and $K_{00}(\pi)(g) = 0$, then
$$[s(p)]_0 - [q]_0 = K_{00}(s)([p]_0 - [q]_0) = (K_{00}(\lambda) \circ K_{00}(\pi))(g) = 0$$
so $[s(p)]_0 = [q]_0$. Thus $g = [p]_0 - [s(p)]_0$, hence (i) is shown.

(ii) Note that I_n is the identity matrix in $\mathcal{M}_n(\mathbb{C})$. Let $p, q \in \mathrm{P}_\infty(\tilde{A})$, and using the stable equivalence $\sim_s$ (see Construction 3.3.2), one has
$$\begin{aligned} [p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0 &\iff [p \oplus s(q)]_0 = [q \oplus s(p)]_0 \\ &\iff p \oplus s(q) \underset{s}{\sim} q \oplus s(p) \\ &\iff p \oplus s(q) \oplus I_n \sim q \oplus s(p) \oplus I_n \text{ for any } n \in \mathbb{N} \\ &\implies p \oplus r_1 \sim q \oplus r_2 \text{ for some } r_1, r_2 \in \mathrm{P}_\infty(\mathbb{C}) \end{aligned} \tag{3.4}$$
So (a) implies (b). If $p \oplus r_1 \sim q \oplus r_2$ for some $r_1, r_2 \in \mathrm{P}_\infty(\mathbb{C})$, then $\mathrm{rank}(r_1) = k$ and $\mathrm{rank}(r_2) = l$ for some $k, l \in \mathbb{N}$, hence by Proposition 3.1.9 (iv), $r_1 \sim I_k$ and $r_2 \sim I_l$ in $\mathrm{P}_\infty(\mathbb{C})$, hence
$$p \oplus I_k \sim p \oplus r_1 \sim q \oplus r_2 \sim q \oplus I_l.$$

So (b) implies (c). Finally, if $p \oplus 1_k \sim q \oplus 1_l$ in $\mathrm{P}_\infty(\tilde{A})$ for some $k, l \in \mathbb{N}$, so $\pi(p) \oplus I_k \sim \pi(q) \oplus I_l$, and by Proposition 3.1.9 (iv), there is a $n \in \mathbb{N}$ such that

$$\pi(p) \oplus I_k \sim I_n \sim \pi(q) \oplus I_l$$

So $s(p) \oplus 1_k \sim 1_n \sim s(q) \oplus 1_l$ by applying λ (refer to (2.1)), and hence

$$p \oplus s(q) \oplus 1_{n+l} \sim p \oplus s(p) \oplus 1_{n+k} \sim q \oplus s(p) \oplus 1_{n+l}$$

thus by our series of equivalences in (3.4), then $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$. So (c) implies (a).

(iii) As $[p]_0 = [s(p)]_0$, the rest follows from Construction 3.3.2 and the Structure of K_{00} 3.3.3 (v).

(iv) Follows from (3.3). ■

The following lemmas are used to prove the exactness properties of K_0 functor.

Lemma 3.4.5. Let $\varphi : A \to B$ be a *-homomorphism between C^*-algebras A and B. Suppose $g \in \ker(K_0(\varphi))$, then there is a $p \in \mathrm{P}_\infty(\tilde{A})$ such that $g = [p]_0 - [s(p)]_0$ and $\tilde{\varphi}(p) \sim_u s(\tilde{\varphi}(p))$ in $\mathcal{U}_\infty(\tilde{B})$. Furthermore if φ is surjective, then p can be chosen such that $\tilde{\varphi}(p) = s(\tilde{\varphi}(p))$.

Proof. By Structure of K_0 3.4.4 (i), there is a $p_1 \in \mathrm{P}_\infty(\tilde{A})$ such that $g = [p_1]_0 - [s(p_1)]_0$, hence

$$0 = K_0(\varphi)(g) = [\tilde{\varphi}(p_1)]_0 - [s(\tilde{\varphi}(p_1))]_0 \quad \text{i.e.} \quad [\tilde{\varphi}(p_1)]_0 = [s(\tilde{\varphi}(p_1))]_0$$

so by the Structure of K_{00} 3.3.3 (v), there is a $n \in \mathbb{N}$ such that $\tilde{\varphi}(p_1) \oplus 1_n \sim s(\tilde{\varphi}(p_1)) \oplus 1_n$, and choose $p_2 = p_1 \oplus 1_n$, so one has $\tilde{\varphi}(p_2) = \tilde{\varphi}(p_1) \oplus 1_n \sim s(\tilde{\varphi}(p_1)) \oplus 1_n = s(\tilde{\varphi}(p_2))$, hence by Diagram 1, there is a $n \in \mathbb{N}$ such that

$$\tilde{\varphi}(p_2) \oplus 0_n \underset{u}{\sim} s(\tilde{\varphi}(p_2)) \oplus 0_n \quad \text{in} \quad \mathcal{U}_\infty\left(\widetilde{\mathcal{M}_k(\tilde{B})}\right)$$

for some $k \in \mathbb{N}$ being the dimension of $\tilde{\varphi}(p_2)$ and $s(\tilde{\varphi}(p_2))$. Let $p = p_2 \oplus 0_n$, so $\tilde{\varphi}(p) \sim_u s(\tilde{\varphi}(p))$. As $\mathcal{M}_k(\tilde{B})$ is a unital, then by Diagram 1, one has

$$\tilde{\varphi}(p) \underset{u}{\sim} s(\tilde{\varphi}(p)) \quad \text{in} \quad \mathcal{U}_\infty(\mathcal{M}_k(\tilde{B}))$$

hence $\tilde{\varphi}(p) \sim_u s(\tilde{\varphi}(p))$ in $\mathcal{U}_\infty(\tilde{B})$. Now

$$[p]_0 - [s(p)]_0 = [p_2]_0 - [s(p_2)]_0 = [p_1]_0 - [s(p_1)]_0 = g,$$

hence (i) is shown. Suppose now φ is surjective, now by the previous part, there is a $n \in \mathbb{N}$, $p_1 \in \mathrm{P}_n(\tilde{A})$, $u \in \mathcal{U}_n(\tilde{B})$ such that $\tilde{\varphi}(p_1) = us(\tilde{\varphi}(p_1))u^*$ with $g = [p_1]_0 - [s(p_1)]_0$. As φ is surjective, then so is $\tilde{\varphi}$, and hence so is the induced map $\tilde{\varphi} : \mathcal{M}_n(\tilde{A}) \to \mathcal{M}_n(\tilde{B})$ from the Functoriality of Matrix Algebras 2.5.2. Thus there is a $v \in \mathcal{U}_{2n}(\tilde{A})$ such that $\tilde{\varphi}(v) = u \oplus u^*$ by Lemma 2.6.4 (iii). Let $p = v\mathrm{diag}(p_1, 0_n)v^* \in \mathrm{P}_{2n}(\tilde{A})$, and

$$\tilde{\varphi}(p) = (u \oplus u^*)(\tilde{\varphi}(p_1) \oplus 0_n)(u^* \oplus u) = s(\tilde{\varphi}(p_1)) \oplus 0_n \in \mathcal{M}_{2n}(\mathbb{C})1_{\tilde{B}}$$

so $s(\tilde{\varphi}(p)) = \tilde{\varphi}(p)$. As $p_1 \sim p_1 \oplus 0_n \sim p$ by Diagram 1, then $g = [p]_0 - [s(p)]_0$, as required. ■

Lemma 3.4.6. If one has an exact sequence of C^*-algebras:

$$0 \longrightarrow A \overset{\varphi}{\hookrightarrow} B \overset{\psi}{\twoheadrightarrow} C \longrightarrow 0$$

then one has an exact sequence:

$$0 \longrightarrow \tilde{A} \overset{\tilde{\varphi}}{\hookrightarrow} \tilde{B} \xrightarrow{(\mathrm{id}_{\tilde{C}} - s_C) \circ \tilde{\psi}} C \longrightarrow 0$$

In particular for each $n \in \mathbb{N}$, one has an exact sequence:

$$0 \longrightarrow \mathcal{M}_n(\tilde{A}) \overset{\tilde{\varphi}}{\hookrightarrow} \mathcal{M}_n(\tilde{B}) \xrightarrow{(\mathrm{id}_{\tilde{C}} - s_C) \circ \tilde{\psi}} \mathcal{M}_n(C) \longrightarrow 0$$

Proof. From the Universality of Unitization 2.2.4, $\tilde{\varphi}$ is an injection, and $\tilde{\psi}$ is a surjection, and as $\mathrm{im}(\mathrm{id}_{\tilde{C}} - s_C) = C$, then it follows that $\mathrm{im}\big((\mathrm{id}_{\tilde{C}} - s_C) \circ \tilde{\psi}\big) = C$, so it suffices to prove that $\mathrm{im}(\tilde{\varphi}) = \ker\big((\mathrm{id}_{\tilde{C}} - s_C) \circ \tilde{\psi}\big)$.

Let $a + \alpha 1_{\tilde{A}} \in \tilde{A}$, then

$$\big((\mathrm{id}_{\tilde{C}} - s_C) \circ \tilde{\psi}\big)(\tilde{\varphi}(a + \alpha 1_{\tilde{A}})) = (\mathrm{id}_{\tilde{C}} - s_C)(\psi(\varphi(a)) + \alpha 1_{\tilde{C}}) = (\mathrm{id}_{\tilde{C}} - s_C)(\alpha 1_{\tilde{C}}) = 0.$$

Suppose $b + \beta 1_{\tilde{B}} \in \ker\big((\mathrm{id}_{\tilde{C}} - s_C) \circ \tilde{\psi}\big)$, thus

$$0 = (\mathrm{id}_{\tilde{C}} - s_C)(\psi(b) + \beta 1_{\tilde{B}}) = \psi(b) + \beta 1_{\tilde{B}} - (\psi(s_B(b)) + \beta 1_{\tilde{B}}) = \psi(b)$$

so $b \in \ker(\psi) = \mathrm{im}(\varphi)$, thus there is a $a \in A$ such that $\varphi(a) = \psi(b)$, hence $\tilde{\varphi}(a + \beta 1_{\tilde{A}}) = \psi(b + \beta 1_{\tilde{B}})$. This shows $\mathrm{im}(\tilde{\varphi}) = \ker\big((\mathrm{id}_{\tilde{C}} - s_C) \circ \tilde{\psi}\big)$, and the rest follows from the Functoriality of Matrix Algebras 2.5.2. ■

It follows that K_0 has a much stronger functoriality properties than K_{00}. We shall finish this chapter with the next three theorems.

Theorem 3.4.7. *(Half Exactness of K_0).* An exact sequence of C^*-algebras:

$$0 \longrightarrow A \overset{\varphi}{\hookrightarrow} B \overset{\psi}{\twoheadrightarrow} C \longrightarrow 0$$

induces an exact sequence of Abelian groups:

$$K_0(A) \xrightarrow{K_0(\varphi)} K_0(B) \xrightarrow{K_0(\psi)} K_0(C).$$

Proof. By functoriality of K_0, one has $K_0(\psi) \circ K_0(\varphi) = K_0(\psi \circ \varphi) = K_0(0) = 0$, so $\mathrm{im}(K_0(\varphi)) \subseteq \ker(K_0(\psi))$.

Suppose $g \in \ker(K_0(\psi))$, so by Lemma 3.4.5, there is a $p \in \mathrm{P}_\infty(\tilde{B})$ such that $\tilde{\psi}(p) = s(\tilde{\psi}(p))$ and $g = [p]_0 - [s(p)]_0$, so as $p \in \ker\big((\mathrm{id}_{\tilde{C}} - s_C) \circ \tilde{\psi}\big)$, then by Lemma 3.4.6, there is a $q \in \mathcal{M}_\infty(\tilde{A})$ such that $\tilde{\varphi}(q) = p$, and as $\tilde{\varphi}$ is injective, then $q \in \mathrm{P}_\infty(\tilde{A})$. Finally by naturality of the scalar map (3.3), one has

$$K_0(\varphi)[q - s(q)]_0 = [p]_0 - [s(p)]_0 = g,$$

hence $\ker(K_0(\psi)) = \mathrm{im}(K_0(\varphi))$, as required. ■

Theorem 3.4.8. *(Split Exactness of K_0).* A split-exact sequence of C^*-algebras:

$$0 \longrightarrow A \overset{\varphi}{\hookrightarrow} B \underset{\lambda}{\overset{\psi}{\rightleftarrows}} C \longrightarrow 0$$

induces a split-exact sequence of Abelian groups:

$$0 \longrightarrow K_0(A) \overset{K_0(\varphi)}{\hookrightarrow} K_0(B) \underset{K_0(\lambda)}{\overset{K_0(\psi)}{\rightleftarrows}} K_0(C) \longrightarrow 0.$$

Proof. By the Half Exactness of K_0 3.4.7, we have exactness at $K_0(B)$, and as $K_0(\lambda)$ is a right-inverse of $K_0(\psi)$ by functoriality, one has $K_0(\psi)$ is surjective, so we have exactness at $K_0(C)$. So it suffices to show that $\ker(K_0(\varphi)) = 0$.
Let $g \in \ker(K_0(\varphi))$, so by Lemma 3.4.5, there is a $n \in \mathbb{N}$, $p \in \mathrm{P}_n(\tilde{A})$, and $u \in \mathcal{U}_n(\tilde{A})$ such that $g = [p]_0 - [s(p)]_0$ and $u\tilde{\varphi}(p)u^* = s(\tilde{\varphi}(p))$. Let $v = (\tilde{\lambda} \circ \tilde{\psi})(u^*)u \in \mathcal{U}_n(\tilde{B})$, thus $\tilde{\psi}(v) = 1_n$, so $\tilde{\psi}(v) = s(\tilde{\psi}(v))$, hence by Lemma 3.4.5, there is a $w \in \mathcal{M}_n(\tilde{A})$ such that $\tilde{\varphi}(w) = v$, and as $\tilde{\varphi}$ is injective, then $w \in \mathcal{U}_n(\tilde{A})$. Note that as $\psi \circ \varphi = 0$, then $\widetilde{\psi \circ \varphi} = \tilde{0}$ is a scalar map, i.e. $\tilde{0}(a + \alpha 1_{\tilde{A}}) = \alpha 1_{\tilde{B}}$ here, so finally,

$$\begin{aligned}
\tilde{\varphi}(wpw^*) = v\tilde{\varphi}(p)v^* &= (\tilde{\lambda} \circ \tilde{\psi})(u^*)s(\tilde{\varphi}(p))(\tilde{\lambda} \circ \tilde{\psi})(u) \\
&= (\tilde{\lambda} \circ \tilde{\psi})(u^* s(\tilde{\varphi}(p))u) \\
&= (\tilde{\lambda} \circ \tilde{\psi})(\tilde{\varphi}(p)) \\
&= \left(\tilde{\lambda} \circ \widetilde{\psi \circ \varphi}\right)(p) = (\tilde{\lambda} \circ \tilde{0})(p) = s(\tilde{\varphi}(p))) = \tilde{\varphi}(s(p)),
\end{aligned}$$

so $wpw^* = s(p)$ by injectivity of $\tilde{\varphi}$. As $p \sim_u s(p)$, then $g = 0$, as required. ■

Theorem 3.4.9. *(Additivity of K_0).* Let A and B be C^*-algebras, then $K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$.

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(A) & \xrightarrow{\alpha} & K_0(A) \oplus K_0(B) & \xrightarrow{\beta} & K_0(B) & \longrightarrow & 0 \\
 & & \| & & \downarrow \xi & & \| & & \\
0 & \longrightarrow & K_0(A) & \xrightarrow[K_0(\iota_A)]{} & K_0(A \oplus B) & \xrightarrow[K_0(\pi_B)]{} & K_0(B) & \longrightarrow & 0
\end{array}$$

where

$$\begin{aligned}
\alpha(g) &= (g, 0), & \beta(g, h) &= h \\
\iota_A : A \to A \oplus B : a &\mapsto (a, 0), & \pi_B : A \oplus B &\to B : (a, b) \mapsto b
\end{aligned}$$

and

$$\xi(g, h) = K_0(\iota_A)(g) + K_0(\iota_B)(h).$$

This makes the diagram commutative, and the top row is clearly exact following from definition, and the bottom row is exact as it is induced from the split exact sequence:

$$0 \longrightarrow A \overset{\iota_A}{\hookrightarrow} A \oplus B \underset{\iota_B}{\overset{\pi_B}{\rightleftarrows}} B \longrightarrow 0$$

which we use the Split Exactness of K_0 3.4.8. Hence by Five Lemma 5.4.5, ξ is an isomorphism, as required. ■

3.5 Computation of K_0-Groups

We finish off this section by providing some common examples of $K_0(A)$. This chapter is rather independent of the thesis; the only main takeaway is the next example: the K_0-groups for finite-dimensional C^*-algebras.

Example 3.5.1. *(The K_0-group for finite-dimensional a C^*-algebra A).* If $A = \mathcal{M}_n(\mathbb{C})$ for some $n \in \mathbb{C}$, then by one has $K_0(A) \cong \mathbb{Z}$ by the Structure of $K_{00}(\mathcal{B}(H))$ 3.3.9 (ii) since $K_0(A) \cong K_{00}(A)$ as A is unital. Now in general, by Theorem 2.1.6, $A \cong \mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_k}(\mathbb{C})$ for $n_1, \ldots, n_k \in \mathbb{N}$ and some $k \in \mathbb{N}$. By the Additivity of K_0 3.4.9, one has that

$$K_0(A) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k \text{ copies}}.$$

In particular, $K_0(\mathbb{C}) \cong \mathbb{Z}$, which gives us the structure of $K_0\left(\tilde{A}\right)$ simply by algebraic arguments.

Lemma 3.5.2. Suppose one has a split-exact sequence of Abelian groups

$$0 \longrightarrow G \overset{\varphi}{\hookrightarrow} H \underset{\nu}{\overset{\mu}{\rightleftarrows}} K \longrightarrow 0.$$

Then $H \cong G \oplus K$.

Proof. Define $\psi : G \oplus K \to H : (g, k) \mapsto \varphi(g) - \nu(k)$, which is a homomorphism.
Let $(g, k) \in \ker(\psi)$, so $\varphi(g) = \nu(k)$, applying μ, we get that $0 = k$. Thus $\varphi(g) = 0$, but $\ker(\varphi) = 0$, so $g = 0$. Hence ψ is injective.
Let $h \in H$, then $\mu(h - \nu(\mu(h))) = \mu(h) - \mu(h) = 0$, so $h - \nu(\mu(h)) \in \ker(\mu) = \operatorname{im}(\varphi)$, so there is a $g \in G$ such that $\varphi(g) = h - \nu(\mu(h))$, hence $\psi(g, -\mu(h)) = h - \nu(\mu(h)) + \nu(\mu(h)) = h$. Thus ψ is surjective, as required. ■

Corollary 3.5.3. Let A be a C^*-algebra, then $K_0\left(\tilde{A}\right) \cong K_0(A) \oplus \mathbb{Z}$.

Proof. Note that the split-exact sequence (2.1):

$$0 \longrightarrow A \overset{\iota}{\hookrightarrow} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

induces another split-exact sequence by Split Exactness of K_0 3.4.8:

$$0 \longrightarrow K_0(A) \overset{K_0(\iota)}{\hookrightarrow} K_0\left(\tilde{A}\right) \underset{K_0(\lambda)}{\overset{K_0(\pi)}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0.$$

So $K_0\left(\tilde{A}\right) \cong K_0(A) \oplus \mathbb{Z}$ by preceding lemma. ■

1⟩ K_0-Groups for $\mathcal{C}(X)$

Construction 3.5.4. *(The trace map for $\mathcal{C}(X)$).* Given a connected locally compact Hausdorff space X and $A = \mathcal{C}(X)$. Then there is an additive map $\operatorname{tr} : \mathrm{P}_\infty(A)/\sim \to \mathbb{N}_0$ such that $\operatorname{tr}([p]) = \operatorname{tr}(p(x))$ for a fixed $x \in X$, where $\operatorname{tr}(p(x))$ is the usual trace for matrices as we make the identification

$\mathcal{M}_n(A) \cong \mathcal{C}(X, \mathcal{M}_n(\mathbb{C}))$ for each $n \in \mathbb{N}$; see Example 2.5.4 (ii). In particular, one has a homomorphism $\dim : K_0(A) \to \mathbb{Z}$ defined as $\dim([p]_0) = \operatorname{tr}(p)$.
For each $p \in \mathrm{P}_n(\mathcal{C}_0(X))$, the map $\operatorname{tr}(p) : x \mapsto \operatorname{tr}(p(x))$ is in $\mathcal{C}_0(X, \mathbb{Z})$; see Proposition 3.1.9 (v). As X is connected, then $\operatorname{tr}(p)$ is a constant map, hence the map $\operatorname{tr} : \mathrm{P}_\infty(\mathcal{C}_0(X)) \to \mathbb{Z}$ is well-defined. As for each $p, q \in \mathrm{P}_\infty(A)$, then $p \sim q$ implies $p(x) \sim q(x)$ for all $x \in X$, thus the map $[p] \mapsto \operatorname{tr}(p)$ is well-defined on $\mathrm{P}_\infty(A)/\sim$, and similarly, tr is additive. Thus the existence of $\dim$ follows from the Universal Property of K_{00}-Groups 3.3.4.

Example 3.5.5. *($K_{00}(\mathcal{C}_0(X)) = 0$ for a connected non-compact locally compact Hausdorff space X).* From Construction 3.5.4, given $p \in \mathrm{P}_\infty(\mathcal{C}_0(X))$ the map $\operatorname{Tr}(p)$ is in $\mathcal{C}_0(X, \mathbb{Z})$.
Note that in general $f \in \mathcal{C}_0(X, \mathbb{Z})$, as X is non-compact, there is a compact $K \subsetneq X$ such that $|f(x)| < \frac{1}{2}$ for all $x \in X \setminus K$. As f is a constant in $\mathbb{Z}$, then $f = 0$.
Hence $\operatorname{Tr}(p) = 0$, thus $\operatorname{tr}(p(x)) = 0$ implies $p(x) = 0$ for all $x \in X$, so $p = 0$.[1] Thus $\mathrm{P}_n(\mathcal{C}_0(X)) = \{0_n\}$, so $K_{00}(\mathcal{C}_0(X)) = 0$.

Example 3.5.6. *($\dim$ is surjective for a connected compact Hausdorff space X).* As $1 \in \mathcal{C}(X)$, then one has $\dim([1]_0) = 1$, so $\dim$ is surjective.

Example 3.5.7. *(The map $\dim$ is an isomorphism for a contractible compact Hausdorff space X).* Recall that X being contractible means that there is a continuous map $F : X \times [0,1] \to X$ and $x_0 \in X$ such that $F(\cdot, 0) = \mathrm{id}_X$ and $F(\cdot, 1) = x_0$.
For each $t \in [0,1]$, define $\varphi_t : \mathcal{C}(X) \to \mathcal{C}(X) : f \mapsto f(F(\cdot, t))$, which is a *-homomorphism such that $\varphi_0(f) = f$ and $\varphi_1(f) = f(x_0)$, and also for each $f \in \mathcal{C}(X)$, the map $t \mapsto \varphi_t(f)$ is continuous. So $\mathrm{id}_X \sim_h \varphi_1$. Define $\mu : \mathcal{C}(X) \to \mathbb{C}$ as $\mu(f) = f(x_0)$ and $\nu : \mathbb{C} \to \mathcal{C}(X) : \alpha \mapsto \alpha 1$, hence $\mu \circ \nu = \mathrm{id}_{\mathbb{C}}$ and $\nu \circ \mu = \varphi_1 \sim_h \mathrm{id}_{\mathbb{C}}$, hence

$$\mathcal{C}(X) \xrightarrow{\ \mu\ } \mathbb{C} \xrightarrow{\ \nu\ } \mathcal{C}(X)$$

is a homotopy. Recall the isomorphism $K_0(\operatorname{tr}) : K_0(\mathbb{C}) \to \mathbb{Z}$ from the Structure of $K_{00}(\mathcal{B}(H))$ 3.3.9. Thus one has a commutative diagram:

$$\begin{array}{ccc} K_0(\mathcal{C}(X)) & \xrightarrow{\ \dim\ } & \mathbb{Z} \\ & \searrow^{\mu} & \uparrow{\scriptstyle K_0(\operatorname{tr})} \\ & & K_0(\mathbb{C}) \end{array}$$

where μ and $K_0(\operatorname{tr})$ are isomorphisms by the Homotopy Invariance of K_0 3.4.3, then $\dim$ is also an isomorphism.

Proposition 3.5.8. Let X be the disjoint union space of locally compact Hausdorff spaces X_1 and X_2. Then $\mathcal{C}_0(X) \cong \mathcal{C}_0(X_1) \oplus \mathcal{C}_0(X_2)$.

Proof. Let $\lambda_1 : X_1 \to X$ and $\lambda_2 : X_2 \to X$ be the canonical inclusion maps. Define

$$\Phi : \mathcal{C}_0(X) \to \mathcal{C}_0(X_1) \oplus \mathcal{C}_0(X_2) : f \mapsto (f \circ \lambda_1, f \circ \lambda_2).$$

Let $f \in \mathcal{C}_0(X)$ and $\varepsilon > 0$, so there is a compact $K \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. Now $K_i = \lambda_i^{-1}(K \cap \lambda_i(K))$ is compact in X_i, and one has $|f(\lambda_i(x))| < \varepsilon$ for $x \in X_i \setminus K_i$, so $f \circ \lambda_i \in \mathcal{C}_0(X_i)$ for each $i \in \{1, 2\}$. Hence Φ is a well-defined map. Now it is clear that Φ is a *-

[1] A projection matrix with zero eigenvalues must be the zero matrix.

homomorphism too, and Φ is surjective by the universal property of disjoint union spaces. Now Φ is also injective as $X = \lambda_1(X_1) \cup \lambda_2(X_2)$. Hence Φ is an isomorphism. ■

Corollary 3.5.9. Let X be a locally connected, locally compact Hausdorff space such that X has finitely many connected components $\{C_i\}_{i \leq n}$. Then

$$K_0(\mathcal{C}(X)) \cong \bigoplus_{i \leq n} K_0(\mathcal{C}(C_i))$$

Proof. As X is locally connected, all of its components are clopen, so X is a disjoint union space of its connected components. The rest follows from Proposition 3.5.8 and the Additivity of K_0 3.4.■.

We now try to compute the K_0-group for $\mathcal{C}_0(\mathbb{R})$. Note that the one-point compactification of $\mathbb{R}$ is $\mathbb{S}^1$, and by Example 2.2.7, the unitization $\mathcal{C}_0(\mathbb{R})$ is isomorphic to $\mathcal{C}(\mathbb{S}^1)$ and one has a split-exact sequence

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}) \hookrightarrow \mathcal{C}(\mathbb{S}^1) \twoheadrightarrow \mathbb{C} \longrightarrow 0$$

and we use the fact that $K_0(\mathbb{S}^1) \cong \mathbb{Z}$ by [2, Example 11.3.3] and Corollary 3.5.3 to conclude that $K_0(\mathcal{C}(\mathbb{S}^1)) \cong K_0(\mathcal{C}_0(\mathbb{R})) \oplus \mathbb{Z}$, hence $K_0(\mathcal{C}_0(\mathbb{R})) = 0$. Let $I \subseteq \mathbb{R}$ be any interval:

- If I is open, then I is homeomorphic to $\mathbb{R}$, so $\mathcal{C}_0(I) \cong \mathcal{C}_0(\mathbb{R})$ as C^*-algebras, hence $K_0(\mathcal{C}_0(I)) = 0$.
- If I is closed and bounded, then I is compact, then $K_0(\mathcal{C}(I)) \cong \mathbb{Z}$ by Example 3.5.7.
- If I is half-closed, i.e. of the form $[a, b)$ or $(a, b]$ (not necessarily bounded), then I is homeomorphic to a bounded half-closed interval by passing it through $\arctan$. So we assume I is half-closed and bounded, then the one point-compactification of I is $\overline{I}$, hence one has a split-exact sequence

$$0 \longrightarrow \mathcal{C}_0(I) \hookrightarrow \mathcal{C}\big(\overline{I}\big) \twoheadrightarrow \mathbb{C} \longrightarrow 0$$

and we know that $K_0\big(\mathcal{C}\big(\overline{I}\big)\big) \cong \mathbb{Z}$, and by Corollary 3.5.3, one has $K_0\big(\mathcal{C}\big(\overline{I}\big)\big) \cong K_0(\mathcal{C}(I)) \oplus \mathbb{Z}$, so one has $K_0(\mathcal{C}(I)) = 0$.

