Real Symplectic Implosion

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Abstract

Symplectic implosion was introduced by Guillemin, Jeffrey, and Sjamaar in [GJS02], and is an abelianisation of symplectic reduction. In implosion, one constructs the imploded cross-section M_{impl} of a Hamiltonian G-space M. It is characterised by the property that the reduction of M by the whole group G is isomorphic to the reduction of the imploded cross-section M_{impl} by a maximal torus T of G.

On the other hand, a real structure on a symplectic manifold is an anti-symplectic involution on the manifold, i.e. an anti-isomorphism in the symplectic category which squares to the identity function. The fixed point set of such an involution is either empty or a Lagrangian submanifold. These structures were generalised to Hamiltonian G-spaces by Duistermaat in [Dui83], in the case that G is an abelian Lie group. The non-abelian case was studied by O'Shea and Sjamaar in [OS00].

In this thesis, we find conditions under which the imploded cross-section of a real Hamiltonian G-space inherits a real Hamiltonian T-structure, where $T \subseteq G$ is a certain maximal torus of G. The fixed point set of the induced real structure on the imploded cross-section is then either empty or a Lagrangian submanifold. Hence we have a method of constructing Lagrangian submanifolds in imploded cross-sections.

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Contents

Chapter 1

Introduction

Symplectic geometry arose in the nineteenth century as the mathematical foundation of classical mechanics. It is a fairly rigid, even dimensional geometry where one measures signed areas of a smooth manifold. This is a divergence from more familiar geometries, such as Riemannian geometry, where one is concerned with measuring lengths and angles. A major consequence of this is that, unlike in the Riemannian case, symplectic geometry has no local invariants. This links the study of symplectic geometry intimately to the study of the differential topology of the underlying space.

The links between symplectic geometry and classical mechanics deepen when one considers the action of a Lie group on the manifold. Classically, the dynamics of a system is governed by a set of differential equations called Hamilton's equations. Suppose there is a vector field on the phase space of the system, whose integral curves are defined by Hamilton's equations. We call such a vector field Hamiltonian. If there exists an action of a Lie group on the phase space, and the action preserves the flow of the Hamiltonian vector field, then the action is said to be by symmetries of the system. Noether's theorem then states that there exists a conserved quantity of the system; a smooth map invariant under the flow of the Hamiltonian vector field.

In mathematics, this conserved quantity is called the moment map; as a classical example of the conserved quantity is (angular) momentum. A symplectic manifold, equipped with Lie group action and associated moment map, is called a Hamiltonian space. A more detailed look at symplectic manifolds and their relation to classical mechanics can be found in the books [GS84b], and [Arn89].

The introduction of Hamiltonian spaces has been a useful tool in symplectic geometry. In [MW74], and [Mey73], Marsden, Weinstein, and Meyer used the moment map to define the procedure of symplectic reduction. Reduction is the symplectic version of the quotient manifold theorem from differential topology, which defines a smooth manifold structure on the quotient space of a smooth manifold by a Lie group acting on it. Such a general procedure had been elusive, due to dimensionality reasons (namely symplectic manifolds being even dimensional). The three authors rectified this situation by using the moment map. They take a regular level set of the moment map, which is smooth submanifold by the inverse function theorem, and consider the quotient of this subset. Thus, in reduction, one does not consider the quotient of the whole manifold but rather a quotient of a subspace.

In this thesis we deal with two distinct notions. The first is symplectic implosion, and the second is real structures.

In [GJS02] Guillemin, Jeffrey, and Sjamaar introduce symplectic implosion as a way to abelianise reduction. Explicitly, they define the imploded cross-section of a symplectic manifold. It is characterised by the property that the reduction of the manifold by the whole Lie group, agrees with the reduction of the imploded cross-section by a maximal torus (a maximal compact, connected, abelian subgroup). The price to abelianisation is that the resulting imploded cross-section is almost always not smooth; it is a singular space. However, it may be decomposed into symplectic submanifolds, which cover the whole space, and fit together in a particularly nice way called a topological stratification.