Using Corollary 3.5.9, we can characterize the K_0-group for all finite unions of intervals of $\mathbb{R}$, for example, given $U = (-\infty, 0) \cup [1, 2] \cup [3, 5) \cup \{37\} \cup (40, 50)$, one has that

$$\begin{aligned} K_0(\mathcal{C}_0(U)) &\cong K_0(\mathcal{C}_0((-\infty, 0))) \oplus K_0(\mathcal{C}([1, 2])) \oplus K_0(\mathcal{C}_0([3, 5))) \\ &\oplus K_0(\mathcal{C}(\{37\})) \oplus K_0(\mathcal{C}_0((40, 50))) \\ &\cong 0 \oplus \mathbb{Z} \oplus 0 \oplus \mathbb{Z} \oplus 0 \cong \mathbb{Z}^2. \end{aligned}$$

2〉 K_0-Groups for $\mathcal{B}(H)$

Let H be an infinite-dimensional Hilbert space, then $K_{00}(\mathcal{B}(H)) = 0$ by the Structure of $K_{00}(\mathcal{B}(H))$ 3.3.9, and by Proposition 3.4.1, one has $K_0(\mathcal{B}(H)) = 0$ as $\mathcal{B}(H)$ is unital. It shall be shown in Example 4.3.7, if H is separable, then $K_0(\mathcal{K}(H)) \cong \mathbb{Z}$.

4 | Classification of Separable AF-Algebras

We now focus our attention towards classifying C^*-algebras via their K_0-groups. It is clear that with our current tools, just by looking into the K_0-group will not provide enough information to encapsulate the structure of the C^*-algebra. For example, all $K_0(\mathcal{M}_n(\mathbb{C})) = \mathbb{Z}$ for all $n \in \mathbb{N}$ yet $\mathbb{C}$ is "not similar to" $\mathcal{M}_2(\mathbb{C})$ at all. However, we can add an 'order' structure to our K_0-groups which in turn gives sufficient condition to classify, in this case, *approximately finite-dimensional algebras* or AF-algebras for short.

We begin our first chapter by building up the necessary theory in order to talk about the 'orderedness' of our Abelian groups, and how the 'stability' of the C^*-algebra relates to the ordered structure of their K_0-groups.

The second chapter will focus on inductive limit constructions on both the C^*-algebra and the K_0-group level, and discuss their properties independently. As all AF-algebras are finite-dimensional C^*-algebras under inductive limits, this chapter will serve as a foundation for all upcoming inductive limit results such as the Inductive Continuity of K_0 4.3.5 and the main theorem, namely Elliott's Theorem 4.5.5.

We shall briefly touch on the definition and properties of AF-algebras in the third chapter. There are a lot more nice properties that AF-algebras possesses; see [7]. However, we shall only focus on the results relevant to the classification theorem.

Finally in the last chapter, we shall prove the celebrated classification theorem of AF-algebras; the Elliott's Theorem 4.5.5.

4.1 The (K_0, K_0^+) Functor

We now begin adding more information to our K_0-group invariant, more specifically, we will consider *ordered Abelian groups with distinguished order units* rather than just Abelian groups. This allows us to finally make distinctions between certain C^*-algebras at the K_0-group level. Firstly, we will discuss the *finiteness* properties of C^*-algebras, which are properties observed in the finite-dimensional case.

Definition 4.1.1. *(Stably Finiteness of C^*-algebras)*. Let A be a C^*-algebra and p be a projection on A. We say p is **infinite** if there is a $q \in \mathrm{P}(A)$ such that $p \sim q < p$, i.e. $p \sim q$, and $\sigma(p - q) \subseteq (0, \infty)$. Otherwise, p is said to be **finite**.
If A is unital, then we say A is **finite** if 1_A is a finite projection, otherwise A is said to be **infinite**. We say A is **stably finite** if $\mathcal{M}_n(A)$ is finite for all $n \in \mathbb{N}$. If A is nonunital, then A is **finite** (resp. **infinite**, or **stably finite**) if $\tilde{A}$ is finite (resp. infinite, or stably finite).

Note that $a \in A$ is an **isometry** if $a^*a = 1$ if A is unital. Feel free to refer to the Chapter 2.5 on positive elements as this lemma relies on the observations made there.

Lemma 4.1.2. Let A be a unital C^*-algebra, then the following conditions are equivalent:
(i) A is finite.
(ii) All isometries in A are unitary.
(iii) All projections in A are finite.
(iv) All left-invertible elements in A are invertible.
(v) All right-invertible elements in A are invertible.
Furthermore, if A is nonunital, then (iii) still holds if A is finite.

Proof. (i)$\Longrightarrow$(ii). Suppose A is finite. If $a \in A$ is an isometry, then aa^* is a projection with $1 = a^*a \sim aa^*$, so as 1 is finite, one has $aa^* \geq 1$. That means $\sigma(aa^*) - 1 \subseteq [0, \infty)$, so $\sigma(aa^*) = \{1\}$ as $\sigma(aa^*) \subseteq \{0, 1\}$. Hence $aa^* = 1$, as required.

(ii)$\Longrightarrow$(i). Suppose all isometries are unitary. Let $p \in \mathrm{P}(A)$ such that $p \sim 1$, then there is a $v \in A$ such that $p = vv^*$ and $1 = v^*v$, hence $p = 1$ as v is unitary. Then 1 is a finite projection, so A is finite.

(ii)$\Longrightarrow$(iii). Suppose A is finite. Let $p, q \in \mathrm{P}(A)$ such that $p \sim q \leq p$. Now there is a $v \in A$ such that $p = v^*v$ and $q = vv^*$, and define $w = v + (1 - p)$. Note that $(1 - p)v^* = 0 = v(1 - p)$, and by Lemma 2.3.2, one has $pq = qp = q$, so

$$w^*w = 1 + v^*(1 - p) + (1 - p)v = 1 + v^*q(1 - p) + (1 - p)qv = 1$$
$$\text{and} \quad ww^* = q + 1 - p$$

thus by (ii), w is unitary, so $1 = q + 1 - p$, hence $p = q$. Thus p is finite.

(iii)$\Longrightarrow$(i). Suppose all projections are finite, then 1 is finite, so A is finite.

(iv)$\Longrightarrow$(v). Suppose all left-invertible elements are invertible. Let $a \in A$ be right-invertible, then there is a $b \in A$ such that $ab = 1$, so b is invertible, hence $ba = babb^{-1} = b1b^{-1} = 1$.

(v)$\Longrightarrow$(iv). Similar as above.

(iv)$\Longrightarrow$(ii). Trivial.

(ii)$\Longrightarrow$(iv). Suppose all isometries are unitary. Let $a \in A$ be left-invertible, by Lemma 2.4.6 a^*a is invertible, so let $v = a(a^*a)^{-\frac{1}{2}}$. Now one has $v^*v = 1$, thus v is unitary hence invertible. Thus $a = v(a^*a)$ is also invertible.

Suppose A is nonunital and is finite, and $p, q \in \mathrm{P}(A)$ such that $p \sim q \leq p$. Then borrowing the defined notations in the proof of (ii)$\Longrightarrow$(iii), where $w = v + (1_{\tilde{A}} - p)$ and we know that $w^*w = 1_{\tilde{A}}$ and $ww^* = q + 1_{\tilde{A}} - p$. As $1_{\tilde{A}}$ is finite, then $1_{\tilde{A}} \sim ww^* \geq 1_{\tilde{A}}$, so $\sigma(ww^*) - 1 \subseteq [0, \infty)$, it follows that $ww^* = 1_{\tilde{A}}$, so $p = q$, as required. ∎

Corollary 4.1.3. Let A and B be finite (resp. stably finite) unital C^*-algebras, then $A \oplus B$ is finite (resp. stably finite).

Proof. Let $(a, b) \in A \oplus B$ be left-invertible, then a and b are left-invertible in A and B respectively, hence they are invertible, and thus so is (a, b). Therefore by preceding lemma, one has $A \oplus B$ is finite. If A and B are stably finite, then by Example 2.5.6, $\mathcal{M}_n(A \oplus B) \cong \mathcal{M}_n(A) \oplus \mathcal{M}_n(B)$ is finite for each $n \in \mathbb{N}$. Thus $A \oplus B$ are stably finite. ∎

Corollary 4.1.4. A finite-dimensional C^*-algebra A is unital and stably finite.

Proof. By Theorem 2.1.6, A is a sum of matrix algebras over $\mathbb{C}$ (which are unital), so from Corollary 4.1.3, it suffices to show that $\mathcal{M}_n(\mathbb{C})$ is stably finite for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, then by Lemma 4.1.2, $\mathcal{M}_n(\mathbb{C})$ is a finite C^*-algebra as all left-invertible matrices over $\mathbb{C}$ are invertible by linear algebra. Now for each $m \in \mathbb{N}$, one has $\mathcal{M}_m(\mathcal{M}_n(\mathbb{C})) \cong \mathcal{M}_{mn}(\mathbb{C})$ is finite, so $\mathcal{M}_n(\mathbb{C})$ is stably finite, as required. ■

Definition 4.1.5. *(Ordered Abelian Groups)*. The pair (G, G^+) is called an **ordered Abelian group** if G is an Abelian group and there is a **positive cone** $G^+ \subseteq G$, which satisfies
OG1. $G^+ + G^+ \subseteq G^+$;
OG2. $G^+ \cap (-G^+) = 0$;
OG3. $G^+ - G^+ = G$.
Hence one can define a relation $\leq$ on G such that $x \leq y$ if $y - x \in G^+$. Now **OG2** implies that $\leq$ is reflexive and antisymmetric, and **OG1** implies $\leq$ is transitive, so $\leq$ is a partial order on G. Note that $x \leq y$ implies $x + z \leq y + z$ for all $z \in G$. Note that **OG1** implies that G^+ is an Abelian semigroup.
We say an element $u \in G^+$ to be an **order unit** if for each $g \in G$, there is a $n \in \mathbb{N}$ such that $-nu \leq g \leq nu$. Then the triple (G, G^+, u) is an **ordered Abelian group with a (distinguished) order unit**. We say (G, G^+) is **simple** if each $u \in G^+ \setminus \{0\}$ is an order unit.
If an Abelian group G has a partial order $\leq$ such that $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in G$. Then the set $G^+ = \{x \in G : x \geq 0\}$ satisfies **OG1** and **OG2**.

From our definition, it is immediately clear that $G = (\mathbb{Z}, \mathbb{N}_0)$ is a simple ordered Abelian group under the usual order. Furthermore, so is $G^k = \left(\mathbb{Z}^k, \mathbb{N}_0^k\right)$ for each $k \in \mathbb{N}$ where we induce the usual ordering to be component-wise. We see that G is a simple ordered Abelian group, while G^k is not necessarily simple for $k > 1$.

Definition 4.1.6. *(Positive Cones)*. Given a C^*-algebra A, the **positive cone** of $K_0(A)$ is

$$K_0(A)^+ = \{[p]_0 : p \in \mathrm{P}_\infty(A)\} \subseteq K_0(A).$$

Note that $K_0(A)^+$ is precisely the image of $\mathrm{P}_\infty(A)/\sim$ under the Grothendieck map.

Proposition 4.1.7. Let A be a C^*-algebra. Then
(i) $K_0(A)^+ + K_0(A)^+ \subseteq K_0(A)^+$.
(ii) If A is unital, then $K_0(A)^+ - K_0(A)^+ = K_0(A)$.
(iii) If A is stably finite, then $K_0(A)^+ \cap (-K_0(A)^+) = 0$.
(iv) If A is unital and stably finite, then $(K_0(A), K_0(A)^+, [1_A]_0)$ is an ordered Abelian group with a distinguished order unit.

Proof.

(i) Given $p, q \in \mathrm{P}_\infty(A)$, one has $p \oplus q \in \mathrm{P}_\infty(A)$, so $[p]_0 + [q]_0 \in K_0(A)^+$, as required.

(ii) Note that $K_0(A)^+$ is an Abelian semigroup of $K_{00}(A)$, so by the Structure of Grothendieck Groups 3.2.1 (iv), one has $K_{00}(A) = K_0(A)^+ - K_0(A)^+$. As $K_{00}(A)$ and $K_0(A)$ are isomorphic via the inclusion map (see Proposition 3.4.1), then one has $K_0(A) = K_0(A)^+ - K_0(A)^+$.

(iii) Suppose $g \in K_0(A)^+ \cap (-K_0(A)^+)$, then $g = [p]_0 = -[q]_0$ for $p, q \in \mathrm{P}_\infty(A)$. As $[p \oplus q]_0 = 0$ in $K_0(A) \subseteq K_{00}\left(\tilde{A}\right)$, so by the Structure of K_{00} 3.3.3 (v), there is a $r \in \mathrm{P}_\infty\left(\tilde{A}\right)$ such that $p \oplus q \oplus r \sim r$. Without loss of generality, suppose $p, q, r \in \mathrm{P}_n(\infty)$ for some $n \in \mathbb{N}$, this can be achieved by direct summing zero matrices. Let $p' = p \oplus 0_n \oplus 0_n$, $q' = 0_n \oplus q \oplus 0_n$, and $r' = 0_n \oplus 0_n \oplus r$, then $p'q' = p'r' = q'r' = 0_{3n}$, so $p' \perp q' \perp r'$ with $p \sim p'$, $q \sim q'$, $r \sim r'$. So by Proposition 3.3.1 (iv), one has $p' + q' + r' \sim r'$, thus as $p' + q' + r'$ is a finite projection, one has $r' \geq p' + q' + r'$, so $0 \geq p' + q'$. Thus $p' + q' = 0$ as $p' + q'$ is a projection, so $p' = 0$ by multiplying by p', as hence $g = [p]_0 = 0$, as required.

(iv) It suffices to show that $[1_A]_0$ is an order unit for $K_0(A)$. Let $g \in K_0(A)$, then as $K_0(A) \cong K_{00}(A)$, by the Structure of K_{00} 3.3.3, one has $g = [p]_0 - [q]_0$ for some $p, q \in \mathrm{P}_n(A)$ and $n \in \mathbb{N}$. We write $1 = 1_A$, and note that $[1_n]_0 = n[1]_0$. Now $1_n - p, 1_n - q \in \mathrm{P}_n(A)$, and note by Construction 3.3.2, one has $p \oplus (1_n - p) \sim 1_n$, so one has

$$-n[1]_0 = -[1_n]_0 = -[q]_0 - [1_n - q]_0 \leq -[q]_0 \leq [p]_0 - [q]_0 = g$$
$$\leq [p]_0 \leq [p]_0 + [1-p]_0 = [1_n]_0 = n[1]_0.$$

Hence $[1]_0$ is an ordered unit for $K_0(A)$, as required. ∎

Defintition 4.1.8. *(Positive Group Homomorphisms).* Let (G, G^+) and (H, H^+) be ordered Abelian groups, then a group homorphism $\varphi : G \to H$ is **positive** if $\varphi(G^+) \subseteq H^+$, and we say φ is an **order isomorphism** if φ is an isomorphism and $\varphi(G^+) = H^+$.
Suppose g and h are now distinguished order units of G and H respectively, then we say φ is **(order) unital** if $\varphi(u) = v$, and hence we say (G, G^+, g) and (H, H^+, h) are **isomorphic (as ordered Abelian groups with units)** or **unital order isomorphic** if φ can be chosen as a unital order isomorphism.

We now have two new categories:

- The category of ordered Abelian groups, OrdAb, whose objects are ordered Abelian groups (G, G^+), and morphisms are positive group homomorphisms $f : (G, G^+) \to (H, H^+)$ (so $f(G^+) \subseteq H^+$). It is clear that positive group homomorphisms are preserved under compositions.
- The category of ordered Abelian groups with distinguished order units, OrdAb_1, whose objects are Abelian groups with distinguished order units (G, G^+, u_G), and morphisms are unital positive group homomorphisms $f : (G, G^+, u_G) \to (H, H^+, u_H)$ (so $f(G^+) \subseteq H^+$ and $f(u_G) = u_H$). It is clear that unital positive group homomorphisms are preserved under compositions.

It is clear that the categories OrdAb and OrdAb_1 contains zero objects, namely the trivial group 0 which has the positive cone $0^+ = \{0\}$ and distinguished order unit 0. We also often omit the G^+ and u_G when stating that f is a unital positive group homomorphism if the context is clear. We shall also observe that the categories OrdAb and OrdAb_1 are also closed under finite products, that is, given triples (G, G^+, u_G) and (H, H^+, u_H), we can make a new ordered Abelian group $(G \times H, G^+ \times H^+)$ with a distinguished order unit (u_G, u_H).

Proposition 4.1.9. Let $\varphi : A \to B$ be a *-homomorphism between C^*-algebras. Then

(i) $K_0(\varphi)(K_0(A)^+) \subseteq K_0(B)^+$.
(ii) If A, B and φ are unital, then $K_0(\varphi)$ is unital, i.e. $K_0(\varphi)[1_A]_0 = [1_B]_0$.
(iii) If φ is a *-isomorphism, then $K_0(\varphi)(K_0(A)^+) = K_0(B)^+$.
(iv) If A is unital and φ is an *-isomorphism, then $K_0(\varphi)$ is an unital order isomorphism between $(K_0(A), K_0(A)^+, [1_A]_0)$ and $(K_0(B), K_0(B)^+, [1_B]_0)$.

Proof.

(i) Follows from the definition: $K_0(\varphi)[p]_0 = [\tilde{\varphi}(p)]_0 = [\varphi(p)]_0$ for $p \in \mathrm{P}_\infty(A)$.
(ii) Indeed, $K_0(\varphi)[1_A]_0 = [\tilde{\varphi}(1_A)]_0 = [\varphi(1_A)]_0 = [1_B]_0$.
(iii) Let $q \in \mathrm{P}_n(B)$ for some $n \in \mathbb{N}$, then there is a $p \in \mathcal{M}_n(A)$ such that $\varphi(p) = q$. Then as $\varphi(p)^* = \varphi(p)^2 = \varphi(p)$, it follows from injectivity that $p \in \mathrm{P}_n(A)$, so $K_0(\varphi)[p]_0 = [q]_0$, as required.
(iv) This follows from (i), (ii), and (iii). ∎

The last statement (statement (iv)) of the preceding proposition states that the triple $(K_0(A), K_0(A)^+, [1_A]_0)$ is indeed an invariant of A. In particular, this triple is sufficient to cover the flaws of $K_0(A)$, which originally cannot distinguish between finite-dimensional algebras as discussed initially, but the triple can as per Proposition 4.1.9.

Proposition 4.1.10. Let $n, m \in \mathbb{N}$, and $A = \mathcal{M}_n(\mathbb{C})$. One has that
(i) $K_0(\mathrm{tr})$ is a unital order isomorphism from $(K_0(A), K_0(A)^+, [1_A]_0)$ to $(\mathbb{Z}, \mathbb{N}_0, n)$.
(ii) Let $G_n = (\mathbb{Z}, \mathbb{N}_0, n)$, then G_n is unital order isomorphic to G_m if, and only if, $n = m$.
(iii) If $A = \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{M}_m(\mathbb{C})$, then $(K_0(A), K_0(A)^+, [1_A]_0) \cong (\mathbb{Z}^2, \mathbb{N}_0^2, (n, m))$.

Proof.
(i) From the Structure of $K_{00}(\mathcal{B}(H))$ 3.3.9, we have the natural isomorphism $K_0(\mathrm{tr}) : K_0(A) \to \mathbb{Z}$ (we make the identification $K_0(A) = K_{00}(A)$ here), and it is clear that $K_0(\mathrm{tr})(K_0(A)^+) = \mathbb{N}_0$, and $K_0(\mathrm{tr})([1_A]_0) = n$, thus $K_0(\mathrm{tr})$ is an unital order isomorphism from $(K_0(A), K_0(A)^+, [1_A]_0)$ to $(\mathbb{Z}, \mathbb{N}_0, n)$.
(ii) It is clear that if $n = m$, then we are done. So now suppose there is a unital order isomorphism φ from G_n to G_m. Note that $\varphi(r) = \varphi(1)r$ for all $r \in \mathbb{Z}$, and since φ is surjective, then $\varphi(1)$ must generate $\mathbb{Z}$. The only generators of $\mathbb{Z}$ are ± 1, and as $\varphi(1) \in \mathbb{N}_0$ since φ is positive, one has $\varphi(1) = 1$. Now $m = \varphi(n) = \varphi(1)n = n$, as required.
(iii) Following from the Additivity of K_0 3.4.9, we have an isomorphism $K_0(\mathrm{tr}) \oplus K_0(\mathrm{tr}) : K_0(A) \to \mathbb{Z}^2$ where we make the identification $K_0(A) = K_0(\mathcal{M}_n(\mathbb{C})) \oplus K_0(\mathcal{M}_m(\mathbb{C}))$. The rest follows similarly as per (i). ■

Denote C*-Alg$_s$ to be the category of stably finite unital C^*-algebras where the morphisms are unital *-homomorphisms, so C*-Alg is a subcategory of C*-Alg. From the results proven above, we can define the functor

$$(K_0, K_0^+) : \text{C*-Alg}_s \to \text{OrdAb}_1$$

which given a unital C^*-algebra A, define $(K_0, K_0^+)(A)$ as an ordered Abelian group with a distinguished order unit $(K_0(A), K_0(A)^+, [1_A]_0)$. Given a unital *-homomorphism $\varphi : A \to B$ between unital C^*-algebras A and B, then define $(K_0, K_0^+)(\varphi)$ as the map $K_0(\varphi)$, which we know $K_0(\varphi)$ is a positive unital group homomorphism by Proposition 4.1.9. The functoriality is given in the next proposition.

Proposition 4.1.11. *(Functoriality of (K_0, K_0^+)).* Let $A, B, C \in$ C*-Alg$_s$ and $\varphi : A \to B$, $\psi : B \to C$ be unital *-homomorphisms. One has:
(i) $(K_0, K_0)^+(\psi \circ \varphi) = (K_0, K_0^+)(\psi) \circ (K_0, K_0^+)(\varphi)$.
(ii) $(K_0, K_0)^+(\mathrm{id}_A) = \mathrm{id}_{(K_0(A), K_0(A)^+)}$.

Proof. This immediate follows from the Functoriality of K_0 3.4.2. ■

Proposition 4.1.12. *(Additivity of (K_0, K_0^+)).* Let A and B be C^*-algebras, then

$$K_0(A \oplus B)^+ = K_0(A)^+ \oplus K_0(B)^+ = \{(g, h) : g \in K_0(A)^+, h \in K_0(B)^+\}$$

is the positive cone of $K_0(A \oplus B)$, where the identification is given by ξ in the Additivity of K_0 3.4.9. In particular, $K_0(\xi)$ is an order isomorphism. If A and B are unital, then $K_0(\xi)$ is also unital.

Proof. In the notation of the Additivity of K_0 3.4.9, one has $K_0(\xi) = K_0(\iota_A) \oplus K_0(\iota_B)$, and it suffices to show that

$$K_0(\xi)(K_0(A)^+ \oplus K_0(B)^+) = K_0(A \oplus B)^+.$$

Let $(g, h) \in K_0(A) \oplus K_0(B)$, and $x = (K_0(\iota_A) \oplus K_0(\iota_B))(g, h)$. As $K_0(\pi_A)$ and $K_0(\pi_B)$ are positive, then $g = K_0(\pi_A)(x) \in K_0(A)^+$ and $h = K_0(\pi_B)(x) \in K_0(B)^+$. If g and h are positive, then as $K_0(\iota_A)$ and $K_0(\iota_B)$ are positive maps, then it follows that $x = K_0(\iota_A)(g) + K_0(\iota_B)(h) \in$

$K_0(A \oplus B)^+$. So $K_0(\xi)$ is an order isomorphism. Now if the units exists, then it is clear that $K_0(\xi)[(1_A, 1_B)]_0 = ([1_A]_0, [1_B]_0)$. ■

To summarize this section, we see that the functor $(K_0, K_0)^+$ when restricted to the subcategory FinAlg_1 yields a **classification functor**, that is, the objects A, B in FinAlg_1 are isomorphic if, and only if, the images $(K_0, K_0^+)(A)$ and $(K_0, K_0^+)(B)$ are isomorphic.

4.2 Inductive Limit Constructions

In this chapter, we first consider the general inductive limit constructions of C^*-algebras and ordered Abelian groups. The main motivation here is that this allows to push our classification theorem Proposition 4.1.10 of finite-dimensional C^*-algebras, to AF-algebras. This gives a larger class of C^*-algebras that can be classified by (K_0, K_0^+). Secondly, we shall show the relevant properties of C^*-algebras that are inherited through taking inductive limits.

1⟩ Inductive Limit of C^*-Algebras

Let I be some index set, and $\{A_\alpha\}_{\alpha\in I}$ be a family of C^*-algebras. Construct the product C^*-algebra A as follows: we treat elements of A to be functions $a : I \to \bigcup_{\alpha\in I} A_\alpha$ with $a_\alpha := a(i) \in A_\alpha$ such that

$$\|a\| = \sup_{\alpha\in I}\|a_\alpha\|_{A_\alpha} < \infty.$$

Let $\|\cdot\|$ be norm of A as defined above, and we equip addition, scalar multiplication, *-operation, and multiplication to be pointwise. As it turns out A is a categorical product of the collection $(A_\alpha)_{\alpha\in I}$, and we will write $A = \prod_{\alpha\in I} A_\alpha$ in reference to this construction. We need a small lemma to compute supremums.

Lemma 4.2.1. Let $f : [0, \infty) \to \mathbb{R}$ be an increasing continuous map, and $A \subseteq [0, \infty)$ be nonempty and bounded. Then $f(\sup A) = \sup f(A)$.

Proof. Let $\alpha = \sup A < \infty$, then $f(\alpha) \geq f(a)$ for all $a \in A$, so $f(\alpha) \geq \sup f(A)$. Let $\varepsilon > 0$, so there is a $\delta > 0$ such that $|f(x) - f(\alpha)| \leq \varepsilon$ for all $x \in (\alpha - \delta, \alpha + \delta)$, and in particular there is a $a \in A \cap (\alpha - \delta, \alpha + \delta)$. Thus $f(\alpha) \leq f(a) + \varepsilon \leq \sup f(A) + \varepsilon$. Taking $\varepsilon \downarrow 0$ to get $f(\alpha) \leq \sup f(A)$. Hence $f(\sup A) = \sup f(A)$, as required. ■

Theorem 4.2.2. The set $A = \prod_{\alpha\in I} A_\alpha$ is the categorical product of C^*-algebras. Furthermore, if A_α is unital for each $\alpha \in I$, then A is the categorical product of C^*-algebras.

Proof. We first need to show that A is a C^*-algebra by showing:

(i) satisfy $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$;
(ii) satisfies C^*-identity;
(iii) completeness;

as it is clear that A is a complex normed space. Let $a, b \in A$. For (i),

$$\|ab\| = \sup_{\alpha\in I}\|a_\alpha b_\alpha\| \leq \sup_{\alpha\in I}\|a_\alpha\|\|b_\alpha\| \leq \|a\|\|b\|.$$

For (ii), using Lemma 4.2.1:

$$\|a^*a\| = \sup_{\alpha\in I}\|a_\alpha^* a_\alpha\| = \sup_{\alpha\in I}\|a_\alpha\|^2 = \|a\|^2$$

For (iii), let $(a^{(n)})_{n\in\mathbb{N}}$ be a Cauchy sequence in A. Fix $\alpha \in I$, and let $n, m \in \mathbb{N}$, observe that

$$\left\|a_\alpha^{(n)} - a_\alpha^{(m)}\right\| = \left\|\left(a^{(n)} - a^{(m)}\right)_\alpha\right\| \leq \left\|a^{(n)} - a^{(m)}\right\|$$

so it follows that $\left(a_\alpha^{(n)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in A_α, so there is a $a_\alpha \in A_\alpha$ such that $\lim_{n\to\infty} a_\alpha^{(n)} = a_\alpha$. Let $a = (a_\alpha)_{\alpha\in I}$, and we need to show that $\|a\| < \infty$ and $\lim_{n\to\infty} a^{(n)} = a$ to conclude (iii). As Cauchy sequences are bounded, there is a $M > 0$ such that $\left\|a^{(n)}\right\| \leq M$ for all $n \in \mathbb{N}$, then fixing $\alpha \in I$, there is a $N \in \mathbb{N}$ such that $\left\|a_\alpha^{(N)} - a_\alpha\right\| \leq M$, so one has

$$\|a_\alpha\| \leq \left\|a_\alpha - a_\alpha^{(N)}\right\| + \left\|a_\alpha^{(N)}\right\| \leq M + \left\|a^{(N)}\right\| \leq 2M,$$

hence $\|a\| \leq 2M$, thus $a \in A$.

Let $\varepsilon > 0$, so there is a $N \in \mathbb{N}$ such that $\left\|a^{(n)} - a^{(m)}\right\| \leq \varepsilon$ for all $m, n \geq N$. Fix $n \geq N$, so there is a $\alpha \in I$ such that $\left\|a_\alpha - a_\alpha^{(n)}\right\| + \varepsilon \geq \left\|a - a^{(n)}\right\|$, so for each $m \geq N$ one has:

$$\left\|a - a^{(n)}\right\| \leq \left\|a_\alpha - a_\alpha^{(n)}\right\| + \varepsilon \leq \left\|a_\alpha - a_\alpha^{(m)}\right\| + \left\|a_\alpha^{(m)} - a_\alpha^{(n)}\right\| + \varepsilon \leq \left\|a_\alpha - a_\alpha^{(m)}\right\| + 2\varepsilon.$$

Take $m \to \infty$ to get $\left\|a - a^{(n)}\right\| \leq 2\varepsilon$, hence $a^{(n)} \to a$, as required.

We now show that A has the universal property. Let $\pi_\alpha : A \to A_\alpha$ be the natural projection maps, and it is clear that they define *-homomorphisms. Let $(f_\alpha : B \to A_\alpha)_{\alpha\in I}$ be another family of *-homomorphisms mapping from a C^*-algebra B, and define

$$f : B \to A : x \mapsto (f_\alpha(x))_{\alpha\in I}$$

then it is clear that f is a *-homomorphism and $f_\alpha = \pi_\alpha \circ f$ for all $\alpha \in I$. Thus A is indeed a categorical product.

If A_α is unital for each $\alpha \in I$, then it is clear that $1_A = \left(1_{A_\alpha}\right)_{\alpha\in I}$ is the multiplicative identity in A, and it is clear that π_α are unital homomorphisms for each $\alpha \in I$. ■

Suppose I is now a directed set, we define

$$B = \left\{a \in \prod_{\alpha\in I} A_\alpha : \text{there is a } \beta \in I \text{ such that } a_\alpha = 0 \text{ for all } i \geq j\right\},$$

it is clear that B is a two-sided ideal in $\prod_{\alpha\in I} A_\alpha$, thus we can define the C^*-algebra

$$\bigoplus_{\alpha\in I} A_\alpha := \overline{B},$$

which will be called the **direct sum of $(A_\alpha)_{\alpha\in I}$**. Note that given a net of real numbers $(a_\alpha)_{\alpha\in I}$, we define

$$\limsup_{\alpha\in I} a_\alpha = \inf_{\beta\in I} \sup_{i\geq j} a_\alpha = \inf\{\sup\{a_\alpha : i \geq j\} : \beta \in I\},$$

and $\lim_{\alpha\in I} a_\alpha$ or just $\lim_\alpha a_\alpha$ to be usual limits of nets if it exists in the extended reals. Note that this behaves exactly like the same in the $\mathbb{N}$ case, that is,

$$\limsup a_\alpha = \lim_\beta \sup_{i\geq j} a_\alpha = \lim_\beta \sup\{a_\alpha : i \geq j\}$$

and if $J \subseteq I$ is a cofinal set, then $\lim_{j\in J} a_\beta = \lim_{\alpha\in I} a_\alpha$. See [8, p. 32] for more details.

Lemma 4.2.3. Let I be a directed set and $(A_\alpha)_{\alpha\in I}$ be a collection of C^*-algebras, and $\pi : \prod_{n\in\mathbb{N}} A_n \to \prod_{\alpha\in I} A_n / \bigoplus_{\alpha\in I} A_n$ be the canonical map. Given $a = (a_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} A_\alpha$, then

$$\|\pi(a)\| = \limsup \|a_\alpha\|.$$

In particular, $a \in \bigoplus_{\alpha\in I} A_\alpha$ if, and only if, $\lim \|a_\alpha\| = 0$.

Proof. We let B be defined as above. As B is dense in $\bigoplus_{\alpha\in I} A_\alpha$, then $\|\pi(a)\| = \inf\{\|a - b\| : b \in B\}$ by continuity of $b \mapsto \|a - b\|$. Let $b = (b_\alpha)_{\alpha\in I} \in B$, then there is a $\beta \in I$ such that $b_\alpha = 0$ for all $i \geq j$, so

$$\|a-b\| = \sup_{\alpha\in I}\|a_\alpha - b_\alpha\| \geq \limsup\|a_\alpha - b_\alpha\| = \limsup\|a_\alpha\|,$$

hence $\|\pi(a)\| \geq \limsup_i\|a_\alpha\|$. For each $\beta \in I$, define $b^{(j)} = \big(b_\alpha^{(j)}\big)_{\alpha\in I} \in B$ as

$$b_\alpha^{(j)} = \begin{cases} 0 & \text{if } i > j \\ a_\alpha & \text{otherwise}\end{cases},$$

so

$$\|\pi(a)\| \leq \inf_{\beta\in I}\big\|a - b^{(j)}\big\| = \inf_{\beta\in I}\sup_{\alpha\in I}\Big\|a_\alpha - b_\alpha^{(j)}\Big\| = \inf_{\beta\in I}\sup_{i>j}\|a_\alpha\| = \limsup_i\|a_\alpha\|.$$

Thus $\|\pi(a)\| = \limsup\|a_\alpha\|$. Now the last part follows because

$$\begin{aligned} a \in \bigoplus_{\alpha\in I} A_\alpha &\iff \pi(a) = 0 \\ &\iff \|\pi(a)\| = 0 \\ &\iff \limsup\|a_\alpha\| = 0 \\ &\iff \lim\|a_\alpha\| \text{ exists and } \lim\|a_\alpha\| = 0. \end{aligned}$$

■

The preceding lemma gives a precise description of the structure of the direct sum, which is given by

$$\bigoplus_{\alpha\in I} A_\alpha = \left\{ a \in \prod_{\alpha\in I} A_\alpha : \lim_\alpha\|a_\alpha\| = 0 \right\}.$$

Theorem 4.2.4. *(Inductive Completeness of C^*-Algebras).* Let I be a directed set and $A^\bullet = \big(A_\alpha, (\varphi_{\alpha\beta})\big)$ be a collection of C^*-algebras indexed by I. Given $\alpha, \beta \in I$, whenever $\alpha \leq \beta$ does not hold, then define $\varphi_{\alpha\beta} : A_\alpha \to A_\beta$ to be the zero map.
Then $A^\bullet$ has an inductive limit $(A, (\psi_\alpha)_{\alpha\in I})$. Moreover:

(i) $A = \overline{\bigcup_{\alpha\in I}\operatorname{im}(\psi_\alpha)}$.
(ii) $\|\psi_\alpha(a)\| = \lim_\beta\big\|\varphi_{\alpha\beta}(a)\big\|$ for all $\alpha \in I$ and $a \in A_\alpha$.
(iii) $\ker(\psi_\alpha) = \big\{a \in A_\alpha : \lim_\beta\big\|\varphi_{\alpha\beta}(a)\big\| = 0\big\}$ for all $\alpha \in I$.
(iv) If $(B, (\mu_\alpha)_{\alpha\in I})$ is another cocone of $A^\bullet$ and $\lambda : A \to B$ is the map obtained by the universal property, then
  (a) $\ker(\psi_\alpha) \subseteq \ker(\mu_\alpha)$ for all $\alpha \in I$.
  (b) λ is injective if, and only if, $\ker(\mu_\alpha) = \ker(\psi_\alpha)$ for all $\alpha \in I$.
  (c) λ is surjective if, and only if, $B = \overline{\bigcup_{\alpha\in I}\operatorname{im}(\mu_\alpha)}$.
(v) If A_α is unital for each $\alpha \in I$, then A is unital.
(vi) Assume (v) and suppose $\varphi_{\alpha\beta}$ are unital for each $\alpha \leq \beta$ in I. Then ψ_α are unital for each $\alpha \in I$. If B, and μ_α are unital for each $\alpha \in I$, then λ is unital.

Proof. *Construction of ψ_α*
Let $\pi : \prod_{\alpha\in I} A_\alpha \to \prod_{\alpha\in I} A_\alpha / \bigoplus_{\alpha\in I} A_\alpha$ be the canonical map. For each $\alpha \in I$, define

$$\nu_\alpha : A_\alpha \to \prod_{\beta\in I} A_\beta : a \mapsto \big(\varphi_{\alpha\beta}(a)\big)_{\beta\in I} \quad \text{and} \quad \psi_\alpha = \pi\circ\nu_\alpha : A_\alpha \to \prod_{\alpha\in I} A_\alpha / \bigoplus_{\alpha\in I} A_\alpha.$$

Note that as $\big\|\varphi_{\alpha\beta}(a)\big\| \leq \|a\|$ for all $\beta \in I$, so $\|\nu_\alpha(a)\| \leq \|a\| < \infty$, hence ν_α is well-defined, and it is clear that ν_α defines a *-homomorphism, thus ψ_α is a *-homomorphism.

Construction of A, and showing cocone and (i)

Let $\alpha, \beta \in I$ and suppose $\alpha \leq \beta$. Let $a \in A_\alpha$, and consider

$$\nu_\alpha(a) - \nu_\beta\big(\varphi_{\alpha\beta}(a)\big) = \big(\varphi_{\alpha\gamma}(a)\big)_{\gamma\in I} - \big(\varphi_{\beta\gamma}\big(\varphi_{\alpha\beta}(a)\big)_{\gamma\in I}. \tag{4.1}$$

If $\gamma \geq \beta$, then $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$, so (4.1) evaluates to 0 at index k, thus

$$\psi_\alpha(a) - \psi_\beta(\varphi_{\alpha\beta}(a)) = \pi(\nu_\alpha(a) - \nu_\beta(\varphi_{\alpha\beta}(a))) = 0,$$

hence $\psi_\alpha = \psi_\beta \circ \varphi_{\alpha\beta}$. In particular, $\mathrm{im}(\psi_\alpha) \subseteq \mathrm{im}(\psi_\beta)$, so given arbitrary $\alpha, \beta \in I$ and $a \in \mathrm{im}(\psi_\alpha)$, $b \in \mathrm{im}(\psi_\beta)$, then there is a $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$, then one has $a, b \in \mathrm{im}(\psi_k)$, so it follows that $A = \overline{\bigcup_{\alpha \in I} \mathrm{im}(\psi_\alpha)}$ defines a C^*-algebra as a subalgebra of $\mathrm{im}(\pi)$. We restrict the codomains of ψ_α to A for each $\alpha \in I$, so thus $(A, (\psi_\alpha)_{\alpha \in I})$ forms a cocone of $A^\bullet$. This shows (i).

Showing (ii)-(iii)

Let $\alpha \in I$ and $a \in A_\alpha$, then by Lemma 4.2.3, one has

$$\|\psi_\alpha(a)\| = \|\pi(\nu_\alpha(a))\| = \limsup_\beta \|\varphi_{\alpha\beta}(a)\| = \lim_\beta \|\varphi_{\alpha\beta}(a)\|,$$

and the limit exists as for each $\gamma \geq \beta \geq \alpha$, one has $\varphi_{\alpha\gamma} = \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta}$, so $\|\varphi_{\alpha\gamma}(a)\| = \|\varphi_{\beta\gamma}(\varphi_{\alpha\beta}(a))\| \leq \|\varphi_{\alpha\beta}(a)\|$, so $(\varphi_{\alpha\beta}(a))_{\beta \in I}$ is a decreasing net. This proves (ii), and (iii) immediate follows.

Universality of $(A, (\psi_\alpha)_{\alpha \in I})$ - Existence of λ and showing (a)

Let $(B, (\mu_\alpha)_{\alpha \in I})$ be another cocone of $A^\bullet$, and let $\alpha \in I$. Suppose $a \in \ker(\psi_\alpha)$, then $\lim_\beta \|\varphi_{\alpha\beta}(a)\| = 0$, so for each $\beta \geq \alpha$, one has

$$\|\mu_\alpha(a)\| = \|\mu_\beta(\varphi_{\alpha\beta}(a))\| \leq \|\varphi_{\alpha\beta}(a)\|$$

hence $\|\mu_\alpha(a)\| = 0$ by taking limits, and thus $\mu_\alpha(a) = 0$, so $a \in \ker(\mu_\alpha)$, i.e. $\ker(\psi_\alpha) \subseteq \ker(\mu_\alpha)$. This shows (a). Then by the First Isomorphism Theorem 2.1.5, there is a unique *-homomorphism $\lambda_\alpha : \mathrm{im}(\psi_\alpha) \to B$ such that $\mu_\alpha = \lambda_\alpha \circ \psi_\alpha$. As $\mathrm{im}(\psi_\alpha) \subseteq \mathrm{im}(\psi_\beta)$, one has $\lambda_\beta|_{\mathrm{im}(\psi_\alpha)} = \lambda_\alpha$ by uniqueness for $\alpha \leq \beta$, and I is a directed set, then using a similar justification for showing that A is a C^*-algebra, one has a *-homomorphism $\lambda' : \bigcup_{\alpha \in I} \mathrm{im}(\psi_\alpha) \to B$ that extends λ_α for all $\alpha \in I$. Since λ' is uniformly continuous (norm-decreasing in fact), then one has a unique uniformly continuous extension $\lambda : A \to B$ of λ', hence $\mu_\alpha = \lambda \circ \psi_\alpha$ for each $\alpha \in I$.

Uniqueness of λ

Suppose $\delta : A \to B$ is another *-homomorphism such that $\mu_\alpha = \delta \circ \psi_\alpha$ for all $\alpha \in I$, then it follows by uniqueness of λ_α, δ extends λ_α for all $\alpha \in I$, hence δ extends λ', and thus $\delta = \lambda$ by uniqueness. Hence $(A, (\psi_\alpha)_{\alpha \in I})$ is an inductive limit of $A^\bullet$.

Properties (b) and (c) of λ

Let $\alpha \in I$, as $\mu_\alpha = \lambda \circ \psi_\alpha$ for each $\alpha \in I$, then one has $\ker(\mu_\alpha) = \ker(\psi_\alpha)$ if λ is injective. If $\ker(\mu_\alpha) = \ker(\psi_\alpha)$ for all $\alpha \in I$, then λ_α is injective. Indeed, $y = \psi_\alpha(x) \in \ker(\lambda_\alpha)$, then $\lambda_\alpha(y) = \mu_\alpha(x) = 0$, so $x \in \ker(\mu_\alpha) = \ker(\psi_\alpha)$, so $y = \psi_\alpha(x) = 0$, as required. Hence λ_α is an isometry for each $\alpha \in I$, thus λ', and hence, λ are isometries. This proves (b).

Note that as $\mathrm{im}(\lambda_\alpha) = \mathrm{im}(\mu_\alpha)$, then $\mathrm{im}(\lambda') = \bigcup_{\alpha \in I} \mathrm{im}(\mu_\alpha)$, hence $\mathrm{im}(\lambda) = \overline{\bigcup_{\alpha \in I} \mathrm{im}(\mu_\alpha)}$ as $\mathrm{im}(\lambda)$ is closed. Thus λ is surjective if, and only if, $B = \overline{\bigcup_{\alpha \in I} \mathrm{im}(\mu_\alpha)}$. This proves (c).

Conditions (v) and (vi)

Assume (v), then by Theorem 4.2.2, the product $\prod_{\alpha \in I} A_\alpha$ has as natural unit 1, and as π is surjective, one has a unital C^*-algebra $\prod_{\alpha \in I} A_\alpha / \bigoplus_{\alpha \in I} A_\alpha$ with the unit given by $\pi(1)$, hence A is a unital algebra as a subalgebra of a unital algebra. Assume (vi). Given $\alpha \in I$, then for each $\beta \geq \alpha$, then the βth component of

$$\nu_\alpha(1_{A_\alpha}) - (1_{A_\gamma})_{\gamma \in I}$$

is zero, so applying π, one obtains $\psi_\alpha\big(1_{A_\alpha}\big) - \pi(1) = 0$, hence ψ_α are unital and in particular $\pi(1) \in \mathrm{im}(\psi_\alpha)$. If μ_α is unital for each $\alpha \in I$, then λ_α is unital for each $\alpha \in I$ as

$$\lambda_\alpha(\pi(1)) = \lambda_\alpha\big(\psi_\alpha\big(1_{A_\alpha}\big)\big) = \mu_\alpha\big(1_{A_\alpha}\big) = 1_B,$$

thus λ', hence λ, are unital. ■

The preceding theorem proves that the category C*-Alg and C*-Alg$_1$ is inductively complete; see Definition 5.3.3. In fact, the category C*-Alg is actually small complete and small cocomplete; see [9]. We shall also take $\varinjlim A^\bullet$ to be the C^*-algebra constructed in the above theorem, which will be *the* inductive limit. It turns out, the connecting morphisms in the inductive limits of C^*-algebras can be assumed to be injective, in particular, isometries.

Proposition 4.2.5. Let I be a directed set and $A^\bullet = \big(A_\alpha, (\varphi_{\alpha\beta})\big)$ be a diagram in C*-Alg indexed by I. Then there is a diagram $B^\bullet$ in C*-Alg indexed by I such that the connecting and boundary morphisms of $B^\bullet$ are isometries and $\varinjlim A^\bullet \cong \varinjlim B^\bullet$. If the connecting morphisms of $A^\bullet$ are unital, then so are the connecting morphisms of $B^\bullet$.

Proof. We consider the setup from Lemma 5.4.7 with $B^\bullet = \big(B_\alpha, (\psi_{\alpha\beta})\big)$, and we have $(B, (\nu_\alpha)_{\alpha\in I})$ is the inductive limit of $B^\bullet$ by the preceding theorem. By (iii) of the theorem, let $\alpha \in I$ and $x \in \ker(\nu_\alpha)$, then $\|x\| = \lim_\beta \|x\| = \lim_\beta \big\|\psi_{\alpha\beta}(x)\big\| = 0$ as $\psi_{\alpha\beta}$ are isometries as they are injective. Thus $x = 0$, hence ν_α is injective and thus an isometry. So by Lemma 5.4.7 (ii) and the theorem's (iv)(b), the universal map $\pi : A \to B$ is injective, and by the lemma (iii) and the theorem's (iv)(c), one has π is surjective. So π is an isomorphism.

Let $\alpha \leq \beta$ in I, if $\varphi_{\alpha\beta}$ is unital (so the A_α's are unital), then so are $\psi_{\alpha\beta}$ as a composition of unital maps $\pi_\beta \circ \varphi_{\alpha\beta}$. ■

Furthermore, the converse of the Inductive Completeness of C^*-Algebras 4.2.4 (vi) holds, that is, if the inductive limit is unital, then it can be recognized as an inductive limit in C*-Alg$_1$, i.e. the connecting morphisms can now be assumed to be unital.

Corollary 4.2.6. Let I be a directed set and $A^\bullet = \big(A_\alpha, (\varphi_{\alpha\beta})\big)$ be a diagram in C*-Alg indexed by I, if the inductive limit of $A^\bullet$ and $(A, (\psi_\alpha)_{\alpha\in I})$ be the inductive limit of $A^\bullet$. If A is unital, then A is an inductive limit in C*-Alg$_1$.

Proof. By the preceding proposition, we may assume that the connecting and boundary maps of $A^\bullet$ are isometries. As $A = \overline{\bigcup_{\alpha\in I} \mathrm{im}(\psi_\alpha)}$, and A has a unit 1_A, then there is a $\alpha \in I$ and a $x \in A_\alpha$ such that $\|1_A - \psi_\alpha(x)\| < 1$. Let

$$J = \{\alpha \in I : \text{there is a } x \in A_\alpha \text{ such that } \|1_A - \psi_\alpha(x)\| < 1\},$$

and note that for each $\alpha \in J$ and a $x \in A_\alpha$ such that $\|1_A - \psi_\alpha(x)\| < 1$, then $\psi_\alpha(x)$ is invertible in A, and by Lemma 2.4.6 (ii), $\psi_\alpha(x)$ is invertible in $\mathrm{im}(\psi_\alpha)$, hence $\mathrm{im}(\psi_\alpha)$ is a unital algebra by Lemma 2.4.6. As ψ_α is an isometry, then A_α is unital, and denote its unit as 1_α. Let $\beta \geq \alpha$ in I, then note that

$$\big\|1_A - \psi_\beta(\varphi_{\alpha\beta}(x))\big\| = \|1_A - \psi_\alpha(x)\| < 1$$

so $\beta \in J$, hence one has

$$\psi_\beta(\varphi_{\alpha\beta}(1_\alpha)) = \psi_\alpha(1_\alpha) = 1_A = \psi_\beta(1_\beta)$$

and by injectivity of ψ_β, one has $\varphi_{\alpha\beta}(1_\alpha) = 1_\beta$, hence $\varphi_{\alpha\beta}$ is unital. As J is cofinal in I, then by Lemma 5.4.6, one has that A is an inductive limit of $B^\bullet = \big(A_\alpha, (\varphi_{\alpha\beta})\big)_{\substack{\alpha,\beta\in J,\\ \alpha\leq\beta}}$, which is a diagram in C*-Alg$_1$. ■

The next proposition provides us a very convenient way of constructing and identifying inductive limits. This result will especially be useful for identifying $\mathcal{K}(H)$ as an inductive limit for a separable Hilbert space H.

Proposition 4.2.7. Let B be a C^*-algebra, and I be a directed set. Suppose $A_\alpha \subseteq B$ is a C^*-subalgebra for each $\alpha, \beta \in I$ such that $A_\alpha \subseteq A_\beta$ for each $\alpha \leq \beta$ in I and $A_\alpha \cap A_\beta = 0$ if α and β are not comparable. Define $\varphi_{\alpha\beta} : A_\alpha \to A_\beta$ to be the inclusion map, then $A = \overline{\bigcup_{\alpha\in I} A_\alpha}$ is an inductive limit of $A^\bullet = \big(A_\alpha, (\varphi_{\alpha\beta})\big)$.

Proof. Define $\psi_\alpha : A_\alpha \to A$ to be inclusion maps for each $\alpha \in I$. Let $\alpha \leq \beta$ in I, so one has $\psi_\beta \circ \varphi_{\alpha\beta} = \psi_\alpha$ is obvious, hence $(A, (\psi_\alpha)_{\alpha\in I})$ defines a cocone of $A^\bullet$. Let $(C, (\mu_\alpha)_{\alpha\in I})$ be another cocone of $A^\bullet$, then define

$$\lambda' : \bigcup_{\alpha\in I} A_\alpha \to C$$

as $\lambda'(x) = \mu_\alpha(x)$ if $x \in A_\alpha$. We claim that $\lambda'(x)$ is independent of the choice of A_α, so if $x \in A_\beta$ also, then we have three cases: $A_\alpha \cap A_\beta = 0$, $A_\alpha \subseteq A_\beta$, or $A_\beta \subseteq A_\alpha$. The case $A_\alpha \cap A_\beta = 0$ is trivial, so–without loss of generality–suppose $A_\alpha \subseteq A_\beta$. Then $\mu_\beta(x) = \mu_\beta\big(\varphi_{\alpha\beta}(x)\big) = \mu_\alpha(x)$, hence λ' is well-defined.
Clearly λ' is a *-homomorphism, so uniformly continuous, and thus we can consider the unique continuous extension $\lambda : A \to C$ of λ', which λ is still a *-homomorphism. Now λ is clearly unique, as $\lambda \circ \psi_\alpha = \mu_\alpha$ for all $\alpha \in I$. Thus $(A, (\psi_\alpha)_{\alpha\in I})$ is universal, as required. ■

In the case of separable AF-algebras, they can realized as inductive limits of separable C^*-algebras indexed by natural numbers.

Proposition 4.2.8. Let I be a directed set, and $A^\bullet$ be a diagram in C*-Alg indexed by I. Let $(A, (\psi_\alpha)_{\alpha\in I})$ be the inductive limit of A. Then A is separable if, and only if, there is an increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ in I such that $A = \overline{\bigcup_{n\in\mathbb{N}} \operatorname{im}\big(\psi_{\alpha_n}\big)}$. In any case, A can be recognized as an inductive limit of separable C^*-algebras where the underlying index set is $\mathbb{N}$.

Proof. Note that we shall use the Inductive Completeness of C^*-Algebras 4.2.4 (i) here, and write A as the inductive limit of $A^\bullet$, and $(\psi_\alpha)_{\alpha\in I}$ be the boundary maps of A, which can be assumed to be isometries for by Proposition 4.2.5.
"$\Longleftarrow$". Suppose the latter holds, then we can assume the objects of $A^\bullet$ to be separable and $I = \mathbb{N}$. Now $\operatorname{im}(\psi_\alpha)$ is separable as ψ_α is an isometry from a separable space, and as $A = \overline{\bigcup_{\alpha\in I} \operatorname{im}(\psi_\alpha)}$, so A is the closure of a countable union of separable subspaces. Let D_α be a countable dense subset of $\operatorname{im}(\psi_\alpha)$ for each $\alpha \in I$, and define $D = \bigcup_{\alpha\in I} D_\alpha$, which is countable. Thus[1]

$$\bigcup_{\alpha\in I} \operatorname{im}(\psi_\alpha) = \bigcup_{\alpha\in I} \overline{D_\alpha}^{\operatorname{im}(\psi_\alpha)} \subseteq \bigcup_{\alpha\in I} \overline{D_\alpha}^{A} \subseteq \overline{D}^{A},$$

in particular,

$$A \subseteq \overline{\overline{D}^{A}} = \overline{D}$$

so A is separable.[2]

[1]The superscripts refers the space that the closure is taken with respect to.