The techniques presented in [GJS02] have found use in other areas of symplectic geometry. For example, implosion comes with a universal object given by the imploded cross-section of the cotangent bundle of the Lie group acting on the symplectic manifold. The idea of the cotangent bundle acting as a universal object has found uses in other areas, such as Martens and Thaddeus' work on non-abelian symplectic cuts [MT12]. Symplectic implosion has also been used by Kirwan in [Kir11] to provide a proposal towards non-reductive geometric invariant theory (GIT) in algebraic geometry. This link to algebraic geometry does not stop here, as recent work by Safronov shows implosion can be viewed algebraically in the framework of derived geometry [Saf17].

Implosion has also been shown to exist for manifolds equipped with additional structures. In [GJS02] the authors show that the implosion of a Kähler manifold is Kähler; and in a series of papers [DKS13a; DKS13b; DKS14; Dan+16] Dancer, Kirwan, and Swann define implosion for hyperkähler manifolds. This has potential implications for physics, as the canonical examples of hyperkähler structures appear in the moduli space of solutions to some gauge theories. In [HJS06] Hurtubise, Jeffrey, and Sjamaar show that implosion also holds in the case of quasi-Hamiltonian spaces; which were introduced by Alekseev et al. in [AMM98]. Example of such spaces includes the moduli space of flat connections on a Riemann surface. Thus, again, we see that implosion has interesting links to moduli problems arising in geometry. Moreover, in [HJS06] Hurtubise et al. show that the stratification of the imploded cross-section generates examples of new quasi-Hamiltonian spaces. Thus, in this case, implosion also aids in answering questions on existence.

On the other hand, a real structure on a Hamiltonian space is a pair of smooth involutions (a function which squares to the identity): one on the symplectic manifold, and one on the Lie group acting on the manifold. The involution on the Lie group is required to be a Lie group automorphism; while the involution on the manifold is required to be anti-symplectic, an anti automorphism in the symplectic category. The reasoning for considering such structures is as follows: consider the fixed point set of the anti-symplectic involution, called the real locus, on the manifold. The anti-symplectic condition forces the real locus to either be empty or a Lagrangian submanifold, which is a special class of submanifolds generalising the idea of the collection of all possible momenta in a physical system.

Such spaces were first studied in the case of an abelian Lie group by Duistermaat in [Dui83]. Duistermaat used this structure to prove a convexity theorem relating the convexity of the images of the manifold, and the real locus, under the moment map. This was generalised to the case of non-abelian Lie groups by O'Shea and Sjamaar in [OS00], who proved their own convexity theorem.

Much of the further work concerning real structures have involved looking at the structure of the real locus. The relationship between the cohomology of the symplectic manifold and its real locus has been investigated in papers such as [HHP05] and [BGH04]. The situation of when the reduction of a real Hamiltonian space inherits a real structure was investigated in O'Shea and Sjamaar's paper [OS00, Section 7]. Subsequently, the analysis of the real loci in the quotients have been investigated in papers such as [Fot05] and [GH04].

The goal of this thesis is to review this theory, and to investigate under what conditions the imploded cross-section of a real Hamiltonian space inherits an induced real Hamiltonian structure relative to the maximal torus we have imploded by. As far as the author is aware, this has not appeared previously in the literature, and would provide a notion of real symplectic implosion. Moreover, supposing the structure does descend, we may take the real locus of the real structure in the imploded cross-section; which is either empty or Lagrangian. As such we also obtain examples of Lagrangian submanifolds in imploded cross-sections.

1.1 Outline of Thesis

The following is an outline of the remainder of this thesis:

In Chapter 2 we provide a brief introduction into the area of symplectic geometry. We also provide the definition of a Hamiltonian space; prove various properties of said spaces, and also provide a bank of examples.

Chapter 3 concerns the theory of symplectic reduction. Here we provide the definition of ordinary, or Marsden-Weinstein-Meyer, reduction; but also its various generalisations like the shifting trick and reduction in stages. The latter half of the chapter introduces advanced topics from the theory of smooth group actions on manifolds, in order to state the singular reduction theory of Sjamaar and Lerman which underpins symplectic implosion.