"$\Longrightarrow$". Suppose A is separable. Now $A' = \bigcup_{\alpha\in I} \operatorname{im}(\varphi_\alpha)$ is separable as a subspace of a separable metric space, so there is a subset $\{e_n : n \in \mathbb{N}\}$ of A' such that $A' = \overline{\{e_n : n \in \mathbb{N}\}}^{A'}$. For each, now there is a $\alpha_1 \in I$ such that $e_1 \in \operatorname{im}\big(\varphi_{\alpha_1}\big)$, and there is a $\alpha' \in I$ such that $e_2 \in \operatorname{im}(\varphi_{\alpha'})$. So there is a $\alpha_2 \in I$ such that $\alpha_2 \geq \alpha_1$ and $\alpha_2 \geq \alpha'$ by directedness of I, hence $e_2 \in \operatorname{im}\big(\varphi_{\alpha_2}\big)$. We can continue this process inductively, to find that for each $n \in \mathbb{N}$, there is a $\alpha_n \in I$ such that $e_n \in \operatorname{im}\big(\varphi_{\alpha_n}\big)$ where $\alpha_n \leq \alpha_{n+1}$, in particular $\operatorname{im}\big(\varphi_{\alpha_n}\big) \subseteq \operatorname{im}\big(\varphi_{\alpha_{n+1}}\big)$. So

$$A' = \overline{\{e_n : n \in \mathbb{N}\}}^{A'} \subseteq \overline{\bigcup_{n\in\mathbb{N}} \operatorname{im}\big(\varphi_{\alpha_n}\big)}^{A'} \subseteq A'$$

hence

$$A = \overline{\bigcup_{n\in\mathbb{N}} \operatorname{im}\big(\varphi_{\alpha_n}\big)}.$$

By Proposition 4.2.7, A is an inductive limit of $\big(\operatorname{im}\big(\varphi_{\alpha_n}\big), (\iota_{nm})\big)$ where $\iota : \operatorname{im}\big(\varphi_{\alpha_n}\big) \to \operatorname{im}\big(\varphi_{\alpha_m}\big)$ is the inclusion map for each $n \leq m$ in $\mathbb{N}$. ■

Example 4.2.9. *(The Space of Compact Operators $\mathcal{K}(H)$).* We assume that H is a separable infinite-dimensional Hilbert space. For each $n \in \mathbb{N}$, let $A_n = \mathcal{M}_n(\mathbb{C})$ and the *-homomorphisms

$$\varphi_n : A_n \to A_{n+1} : x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

We shall make the claim here that $\mathcal{K}(H)$ can be recognized as an inductive limit of the following diagram:

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

More specifically, if we define

$$\varphi_{nm} : A_n \to A_m = \begin{cases} \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n & \text{if } m > n \\ \mathrm{id}_{A_n} & \text{if } m = n \\ 0 & \text{if } m < n \end{cases}$$

then we claim that $\mathcal{K}(H)$ is an inductive limit of the diagram $A^\bullet = (A_n, (\varphi_{nm}))$ indexed by $\mathbb{N}$. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis H and let $n \in \mathbb{N}$. Let $P_n : H \to E_n$ be the orthogonal projection onto $E_n = \operatorname{span}\{e_k : k \leq n\}$, and define $B_n = P_n \mathcal{B}(H) P_n$ as a subalgebra of $\mathcal{B}(H)$. For each $a \in A_n$, define $a' : H \to H$ as a linear map such that $a'|_{E_n}$ is a linear map with associated matrix a with respect to the basis $\{e_k : k \leq n\}$, and $a' e_k = 0$ for all $k > n$. So $a' \in \mathcal{B}(H)$ with $\|a'\| = \|a\|$ and $a' = P_n a' P_n$, and

$$\alpha_n : A_n \to B_n : a \mapsto a'.$$

Clearly α_n is a *-embedding. Let $T \in \mathcal{B}(H)$, and let a be the associated matrix of $P_n T P_n|_{E_n}$ with respect the basis $\{e_k : k \leq n\}$, then it is clear that $\alpha_n(a) = P_n T P_n$, so α_n is an isomorphism. Note that $P_{n+1} P_n = P_n = P_n P_{n+1}$, so $B_n \subseteq B_{n+1}$, thus one can define inclusion maps $\iota_n : B_n \to B_{n+1}$, hence one has $\iota_{n+1} \circ \alpha_n = \alpha_{n+1} \circ \varphi_n$, thus one has a commutative diagram:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\varphi_1} & A_2 & \xrightarrow{\varphi_2} & A_3 & \xrightarrow{\varphi_3} & \cdots \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ B_1 & \xrightarrow{\iota_1} & B_2 & \xrightarrow{\iota_2} & B_3 & \xrightarrow{\iota_3} & \cdots \end{array}$$

[2] Without the superscripts, we assume the closure is taken with respect to the whole space A.

By Proposition 4.2.10, one has that $\mathcal{K}(H) = \overline{\bigcup_{n\in\mathbb{N}} P_n\mathcal{B}(H)P_n}$. Thus by Proposition 4.2.7, $\mathcal{K}(H)$ is an inductive limit of $(B_n, (\iota_{nm}))$, and by the Elliott's Intertwining Argument 5.4.8, one has that $A \cong \mathcal{K}(H)$.

Proposition 4.2.10. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded operators on a Hilbert space H, and let $S \in \mathcal{K}(H)$. Suppose $\sup_{n\in\mathbb{N}}\|T_n\| < \infty$, and $T_n \to \mathrm{id}_H$ pointwise, then the following holds:

(i) $T_nS \to S$ (ii) $ST_n \to S$ (iii) $T_nST_n \to S$.

where the convergence is under the operator norm.

Proof.

(i) As the image of S under the closed unit ball is relatively compact, it is in particular totally bounded. So given a $\varepsilon > 0$, there is a $n \in \mathbb{N}$, $x_1, ..., x_n \in H$ with $\|x_i\| \leq 1$ for $i \leq n$ such that for each $y \in H$ with $\|y\| \leq 1$, there is a $i \leq n$ with $\|Sy - Sx_i\| \leq \varepsilon$. As $T_nSx_i \to Sx_i$ for each $i \leq n$, then there is a $N \in \mathbb{N}$ such that $\|T_kSx_i - Sx_i\| \leq \varepsilon$ for all $k \geq N$ and $i \leq n$. Thus given a unit vector $y \in H$ and $k \geq N$, there is a $i \leq n$ such that $\|Sy - Sx_i\| \leq \varepsilon$, so one has

$$\begin{aligned}\|T_kSy - Sy\| &\leq \|T_kSy - T_kSx_i\| + \|T_kSx_i - Sx_i\| + \|Sx_i - Sy_i\| \\ &\leq \|T_k\|\varepsilon + 2\varepsilon \leq M\varepsilon\end{aligned}$$

where $M = \sup_{k\in\mathbb{N}}\|T_k\| + 2 < \infty$. Hence $\|T_kS - S\| \leq M\varepsilon$, thus $T_kS \to S$.

(ii) Let $x \in H$, so

$$\begin{aligned}\|T_n^*x - x\|^2 &= \|T_n^*x\|^2 - 2\mathfrak{R}\langle T_n^*x, x\rangle + \|x\|^2 \\ &\leq \|T_n\|^2\|x\|^2 - 2\mathfrak{R}\langle x, T_nx\rangle + \|x\|^2 \\ &\xrightarrow{n\to\infty} \|\mathrm{id}_H\|^2\|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0.\end{aligned}$$

Thus $T_n^* \to \mathrm{id}_H^* = \mathrm{id}_H$ pointwise, and as $*$ is an isometry, one has that $(T_n^*)_{n\in\mathbb{N}}$ is also a uniformly bounded sequence. As S^* is also a compact operator, one has that $T_n^*S^* \to S^*$ in the operator norm by part (i). By continuity of $*$, one has $ST_n = (T_n^*S^*)^* \to (S^*)^* = S$.

(iii) Let $M = \sup_{n\in\mathbb{N}}\|T_n\|$. For each $n \in \mathbb{N}$, observe that

$$\begin{aligned}\|T_nST_n - S\| &\leq \|T_nST_n - T_nS\| + \|T_nS - S\| \\ &\leq \|T_n\|\|ST_n - S\| + \|T_nS - S\| \\ &\leq M\|ST_n - S\| + \|T_nS - S\|,\end{aligned}$$

then the rest follows from part (i) and part (ii) by taking $n \to \infty$. ■

2⟩ Inductive Limit of Groups

We have a similar result in the category of Grp and OrdAb.

Lemma 4.2.11. Let $\varphi : G \to H$ be a group homomorphism between Abelian groups G and H. Let $G^+ \subseteq G$. Then one has

(i) If $G^+ + G^+ \subseteq G^+$, then $\varphi(G^+) + \varphi(G^+) \subseteq \varphi(G^+)$.

(ii) If $G^+ - G^+ = G$, then $\varphi(G^+) - \varphi(G^+) = \mathrm{im}(\varphi)$.

Proof. Trivial. ■

Theorem 4.2.12. *(Inductive Completeness of Various Groups)*. Let I be a directed set, and $G^\bullet = (G_\alpha, (\varphi_{\alpha\beta})$ be a collection of groups indexed by I. Then $\varinjlim G^\bullet = (G, (\psi_\alpha)_{\alpha\in I})$ exists. Furthermore:

(i) $G = \bigcup_{\alpha\in I} \operatorname{im}(\psi_\alpha)$.

(ii) $\ker(\psi_\alpha) = \bigcup_{\substack{\beta\in I\\ \beta\geq\alpha}} \ker(\varphi_{\alpha\beta})$ for each $\alpha \in I$.

(iii) Let $(H, (\mu_\alpha)_{\alpha\in I})$ be another cocone of $G^\bullet$ and $\lambda : G \to H$ is the map obtained by the universal property, then

  (a) $\ker(\psi_\alpha) \subseteq \ker(\mu_\alpha)$ for all $\alpha \in I$.

  (b) λ is injective if, and only if, $\ker(\mu_\alpha) = \ker(\psi_\alpha)$ for all $\alpha \in I$.

  (c) λ is surjective if, and only if, $H = \bigcup_{\alpha\in I} \operatorname{im}(\mu_\alpha)$.

(iv) Suppose for each $\alpha \in I$, G_α is now an Abelian group with a positive cone G_α^+, and $\varphi_{\alpha\beta} : (G_\alpha, G_\alpha^+) \to \big(G_\beta, G_\beta^+\big)$ are now positive group homomorphism for each $\alpha \leq \beta$ in I. Then $G^+ = \bigcup_{\alpha\in I} \psi_\alpha(G_\alpha^+)$ is a positive cone for G, and ψ_α are positive group homomorphisms for each $\alpha \in I$. In particular, $((G, G^+), (\psi_\alpha)_{\alpha\in I})$ is the inductive limit of $\big((G_\alpha, G_\alpha^+), (\varphi_{\alpha\beta})\big)$ in OrdAb.

Proof. Define $P = \prod_{\alpha\in I} G_\alpha$ to be the usual products of groups; see Example 5.2.4. Define

$$Q = \{g \in P : \text{there is a } \beta \in I \text{ such that } a_\alpha = 0 \text{ for all } \alpha \geq \beta\},$$

then it is clear that Q is a normal subgroup of P. Consider the canonical map $\pi : P \to P/Q$, and define $\varphi_{\alpha\beta} : G_\alpha \to G_\beta$ to be the zero map whenever $\alpha \leq \beta$ does not hold for $i, \beta \in I$. For each $\alpha \in I$, define

$$\nu_\alpha : G_\alpha \to P : g \mapsto \big(\varphi_{\alpha\beta}(g)\big)_{\beta\in I} \quad \text{and} \quad \psi_\alpha = \pi \circ \nu_\alpha : G_\alpha \to P/Q.$$

Thus ψ_α are homomorphisms, and observe that for each $\alpha \leq \beta$, and $g \in G_\alpha$, one has

$$\nu_\alpha(g) - \nu_\beta\big(\varphi_{\alpha\beta}(g)\big) = \big(\varphi_{\alpha\gamma}(g)\big)_{\gamma\in I} - \big(\varphi_{\beta\gamma}\big(\varphi_{\alpha\beta}(g)\big)\big)_{\gamma\in I}$$

which evaluates to zero for at index $\gamma \geq \beta$ as $\varphi_{\alpha\gamma} = \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta}$. Hence

$$\psi_\alpha(g) - \psi_\beta\big(\varphi_{\alpha\beta}(g)\big) = \pi\big(\nu_\alpha(g) - \nu_\beta\big(\varphi_{\alpha\beta}(g)\big)\big) = 0,$$

thus $\psi_\alpha = \psi_\beta \circ \varphi_{\alpha\beta}$. This also shows that $\operatorname{im}(\psi_\alpha) \subseteq \operatorname{im}\big(\psi_\beta\big)$, hence we can define the subgroup $G = \bigcup_{\alpha\in I} \operatorname{im}(\psi_\alpha)$ of P/Q as I is directed. We restrict the codomains of ψ_α to G for each $\alpha \in I$, so we have shown that $(G, (\psi_\alpha)_{\alpha\in I})$ forms a cocone of $G^\bullet$. This shows (i).

Let $\alpha \leq \beta$ in I, then as $\psi_\alpha = \psi_\beta \circ \varphi_{\alpha\beta}$, then it is clear that $\ker\big(\varphi_{\alpha\beta}\big) \subseteq \ker(\psi_\alpha)$. Let $x \in \ker(\psi_\alpha)$, then $0 = \psi_\alpha(x) = \pi(\nu_\alpha(x))$ shows that $\nu_\alpha(x) \in Q$, so there is a $\beta \geq \alpha$ such that $\varphi_{\alpha\beta}(x) = 0$, so $x \in \ker\big(\varphi_{\alpha\beta}\big)$. This shows (ii). From (iii), as $\mu_\alpha = \mu_\beta \circ \varphi_{\alpha\beta}$, then $x \in \ker(\mu_\alpha)$, hence $\ker(\psi_\alpha) \subseteq \ker(\mu_\alpha)$. This shows (a). By the first isomorphism theorem, it follows we can define a unique homomorphism $\lambda_\alpha : \operatorname{im}(\psi_\alpha) \to H$ such that $\mu_\alpha = \lambda_\alpha \circ \psi_\alpha$. As $\operatorname{im}(\psi_\alpha) \subseteq \operatorname{im}\big(\psi_\beta\big)$, then $\lambda_\beta|_{\operatorname{im}(\psi_\alpha)} = \lambda_\alpha$ by uniqueness, thus we can define a homomorphism $\lambda : G \to H$ which extends λ_α for all $\alpha \in I$. In particular, $\mu_\alpha = \lambda \circ \psi_\alpha$ for each $\alpha \in I$, and note that λ is unique as λ_α is unique for each $\alpha \in I$. This shows that $(G, (\psi_\alpha)_{\alpha\in I})$ is indeed an inductive limit of $G^\bullet$.

For (iii)(b). It is clear that if λ is injective, then $\ker(\mu_\alpha) = \ker(\psi_\alpha)$ as $\mu_\alpha = \lambda \circ \psi_\alpha$ for each $\alpha \in I$. If $\ker(\mu_\alpha) = \ker(\psi_\alpha)$ for each $\alpha \in I$, then λ_α is injective for each $\alpha \in I$ see the proof Inductive Completeness of C^*-Algebras 4.2.4 (iv)(b), thus λ is also injective.

For (iii)(c). Note that $\operatorname{im}(\lambda_\alpha) = \operatorname{im}(\mu_\alpha)$, thus $\operatorname{im}(\lambda) = \bigcup_{\alpha\in I} \operatorname{im}(\mu_\alpha)$. So λ is surjective if, and only if, $H = \bigcup_{\alpha\in I} \operatorname{im}(\mu_\alpha)$.

For (iv). We first show that G^+ is a positive cone of G. For **OG1**; see Definition 4.1.5. Let $g, h \in G^+$, and by directedness of I, we can assume that $g, h \in \psi_\alpha(G^+)$ for some $\alpha \in I$, hence by Lemma 4.2.11

(i), it follows that $g + h \in \psi_\alpha(G^+) \subseteq G^+$. This shows **OG1**.
For **OG3**. Let $z \in G$, so $z = \psi_\alpha(g) \in \operatorname{im}(\psi_\alpha)$ for some $\alpha \in I$, then by Lemma 4.2.11 (ii), $\psi_\alpha(g) = \psi_\alpha(x) - \psi_\alpha(y) \in \psi_\alpha(G_\alpha^+) - \psi_\alpha(G_\alpha^+) \subseteq G^+ - G^+$ for some $x, y \in G_\alpha^+$. This shows **OG3**.
For **OG2**. Let $z \in G^+ \cap (-G^+)$, so by directedness of I, there is a $\alpha \in I$, and $x, y \in G_\alpha^+$ such that $\psi_\alpha(x) = -\psi_\alpha(y) = z$. Now $\pi(\nu_\alpha(x+y)) = \psi_\alpha(x+y) = 0$, so there is a $\beta \in I$ with $\beta \geq \alpha$ such that for each $k \geq j$, one has $\varphi_{\alpha\gamma}(x+y) = 0$, i.e. $\varphi_{\alpha\gamma}(x) = -\varphi_{\alpha\gamma}(y)$. As $\varphi_{\alpha\gamma}(x), \varphi_{\alpha\gamma}(y) \in G_k^+$, then $\varphi_{\alpha\gamma}(x) \in G_k^+ \cap (-G_k^+) = 0$, so $0 = \varphi_{\alpha\gamma}(x)$. This holds for all $k \geq j$, thus $0 = \pi(\nu_\alpha(x)) = \psi_\alpha(x) = z$. Hence $G^+ \cap (-G^+) = 0$. Thus G^+ is a positive cone.
Then it is clear that $\psi_\alpha(G_\alpha^+) \subseteq G^+$ for each $\alpha \in I$ by definition, so ψ_α are positive. Hence $((G, G^+), (\psi_\alpha)_{\alpha\in I})$ is now a cocone of $\left((G_\alpha, G_\alpha^+), (\varphi_{\alpha\beta})\right)$ in OrdAb. To prove universality, suppose now H has a positive cone H^+, and all of the maps μ_α are now positive. Let $g \in G^+$, so $g = \psi_\alpha(x) \in \psi_\alpha(G_\alpha^+)$ for some $\alpha \in I$ and $x \in G_\alpha^+$, thus $\lambda_\alpha(g) = \lambda_\alpha(\psi_\alpha(x)) = \mu_\alpha(x) \in H^+$, hence $\lambda(g) = \lambda_\alpha(g) \in H^+$. Thus $\lambda(G^+) \subseteq H^+$, and the rest follows. ■

4.3 Continuity of the K_0 Functor

We shall prove the inductive continuity of taking matrix algebras and unitizations. Keep in mind that this proof is possible as per the construction and conclusions laid out by the Inductive Completeness of C^*-Algebras 4.2.4 (iii) and (iv).

Lemma 4.3.1. *(Inductive Continuity of Matrix Algebras)*. Let I be a directed set, and $A^\bullet = \left(A_\alpha, (\varphi_{\alpha\beta})\right)$ be a diagram in C*-Alg indexed by I. Let $n \in \mathbb{N}$, then one has $\mathcal{M}_n\left(\varinjlim A^\bullet\right)$ is an inductive limit of $\mathcal{M}_n(A^\bullet)$.

Proof. Given $\left(\varinjlim A^\bullet, (\mu_\alpha)_{\alpha\in I}\right)$ to be the inductive limit of $A^\bullet$, then $\left(\mathcal{M}_n\left(\varinjlim A^\bullet\right), (\mathcal{M}_n(\mu_\alpha))_{\alpha\in I}\right)$ is a cocone of $\mathcal{M}_n(A^\bullet)$, which $\mathcal{M}_n(A^\bullet)$ has the inductive limit $(M, (\nu_\alpha)_{\alpha\in I})$, such that $\nu_\alpha = \nu_\beta \circ \mathcal{M}_n\left(\varphi_{\alpha\beta}\right)$ for each $\alpha \leq \beta$ in I. By universality, there is a unique *-homomorphism $\lambda : M \to \mathcal{M}_n\left(\varinjlim A^\bullet\right)$ such that $\mathcal{M}_n(\mu_\alpha) = \lambda \circ \nu_\alpha$ for each $\alpha \in I$.
As

$$
\begin{aligned}
x = (x_{ij}) \in \ker(\mathcal{M}_n(\mu_\alpha)) &\implies x_{ij} \in \ker(\mu_\alpha) \\
&\implies \lim_\beta \|\varphi_{\alpha\beta}(x_{ij})\| = 0 \\
&\implies \lim_\beta \|\mathcal{M}_n(\varphi_{\alpha\beta})(x)\| = 0 \\
&\implies x \in \ker(\nu_\alpha)
\end{aligned}
$$

where the third implication comes from Lemma 2.5.3. Thus $\ker(\mathcal{M}_n(\mu_\alpha)) \subseteq \ker(\nu_\alpha)$, hence λ is injective.
Note that $\varinjlim A^\bullet = \overline{\bigcup_{\alpha\in I} \operatorname{im}(\mu_\alpha)}$, so given $x = (x_{ij}) \in \mathcal{M}_n\left(\varinjlim A^\bullet\right)$, each x_{ij} can be approximated by a sequence $\left(x_{ij}^{(k)}\right)_{k\in\mathbb{N}}$ in $\bigcup_{\alpha\in I} \operatorname{im}(\mu_\alpha)$. Define $x^{(k)} = \left(x_{ij}^{(k)}\right) \in \bigcup_{\alpha\in I} \operatorname{im}(\mathcal{M}_n(\mu_\alpha))$, hence $\lim_{k\to\infty} x^{(k)} = x$ by Lemma 2.5.3, thus $\mathcal{M}_n\left(\varinjlim A^\bullet\right) = \overline{\bigcup_{\alpha\in I} \operatorname{im}(\mathcal{M}_n(\mu_\alpha))}$. Hence λ is surjective. Thus λ is an isomorphism, as required. ■

Lemma 4.3.2. *(Inductive Continuity of Unitization)*. Let I be a directed set, and $A^\bullet = \left(A_\alpha, (\varphi_{\alpha\beta})\right)$ be a diagram in C*-Alg indexed by I. Then $\left(\varinjlim A^\bullet\right)^\sim$ is an inductive limit of $\widetilde{A}^\bullet$ in C*-Alg$_1$.

Proof. Given $\left(\varinjlim A^\bullet, \mu_\alpha\right)$ to be the inductive limit of $A^\bullet$, then $\left(\left(\varinjlim A^\bullet\right)^\sim, (\widetilde{\mu_\alpha})_{\alpha\in I}\right)$ is a cocone of $\widetilde{A}^\bullet$, which $\widetilde{A}^\bullet$ has the inductive limit $(B, (\nu_\alpha)_{\alpha\in I})$, such that $\nu_\alpha = \nu_\beta \circ \widetilde{\varphi_{\alpha\beta}}$ for each $\alpha \le \beta$ in I. By universality, there is a unique unital *-homomorphism $\lambda : B \to \left(\varinjlim A^\bullet\right)^\sim$ such that $\widetilde{\mu_\alpha} = \lambda \circ \nu_\alpha$ for each $\alpha \in I$.
As

$$\begin{aligned} x = a + c1 \in \ker(\widetilde{\mu_\alpha}) &\implies a \in \ker(\mu_\alpha) \text{ and } c = 0 \\ &\implies \lim_\beta \left\|\varphi_{\alpha\beta}(a)\right\| = 0 \quad \text{and} \quad \widetilde{\varphi_{\alpha\beta}}(x) = \varphi_{\alpha\beta}(a) \quad \forall \beta \in I \\ &\implies \lim_\beta \left\|\widetilde{\varphi_{\alpha\beta}}(x)\right\| = 0 \\ &\implies x \in \ker(\nu_\alpha). \end{aligned}$$

Thus $\ker(\widetilde{\mu_\alpha}) \subseteq \ker(\nu_\alpha)$, hence λ is injective.
Note that $\varinjlim A^\bullet = \overline{\bigcup_{\alpha\in I} \operatorname{im}(\mu_\alpha)}$, so given $x = a + c1 \in \left(\varinjlim A^\bullet\right)^\sim$, where a can be approximated by a sequence $(a_n)_{n\in\mathbb{N}}$ in $\bigcup_{\alpha\in I} \operatorname{im}(\mu_\alpha)$ and $c \in \mathbb{C}$. Let $x_n = a_n + c1$ for each $n \in \mathbb{N}$, so one has a sequence $(x_n)_{n\in\mathbb{N}}$ in $\bigcup_{\alpha\in I} \operatorname{im}(\widetilde{\mu_\alpha})$ such that $x_n \to x$, thus $\left(\varinjlim A^\bullet\right)^\sim = \overline{\bigcup_{\alpha\in I} \operatorname{im}(\widetilde{\mu_\alpha})}$. Hence λ is surjective. Thus λ is an isomorphism, as required. ■

We now need a lemma to prove inductive continuity of K_0. First we have a lemma for our lemma.

Lemma 4.3.3. Let A be a C^*-algebra.

(i) If $a \in A$ be self-adjoint with $\delta = \left\|a - a^2\right\| < 1/4$, then there is a projection $p \in A$ with $\|a - p\| \le 2\delta$.

(ii) Let $p, q \in \mathrm{P}(A)$. If there is a $x \in A$ with $\|x^*x - p\| < 1/2$, and $\|xx^* - q\| < 1/2$, then $p \sim q$.

Proof.

(i) If $t \in \sigma(a)$, then $t - t^2 \in \sigma(a - a^2)$ by the Continuous Functional Calculus 2.4.3 (ii), and if $\left|t - t^2\right| \le \delta < 1/4$ for $t \in \mathbb{R}$, then $t \in [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta]$; see [2, 6.3.1] for confirmation. Hence if $\left\|a - a^2\right\| = \delta < 1/4$, then

$$\sigma(a) \subseteq \left\{t \in \mathbb{R} : \left|t - t^2\right| \le \delta\right\} \subseteq [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta],$$

then one can define a continuous map

$$f : [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta] \to \mathbb{C} : t \mapsto \begin{cases} 0 & \text{if } t \in [-2\delta, 2\delta] \\ 1 & \text{if } t \in [1 - 2\delta, 1 + 2\delta] \end{cases}$$

as the domain is a disjoint union. Hence $p = f(a)$ is a projection as $f = f^2 = f^*$, and

$$\|a - p\| = \left\|\mathrm{id}_{\sigma(a)} - f\right\|_\infty = \max\left\{\sup_{t\in[-2\delta,2\delta]} |t|, \sup_{t\in[1-2\delta,1+2\delta]} |t - 1|\right\} \le 2\delta.$$

(ii) Let $\delta = 1/2 \max\{\|x^*x - p\|, \|xx^* - q\|\}$, so $\delta < 1/4$. Let $K = \sigma(x^*x) \cup \sigma(xx^*)$, which satisfies $K \subseteq [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta]$ by Lemma 3.1.6. Define $f \in \mathcal{C}(K)$ as above, and let $p_0 = f(x^*x)$ and $q_0 = f(xx^*)$, now $\|p - p_0\| \le 4\delta < 1$ and $\|q - q_0\| \le 4\delta < 1$, so $p \sim p_0$ and $q \sim q_0$ by Diagram 1. Given a polynomial $P(z) \in \mathbb{C}[z]$, one has $xP(x^*x)x^* = P(xx^*)xx^*$ as $x(x^*x)^n x^* = (xx^*)^n xx^*$ for each $n \in \mathbb{N}$ and the rest follows from linearity. Thus by Stone-Weierstrass, $xg(x^*x)x^* = g(xx^*)xx^*$ for each $g \in \mathcal{C}(K)$. Define $g \in \mathcal{C}(K, [0, \infty))$ such that $tg(t)^2 = f(t)$ for each $t \in K$, and let $v = xg(x^*x)$, so one has

$$\begin{aligned} v^*v &= g(x^*x)x^*xg(x^*x) = x^*xg(x^*x)g(x^*x) = f(x^*x) = p_0 \\ vv^* &= xg(x^*x)^2 x^* = g(xx^*)^2 xx^* = f(xx^*) = q_0, \end{aligned}$$

thus $p_0 \sim q_0$, hence $p \sim q$. ■

This technical lemma here, which will be the crux for proving the continuity of K_0, essentially states that the elements satisfying the Murray-von Neumann relation $\sim$ in the inductive limit can be 'approximated' by elements in the constituents that makes up the inductive limit.

Lemma 4.3.4. Let I be a directed set, and $A^\bullet = \big(A_\alpha, (\varphi_{\alpha\beta})\big)$ be a diagram in C*-Alg indexed by I, and $(A, (\psi_\alpha)_{\alpha\in I})$ be the inductive limit of $A^\bullet$. Let $n \in \mathbb{N}$. Then one has the following:

(i) If $p \in \mathrm{P}_n(A)$, then there is an $\alpha \in I$ and a $q \in \mathrm{P}_n(A_\alpha)$ such that $\psi_\alpha(q) \sim p$.

(ii) Let $\alpha \in I$, and $p, q \in \mathrm{P}_n(A_\alpha)$ such that $\psi_\alpha(p) \sim \psi_\alpha(q)$, then there is a $\gamma \geq \alpha$ such that $\varphi_{\alpha\gamma}(p) \sim \varphi_{\alpha\gamma}(q)$.

Proof. We first assume $n = 1$. Recall that by the Inductive Completeness of C^*-Algebras 4.2.4, one has $A = \overline{\bigcup_{\alpha\in I} \mathrm{im}(\psi_\alpha)}$, and $\|\psi_\alpha(x)\| = \lim_\beta \|\varphi_{\alpha\beta}(x)\|$ for each $\alpha \in I$ and $x \in A_\alpha$.

(i) There is a $\alpha \in I$ and $b \in A_\alpha$ such that $\|p - \psi_\alpha(b)\| < 1/5$. Let $a_\alpha = (b + b^*)/2$, so a_α is self-adjoint and so is $a_\beta = \varphi_{\alpha\beta}(a_\alpha)$ with

$$\|p - \psi_\beta(a_\beta)\| = \|p - \psi_\beta(\varphi_{\alpha\beta}(a_\alpha))\| = \|p - \psi_\alpha(a_\alpha)\| \leq \left\|\frac{p}{2} - \frac{b}{2}\right\| + \left\|\frac{p}{2} - \frac{b^*}{2}\right\| < \frac{1}{5}$$

for each $\beta \geq \alpha$ in I. Now Lemma 3.1.6 implies

$$\sigma(\psi_\alpha(a_\alpha)) \subseteq \left[-\frac{1}{5}, \frac{1}{5}\right] \cup \left[\frac{4}{5}, \frac{6}{5}\right],$$

so using any graphing calculator, one observes that

$$\|\psi_\alpha(a_\alpha - a_\alpha^2)\| = \max\{|t - t^2| : t \in \sigma(\psi_\alpha(a_\alpha))\} < \frac{1}{4}.$$

As $\|\psi_\alpha(x)\| = \lim_\beta \|\varphi_{\alpha\beta}(x)\|$ for $x \in A_\alpha$, there is a $\beta \geq \alpha$ in I such that $\|a_\beta - a_\beta^2\| < 1/4$, so by Lemma 4.3.3 (i), there is a $q \in A_\beta$ such that $\|a_\beta - q\| < 1/2$. Thus noting that *-homomorphisms are norm-decreasing, one has

$$\|\psi_\beta(q) - p\| \leq \|\psi_\beta(q - a_\beta)\| + \|\psi_\beta(a_\beta) - p\| \leq \frac{1}{2} + \frac{1}{5} < 1,$$

hence $\psi_\beta(q) \sim p$ by Diagram 1.

(ii) Let $v \in A$ such that $\psi_\alpha(p) = v^*v$ and $\psi_\alpha(q) = vv^*$, hence there is a $\beta \in I$ and $x \in A_\beta$ such that $\|v - \psi_\beta(x)\| < \varepsilon$ for a fixed $\varepsilon > 0$. Choose a $\gamma \geq \alpha, \beta$, then one has

$$\|v - \psi_\gamma(\varphi_{\beta\gamma}(x))\| = \|v - \psi_\beta(x)\| < \varepsilon.$$

Relabel γ as β and $\varphi_{\beta\gamma}(x)$ as x, then we can find a $\beta \geq \alpha$ such that $\|v - \psi_\beta(x)\| < \varepsilon$. Now

$$\begin{aligned}\|v^*v - \psi_\beta(x^*x)\| &= \big\|(v^* - \psi_\beta(x^*))(v - \psi_\beta(x)) + v^*\psi_\beta(x) + \psi_\beta(x^*)v\big\| \\ &\leq \varepsilon^2 + 2\|v^*\psi_\beta(x)\|,\end{aligned}$$

and similarly,

$$\|vv^* - \psi_\beta(xx^*)\| \leq \varepsilon^2 + 2\|v\psi_\beta(x)^*\|.$$

As the maps $(x, y) \mapsto x^*y$ and $(x, y) \mapsto xy^*$ are continuous in a C^*-algebra setting, then we can choose an $\varepsilon > 0$ small enough such that

$$\|\psi_\alpha(p) - \psi_\beta(x^*x)\| < \frac{1}{2} \quad \text{and} \quad \|\psi_\alpha(q) - \psi_\beta(xx^*)\| < \frac{1}{2}$$

for some $\beta \geq \alpha$ and $x \in A_\beta$. As $\psi_\alpha = \psi_\beta \circ \varphi_{\alpha\beta}$, then one has

$$\max\{\|\psi_\beta(\varphi_{\alpha\beta}(p) - x^*x)\|, \|\psi_\beta(\varphi_{\alpha\beta}(q) - xx^*)\|\} < \frac{1}{2},$$

hence there is a $\gamma \geq \beta$ such that

$$\max\{\|\varphi_{\beta\gamma}(\varphi_{\alpha\beta}(p) - x^*x)\|, \|\varphi_{\beta\gamma}(\varphi_{\alpha\beta}(q) - xx^*)\|\} < \frac{1}{2},$$

i.e.

$$\|\varphi_{\alpha\gamma}(p) - y^*y\| < \frac{1}{2} \quad \text{and} \quad \|\varphi_{\alpha\gamma}(q) - yy^*\| < \frac{1}{2}$$

where $y = \varphi_{\beta\gamma}(x)$. Hence by Lemma 4.3.3 (ii), one has that $\varphi_{\alpha\gamma}(p) \sim \varphi_{\alpha\gamma}(q)$, as required. Now assume for any $n \in \mathbb{N}$. Note that the contents of the statements (i) and (ii) are exactly when you replace A to $\mathcal{M}_n(A)$. By the Inductive Continuity of Matrix Algebras 4.3.1, we can assume $\mathcal{M}_n(A)$ is the inductive limit of $\mathcal{M}_n(A^\bullet)$, thus (i) and (ii) follows. ■

Theorem 4.3.5. *(Inductive Continuity of K_0).* Let I be a directed set, and $A^\bullet = (A_\alpha, (\varphi_{\alpha\beta}))$ be a diagram in C*-Alg indexed by I, and $(A, (\psi_\alpha)_{\alpha\in I})$ be the inductive limit of $A^\bullet$. Then $(K_0(A), (K_0(\psi_\alpha))_{\alpha\in I})$ is an inductive limit of $K_0(A^\bullet)$. Moreover:

(i) $K_0(A) = \bigcup_{\alpha\in I} \operatorname{im}(K_0(\psi_\alpha))$.

(ii) $K_0(A)^+ = \bigcup_{\alpha\in I} K_0(\psi_\alpha)\big(K_0(A_\alpha)^+\big)$.

(iii) $\ker(K_0(\psi_\alpha)) = \bigcup_{\substack{\beta\in I\\ \beta\geq\alpha}} \ker\big(K_0(\varphi_{\alpha\beta})\big)$ for each $\alpha \in I$.

(iv) If $\big(K_0(A_\alpha), K_0(A_\alpha)^+\big)$ is an ordered Abelian group for each $\alpha \in I$, then $((K_0(A), K_0(A)^+), (K_0(\psi_\alpha))_{\alpha\in I})$ is an inductive limit of $\Big(\big(K_0(A_\alpha), K_0(A_\alpha)^+\big), \big(K_0(\varphi_{\alpha\beta})\big)\Big)$ in OrdAb.

(v) If A_α is unital with unit 1_α, $\big(K_0(A_\alpha), K_0(A_\alpha)^+, [1_\alpha]_0\big)$ is an ordered Abelian group with distinguished order unit for each $\alpha \in I$, and $A^\bullet$ is a diagram in C*-Alg$_1$. Then $((K_0(A), K_0(A)^+, [1_A]_0), (K_0(\psi_\alpha))_{\alpha\in A})$ is an inductive limit of $K^\bullet = \Big(\big(K_0(A_\alpha), K_0(A_\alpha)^+, [1_\alpha]_0\big), K_0(\varphi_{\alpha\beta})\Big)$ in OrdAb$_1$.

Proof. For simplicity of notations, for each $\alpha \leq \beta$ in I, we shall note that $\tilde{\varphi}_{\alpha\beta}$ defines the induced *-homomorphism from $\mathcal{M}_n\big(\widetilde{A_\alpha}\big)$ to $\mathcal{M}_n\big(\widetilde{A_\beta}\big)$ (which is achieved by applying the matrix algebra functor after the unitization functor) for each $n \in \mathbb{N}$. We have similar conventions for ψ_α and $\tilde{\psi}_\alpha$ respectively. Thus by the Inductive Continuity of Matrix Algebras 4.3.1 and the Inductive Continuity of Unitization 4.3.2, one has that $\big(\mathcal{M}_n(\tilde{A}), (\tilde{\psi}_\alpha)_{\alpha\in I}\big)$ is an inductive limits of $\mathcal{M}_n\big(\widetilde{A^\bullet}\big)$ in C*-Alg$_1$.

(i) Let $g \in K_0(A)$, then by Structure of K_0 3.4.4, there is a $k \in \mathbb{N}$ and $p \in \mathrm{P}_n\big(\tilde{A}\big)$ such that $g = [p]_0 - [s(p)]_0$. By Lemma 4.3.4 (i), there is a $\alpha \in I$ and $q \in \mathrm{P}_n\big(\widetilde{A_\alpha}\big)$ such that $\tilde{\psi}_\alpha(q) \sim p$. So by Structure of K_0 3.4.4 (iv),

$$g = [p] - [s(p)]_0 = \big[\tilde{\psi}_\alpha(q)\big]_0 - \big[s(\tilde{\psi}_\alpha(q))\big]_0 = K_0(\psi_\alpha)[q]_0 \in \operatorname{im}(K_0(\psi_\alpha)).$$

Hence (i) is shown.

(ii) Note that $\supseteq$ is obvious as $K_0(\psi_\alpha)$ is positive for each $\alpha \in I$. Let $g \in K_0(A)^+$, then $g = [p]_0$ for some $p \in \mathrm{P}_n(A)$ and $n \in \mathbb{N}$. By Lemma 4.3.4 (i), there is a $q \in \mathrm{P}_n(A_\alpha)$ for some $\alpha \in I$ such that $\psi_\alpha(q) \sim p$. Thus $g = [\psi_\alpha(q)]_0 = K_0(\psi_\alpha)(q) \in K_0(\psi_\alpha)\big(K_0(A_\alpha)^+\big)$. Hence (ii) is shown.

(iii) This part uses the Inductive Completeness of C^*-Algebras 4.2.4 (i) and (ii). Let $\alpha \in I$. As $\psi_\alpha = \psi_\beta \circ \varphi_{\alpha\beta}$, then $K_0(\psi_\alpha) = K_0(\psi_\beta) \circ K_0(\varphi_{\alpha\beta})$, hence $\ker\big(K_0(\varphi_{\alpha\beta})\big) \subseteq \ker(K_0(\psi_\alpha))$ for each $\beta \geq \alpha$ in I. Thus $\supseteq$ is shown. Let $g \in \ker(K_0(\psi_\alpha))$, then there is a $n \in \mathbb{N}$ such that $p \in \mathrm{P}_n\big(\widetilde{A_\alpha}\big)$ such that $g = [p]_0 - [s(p)]_0$, and so $\tilde{\psi}_\alpha(p) \sim \tilde{\psi}_\alpha(s(p))$. By Lemma 4.3.4 (ii), there is a $\gamma \geq \alpha$ in I such that $\tilde{\varphi}_{\alpha\gamma}(p) \sim \tilde{\varphi}_{\alpha\gamma}(s(p)) = s(\tilde{\varphi}_{\alpha\gamma}(p))$, hence by Structure of K_0 3.4.4 (iv), one has

$$K_0(\varphi_{\alpha\gamma})(g) = [\tilde{\varphi}_{\alpha\gamma}(p)] - [s(\tilde{\varphi}_{\alpha\gamma}(p))] = 0$$

thus $g \in \ker(K_0(\varphi_{\alpha\gamma}))$, as required.

Continuity of K_0

This part uses the Inductive Completeness of Various Groups 4.2.12 (ii), and (iii)(b) and (iii)(c). Note that $K_0(A^\bullet)$ is a diagram in Ab indexed by I, thus referring to the Inductive Completeness of Various Groups 4.2.12, we obtain the inductive limit $(G, (\mu_\alpha)_{\alpha \in I})$ of $K_0(A^\bullet)$ such that $\mu_\alpha = \mu_\beta \circ K_0(\varphi_{\alpha\beta})$ for all $\beta \geq \alpha$ in I. As $(K_0(A), (K_0(\psi_\alpha))_{\alpha \in I})$ is a cocone of $K_0(A^\bullet)$, then there is a unique homomorphism $\lambda : G \to K_0(A)$ such that $K_0(\psi_\alpha) = \lambda \circ \mu_\alpha$ for each $\alpha \in I$. By part (i), we have that λ is surjective. By part (iii), one has $\ker(K_0(\psi_\alpha)) = \ker(\mu_\alpha)$ for each $\alpha \in I$, so λ is injective. Thus λ is an isomorphism, as required.

(iv) Note that $G^+ = \bigcup_{\alpha \in I} \mu_\alpha\big(K_0(A_\alpha)^+\big)$, so by part (ii), one has

$$\lambda(G^+) = \bigcup_{\alpha \in I} \lambda\Big(\mu_\alpha\big(K_0(A_\alpha)^+\big)\Big) = \bigcup_{\alpha \in I} K_0(\psi_\alpha)\big(K_0(A_\alpha)^+\big) = K_0(A)^+,$$

thus λ is a positive isomorphism, as required.

(v) Firstly, we show that the objects in question are well-defined. By the Inductive Completeness of C^*-Algebras 4.2.4 (v), A is unital, so 1_A exists, and ψ_α is unital for each $\alpha \in I$. By Proposition 4.1.9, $K_0(\varphi_{\alpha\beta})$ and $K_0(\psi_\alpha)$ are unital positive homomorphisms, thus the context of question is well-defined, such as that $K^\bullet$ is indeed a diagram in OrdAb_1. It suffices to show that $L = ((K_0(A), K_0(A)^+, [1_A]_0), (K_0(\psi_\alpha))_{\alpha \in I})$ is a universal cocone of $K^\bullet$, which we note L is indeed a cocone. Let $((G, G^+, u), (\mu_\alpha)_{\alpha \in I})$ be a cocone of $K^\bullet$, and we note that $((G, G^+), (\mu_\alpha)_{\alpha \in I})$ is a cocone of $K^\bullet$ in OrdAb where we identified $K^\bullet$ via the forgetful functor $\mathsf{OrdAb}_1 \hookrightarrow \mathsf{OrdAb}$. Thus there is a unique positive homomorphism $\lambda : K_0(A) \to G$ such that $\mu_\alpha = \lambda \circ K_0(\psi_\alpha)$ for each $\alpha \in I$. Fix any $\alpha \in I$, then in particular,

$$\lambda([1_A]_0) = \lambda(\psi_\alpha([1_\alpha]_0)) = \mu_\alpha([1_\alpha]_0) = u,$$

so λ is unital. Hence L is universal in OrdAb_1, as required. ■

The part (v) of the preceding theorem can be rephrased as follows.

Theorem 4.3.6. *(Inductive Continuity of (K_0, K_0^+)).* Let I be a directed set, and $A^\bullet$ be a diagram in C*-Alg$_s$ indexed by I. Suppose $\varinjlim A^\bullet$ exists in C*-Alg$_s$, then $(K_0, K_0^+)\big(\varinjlim A^\bullet\big)$ is an inductive limit of $(K_0, K_0^+)(A^\bullet)$.

As a consequence of the continuity of K_0, we can now compute the K_0-group for $\mathcal{K}(H)$ when H is a separable Hilbert space.

Example 4.3.7. *($K_0(\mathcal{K}(H))$).* Recall from Example 4.2.9, $\mathcal{K}(H)$ can realized as the inductive limit of the sequence:

$$\mathbb{C} \xrightarrow{\varphi_1} \mathcal{M}_2(\mathbb{C}) \xrightarrow{\varphi_2} \mathcal{M}_3(\mathbb{C}) \xrightarrow{\varphi_3} \cdots$$

where $\varphi_n : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_{n+1}(\mathbb{C})$ is defined as $\varphi_n(x) = x \oplus 0_1$. In particular, one has the following commutative diagram using $K_0(\mathrm{tr})$ from the Structure of $K_{00}(\mathcal{B}(H))$ 3.3.9,

$$\begin{array}{ccc} K_0(\mathcal{M}_n(\mathbb{C})) & \xrightarrow{K_0(\varphi_n)} & K_0(\mathcal{M}_{n+1}(\mathbb{C})) \\ {\scriptstyle K_0(\mathrm{tr})}\downarrow & & \downarrow{\scriptstyle K_0(\mathrm{tr})} \\ \mathbb{Z} & = & \mathbb{Z} \end{array}$$

where $=$ arrow means the $\mathrm{id}_{\mathbb{Z}}$ map. As $K_0(\mathrm{tr})$ is an isomorphism, then on the K_0-group level, the sequence

$$K_0(\mathbb{C}) \xrightarrow{K_0(\varphi_1)} K_0(\mathcal{M}_2(\mathbb{C})) \xrightarrow{K_0(\varphi_2)} K_0(\mathcal{M}_3(\mathbb{C})) \xrightarrow{K_0(\varphi_3)} \cdots \tag{4.2}$$

can be realized as the sequence

$$\mathbb{Z} = \mathbb{Z} = \mathbb{Z} = \cdots$$

which clearly has an inductive limit of $\mathbb{Z}$. Since by the Inductive Continuity of K_0 4.3.5, the K_0-group for $\mathcal{K}(H)$ is the inductive limit for the sequence (4.2), so $K_0(\mathcal{K}(H)) \cong \mathbb{Z}$.

4.4 Approximately Finite-Dimensional Algebras

We shall first introduce the notion of AF-algebras and state the relevant general properties that they have and inherit from the finite-dimensional case. This will help us to prove the classification theorem next chapter.

For clarity of the definitions, we define the following categories:

- The category of finite-dimensional C^*-algebras, FinC*-Alg, whose objects are finite-dimensional C^*-algebras, and morphisms are *-homomorphisms.
- The category of FinC*-Alg_1, whose objects are finite-dimensional C^*-algebras, and morphisms are *unital* *-homomorphisms.

Both FinC*-Alg and FinC*-Alg_1 are subcategories of C*-Alg and C*-Alg_1 respectively, thus given a diagram in the subcategories, we may identify their inductive limits as objects in the larger categories.

Definition 4.4.1. *(AF-Algebras)*. We say a C^*-algebra A is an **approximately finite-dimensional algebra** or **AF-algebra** if A is an inductive limit in C*-Alg of a diagram $A^\bullet$ in FinC*-Alg indexed by some directed set I.

The next lemma shows that the category of unital AF-algebras AF-Alg as a subcategory of C*-Alg_1 is actually a subcategory of unital and stably finite C^*-algebras C*-Alg_s, which means we can associate a unital AF-algebra A with its ordered Abelian K_0-group.

Lemma 4.4.2. An AF-algebra A is stably finite.

Proof. Write $A^\bullet = \left(A_\alpha, (\varphi_{\alpha\beta})\right)$ be a diagram in FinC*-Alg, and $(A, (\psi_\alpha)_{\alpha\in I})$ be an inductive limit of $A^\bullet$.

Case 1: If A is unital.

Let $s \in A$ be an isometry, i.e. $s^*s = 1$, then as $A = \overline{\bigcup_{\alpha\in I} \operatorname{im}(\psi_\alpha)}$, then there is a $\alpha \in I$ and a $x \in A_\alpha$ such that $\|\psi_\alpha(x) - s\| < 1$. Thus

$$\|s^*\psi_\alpha(x) - 1\| = \|s^*(\psi_\alpha(x) - s)\| < \|s^*\| = 1$$

so by Lemma 3.1.3, $s^*\psi_\alpha(x)$ is invertible in A, in particular, $\psi_\alpha(x)$ is left-invertible with left-inverse $(s^*\psi_\alpha(x))^{-1}s^*$, thus $\psi_\alpha(x)^*\psi_\alpha(x)$ is invertible in $\operatorname{im}(\psi_\alpha)$ by Lemma 2.4.6. As A_α is finite-dimensional, then so is $\operatorname{im}(\psi_\alpha)$, in particular, $\operatorname{im}(\psi_\alpha)$ is finite, thus $\psi_\alpha(x)$ is invertible by Lemma 4.1.2. Thus $s^* = (s^*\psi_\alpha(x))\psi_\alpha(x)^{-1} \in \mathrm{GL}(A)$, so s^* is invertible, and as s is a right-inverse of s^*, then s is the inverse of s^*. So $ss^* = 1$, i.e. s is unitary, so by Lemma 4.1.2, A is finite.

Case 2: If A is nonunital.

Let $n \in \mathbb{N}$, and it suffices to show that $\mathcal{M}_n\left(\tilde{A}\right)$ is finite. By the Inductive Continuity of Matrix Algebras 4.3.1 and the Inductive Continuity of Unitization 4.3.2, $\mathcal{M}_n\left(\tilde{A}\right)$ can be realized as an inductive limit of $\mathcal{M}_n\left(\widetilde{A^\bullet}\right)$, and as each object in $\mathcal{M}_n\left(\widetilde{A^\bullet}\right)$ is finite-dimensional, then $\mathcal{M}_n\left(\tilde{A}\right)$ is a AF-algebra, thus $\mathcal{M}_n\left(\tilde{A}\right)$ is finite by Case 1. ■

Proposition 4.4.3. An AF-algebra A is separable if, and only if, A can be recognized as an inductive limit of finite-dimensional C^*-algebras where the underlying index set is $\mathbb{N}$.

Proof. Follows from Proposition 4.2.8 as each $\operatorname{im}(\psi_\alpha)$ in the proposition is finite-dimensional, hence separable. ■

4.5 Elliott's Classification of Separable AF-Algebras

In this chapter, we shall prove the Elliot's classification theorem of AF-algebras. Before we proceed, we shall make some observations regarding finite-dimensional algebras. We make the remark that given a sum of matrix algebras

$$A = \mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_m}(\mathbb{C})$$

then we have a natural basis for A, which for $1 \leq r \leq m$ and $1 \leq i, j \leq n_r$, we define $e_{ij} \in \mathcal{M}_{n_r}$ such that the (i,j)th-entry of e_{ij} is 1 and 0 everywhere else, and let

$$e_{ij}^{(r)} = \big(0, ..., 0, e_{ij}, 0, ..., 0\big)$$

where the e_{ij} is at the rth position. Note that $\{e_{ij} : 1 \leq r \leq m, 1 \leq i, j \leq n_r\}$ forms a basis for A and we say they are the **standard basis for A**, and they satisfy the following properties:

MU1. $e_{ij}^{(r)} e_{kl}^{(s)} = e_{il}^{(r)}$ if $r = s$ and $j = k$, and 0 otherwise.

MU2. $\left(e_{ij}^{(r)}\right)^* = e_{ji}^{(r)}$.

MU3. $\left\{e_{ij}^{(r)} : 1 \leq r \leq m, 1 \leq i, j \leq n_r\right\}$ is a basis for A.

In general, if A is an arbitrary C^*-algebra such that there exists a $m \in \mathbb{N}$, $n_1, ..., n_m \in \mathbb{N}$, and $e_{ij}^{(r)} \in A$ for $1 \leq r \leq m$ and $1 \leq i, j \leq n_r$, such that the collection $\left\{e_{ij}^{(r)} : 1 \leq r \leq m, 1 \leq i, j \leq n_r\right\}$ satisfies **MU1** and **MU2**, to be a collection of **matrix units**, if **MU3** is also satisfied, then we say a **basis of matrix units**. This is in our interest as we can make comments about the 'natural basic' elements in our C^*-algebras which has 'finite-dimensional' subalgebras without explicitly stating their isomorphism to the sum of matirx algebras. For simplification of notation, we shall omit the inequalities for the indices, and their meaning will be reflected in our definition above if the context is clear.

Proposition 4.5.1. Let A and B be C^*-algebras and $\left(e_{ij}^{(r)}\right)$ and $\left(f_{ij}^{(r)}\right)$ be collections of matrix units for A and B respectively where $1 \leq r \leq m$ and $1 \leq i, j \leq n_r$. Then:

(i) The collection $\left(e_{ij}^{(r)}\right)$ is linearly independent if $e_{ij}^{(r)} \neq 0$ for all i, j, r.

Suppose $\left(e_{ij}^{(r)}\right)$ is now a basis of matrix units.

(ii) There is a unique *-homomorphism $\varphi : A \to B$ such that $\varphi\left(e_{ij}^{(r)}\right) = f_{ij}^{(r)}$ for all i, j, r.

(iii) If $\left(f_{ij}^{(r)}\right)$ is a basis of matrix units. Then φ is a *-isomorphism. In particular, $\left(e_{ij}^{(r)}\right)$ is a basis for A, and A is isomorphic to

$$\mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_m}(\mathbb{C}).$$

Proof. Let $E = \left\{e_{ij}^{(r)} : i, j, r\right\}$, and consider

$$0 = \sum_{r=1}^{m} \sum_{i,j=1}^{n_r} a_{ij}^{(r)} e_{ij}^{(r)} \quad \text{for } a_{ij}^{(r)} \in \mathbb{C}.$$

Fix $s \leq m$ and $k, l \leq n_s$. Then

$$0 = e_{kk}^{(s)} a e_{ll}^{(s)} = \sum_{ij=1}^{n_s} a_{ij}^{(s)} e_{kk}^{(s)} e_{ij}^{(s)} e_{ll}^{(s)} = a_{kl}^{(s)} e_{kl}^{(s)},$$

thus $a_{kl}^{(s)} = 0$. So E is linearly independent; this shows (i).

Assume (ii). We can define a map $\varphi : E \to B$ such that $\varphi\left(e_{ij}^{(r)}\right) = f_{ij}^{(r)}$ for all i, j, r, which has a unique linear onto A–still denoted as φ. By **MU1** and **MU2**, φ is a *-homomorphism. So if (iii) is assumed, then φ is bijective hence an isomorphism. In particular, we can choose B to be $\mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_m}(\mathbb{C})$ which establishes the isomorphism type of A. ■

Lemma 4.5.2. Suppose $E = \left\{e_{ii}^{(r)} : 1 \leq r \leq m, 1 \leq i \leq n_r\right\}$ is a set of mutually orthogonal projections in a C^*-algebra A. Suppose

$$e_{11}^{(r)} \sim e_{22}^{(r)} \cdots \sim e_{n_r n_r}^{(r)}$$

for each r. Then E can be extended to a collection of matrix units $\left(e_{ij}^{(r)}\right)$ in A.