Chapter 4 is dedicated to symplectic implosion. This chapter mainly follows the original paper [GJS02] by Guillemin, Jeffrey, and Sjamaar. However, we have added sections recalling the construction of symplectic cross-sections, and fundamental Weyl chambers, to help with understanding the construction of the imploded cross-section. Moreover, we have expanded on some proofs in [GJS02] which where terse in places. The chapter also ends with a hands on construction of the imploded cross-section of $T^* \mathrm{SU}(2)$, and shows that it is isomorphic to $\mathbb{C}^2$.

In Chapter 5 we introduce real structures on Hamiltonian spaces. We put a large emphasis on descending real structures to symplectic quotients/reductions. We then answer the main question of this thesis: When does a real Hamiltonian structure descend to the imploded cross-section? We also show that real structure does descend to the imploded cross-section in the case of the universal objects for implosion. We finally end by computing the induced real Hamiltonian structure on the imploded cross-section for $T^* \mathrm{SU}(2)$, and find the corresponding real Hamiltonian structure under the isomorphism with $\mathbb{C}^2$.

Appendix A reviews basic results and ideas from smooth group actions on manifolds such as: proper actions, orbits and stabilisers, principal G-bundles, etc. It also clearly states the notations and conventions used in this thesis.

Throughout this thesis we assume familiarity with the theory of smooth manifolds, of the level presented in books such as [Lee12], or [Tu17].

Chapter 2

Symplectic Geometry

In this chapter we provide an introduction to symplectic geometry, which forms the backbone of this thesis. The material presented in this chapter can be found in any book on symplectic geometry. For example see [Sil01], [GS84b], or [Lee12].

2.1 Symplectic Manifolds

Let M be a smooth manifold, and $\omega \in \Omega^2(M)$ a smooth 2-form on M. Define the **kernel** of ω at a point $p \in M$ to be

$$\ker \omega_p = \{X \in T_pM : i(X)\omega_p = \omega_p(X, -) = 0\}.$$

We say that a 2-form ω is non-degenerate if $\ker \omega_p = \{0\}$ for every $p \in M$. This is equivalent to stating that the induced bundle map $\hat{\omega} : TM \to T^*M$ defined by $\hat{\omega}(X) = i(X)\omega$ is a bundle isomorphism.

Definition 2.1.1. Let M be a smooth manifold. A closed non-degenerate smooth 2-form ω on M is called a **symplectic form**. A pair (M, ω) with M a smooth manifold and ω a symplectic form is called a **symplectic manifold**. Two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) are **symplectomorphic** if there exists a diffeomorphism $F : M_1 \to M_2$ such that $F^*\omega_2 = \omega_1$. In such a case, F is a **symplectomorphism**.

Example 2.1.2. View $\mathbb{R}^{2n}$ as a smooth $2n$-manifold with global coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$, and let $\omega_0 \in \Omega^2(\mathbb{R}^{2n})$ be defined by

$$\omega_0 = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{2.1.1}$$

It is clear that ω is a closed 2-form whose action on the basis vectors $\partial/\partial x^i, \partial/\partial y^j \in T_p\mathbb{R}^{2n}$ is

$$\omega_0\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \omega_0\Big(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\Big) = 0, \quad \omega_0\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\Big) = -\omega_0\Big(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^i}\Big) = \delta_{ij}. \tag{2.1.2}$$

Suppose that

$$X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i} \in T_p\mathbb{R}^{2n}$$

satisfies $\omega_0(X, Y) = 0$ for all $Y \in T_p\mathbb{R}^{2n}$. Then $0 = \omega_0(X, \partial/\partial Y_i) = a^i$ and $0 = \omega_0(X, \partial/\partial x^i) = b^i$, so $X = 0$ and ω_0 is non-degenerate. Thus ω_0 is a symplectic form, and $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic manifold. We refer to ω_0 as the standard symplectic form on $\mathbb{R}^{2n}$. ◀