Proof. Fix r and i, j, then as $e_{11}^{(r)} \sim e_{ii}^{(r)}$, then there exists a $e_{1i}^{(r)} \in A$ such that $e_{1i}^{(r)*} e_{1i}^{(r)} = e_{11}^{(r)}$ and $e_{1i}^{(r)} e_{1i}^{(r)*} = e_{ii}^{(r)}$. Define $e_{ij}^{(r)} = e_{1i}^{(r)*} e_{1j}^{(r)}$. Now $e_{ij}^{(r)*} = e_{1j}^{(r)*} e_{1i}^{(r)} = e_{ji}^{(r)}$, so **MU2** is satisfied.
Fix s and k, l. We note Proposition 3.1.1 (i). If $r \neq s$ or $j \neq k$, then

$$e_{ij}^{(r)} e_{kl}^{(s)} = e_{1i}^{(r)*} e_{1j}^{(r)} e_{1k}^{(s)*} e_{1l}^{(s)} = e_{1i}^{(r)*} e_{j1}^{(r)*} e_{k1}^{(s)} e_{1l}^{(s)} = e_{1i}^{(r)*} e_{1j}^{(r)} e_{jj}^{(r)} e_{kk}^{(s)} e_{1k}^{(s)*} e_{1l}^{(s)} = 0,$$

as $e_{jj}^{(r)} e_{kk}^{(s)} = 0$. Otherwise if $r = s$ and $j = k$, then

$$e_{ij}^{(r)} e_{kl}^{(s)} = e_{1i}^{(r)*} e_{1j}^{(r)} e_{1j}^{(r)*} e_{1l}^{(r)} = e_{1i}^{(r)*} e_{11}^{(r)} e_{1l}^{(r)} = e_{1i}^{(r)*} e_{1l}^{(r)} = e_{il}^{(r)},$$

so **MU1** is satisfied, as required. ■

Where $1 \leq r, s \leq m$, $1 \leq i, j \leq n_r$ and $1 \leq k, l \leq n_s$. We now observe that the K_0-group for A has the following structure by Proposition 4.2.10:

$$\begin{aligned} K_0(A) &= \mathbb{Z}\left[e_{11}^{(1)}\right]_0 \oplus \mathbb{Z}\left[e_{11}^{(2)}\right]_0 \oplus \cdots \oplus \mathbb{Z}\left[e_{11}^{(m)}\right]_0 \cong \mathbb{Z}^m \\ K_0(A)^+ &= \mathbb{Z}^+\left[e_{11}^{(1)}\right]_0 \oplus \mathbb{Z}^+\left[e_{11}^{(2)}\right]_0 \oplus \cdots \oplus \mathbb{Z}^+\left[e_{11}^{(m)}\right]_0 \cong (\mathbb{Z}^m)^+ \\ [1_A]_0 &= n_1\left[e_{11}^{(1)}\right]_0 + n_2\left[e_{11}^{(2)}\right]_0 + \cdots + n_m\left[e_{11}^{(m)}\right]_0. \end{aligned} \tag{4.3}$$

where $(\mathbb{Z}^m)^+ = \{(x_1, \ldots, x_m) \in \mathbb{Z}^m : x_i \geq 0 \text{ for all } i \leq m\}$, and note that $e_{11}^{(r)} \sim e_{ii}^{(r)}$, which is given by $e_{1i}^{(r)*} e_{1i}^{(r)} = e_{ii}^{(r)}$ and $e_{1i}^{(r)} e_{1i}^{(r)*} = e_{ii}^{(r)}$, and $1_A = \sum_{r=1}^{m} \sum_{i=1}^{n_r} e_{ii}^{(r)}$.

We first prove a very strong lemma, which essentially gives us sufficient conditions on when homomorphisms $\alpha : K_0(A) \to K_0(B)$ can be lifted to *-homomorphisms $\varphi : A \to B$ such that $K_0(\varphi) = \alpha$. We shall keep in mind of the structure of $K_0(A)$ for a finite-dimensional (4.3). Recall that C^*-algebra A has the cancellation property if $[p]_0 = [q]_0$ implies $p \sim q$ for each $p, q \in \mathrm{P}_\infty(A)$.

Lemma 4.5.3. Let A be a finite-dimensional C^*-algebra, and B be a unital C^*-algebra with the cancellation property. Then

(i) Given a positive group homomorphism $\alpha : K_0(A) \to K_0(B)$ with $\alpha([1_A]_0) \leq [1_B]_0$, there is a *-homomorphism $\varphi : A \to B$ with $K_0(\varphi) = \alpha$. Furthermore, α is unital if, and only if, φ is unital.

(ii) Let $\varphi, \psi : A \to B$ be *-homomorphisms. Then $K_0(\varphi) = K_0(\psi)$ if, and only if, $\psi = \mathrm{ad}\, u \circ \varphi$ for some $u \in B$.

Proof. Claim: If $p \in \mathrm{P}(B)$, and $g \in K_0(B)^+$ such that $g \leq [1_B]_0 - [p]_0$. Then there is a $q \in \mathrm{P}(B)$ such that $p \perp q$ and $g = [q]_0$.
Let $n \in \mathbb{N}$ and $e, f \in \mathrm{P}_n(B)$ such that $[e]_0 = g$ and $[f]_0 = [1_B]_0 - [p]_0 - g$. By cancellation property, $[e \oplus f]_0 = [1_B - p]_0$ implies $e \oplus f \sim 1_B - p$. See proof of Proposition 4.1.7 (iv) for $[1_B - p]_0 = [1_B]_0 - [p]_0$. Let $v \in \mathcal{M}_{1,2n}(B)$ such that $e \oplus f = v^* v$ and $1_B - p = v v^*$, then $q = v(e \oplus 0_n)v^*$. Now $q = q^*$, and

$$q^2 = v(e \oplus 0_n)(e \oplus f)v^* = v(e \oplus 0_n)v^*$$

so q is a projection, and

$$pq = (1_B - vv^*)v(e \oplus 0_n)v^* = q - v(e \oplus f)(e \oplus 0_n)v^* = 0,$$

so $p \perp q$, and finally $q \sim e$, so $[q]_0 = [e]_0 = g$, as required.
Claim: If $g_1, ..., g_n \in K_0(B)^+$ satisfies $\sum_{i \le n} g_i \le [1_B]_0$, then there are mutually orthogonal projections $p_1, ..., p_n \in B$ such that $[p_i]_0 = g_i$ for each $i \le n$.
Clearly the statement holds for $n = 1$, so by induction on $n > 1$, suppose $g_1, ..., g_{n-1}, g_n \in K_0(B)^+$ with $\sum_{i \le n-1} g_i \le [1_B]_0$ with mutually orthogonal projections $p_1, ..., p_{n-1} \in B$ such that $[p_i]_0 = g_i$ for $i \le n-1$. By Lemma 2.3.3, $g_n \le [1_B]_0 - [p_1 + \cdots + p_{n-1}]$, so by preceding claim, there is a $p_n \in \mathrm{P}(B)$ such that $p_1 + \cdots + p_{n-1} \perp p_n$ and $[p_n]_0 = g_n$. So for each $i \le n-1$, one has

$$0 \le p_i p_n \le p_1 p_n + \cdots + p_{n-1} p_n = 0$$

so $p_i p_n = 0$, hence $p_1, ..., p_n$ are mutually orthogonal. Hence claim is proven.

(i) Let $\left(e_{ij}^{(r)}\right)$ be the standard basis for A for $1 \le r \le m$ and $1 \le i, j \le n_r$, and as $1_A = \sum_{r \le m} \sum_{i \le n_r} e_{ii}^{(r)}$, then by preceding claim, there are mutually orthogonal collection of projections $\left\{f_{ii}^{(r)} : 1 \le r \le m, 1 \le i \le n_r\right\}$ in B such that $\alpha\left(\left[e_{ii}^{(r)}\right]_0\right) = \left[f_{ii}^{(r)}\right]_0$ for all $1 \le r \le m$ and $1 \le i \le n_r$. By cancellation property of B, one has

$$e_{ii}^{(r)} \sim e_{jj}^{(r)} \Rightarrow \left[e_{ii}^{(r)}\right]_0 = \left[e_{jj}^{(r)}\right]_0 \Rightarrow \left[f_{ii}^{(r)}\right]_0 = \left[f_{jj}^{(r)}\right]_0 \Rightarrow f_{ii}^{(r)} \sim f_{jj}^{(r)}$$

for each r and i. By Lemma 4.4.2, the collection extends to a collection of matrix units $\left(f_{ij}^{(r)}\right)$ in B. By Proposition 4.5.1 (ii), there is a *-homomorphism $\varphi : A \to B$ such that $\varphi\left(e_{ij}^{(r)}\right) = f_{ij}^{(r)}$ for all i, j, r, and hence $K_0(\varphi) = \alpha$ as $\left\{\left[e_{ij}^{(r)}\right]_0 : i, j, r\right\}$ generates $K_0(A)$.
Suppose $\alpha([1_A]_0) = [1_B]_0$. Let $p = \sum_{r \le m} \sum_{i \le n_r} f_{ii}^{(r)}$, which is a projection by Lemma 2.3.3 with $\varphi(1_A) = p$. Thus

$$[1_B - p]_0 = [1_B]_0 - [p]_0 = \alpha([1_A]_0) - K_0(\varphi)([1_A]_0) = 0$$

so $1_B - p \sim 0$ by cancellation property, thus $1_B - p = 0$. Hence $\varphi(1_A) = p = 1_B$. So φ is unital. If φ is unital, then $K_0(\varphi) = \alpha$ is automatically unital.

(ii) Suppose $K_0(\varphi) = K_0(\psi)$. As

$$\left[\varphi\left(e_{11}^{(r)}\right)\right]_0 = K_0(\varphi)\left[e_{11}^{(r)}\right]_0 = K_0(\psi)\left[e_{11}^{(r)}\right]_0 = \left[\psi\left(e_{11}^{(r)}\right)\right]_0$$
$$[1_B - \varphi(1_A)]_0 = [1_B]_0 - K_0(\varphi)[1_A]_0 = [1_B]_0 - K_0(\psi)[1_B]_0 = [1_B - \psi(1_A)]_0,$$

then by cancellation properties of B, there are $v_1, ..., v_m, w \in B$ such that

$$v_r^* v_r = \varphi\left(e_{11}^{(r)}\right), \qquad v_r v_r^* = \psi\left(e_{11}^{(r)}\right)$$
$$w^* w = 1_B - \varphi(1_A), \quad ww^* = 1_B - \psi(1_A).$$

Note that $\psi\left(e_{i1}^{(r)}\right) v_r \varphi\left(e_{1i}^{(r)}\right)$ is a partial isometry for each $1 \le r \le m$ and $1 \le i \le n_r$, and

$$w^* w + \sum_{r \le m} \sum_{i \le n_r} \left(\psi\left(e_{i1}^{(r)}\right) v_r \varphi\left(e_{1i}^{(r)}\right)\right)^* \left(\psi\left(e_{i1}^{(r)}\right) v_r \varphi\left(e_{1i}^{(r)}\right)\right)$$
$$= w^* w + \sum_{r \le m} \sum_{i \le n_r} \varphi\left(e_{i1}^{(r)}\right) v_r^* \psi\left(e_{11}^{(r)}\right) v_r \varphi\left(e_{1i}^{(r)}\right)$$
$$= w^* w + \sum_{r \le m} \sum_{i \le n_r} \varphi\left(e_{i1}^{(r)}\right) v_r^* v_r v_r^* v_r \varphi\left(e_{1i}^{(r)}\right)$$
$$= w^* w + \sum_{r \le m} \sum_{i \le n_r} \varphi\left(e_{i1}^{(r)}\right) \varphi\left(e_{11}^{(r)}\right) \varphi\left(e_{1i}^{(r)}\right)$$

$$= w^*w + \sum_{r\le m}\sum_{i\le n_r}\varphi\left(e_{ii}^{(r)}\right)$$
$$= 1_B - \varphi(1_A) + \varphi(1_A) = 1_B$$

and similarly,

$$ww^* + \sum_{r\le m}\sum_{i\le n_r}\left(\psi\left(e_{i1}^{(r)}\right)v_r\varphi\left(e_{1i}^{(r)}\right)\right)\left(\psi\left(e_{i1}^{(r)}\right)v_r\varphi\left(e_{1i}^{(r)}\right)\right)^* = 1_B,$$

so by Lemma 2.3.4, the element

$$u = w + \sum_{r\le m}\sum_{i\le n_r}\psi\left(e_{i1}^{(r)}\right)v_r\varphi\left(e_{1i}^{(r)}\right)$$

is unitary. In particular for $s \le m$ and $k, l \le n_s$, and using $w = ww^*w$ to obtain

$$w\varphi\left(e_{kl}^{(s)}\right) = w(1_B - \varphi(1_A))\varphi\left(e_{kl}^{(s)}\right) = 0 = \psi\left(e_{kl}^{(s)}\right)w,$$

hence

$$u\varphi\left(e_{kl}^{(s)}\right) = w\varphi\left(e_{kl}^{(s)}\right) + \sum_{r\le m}\sum_{i\le n_r}\psi\left(e_{i1}^{(r)}\right)v_r\varphi\left(e_{1i}^{(r)}e_{kl}^{(s)}\right) = \psi\left(e_{k1}^{(s)}\right)v_s\varphi\left(e_{1l}^{(s)}\right)$$

and similarly

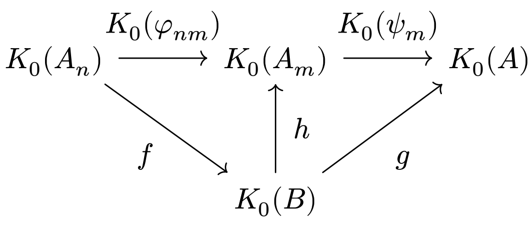
$$\psi\left(e_{kl}^{(s)}\right)u = \psi\left(e_{k1}^{(s)}\right)v_s\varphi\left(e_{1l}^{(s)}\right).$$

Thus $\psi = \mathrm{ad}u \circ \varphi$ as they agree on the basis of A.
Suppose $\psi = \mathrm{ad}u \circ \varphi$ for some $u \in \mathcal{U}(A)$, and since $K_0(\mathrm{ad}u) = \mathrm{id}_{K_0(A)}$ as $\mathrm{ad}u(x) \sim_u x$ for all $x \in A$, then

$$K_0(\psi) = K_0(\mathrm{ad}u) \circ K_0(\varphi) = K_0(\varphi),$$

as required. ■

Lemma 4.5.4. Let $I = \mathbb{N}$, and $A^\bullet = (A_n, (\varphi_{nm}))$ be a diagram in FinC*-Alg indexed by I, and $(A, (\psi_n)_{n\in I})$ be the inductive limit of $A^\bullet$. Let B be a finite-dimensional C^*-algebra, and suppose there is a $n \in I$, such that there are positive group homomorphisms $f : K_0(A_n) \to K_0(B)$, and $g : K_0(B) \to K_0(A)$ with $g \circ f = K_0(\psi_n)$. Then there is a $m > n$ in I, and a positive group homomorphism $h : K_0(B) \to K_0(A_m)$ making the diagram



commutative. Furthermore, if the connecting maps of $A^\bullet$ are unital and f is unital, then so is h.

Proof. Let $\left(e_{ij}^{(r)}\right)$ be the standard basis for B with $r \le m$ and $i, j \le n_r$, and let $y_r = g\left(\left[e_{11}^{(r)}\right]_0\right) \in K_0(A)^+$ for all $r \le m$. From Inductive Continuity of K_0 4.3.5, one has $K_0(A)^+ = \bigcup_{\gamma\in I} K_0(\psi_\gamma)\left(K_0(A_k)^+\right)$, and so there is a $k \in I$ with $k \ge n$ with $x_r \in K_0(A_k)^+$ such that $y_r = K_0(\psi_k)(x_r)$ for all $r \le m$. Note that $K_0(B)$ is the free Abelian group generated by $\left[e_{11}^{(1)}\right]_0, \left[e_{11}^{(2)}\right]_0, \ldots, \left[e_{11}^{(m)}\right]_0$, and so there is a unique homomorphism $h' : K_0(B) \to K_0(A_k)$ such that $h'\left(\left[e_{11}^{(r)}\right]\right) = x_r$ for each $r \le m$. Given $g \in K_0(B)^+$, one has

$$g = \sum_{r \le m} i_r \left[e_{11}^{(r)} \right]_0 \quad \text{for } i_r \in \mathbb{N}_0$$

hence

$$h'(g) = \sum_{r \le m} i_r x_r \in K_0(B)^+$$

thus h' is a positive. Since

$$(K_0(\psi_k) \circ h') \left[e_{11}^{(r)} \right]_0 = K_0(\psi_k) x_r = y_r = g\left(\left[e_{11}^{(r)} \right]_0 \right)$$

then $K_0(\psi_k) \circ h' = g$. Let $\{g_1, ..., g_s\}$ be a set of generators of a finitely generated Abelian group $K_0(A_n)$, and observe that as

$$K_0(\psi_k) \circ (h' \circ f - K_0(\varphi_{nk})) = g \circ f - K_0(\psi_n) = 0$$

then $(h' \circ f - K_0(\varphi_{nk}))(g_i)$ is in the set

$$\ker(K_0(\psi_k)) = \bigcup_{m \ge k} \ker(K_0(\varphi_{km}))$$

for each $i \le s$, so there is a $m > k$ such that

$$(h' \circ f - K_0(\varphi_{nk}))(g_i) \in \ker(\varphi_{km}) \quad \text{for all } i \le s.$$

Let $h = K_0(\varphi_{km}) \circ h'$, so

$$(h \circ f - K_0(\varphi_{nm}))(g_i) = (K_0(\varphi_{km}) \circ (h' \circ f - K_0(\varphi_{nk})))(g_i) = 0$$

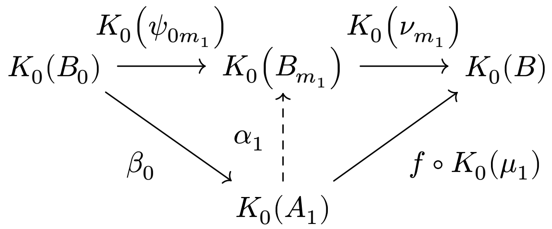
for each $i \le s$, thus $h \circ f = K_0(\varphi_{nm})$. Furthermore,

$$g = K_0(\psi_k) \circ h' = K_0(\psi_m) \circ K_0(\varphi_{km}) \circ h' = K_0(\psi_m) \circ h,$$

so the diagram is commutative. Finally, the last statement follows from the commutativity of the diagram. ∎

Theorem 4.5.5. *(Elliott).* The unital separable AF-algebras A and B are isomorphic if, and only if, the triples $(K_0(A), K_0(A)^+, [1_A]_0)$ and $(K_0(B), K_0(B)^+, [1_B]_0)$ are isomorphic. In particular, if there is an isomorphism $f : (K_0(A), K_0(A)^+, [1_A]_0) \to (K_0(B), K_0(B)^+, [1_B]_0)$, then there is an *-isomorphism $\varphi : A \to B$ with $K_0(\varphi) = f$.

Proof. By Proposition 4.1.9, if A and B are isomorphic, then so are $(K_0(A), K_0(A)^+, [1_A]_0)$ and $(K_0(B), K_0(B)^+, [1_B]_0)$.
So now we suppose there is an isomorphism $f : K_0(A) \to K_0(B)$ with $f(K_0(A)^+) = K_0(B)^+$ and $f([1_A]_0) = [1_B]_0$. By Corollary 4.2.6 and Proposition 4.4.3, we may assume $I = \mathbb{N}$ such that A and B are inductive limits of diagrams $A^\bullet = (A_n, (\varphi_{nm}))$ and $B^\bullet = (B_n, (\psi_{nm}))$ in FinC*-Alg$_1$ indexed by I, also let $\mu_n : A_n \to A$ and $\nu_n : B_n \to B$ be boundary maps of A and B respectively for each $n \in \mathbb{N}$. Let $B_0 = \mathbb{C}$, so one has a *-homomorphism $\varphi' : B_0 \to A_1$ given by $\varphi'(\lambda) = \lambda 1_{A_1}$, similarly, one has unique *-homomorphisms $\psi_{01} : B_0 \to B_1$, and $\nu_0 : B_0 \to B$. Let $\beta_0 = K_0(\varphi')$, and note that $(f \circ K_0(\mu_1)) \circ \beta_0 = K_0(\nu_0)$, so by Lemma 4.5.4, there is a $m_1 \in \mathbb{N}$ and a unital positive group homomorphism $\alpha_1 : A_1 \to B_{m_1}$ such that one has a commutative diagram:



where the existence of α_1 is guaranteed by Lemma 4.5.4. Note that

$$\left(f^{-1} \circ K_0\left(\nu_{m_1}\right)\right) \circ \alpha_1 = f^{-1} \circ (f \circ K_0(\mu_1)) = K_0(\mu_1)$$

so by Lemma 4.5.4, there is a $n_2 > 1$, and a unital positive group homomorphism $\beta_1 : K_0\left(B_{m_1}\right) \to K_0\left(A_{n_2}\right)$ such that one has the following commutative diagram:

$$\begin{array}{ccccc} K_0(A_1) & \xrightarrow{K_0(\varphi_{12})} & K_0\left(A_{n_2}\right) & \xrightarrow{K_0\left(\mu_{n_2}\right)} & K_0(A) \\ & \searrow^{\alpha_1} & \uparrow{\scriptstyle \beta_1} & \nearrow_{f^{-1} \circ K_0\left(\nu_{m_1}\right)} & \\ & & K_0\left(B_{m_1}\right) & & \end{array}$$

We play the same game to obtain a $m_2 > m_1$ and a unital positive group homomorphism $\alpha_2 : K_0\left(A_{n_2}\right) \to K_0\left(B_{m_2}\right)$ such that one has a commutative diagram:

$$\begin{array}{ccccc} K_0\left(B_{m_1}\right) & \xrightarrow{K_0\left(\varphi_{m_1 m_2}\right)} & K_0\left(B_{m_2}\right) & \xrightarrow{K_0\left(\mu_{m_2}\right)} & K_0(B) \\ & \searrow^{\beta_1} & \uparrow{\scriptstyle \alpha_2} & \nearrow_{f \circ K_0\left(\mu_{n_2}\right)} & \\ & & K_0\left(A_{n_2}\right) & & \end{array}$$

Hence continuing this process inductively, we obtain strictly increasing sequences of natural numbers $(n_k)_{k\in\mathbb{N}}$ (where $n_1 = 1$) and $(m_k)_{k\in\mathbb{N}}$ such that one has a commutative diagram

$$\begin{array}{ccccccc} & K_0\left(A_{n_k}\right) & \longrightarrow & K_0\left(A_{n_{k+1}}\right) & \to \cdots \to & K_0(A) \\ {\scriptstyle \beta_{k-1}} \nearrow & & \searrow {\scriptstyle \alpha_k} \quad {\scriptstyle \beta_k} \nearrow & & \searrow {\scriptstyle \alpha_{k+1}} & f \downarrow \uparrow f^{-1} \\ K_0\left(B_{m_{k-1}}\right) & \longrightarrow & K_0\left(B_{m_k}\right) & \longrightarrow & \cdots \to & K_0(B) \end{array}$$

for each $k \in \mathbb{N}$. By Lemma 5.4.6, A and B are inductive limits of the diagrams $\left(A_{n_k}, \left(\varphi_{n_k n_{k+1}}\right)\right)$ and $\left(B_{m_k}, \left(\psi_{m_k m_{k+1}}\right)\right)$ indexed by $k \in \mathbb{N}$, then we may assume that $n_k = m_k = k$ for each $k \in \mathbb{N}$. By Lemma 4.5.3 (i), we can find unital *-homomorphisms $\varepsilon_k' : A_k \to B_k$ and $\eta_{k-1}' : B_{k-1} \to A_k$ such that $K_0(\varepsilon_k') = \alpha_k$ and $K_0(\eta_{k-1}') = \beta_{k-1}$ for each $k \in \mathbb{N}$. As

$$K_0(\varphi_k) = \beta_k \circ \alpha_k = K_0(\eta_k' \circ \varepsilon_k')$$
$$K_0(\psi_k) = \alpha_{k+1} \circ \beta_k = K_0(\varepsilon_{k+1}' \circ \eta_k')$$

where $\varphi_k = \varphi_{k,k+1}$ and $\psi_k = \psi_{k,k+1}$. Note that $\eta_0{}' : B_0 \to A_1$, so $\eta_0{}' = \varphi'$, and one has $\psi_0 = \varepsilon_1' \circ \eta_0'$, and choose $\varepsilon_1 = \varepsilon_1'$. As $K_0(\varphi_1) = K_0(\eta_1' \circ \varepsilon_1')$, then by Lemma 4.5.3 (ii), there is a $u_1 \in \mathcal{U}(A_2)$ such that

$$\varphi_1 = \mathrm{ad} u_1 \circ \eta_1' \circ \varepsilon_1'$$

and hence choose $\eta_1 = \mathrm{ad} u_1 \circ \eta_1'$, so one has $\varphi_1 = \eta_1 \circ \varepsilon_1$. As $K_0(\eta_1') = K_0(\eta_1)$, so $K_0(\psi_1) = K_0(\varepsilon_2' \circ \eta_1)$, then by Lemma 4.5.3 (ii), there is a $v_2 \in \mathcal{U}(B_2)$ such that

$$\psi_1 = \mathrm{ad} v_2 \circ \varepsilon_2' \circ \eta_1$$

then choose $\varepsilon_2 = \mathrm{ad} v_2 \circ \varepsilon_2'$. Continuing this applying Lemma 4.5.3 (ii) inductively, there are $v_k \in \mathcal{U}(B_k)$ (with $v_1 = 1_{B_1}$), and $u_k \in \mathcal{U}(A_{k+1})$ such that

$$\text{given} \quad \eta_k = \mathrm{ad} u_k \circ \eta_k' \quad \text{and} \quad \varepsilon_k = \mathrm{ad} v_k \circ \varepsilon_k',$$
$$\text{one has} \quad \varphi_k = \eta_k \circ \varepsilon_k \quad \text{and} \quad \psi_k = \varepsilon_{k+1} \circ \eta_k.$$

So one has a commutative diagram

$$\begin{array}{ccccccccc} & & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A \\ & \eta_0 \nearrow & & \searrow \varepsilon_1 & \eta_1 \nearrow & \searrow \varepsilon_2 & & & \varphi \downarrow \uparrow \varphi^{-1} \\ B_0 & & \longrightarrow & B_1 & & \longrightarrow & \cdots & \longrightarrow & B \end{array}$$

where the universal *-isomorphism $\varphi : A \to B$ is exists by Elliott's Intertwining Argument 5.4.8, and $K_0(\varepsilon_k) = K_0(\varepsilon'_k) = \alpha_k$. Hence A and B are isomorphic AF-algebras. Note that the two diagrams

$$\begin{array}{ccc} K_0(A_k) & \xrightarrow{K_0(\mu_k)} & K_0(A) \\ {\scriptstyle \alpha_k = K_0(\varepsilon_k)} \downarrow & & \downarrow {\scriptstyle K_0(\varphi)} \\ K_0(B_k) & \xrightarrow{K_0(\nu_k)} & K_0(B) \end{array} \qquad \begin{array}{ccc} K_0(A_k) & \xrightarrow{K_0(\mu_k)} & K_0(A) \\ {\scriptstyle \alpha_k} \downarrow & & \downarrow {\scriptstyle f} \\ K_0(B_k) & \xrightarrow{K_0(\nu_k)} & K_0(B) \end{array}$$

are commutative, so one has $K_0(\varphi)|_{\mathrm{im}(K_0(\mu_k))} = f|_{\mathrm{im}(K_0(\mu_k))}$ for each $k \in \mathbb{N}$. By Inductive Continuity of (K_0, K_0^+) 4.3.6, as

$$K_0(A) = \bigcup_{k\in\mathbb{N}} \mathrm{im}(K_0(\mu_k))$$

then $K_0(\varphi) = f$, as required. ■

By the classification theorem, we have that the functor (K_0, K_0^+) when restricted to the subcategory of unital separable AF-algebras (with morphisms being unital *-homomorphisms), is also a classification functor. We also have a similar proof for the classification of nonunital AF-algebras, which relies on a different, but similar, variant called the dimension range. To be precise, if A is a C^*-algebra, then the **dimension range** of A is the set

$$D(A) = \{[p]_0 : p \in \mathrm{P}(A)\} \subseteq K_0(A)$$

and one can consider the following invariant

$$(K_0(A), D(A))$$

of A called **scaled (pre)ordered (Abelian) groups**.[1] A homomorphism f between scaled (pre)ordered groups (G, G^+, D_G) and (H, H^+, D_H) is a group homomorphism that satisfies $\alpha(G^+) \subseteq H^+$ and $\alpha(D_G) \subseteq D_H$.

Theorem 4.5.6. Let A and B be AF-algebras. If there is an isomorphism $\alpha : K_0(A) \to K_0(B)$ of scaled ordered groups, i.e. α is a group isomorphism such that $\alpha(K_0(A)^+) = K_0(B)^+$ and $\alpha(D(A)) = D(B)$, then there is a *-isomorphism $\varphi : A \to B$ such that $K_0(\varphi) = \alpha$.

Proof. See [5, Theorem 7.3.2]. ■

It turns out Elliott's Theorem 4.5.5 does not hold with nonseparable AF-algebras; see [10]. One of the immediate applications of Elliott's classification theorem is allowing us to easily classify **UHF-algebras**, which are AF-algebras that are countable inductive limits of simple C^*-algebras in FinAlg_1.

Definition 4.5.7. A **uniformly hyperfinite algebra** (UHF) A is an inductive limit of sequences of the form

$$\mathcal{M}_{k_1}(\mathbb{C}) \xrightarrow{\varphi_1} \mathcal{M}_{k_2}(\mathbb{C}) \xrightarrow{\varphi_2} \mathcal{M}_{k_3}(\mathbb{C}) \longrightarrow \cdots \tag{4.4}$$

where $k_1, k_2, \ldots \in \mathbb{N}$ and $\varphi_1, \varphi_2, \ldots$ are unital *-homomorphisms.

[1] The *pre*-prefix in preordered refers to to how the pair $(K_0(A), K_0(A)^+)$ does not actually have a partial order. We we know it will when A is unital and stably finite.

We can associate each UHF-algebra A with a *supernatural number*. Let $\{p_1, p_2, p_3, \dots\}$ be an increasing sequence of all prime numbers, and for each $i, k \in \mathbb{N}$, define $\operatorname{ord}_{p_i}(k)$ to be a number $m \in \mathbb{N}_0$ such that $p_i^m \mid k$ but $p_i^{m+1} \nmid k$. Now given the sequence of numbers $(k_i)_{i\in\mathbb{N}}$ from (4.4), define $n_i = \sup\left\{\operatorname{ord}_{p_i}(k_j) : j \in \mathbb{N}\right\} \in \mathbb{N}_0 \cup \{\infty\}$, and the set $n = \{n_i\}_{i\in\mathbb{N}}$ is called a **supernatural number** associated to the UHF-algebra A. Define $\mathbb{Q}(n) \subseteq \mathbb{Q}$ to be the set of rational numbers $\frac{x}{y}$ such that $x \in \mathbb{Z}$, and $y = \prod_{i\geq 1} p_i^{m_i}$ for $0 \leq m_i < n_i + 1$ where the number i such that $m_i > 0$ is finite, then by [2, Lemma 7.4.4], one has that

$$(K_0(A), [1_A]_0) \cong \mathbb{Q}(n) = \bigcup_{i\geq 1} k_i^{-1}\mathbb{Z}.$$

In particular, one has the following classification theorem.

Theorem 4.5.8. Let A and B be UHF-algebras with associated supernatural numbers n_A and n_B respectively. Then the following are equivalent:

(i) A and B are isomorphic.

(ii) $n_A = n_B$.

(iii) There is an isomorphism $\varphi : K_0(A) \to K_0(A')$ such that $\varphi([1_A]_0) = [1_{A'}]_0$.

(iv) There is a unital order isomorphism $\varphi : (K_0(A), K_0(A)^+, [1_A]_0) \to \left(K_0(A'), K_0(A')^+, [1_{A'}]_0\right)$.

Furthermore, for each supernatural number n, there is a UHF-algebra A whose associated supernatural number n.

For the details of the theorem, refer to [2, Theorem 7.4.5]. One example of the theorem that consider the UHF-algebra A, which is an inductive limit of the sequence:

$$\mathbb{C} \xrightarrow{\varphi_1} \mathcal{M}_2(\mathbb{C}) \xrightarrow{\varphi_2} \mathcal{M}_4(\mathbb{C}) \xrightarrow{\varphi_3} \mathcal{M}_8(\mathbb{C}) \xrightarrow{\varphi_4} \cdots$$

where $\varphi_i : \mathcal{M}_{2^{i-1}}(\mathbb{C}) \to \mathcal{M}_{2^i}(\mathbb{C})$ defined as $\varphi(x) = x \oplus x$. One has that $K_0(A)$ is isomorphic to the group of dyadic rationals. We can also admit a UHF-algebra A such that $K_0(A) \cong \mathbb{Q}$. Consider (4.4) where we choose $k_i = i!$ for each $i \in \mathbb{N}$, so the map $\varphi_i : \mathcal{M}_{k_i}(\mathbb{C}) \to \mathcal{M}_{k_{i+1}}(\mathbb{C})$ defined as $\varphi(x) = \underbrace{x \oplus \cdots \oplus x}_{i+1 \text{ times}}$ is unital. Let n be the associated supernatural number to A, so we know that

$$K_0(A) \cong \mathbb{Q}(n) = \bigcup_{i\geq 1} \frac{1}{i!}\mathbb{Z}.$$

Now given any $y \in \mathbb{N}$, then $1/y = (y-1)!/y! \in \mathbb{Q}(n)$, so it follows that $\mathbb{Q}(n)$ contains every rational number, hence $K_0(A) \cong \mathbb{Q}$. This shows that the K_0-groups for separable C^*-algebras need not be finitely-generated.

This is only the beginning of Elliott's classification program, as much more research was done to classify other classes of C^*-algebras with a much more richer invariant involving K_1-groups. Where we associate each C^*-algebra A with a sextuple $(K_0(A), K_0(A)^+, D(A), K_1(A), T^+(A), \rho_A)$ called the *Elliot invariant* and the question asks if the invariant is sufficient to classify separable amenable C^*-algebras. The details will not be discussed here, see [11] for the recent survey of the program as of 2023.

5 | Appendix: Category Theory

5.1 Category and Functors

We will give a soft introduction towards category theory. The aim is to build up terminologies and tools relevant to this paper, so we only shift our attention towards locally small categories as one might find greater comfort with the axioms that shall be introduced below. Though do note that all results and definitions mentioned will hold in general categories unless stated otherwise, and we shall only state the definitions and results required for our thesis.

Definition 5.1.1. *(Category).* A (locally small) category $\mathscr{A}$ consists a class[1] of **objects** $\mathrm{Obj}(\mathscr{A})$ and a class of **morphisms** $\mathrm{Mor}(\mathscr{A})$. Such that for each $A, B \in \mathrm{Obj}(\mathscr{A})$, one has a set of morphisms $\mathrm{Mor}_{\mathscr{A}}(A, B)$, which satisfies the following:

C1. There is an associative binary operator, called the **composition**,

$$\circ : \mathrm{Mor}_{\mathscr{A}}(B, C) \times \mathrm{Mor}_{\mathscr{A}}(A, B) \to \mathrm{Mor}_{\mathscr{A}}(A, C).$$

That means, for $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$, $g \in \mathrm{Mor}_{\mathscr{A}}(B, C)$, and $h \in \mathrm{Mor}_{\mathscr{A}}(C, D)$, one has

$$(h \circ g) \circ f = h \circ (g \circ f).$$

C2. For each $A \in \mathrm{Obj}(\mathscr{A})$, there is an **identity** $\mathrm{id}_A \in \mathrm{Mor}_{\mathscr{A}}(A, A)$ such that

$$f \circ \mathrm{id}_A = f \quad \text{and} \quad \mathrm{id}_A \circ g = g$$

for all $f \in \mathrm{Mor}_{\mathscr{A}}(B, A)$ and $g \in \mathrm{Mor}_{\mathscr{A}}(A, B)$.

It is common to notate $f : A \to B$ to infer that $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$.

Given two categories $\mathscr{A}$ and $\mathscr{B}$, we say that $\mathscr{A}$ is a **subcategory** of $\mathscr{B}$ if $\mathrm{Obj}(\mathscr{A}) \subseteq \mathrm{Obj}(\mathscr{B})$ and for each $A, B \in \mathrm{Obj}(\mathscr{A})$, one has $\mathrm{Mor}_{\mathscr{A}}(A, B) \subseteq \mathrm{Mor}_{\mathscr{B}}(A, B)$.

Definition 5.1.2. *(Morphisms).* Let $f : A \to B$ be a morphism in some category $\mathscr{A}$. We say f is a **split-monic** (resp. **split-epic**) if f has a **left-inverse** (resp. **right-inverse**), that is, there is a morphism $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ (resp. $f \circ g = \mathrm{id}_B$). Finally, we say that f is an **isomorphism** if f has a left-inverse and a right-inverse, which is readily verified to be the same morphism and is unique.

We say that A is the **domain** of f and B is the **codomain** of f.

Example 5.1.3. We have some examples of categories we are interested in:

- The category of sets, Set, where the objects are sets and the morphisms are functions.
- The category of groups, Grp, where the objects are groups and the morphisms are group homomorphisms.
- The category of Abelian groups, where the objects are Abelian groups and the morphisms are group homomorphisms. As it turns out, Ab is a *full subcategory* of Grp; refer to Definition 5.1.4 below.
- The category of C^*-algebras, C*-Alg, where the objects are C^*-algebras and the morphisms are *-homomorphisms. When discussing about the category of unital C^*-algebras, C*-Alg_1, the morphisms are now unital.

[1]See Wikipedia for definition of class.

Definition 5.1.4. *(Functors)*. Given categories $\mathscr{A}$ and $\mathscr{B}$, a **covariant functor** F is a well-defined mapping from the objects and morphisms of $\mathscr{A}$ into the objects and morphisms of $\mathscr{B}$ such that the following are satisfied:
F1. $F(A) \in \mathrm{Obj}(\mathscr{B})$ for all $A \in \mathrm{Obj}(\mathscr{A})$.
F2. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for each $A \in \mathrm{Obj}(\mathscr{A})$.
F3. For each $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$, one has $F(f) \in \mathrm{Mor}_{\mathscr{B}}(F(A), F(B))$.
F4. $F(g \circ f) = F(g) \circ F(f)$ for $f : A \to B$ and $g : B \to C$ in $\mathscr{A}$.
We write $F : \mathscr{A} \to \mathscr{B}$ to infer that F is a functor.

We say F is **faithful** (resp. **full**) if F is an injection (resp. surjection) from $\mathrm{Mor}_{\mathscr{A}}(A, B)$ into $\mathrm{Mor}_{\mathscr{B}}(F(A), F(B))$ for each $A, B \in \mathrm{Obj}(\mathscr{A})$. If F is faithful and full, we say that F is **fully faithful** and that $\mathscr{A}$ is a **full subcategory** of $\mathscr{B}$.

Example 5.1.5. *(The Identity Functor)*. Every category $\mathscr{A}$ has a covariant functor from $\mathscr{A}$ to itself, namely the **identity functor** $\mathrm{id}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ which does the following:
- $\mathrm{id}_{\mathscr{A}}(A) = A$ for all $A \in \mathrm{Obj}(\mathscr{A})$.
- $\mathrm{id}_{\mathscr{A}}(f) = f$ for all $f \in \mathrm{Mor}(\mathscr{A})$.

It is clear that $\mathrm{id}_{\mathscr{A}}$ is a fully faithful functor.

Definition 5.1.6. Given a category $\mathscr{A}$, we say an object A of $\mathscr{A}$ is **initial** (resp. **final**) if for all $B \in \mathrm{Obj}(\mathscr{A})$, the set $\mathrm{Mor}(A, B)$ (resp. $\mathrm{Mor}(B, A)$) only has 1 element. We say A is a **zero object** if A is both initial and final. Which we denote A as 0.
Now note that give objects A and B, one can construct a unique **zero morphism** $0 : A \to B$ by considering the composition $A \longrightarrow 0 \longrightarrow B$.

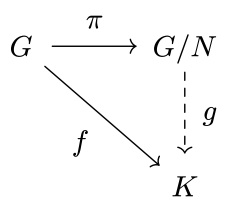
Proposition 5.1.7. Given a category $\mathscr{A}$, then if $A \in \mathrm{Obj}(\mathscr{A})$ is initial (resp. final), then all initial (resp. final) objects in $\mathscr{A}$ are isomorphic to A

Proof. Note that this proof works regardless if A is initial or final. Let $B \in \mathrm{Obj}(\mathscr{A})$ be another initial object, then one obtains $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, A)$. Thus $f \circ g \in \mathrm{Mor}(B, B) = \{\mathrm{id}_B\}$ and $g \circ f \in \mathrm{Mor}(A, A) = \{\mathrm{id}_A\}$, thus f and g are isomorphisms, as required. ■

5.2 Universal Constructions

We begin with an example. Let N be a normal subgroup of a group G, then one has a canonical map $\pi : G \to G/N$. Then one has the following property regarding this construction:

For any homomorphism $f : G \to K$ to another group K, if $N \subseteq \ker(f)$, then there is a unique homomorphism $g : G/N \to K$ such that $f = g \circ \pi$, i.e. one has the following commutative diagram:



We say such a pair $(G/N, \pi)$ is a *universal construction*. In particular, consider the category $\mathscr{A}$ where the objects are all homomorphisms f with domain G such that $N \subseteq \ker(f)$, and given $f : G \to H$ and $g : G \to K$ being objects in $\mathscr{A}$, then a morphism $p \in \mathrm{Mor}_{\mathscr{A}}(f, g)$ is a homomorphism from H to K such that $g = p \circ f$. We shall observe that $\mathscr{A}$ forms a category with composition in a *natural way*; see Construction 5.2.1 where $F = \mathrm{id}_{\mathsf{Grp}}$.

Observe that π is an initial object in $\mathscr{A}$. Thus by Proposition 5.1.7, if there is another group Q and a homomorphism $\tau : G \to Q$ (with $N \subseteq \ker(\tau)$) such that τ is initial in $\mathscr{A}$, then there are morphisms $f \in \mathrm{Mor}_{\mathscr{A}}(G/N, Q)$ and $g \in \mathrm{Mor}_{\mathscr{A}}(Q, G/N)$ such that $f \circ g = \mathrm{id}_\tau = \mathrm{id}_Q$ and $g \circ f = \mathrm{id}_\pi = \mathrm{id}_{G/N}$, and $\pi = g \circ \tau$ and $\tau = f \circ \pi$. Observe that f and g are isomorphisms, so G/N and Q are isomoprhic groups, and π and τ are related by an isomorphism.

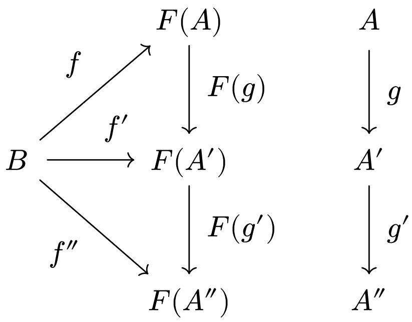
This inspires us with the following construction and definition.

Construction 5.2.1. *(Comma Categories)*. Let F be a covariant functor between categories $\mathscr{A}$ and $\mathscr{B}$, and $B \in \mathrm{Obj}(\mathscr{B})$. Define the **comma category** $B \downarrow F$ as follows:

- The objects of $B \downarrow F$ are morphisms of the form $B \to F(A)$ in $\mathscr{B}$ where $A \in \mathrm{Obj}(\mathscr{A})$.
- Given objects $f : B \to F(A)$ and $f' : B \to F(A')$ where $A, A' \in \mathrm{Obj}(\mathscr{A})$, a morphism $g : f \to f'$ in $B \downarrow F$ is a morphism $g \in \mathrm{Mor}_{\mathscr{A}}(A, A')$ such that $f' = F(g) \circ f$.

The composition in $B \downarrow F$ is given as follows:

Suppose $f : B \to F(A)$, $f' : B \to F(A')$, $f'' : B \to F(A'')$ are morphisms in $\mathscr{B}$ where $A, A', A'' \in \mathrm{Obj}(\mathscr{A})$ with morphisms $g : A \to A'$ and $g' : A' \to A''$ in $\mathscr{A}$ such that $f'' = g' \circ f'$ and $f' = g \circ f$. Then define the composition in $B \downarrow F$ as the composition $g' \circ g : A \to A''$ given in $\mathscr{A}$, so $f'' = (g' \circ g) \circ f$. Thus one has the following commutative diagram:



Hence one has a category $B \downarrow F$.

Similarly, one can define the comma category $F \downarrow B$ as follows:

- The objects of $F \downarrow B$ are morphisms of the form $F(A) \to B$ in $\mathscr{B}$ where $A \in \mathrm{Obj}(\mathscr{A})$.
- Given objects $f : F(A) \to B$ and $f' : F(A') \to B$ where $A, A' \in \mathrm{Obj}(\mathscr{A})$, a morphism $g : f \to f'$ in $F \downarrow B$ is a morphism $g \in \mathrm{Mor}_{\mathscr{A}}(A, A')$ such that $f' = f \circ g$.

The composition is given a natural way similarly as above.

Definition 5.2.2. *(Universal Constructions)*. Let F be a covariant functor between categories $\mathscr{A}$ and $\mathscr{B}$. Given an object $A \in \mathrm{Obj}(\mathscr{A})$ and $B \in \mathrm{Obj}(\mathscr{B})$, and a morphism $f : B \to F(A)$ in $\mathscr{B}$. We say the pair $(F(A), f)$ is a **universal construction** if f is an initial or final object in a subcategory $\mathscr{D}$ of the comma category $B \downarrow F$. Assuming the pair is initial, then the pair $(F(A), f)$ has the following **universal property**:

Let $A' \in \mathrm{Obj}(\mathscr{A})$ and suppose $f' : B \to F(A')$ is a morphism in $\mathscr{B}$ and an object in $\mathscr{D}$, then there is a unique moprhism $g : A \to A'$ in $\mathscr{A}$ such that $f' = F(g) \circ f$. We say g and $F(g)$ to be **universal**.

In general, a **universal construction** would be a suitable pair $(F(A), f)$ that is an initial or final object in a subcategory of any of the comma categories.

Note that by Proposition 5.1.7, universal constructions are unique up to isomorphism, that is for example if $(F(A), f)$ and $(F(A'), f')$ are both initial objects in $B \downarrow F$, then there is an isomorphism $g : A \to A'$ such that $f' = F(g) \circ f$. So we say $(F(A), f)$ and $(F(A'), f')$ are **isomorphic**, and note that it is insufficient to have that an isomorphism g from A to A' to conclude that A and A' are isomorphic as universal constructions, as we also require that $f' = F(g) \circ f$. Hence $(F(A), f)$ and $(F(A'), f')$ are isomorphic if the associated universal morphism is an isomorphism.

The product (e.g. product of groups) is perhaps one of the most common universal constructions. To realize the product in the categorical setting, we first need to consider the following category. Let $\mathscr{A}$ be a category and I be a nonempty set, define the category $\prod_I \mathscr{A}$ as follows:

- Let the objects of $\prod_I \mathscr{A}$ be tuples of the form $(A_i)_{i\in I}$ such that $A_i \in \mathrm{Obj}(\mathscr{A})$ for each $i \in I$.
- A morphism f in $\prod_I \mathscr{A}$ from $(A_i)_{i\in I}$ to $(B_i)_{i\in I}$ is a collection of moprhisms $(f_i)_{i\in I}$ such that $f_i \in \mathrm{Mor}_{\mathscr{A}}(A_i, B_i)$ for each $i \in I$.
- Given morphisms $f = (f_i)_{i\in I}$ and $g = (g_i)_{i\in I}$ such that the codomain of f is the domain of g, define $g \circ f$ as the morphism $(g_i \circ f_i)_{i\in I}$.

The category $\prod_I \mathscr{A}$ is called a **product category**. We also have a **diagonal functor** $\Delta_{\mathscr{A}} : \mathscr{A} \to \prod_I \mathscr{A}$ which maps objects A to the tuple $(A)_{i\in I}$ and morphisms $f : A \to B$ to $(f)_{i\in I}$. Note that $\Delta_{\mathscr{A}}$ defines a faithful covariant functor.

Definition 5.2.3. *(Categorical Product)*. Given a nonempty set I and a category $\mathscr{A}$, and suppose $(A_i)_{i\in I}$ is a collection of objects in $\mathscr{A}$. Then given a object $P \in \mathscr{A}$ and morphisms $\pi_i : P \to A_i$ in $\mathscr{A}$, we say the pair $(P, (\pi_i)_{i\in I})$ is the **(categorical) product** of $(A_i)_{i\in I}$ if $(\pi_i)_{i\in I}$ is final in the category $\Delta_{\mathscr{A}} \downarrow (A_i)_{i\in I}$. That means, given $P' \in \mathrm{Obj}(\mathscr{A})$ and morphisms $f_i : P' \to A_i$, so $(f_i)_{i\in I}$ is an object in $\prod_I \mathscr{A}$, there is a unique morphism $g : P' \to P$ such that $(f_i)_{i\in I} = (\pi_i)_{i\in I} \circ \Delta_{\mathscr{A}}(g)$. That is, $f_i = \pi_i \circ g$ for all $i \in I$. We typically denote the object of the categorical product as $\prod_{i\in I} A_i$. We say the category $\mathscr{A}$ **have categorical products** if a categorical product exists for any collection $(A_i)_{i\in I}$ for any index set I.

Example 5.2.4. *(Product of Groups)*. In the category of groups, Grp, it has a categorical products. Indeed, let I be any index set and $(G_i)_{i\in I}$, and define the usual product group:

$$\prod_{i\in I} G_i = \left\{ f : I \to \bigcup_{i\in I} G_i : f(i) \in G_i \text{ for all } i \in I \right\},$$

and define *canonical projection maps* $\pi_i : G \to G_i$ as $\pi_i(f) = f(i)$, which are homomorphisms for each $i \in I$. Thus for any group P such that one has homomorphisms $f_i : P \to G_i$ for each $i \in I$. Then one can define the homomorphism

$$g : P \to \prod_{i\in I} G_i : p \mapsto (i \mapsto f_i(p))$$

which satisfies $f_i = \pi_i \circ g$ for each $i \in I$. So indeed $\prod_{i\in I} G_i$ is the categorical product.

5.3 Categorical Limits

Categorical limits are one of the most important universal constructions in category theory, which even generalizes the categorical product. However, we shall be interested in *inductive limits* in categories as that is the only main categorical concept mentioned in our classification theorem. Some other definitions are introduced for the sake of completeness.

Let I be a small category, that is, $\mathrm{Obj}(I)$ is a set, and given a category $\mathscr{A}$, we define the **functor category** $\mathscr{A}^I$ as follows:

- The objects of $\mathscr{A}^I$ are covariant functors from I to $\mathscr{A}$.

- The morphisms between objects $F, G : I \to \mathscr{A}$ are natural transformations $\eta : F \to G$, that is, for each $i \in I$, there is a morphism $\eta(i) : F(i) \to F(j)$, such that for each morphism $m : i \to j$ in I, one has $\eta(j) \circ F(m) = G(m) \circ \eta(i)$, i.e. one has the commutative diagram:

$$\begin{array}{ccc} F(i) & \xrightarrow{F(m)} & F(j) \\ {\scriptstyle \eta(i)}\big\downarrow & & \big\downarrow{\scriptstyle \eta(j)} \\ G(i) & \xrightarrow{G(m)} & G(j) \end{array}$$

- Given two natural transformations $\eta : F \to G$ and $\varepsilon : G \to H$, we define the composition $\varepsilon \circ \eta$ such that $(\varepsilon \circ \eta)(i) = \varepsilon(i) \circ \eta(i)$ for each $i \in I$, which is indeed a natural transformation from F to H.

In this case, we say I is an **index category** as we only care about the objects and the morphisms between them rather then their intrinsic properties. Hence if I is a small index category, then I admits a natural preordering $\leq$ (transitive and reflexive) such that $i \leq j$ if, and only if, there is a morphism $i \to j$, and we shall realize this as the canonical ordering on I. Hence we realize small index categories are just preordered sets.

Note that any partially ordered set $(I, \leq)$ can be realized as a small index category in a natural way:

- Let the objects of I to be the elements of I.
- Given $i, j \in I$, we have a morphism $i \to j$ if $i \leq j$.
- Given morphisms $i \to j$ and $j \to k$, we have a morphism $i \to k$ by transitivity of $\leq$.

In particular, we are interested in a type of ordering called *directed ordering*, that is:

- $\leq$ is a partial ordering on I.
- For each $i, j \in I$, there is a $k \in I$ such that $i \leq k$ and $j \leq k$.

Given any small index category I and a covariant functor $F : I \to \mathscr{A}$, we say the collection $A^\bullet = \big(A_i, (f_{ij})_{i,j \in I}\big)$ is a **diagram indexed by F** if $A_i = F(i)$ for each $i \in I$, and given $i, j \in I$ such that $i \leq j$, then $f_{ij} = F(i \to j)$.[1] In general, we say **$A^\bullet$ is a diagram indexed by I in $\mathscr{A}$**. We say the morphisms f_{ij} are **connecting morphisms** (or **connecting maps**) of $A^\bullet$.

If one has an object $A \in \mathscr{A}$ and morphisms $\varphi_i : A \to A_i$ for each $i \in I$ such that for all $i, j \in I$ with $i \to j$, then $\varphi_j = f_{ij} \circ \varphi_i$, i.e. one has the commutative diagram:[2]

$$\begin{array}{ccc} A & \xrightarrow{\varphi_i} & A_i \\ & {\scriptstyle \varphi_j}\searrow & \big\downarrow{\scriptstyle f_{ij}} \\ & & A_j \end{array}$$

Then we say the collection $(A, (\varphi_i)_{i \in I})$ is a **cone** of $A^\bullet$. Note that $(\varphi_i)_{i \in I}$ defines a natural transformation from the constant functor c_A to F since we have a commutative diagram:[3]

[1]We write $i \to j$ to mean a morphism $m : i \to j$ in I. However, if there are two distinct morphisms $m, m' : i \to j$, then there will be two distinct corresponding f_{ij}'s, we shall make this distinction if necessary.

[2]To be more specific, if $m : i \to j$ is a morphism in I, then $\varphi_j = F(m) \circ \varphi_i$.

[3]That is for each $i, j \in I$, one has $c_A(i) = A$ and $c_A(i \to j) = \mathrm{id}_A$.

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle\varphi_i}\big\downarrow & & \big\downarrow{\scriptstyle\varphi_j} \\ A_i & \xrightarrow{f_{ij}} & A_j \end{array}$$

If one has morphisms $\varphi_i : A_i \to A$ for each $i \in I$ instead, such that for all $i, j \in I$ with $i \to j$, then $\varphi_i = \varphi_j \circ f_{ij}$, i.e. one has the commutative diagram:

$$\begin{array}{ccc} A & \xleftarrow{\varphi_i} & A_i \\ & {\scriptstyle\varphi_j}\nwarrow & \big\downarrow{\scriptstyle f_{ij}} \\ & & A_j \end{array}$$

Then we say the collection $(A, (\varphi_i)_{i\in I})$ is a **cocone** of $A^\bullet$, and hence $(\varphi_i)_{i\in I}$ defines a natural transformation from F to c_A.

Suppose $(A, (\varphi_i)_{i\in I})$ is a cone (resp. cocone), then we say the pair is **universal** if $(\varphi_i)_{i\in I}$ it is final (resp. initial) in the category $\Delta_{\mathscr{A}} \downarrow F$ (resp. $F \downarrow \Delta_{\mathscr{A}}$) where $\Delta_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^I$ is the **diagonal functor** which for each $A \in \mathrm{Obj}(\mathscr{A})$, one has $\Delta_{\mathscr{A}}(A) = c_A$. So if $(A, (\varphi_i)_{i\in I})$ is a universal cone, then for each cone $(B, (\psi_i)_{i\in I})$ of $A^\bullet$, there is a unique morphism $\mu : B \to A$ such that $\psi_i = \varphi_i \circ \mu$. Diagrammatically speaking:

$$\begin{array}{ccccc} & & \xrightarrow{\psi_i} & & \\ B & \overset{\exists!\mu}{\dashrightarrow} & A & \xrightarrow{\varphi_i} & A_i \\ & & & {\scriptstyle\varphi_j}\searrow & \big\downarrow{\scriptstyle f_{ij}} \\ & & \xrightarrow{\psi_j} & & A_j \end{array}$$

Similarly, if $(A, \varphi_i)_{i\in I}$ is a universal cocone of $A^\bullet$, then for any cocone $(B, (\psi_i)_{i\in I})$. there is a unique morphism $\mu : A \to B$ such that $\psi_i = \mu \circ \varphi_i$. We say the morphisms φ_i are **boundary morphisms** (or **boundary maps**) of $A^\bullet$.

Definition 5.3.1. *(Categorical Limits).* Given a small index category I, and a category $\mathscr{A}$. Let $A^\bullet$ be a diagram indexed by I in $\mathscr{A}$, then we say a cone (resp. cocone) $(A, (\varphi_i)_{i\in I})$ is a **limit** (resp. **colimit**) of A if $(A, (\varphi_i)_{i\in I})$ is universal, and we write $(A, (\varphi_i)_{i\in I}) \cong \varprojlim A^\bullet$ (resp. $(A, (\varphi_i)_{i\in I}) \cong \varinjlim A^\bullet$). We say the limit is **inductive** if I is directed and $(A, (\varphi_i)_{i\in I})$ is a colimit.

Note that $\cong$ is used instead of $=$, since limits are universal constructions and thus they are all isomorphic to each other, we shall use $=$ if there is a canonical construction of the limits.

Observe that the categorical product of two objects is just the limit of the diagram:

$$\bullet \qquad \bullet$$

Hence categorical products are just limits of small index categories I where the only morphisms in I are identity morphisms. Note that we are particularly interested in inductive limits in this paper, thus the other definitions need not matter too much.

Definition 5.3.2. *(Continuity of Functors)*. Given a covariant functor $F : \mathscr{A} \to \mathscr{B}$ between categories $\mathscr{A}$ to $\mathscr{B}$, and an index category I. We say F is $\boldsymbol{I}$**-continuous** (resp. $\boldsymbol{I}$**-cocountinuous**) if for all diagrams $A^\bullet$ indexed by I such that the limit (resp. colimit) of $A^\bullet$ exists, then one has $F\Big(\varprojlim A^\bullet\Big) \cong \varprojlim F(A^\bullet)$ (resp. $F\Big(\varinjlim A^\bullet\Big) \cong \varinjlim F(A^\bullet)$) where $F(A^\bullet)$ is a diagram in $\mathscr{B}$ indexed by the composition of I and F. We say F is **inductively continuous** if F is I-cocontinuous for every directed set I.

Definition 5.3.3. *(Complete Categories)*. Given an index category I, we say a category $\mathscr{A}$ is $\boldsymbol{I}$**-complete** (resp. $\boldsymbol{I}$**-cocomplete**) if all diagrams in $\mathscr{A}$ indexed by I has a limit (resp. colimit). We say A is:

- **Inductive complete** if $\mathscr{A}$ is $\boldsymbol{I}$**-cocomplete** for all directed sets I.
- **Small complete** (resp. **small cocompelete**) if $\mathscr{A}$ is I-complete (resp. I-cocomplete) for all small index category I.

5.4 Diagram Chasing Lemmas

The purpose of this chapter is to produce general diagram-related lemmas that will be used in this paper. As we are working with groups and C^*-algebras, we shall give a definition that encapsulates the properties of those categories which is sufficient for our lemmas.

Definition 5.4.1. *(Algebraic Categories)*. We say a category $\mathscr{A}$ is **algebraic** if the following condition holds:

AC1. $\mathscr{A}$ has a zero object, which we denote as 0.

AC2. For each $A \in \mathrm{Obj}(\mathscr{A})$, A is a set equipped with a binary operation $+ : A \times A \to A$ that satisfies the group axioms.

AC3. For each $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$, f is a map from A to B that preserves the $+$, i.e. $f(x + y) = f(x) + f(y)$ for each $x, y \in A$.

AC4. For each $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$, the sets $\ker(f) := \{x \in A : f(x) = 0\}$ and $\mathrm{im}(f) := \{f(x) : x \in A\}$ are in $\mathrm{Obj}(\mathscr{A})$.

AC5. If a map $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ is bijective, then it is an isomorphism.

AC6. For each $f \in \mathrm{Mor}_{\mathscr{A}}(A, B)$, the set $A/\ker(f) := \{x + \ker(f) : x \in A\}$ is in $\mathscr{A}$, where $x + \ker(f) := \{x + y : y \in \ker(f)\}$ for each $x \in A$. Furthermore, there is a map $\pi \in \mathrm{Mor}_{\mathscr{A}}(A, A/\ker(f))$ defined as $\pi(x) = x + \ker(f)$ such that for each $g \in \mathrm{Mor}_{\mathscr{A}}(A, B)$ with $\ker(f) \subseteq \ker(g)$, there is a unique $h \in \mathrm{Mor}_{\mathscr{A}}(A/\ker(f), B)$ such that $g = h \circ \pi$.

To elaborate on **AC2**, we mean that $+$ is associative; there is a $0 \in A$ such that $x + 0 = 0 + x = x$ for all $x \in A$; for each $x \in A$, there is a $y \in A$ such that $x + y = y + x = 0$.

It is clear that the categories, such as Grp, and Ab are algebraic by our definition. Fortunately, the category of C^*-algebras, both C*-Alg and C*-Alg$_1$, are also algebraic; see Chapter 2.1. It should be no surprise that a map $f : A \to B$ in an algebraic category is injective if, and only if, $\ker(f) = 0$, and f is surjective if, and only if, $\mathrm{im}(f) = B$.

Definition 5.4.2. *(Exact Sequences)*. Let $\mathscr{A}$ be an algebraic category. Let $A, B, C \in \mathscr{A}$, and $f : A \to B$, and $g : B \to C$ be morphisms. We denote this relation as a **sequence**:

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C$$

to which we say this sequence is **exact at B** if $\mathrm{im}(f) = \ker(g)$. In particular, we say the sequence

$$0 \longrightarrow A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \longrightarrow 0 \tag{5.1}$$

is **exact** if it is exact at A, B, *and* C. We say such sequences are exact sequences. We shall use the arrow $\hookrightarrow$ if the corresponding morphism is an injection, and $\twoheadrightarrow$ if the corresponding morphism is a surjection. We say (5.1) is **split-exact** if there is a $h : C \to B$ that is the right-inverse of g such that one has the sequence:

$$0 \longrightarrow A \overset{f}{\hookrightarrow} B \underset{h}{\overset{g}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

We shall also use the arrow $=$ between two of the same objects A to mean the underlying morphism is id_A.

1⟩ Five Lemma

From here on out, we assume that $\mathscr{A}$ is always an algebraic category.

Lemma 5.4.3. Let A, B, C, D, A', B', C', D', be objects in $\mathscr{A}$, and suppose one has a commutative diagram:

$$\begin{array}{ccccccc}
A & \xrightarrow{\;f\;} & B & \xrightarrow{\;g\;} & C & \xrightarrow{\;h\;} & D \\
{\scriptstyle l}\downarrow & & {\scriptstyle m}\downarrow & & {\scriptstyle n}\downarrow & & {\scriptstyle o}\downarrow \\
A' & \xrightarrow{\;i\;} & B' & \xrightarrow{\;j\;} & C' & \xrightarrow{\;k\;} & D'
\end{array}$$

If the rows are exact, m and o are injective, and l is surjective, then n is injective.

Proof. Let $c \in \ker(n)$, we shall argue that $c = 0$ to conclude the proof.

- As $(o \circ h)(c) = (k \circ n)(c) = k(0) = 0$, then $h(c) \in \ker(o)$.
- As $\ker(o) = 0$, then $h(c) = 0$, so $c \in \ker(h)$.
- As $\mathrm{im}(g) = \ker(h)$, there is a $b \in B$ such that $g(b) = c$.
- As $(j \circ m)(b) = (n \circ g)(b) = n(c) = 0$, so $m(b) \in \ker(j)$.
- As $\mathrm{im}(i) = \ker(j)$, there is a $a' \in A'$ such that $i(a') = m(b)$.
- As $\mathrm{im}(l) = A'$, there is a $a \in A$ such that $l(a) = a'$.
- As $(m \circ f)(a) = (i \circ l)(a) = i(a') = m(b)$ and $\ker(m) = 0$, then $f(w) = b$.
- As $\mathrm{im}(f) = \ker(g)$, then $b \in \ker(g)$, so $c = g(b) = 0$, as required. ■

Lemma 5.4.4. Let A, B, C, D, A', B', C', D', be objects in $\mathscr{A}$, and suppose one has a commutative diagram:

$$\begin{array}{ccccccc}
A & \xrightarrow{\;f\;} & B & \xrightarrow{\;g\;} & C & \xrightarrow{\;h\;} & D \\
{\scriptstyle l}\downarrow & & {\scriptstyle m}\downarrow & & {\scriptstyle n}\downarrow & & {\scriptstyle o}\downarrow \\
A' & \xrightarrow{\;i\;} & B' & \xrightarrow{\;j\;} & C' & \xrightarrow{\;k\;} & D'
\end{array}$$

If the rows are exact, l and n are surjective, and o is injective, then m is surjective.

Proof. Let $b' \in B'$, we shall argue that $b' \in \mathrm{im}(m)$ to conclude the proof.

- As $\mathrm{im}(j) = \ker(k)$, then $j(b') \in \ker(k)$.
- As $\mathrm{im}(n) = C'$, then there is a $c \in C$ such that $n(c) = j(b')$.

- As $(o \circ h)(c) = (k \circ n)(c) = k(j(b')) = 0$, then $h(c) \in \ker(o)$.
- As $\ker(o) = 0$, then $h(c) = 0$, so $c \in \ker(h)$.
- As $\operatorname{im}(g) = \ker(h)$, then there is a $b \in B$ such that $g(b) = c$.
- As $(j \circ m)(b) = (n \circ g)(b) = n(c) = j(b')$, hence $j(b' - m(b)) = 0$, so $b' - m(b) \in \ker(j)$.
- As $\operatorname{im}(i) = \ker(j)$, there is a $a' \in A'$ such that $i(a') = b' - m(b)$.
- As $\operatorname{im}(l) = A'$, there is a $a \in A$ such that $l(a) = a'$.
- As and $(m \circ f)(a) = (i \circ l)(a) = i(a') = b' - m(b)$, one has
$$m(f(a) + b) = b' - m(b) + m(b) = b',$$
as required. ■

Lemma 5.4.5. *(Five)*. Let A, B, C, D, E, A', B', C', D', E' be objects in $\mathscr{A}$, and suppose one has a commutative diagram:

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle o} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\ A' & \xrightarrow{j} & B' & \xrightarrow{k} & C' & \xrightarrow{l} & D' & \xrightarrow{m} & E' \end{array}$$

If the rows are exact, o and q are isomorphisms, n is surjective, r is injective, then p is an isomorphism.

Proof. By viewing p as n in Lemma 5.4.3, we see that p is injective. By viewing p as m in Lemma 5.4.4, we see that p is an surjective. So p is bijective, hence an isomorphism. ■

2⟩ Lemmas on Inductive Limits

We say a directed set J is **cofinal** in a directed set I if there is a mapping $f : J \to I$ such that

(i) For all $x, y \in J$, if $x \leq y$ in J, then $f(x) \leq f(y)$ in I.

(ii) For each $y \in I$, there is a $x \in J$ such that $f(x) \geq y$ in I.

We say such a map f to be a **cofinal map**. Given a diagram $A^\bullet = \left(A_\alpha, (\varphi_{\alpha\beta})\right)$ indexed by I in a category $\mathscr{A}$. Denote $A^{f(\bullet)}$ to a diagram $\left(A_{f(\alpha)}, \left(\varphi_{f(\alpha)f(\beta)}\right)\right)$ be a diagram in $\mathscr{A}$ indexed by J.

Lemma 5.4.6. Let $\mathscr{A}$ be an inductively complete category. Let I be a directed set, and $A^\bullet = \left(A_\alpha, (\varphi_{\alpha\beta})\right)$ be a diagram in $\mathscr{A}$ indexed by I with inductive limit $(A, (\mu_\alpha)_{\alpha\in I})$. If $f : J \to I$ be a cofinal map, then A can be recognized as an inductive limit of $A^{f(\bullet)}$. That is, $\left(A, \left(\mu_{f(\alpha)}\right)_{\alpha\in J}\right)$ is a universal cocone of $A^{f(\bullet)}$.

Proof. Let $\left(A_f, \left(\nu_{f(\alpha)}\right)_{\alpha\in J}\right)$ be an inductive limit of $A^{f(\bullet)}$, as $\left(A, \left(\mu_{f(\alpha)}\right)_{\alpha\in J}\right)$ is a cocone of $A^{f(\bullet)}$, then there is a morphism $\Phi : A_f \to A$ such that $\Phi \circ \nu_{f(\alpha)} = \mu_{f(\alpha)}$ for each $\alpha \in J$. Let $\alpha \in I$, then there is a $g(\alpha) \in J$ such that $f(g(\alpha)) \geq \alpha$, and we define $\varepsilon_\alpha = \nu_{f(g(\alpha))} \circ \varphi_{\alpha f(g(\alpha))}$. Suppose there is another $\beta \in J$ such that $f(\beta) \geq \alpha$, and so there is a $\gamma \in J$ such that $\gamma \geq \beta$ and $\gamma \geq g(\alpha)$, then one has
$$\begin{aligned} \varepsilon_\alpha &= \nu_{f(g(\alpha))} \circ \varphi_{\alpha f(g(\alpha))} = \nu_{f(\gamma)} \circ \varphi_{f(g(\alpha))f(\gamma)} \circ \varphi_{\alpha f(g(\alpha))} \\ &= \nu_{f(\gamma)} \circ \varphi_{\alpha f(\gamma)} = \nu_{f(\gamma)} \circ \varphi_{f(\beta)f(\gamma)} \circ \varphi_{\alpha f(\beta)} = \nu_{f(\beta)} \circ \varphi_{\alpha f(\beta)} \end{aligned}$$
so ε_α does not depend on the choice of $g(\alpha)$, i.e. ε_α is well-defined. In particular, if $\alpha \in J$, then we can choose $g(\alpha) = \alpha$, so $\varepsilon_{f(\alpha)} = \nu_{f(\alpha)} \circ \varphi_{f(\alpha)f(\alpha)} = \nu_{f(\alpha)}$. Now for each $\alpha \leq \beta$ in I, one has
$$\varepsilon_\beta \circ \varphi_{\alpha\beta} = \nu_{f(\beta)} \circ \varphi_{\beta f(\beta)} \circ \varphi_{\alpha\beta} = \nu_{f(\beta)} \circ \varphi_{\alpha f(\beta)} = \nu_{f(\beta)} \circ \varphi_{f(\alpha)f(\beta)} \circ \varphi_{\alpha f(\alpha)} = \nu_{f(\alpha)} \circ \varphi_{\alpha f(\alpha)} = \varepsilon_\alpha,$$
so $\left(A_f, (\varepsilon_\alpha)_{\alpha\in I}\right)$ is a cocone of $A^\bullet$, thus there is a morphism $\Psi : A \to A_f$ such that $\Psi \circ \mu_\alpha = \varepsilon_\alpha$ for each $\alpha \in I$. Let $\alpha \in J$, one has

$$\Psi \circ \Phi \circ \nu_{f(\alpha)} = \Psi \circ \mu_{f(\alpha)} = \varepsilon_{f(\alpha)} = \nu_{f(\alpha)}$$

so by uniqueness, one has $\Psi \circ \Phi = \mathrm{id}_{A_f}$. Let $\alpha \in I$, one has

$$\Phi \circ \Psi \circ \mu_\alpha = \Phi \circ \varepsilon_\alpha = \Phi \circ \nu_{f(g(\alpha))} \circ \varphi_{\alpha f(g(\alpha))} = \mu_{f(g(\alpha))} \circ \varphi_{\alpha f(g(\alpha))} = \mu_\alpha,$$

so by uniqueness, one has $\Phi \circ \Psi = \mathrm{id}_A$. Hence Φ is a universal isomorphism, thus $\big(A, \big(\mu_{f(\alpha)}\big)_{\alpha \in I}\big)$ is an inductive limit of $A^{f(\bullet)}$. ■

Lemma 5.4.7. Let I be a directed set, and $A^\bullet = \big(A_\alpha, (\varphi_{\alpha\beta})\big)$ be a diagram in $\mathscr{A}$ indexed by I and suppose $A^\bullet$ has an inductive limit $(A, (\mu_\alpha)_{\alpha \in I})$. For each $\alpha \in I$, define $B_\alpha = A_\alpha / \ker(\mu_\alpha)$, and let $\pi_\alpha : A_\alpha \to B_\alpha$ be the canonical map identified in **AC6**. Then there are injective maps $\psi_{\alpha\beta} : B_\alpha \to B_\beta$ for each $\alpha \leq \beta$ in I such that $\pi_\beta \circ \varphi_{\alpha\beta} = \psi_{\alpha\beta} \circ \pi_\alpha$. Let $B^\bullet = \big(B_\alpha, (\psi_{\alpha\beta})\big)$ be a diagram indexed by I, and suppose $(B, (\nu_\alpha)_{\alpha \in I})$ is an inductive limit of $B^\bullet$. Then there is a map $\pi : A \to B$, such that the following diagram commutes

$$\begin{array}{ccccccc}
A_\alpha & \xrightarrow{\varphi_{\alpha\beta}} & A_\beta & \longrightarrow & \cdots & \longrightarrow & A \\
\pi_\alpha \downarrow & & \downarrow \pi_\beta & & & & \downarrow \pi \\
B_\alpha & \xrightarrow{\psi_{\alpha\beta}} & B_\beta & \longrightarrow & \cdots & \longrightarrow & B
\end{array}$$

Where the $\cdots \longrightarrow$ arrows refers to the μ_α and ν_α morphisms respectively. Moreover, if we define $\gamma_\alpha = \nu_\alpha \circ \pi_\alpha : A_\alpha \to B$ for each $\alpha \in I$, then:

(i) π is the universal map induced by the cocone $(B, (\gamma_\alpha)_{\alpha \in I})$ of $A^\bullet$.

(ii) If ν_α is injective, then $\ker(\gamma_\alpha) = \ker(\mu_\alpha)$ for each $\alpha \in I$.

(iii) $\mathrm{im}(\gamma_\alpha) = \mathrm{im}(\nu_\alpha)$ for each $\alpha \in I$.

Proof. Let $\alpha \leq \beta$ in I, and define $\tilde{\varphi}_{\alpha\beta} = \pi_\beta \circ \varphi_{\alpha\beta} : A_\alpha \to B_\beta$, then given $x \in \ker(\mu_\alpha)$, one has $0 = \mu_\alpha(x) = \mu_\beta\big(\varphi_{\alpha\beta}(x)\big)$, so $\varphi_{\alpha\beta}(x) \in \ker\big(\mu_\beta\big)$, so $\tilde{\varphi}_{\alpha\beta}(x) = 0$. Thus $\ker(\mu_\alpha) \subseteq \ker\big(\tilde{\varphi}_{\alpha\beta}\big)$, hence by **AC6**, there is a unique map $\psi_{\alpha\beta} : B_\alpha \to B_\beta$ such that $\tilde{\varphi}_{\alpha\beta} = \psi_{\alpha\beta} \circ \pi_\alpha$, in particular,

$$\psi_{\alpha\beta} \circ \pi_\alpha = \tilde{\varphi}_{\alpha\beta} = \pi_\beta \circ \varphi_{\alpha\beta}.$$

Let $\pi_\alpha(x) \in \ker\big(\psi_{\alpha\beta}\big)$, so

$$0 = \psi_{\alpha\beta}(\pi_\alpha(x)) = \pi_\beta\big(\varphi_{\alpha\beta}(x)\big)$$

thus $\varphi_{\alpha\beta}(x) \in \ker\big(\mu_\beta\big)$, so $\mu_\alpha(x) = \mu_\beta\big(\varphi_{\alpha\beta}(x)\big) = 0$, hence $x \in \ker(\mu_\alpha)$, thus $\pi_\alpha(x) = 0$. So $\psi_{\alpha\beta}$ is injective.

Suppose $B^\bullet$ has an inductive limit given in the statement. Let $\alpha \in I$, and define $\gamma_\alpha = \nu_\alpha \circ \pi_\alpha : A_\alpha \to B$, and observe that if $\beta \geq \alpha$, then

$$\gamma_\beta \circ \varphi_{\alpha\beta} = \nu_\beta \circ \pi_\beta \circ \varphi_{\alpha\beta} = \nu_\beta \circ \psi_{\alpha\beta} \circ \pi_\alpha = \nu_\alpha \circ \pi_\alpha = \gamma_\alpha$$

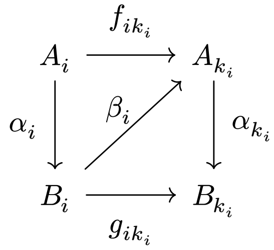
so $(B, (\gamma_\alpha)_{\alpha \in I})$ defines a cocone of $A^\bullet$, hence by universality, there is a unique map $\pi : A \to B$ such that $\gamma_\alpha = \pi \circ \mu_\alpha$. This shows (i).

For (ii). Let $\alpha \in I$. It is clear that $\ker(\mu_\alpha) \subseteq \ker(\gamma_\alpha)$ as $\gamma_\alpha = \pi \circ \mu_\alpha$. Let $x \in \ker(\gamma_\alpha)$, so $\pi_\alpha(x) \in \ker(\nu_\alpha)$. If ν_α is injective, then $\pi_\alpha(x) = 0$, so $x \in \ker(\mu_\alpha)$. Hence $\ker(\mu_\alpha) = \ker(\gamma_\alpha)$.

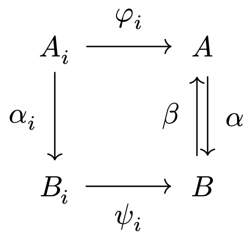
For (iii). As π_α is surjective, then $\mathrm{im}(\gamma_\alpha) = \mathrm{im}(\nu_\alpha)$ follows. ■

Lemma 5.4.8. *(Elliott's Intertwining Argument)*. Let I be a directed set, and $A^\bullet = \big(A_i, (f_{ij})\big)$ and $B^\bullet = \big(B_i, (g_{ij})\big)$ be diagrams in any category $\mathscr{A}$ indexed by I. Suppose:

- There is a mapping $i \mapsto k_i$ from I to I such that $k_i \geq i$ and $k_j \geq k_i$ for all $j \geq i$ in I.
- For each $i \in I$, there are morphisms $\alpha_i : A_i \to B_i$ morphisms $\beta_i : B_i \to A_{k_i}$ such that one has a commutative diagram:



If $(A, (\varphi_i)_{i\in I})$ and $(B, (\psi_i)_{i\in I})$ are inductive limits of $A^\bullet$ and $B^\bullet$ respectively, then there are isomorphisms $\alpha : A \to B$ and $\beta : B \to A$ with $\alpha^{-1} = \beta$ such that one has a commutative diagram:



for each $i \in I$.

Proof. Let $i, j \in I$ with $i \leq j$. Define $\gamma_i = \psi_i \circ \alpha_i : A_i \to B$, and observe that

$$\gamma_j \circ f_{ij} = \psi_j \circ \alpha_j \circ f_{ij} = \psi_j \circ g_{ij} \circ \alpha_i = \psi_i \circ \alpha_i = \gamma_i,$$

so $(B, (\gamma_i)_{i\in I})$ is a cocone of $A^\bullet$. Thus by universality of $\varinjlim A^\bullet$ there is a morphism $\alpha : A \to B$ such that $\gamma_i = \alpha \circ \varphi_i$ for all $i \in I$.

Define $\delta_i = \varphi_{k_i} \circ \beta_i : B_i \to A$, and observe that

$$\delta_j \circ g_{ij} = \varphi_{k_j} \circ \beta_j \circ g_{ij} = \varphi_{k_j} \circ f_{k_i k_j} \circ \beta_i = \varphi_{k_i} \circ \beta_i = \delta_i,$$

so $(A, (\delta_i)_{i\in I})$ is a cocone of $B^\bullet$. Thus by universality of $\varinjlim B^\bullet$ there is a morphism $\beta : B \to A$ such that $\delta_i = \beta \circ \psi_i$ for each $i \in I$.

Now observe that

$$(\beta \circ \alpha) \circ \varphi_i = \beta \circ \gamma_i = \beta \circ \psi_i \circ \alpha_i = \delta_i \circ \alpha_i = \varphi_{k_i} \circ \beta_i \circ \alpha_i = \varphi_{k_i} \circ f_{ik_i} = \varphi_i,$$

so by uniqueness, it follows that $\beta \circ \alpha = \mathrm{id}_A$. Also

$$(\alpha \circ \beta) \circ \psi_i = \alpha \circ \delta_i = \alpha \circ \varphi_{k_i} \circ \beta_i = \gamma_{k_i} \circ \beta_i = \psi_{k_i} \circ \alpha_{k_i} \circ \beta_i = \psi_{k_i} \circ g_{ik_i} = \psi_i$$

so by uniqueness, it follows that $\alpha \circ \beta = \mathrm{id}_B$. Thus $\alpha^{-1} = \beta$ and the rest follows. ∎

6 | References

[1] G. A. Elliott, "On the classification of inductive limits of sequences of semisimple finite-dimensional algebras," *Journal of Algebra*, vol. 38, no. 1, pp. 29–44, Jan. 1976, doi: 10.1016/0021-8693(76)90242-8.

[2] M. Rørdam, F. Larsen, and N. J. Laustsen, *An introduction to K-theory for C*-algebras*, Transferred to digital print., no. 49. in London Mathematical Society student texts. Cambridge: Cambridge Univ. Press, 2007.

[3] B. Blackadar, *Operator algebras: theory of C*-algebras and von Neumann algebras*, no. v. 122. 3. in Encyclopaedia of mathematical sciences, Operator algebras and non-commutative geometry. Berlin ; New York: Springer, 2006.

[4] M. Takesaki, Ed., *Theory of Operator Algebras I*. New York, NY: Springer New York, 1979. doi: 10.1007/978-1-4612-6188-9.

[5] B. Blackadar, *K-Theory for Operator Algebras*, vol. 5. in Mathematical Sciences Research Institute Publications, vol. 5. New York, NY: Springer New York, 1986. doi: 10.1007/978-1-4613-9572-0.

[6] S. Lang, *Algebra*, vol. 211. in Graduate Texts in Mathematics, vol. 211. New York, NY: Springer New York, 2002. doi: 10.1007/978-1-4613-0041-0.

[7] H. O. Milhøj, "AF-algebras and their invariants," *Department of Mathemaical Sciences, University of Copenhagen*, 2018.

[8] *Infinite Dimensional Analysis*. Berlin/Heidelberg: Springer-Verlag, 2006. doi: 10.1007/3-540-29587-9.

[9] M. Khoshkam and J. Tavakoli, "Categorical constructions in C *-algebra theory," *Journal of the Australian Mathematical Society*, vol. 73, no. 1, pp. 97–114, Aug. 2002, doi: 10.1017/S1446788700008491.

[10] T. Katsura, "Non-Separable AF-Algebras," *Operator Algebras*. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 165–173, 2006. doi: 10.1007/978-3-540-34197-0_9.

[11] G. Gong, H. Lin, and Z. Niu, "A review of the Elliott program of classification of simple amenable C*-algebras." Accessed: Nov. 16, 2025. [Online]. Available: https://arxiv.org/abs/2311.14238