Example 2.1.3. View $\mathbb{C}^n$ as a complex n-manifold with coordinates $(z^1, \dots, z^n)$. We can view $\mathbb{C}^n$ as a real $2n$-manifold by taking the coordinates $x^i = \mathrm{Re}(z^i)$, and $y^i = \mathrm{Im}(z^i)$. Consider the 2-form on $\mathbb{C}^n$ defined by

$$\omega_0 = \frac{i}{2} \sum_{i=1}^{n} dz^i \wedge d\bar{z}^i = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

By Example 2.1.2, ω_0 is a symplectic form on $\mathbb{C}^n$ called the standard symplectic form on $\mathbb{C}^n$. Note that for any $v, w \in \mathbb{C}^n$ we have

$$\omega_0(z, w) = \mathrm{Im}(z^* w) \tag{2.1.3}$$

where z^* denotes the conjugate transpose of z. ◀

Not every smooth manifold is symplectic. One can show that every symplectic manifold is even dimensional, immediately ruling out examples such as $\mathbb{R}^{2n+1}$ or $\mathbb{R}P^{2n+1}$. Further, one can show if ω is a symplectic form for a manifold of dimension $2n$, then ω^n (the wedge product of n copies of ω) is a non-zero volume form. Hence every symplectic manifold is orientable. (Thus the Möbius strip is not a symplectic surface. This also shows not every even dimensional smooth manifold is symplectic.) Proof of these statements can be found in [Lee12, Chapter 22].

Moreover, there also exists a cohomological obstruction to a smooth manifold being symplectic. As a rough outline; Let M be a closed symplectic manifold of dimension $2n$ with symplectic form ω. Then, as stated in the previous paragraph, ω^n is non-zero volume form. Moreover, ω^n cannot be exact as this would imply M has zero volume by Stokes' theorem. Hence ω^n defines a non-zero cohomology class $[\omega^n]$ in $H^{2n}_{dR}(M)$. However, $[\omega^n] = [\omega]^n$ which implies that $0 \neq [\omega] \in H^2_{dR}(M)$ further implying $H^2_{dR}(M) \neq 0$. As a corollary, we find that S^2 is the only sphere which admits a symplectic structure.

For any $p \in M$, and any subspace $W \subseteq T_pM$, define the **symplectic complement** of W as

$$W^\perp = \{v \in T_pM : \omega(v, w) = 0 \text{ for all } w \in W\}.$$

It is the symplectic analogue to the orthogonal complement in an inner product space. We also may denote the symplectic complement by W^ω, to avoid confusion with the orthogonal complement.

As is the case for an orthogonal complement, the dimension of $W^\perp$ is the codimension of W, i.e. $\dim W + \dim W^\perp = \dim T_pM$. Using the relationship between W and $W^\perp$, we can define different types of subspaces in T_pM.

Definition 2.1.4. A subspace $W \subseteq T_pM$ is said to be:

- **Symplectic** if $W \cap W^\perp = \{0\}$.
- **Isotropic** if $W \subseteq W^\perp$.
- **Coisotropic** if $W^\perp \subseteq W$.
- **Lagrangian** if $W = W^\perp$.

More generally, an (immersed or embedded) submanifold $S \subseteq M$ is said to be a **symplectic**, **isotropic**, **coisotropic**, or **Lagrangian submanifold** if T_pS (viewed as a subspace of T_pM) has the respective property for all $p \in S$.

From the definitions, we have an easy criterion to determine whether a given subspace has one of the stated properties.

Proposition 2.1.5. *Let (M, ω) be a symplectic manifold, $p \in M$, and $W \subseteq T_pM$ a subspace. Then*

I) *W is symplectic if, and only if, $W^\perp$ is symplectic.*

II) *W is symplectic if, and only if, $\omega|_W$ is non-degenerate.*

III) *W is isotropic if, and only if, $\omega|_W = 0$.*

IV) *W is coisotropic if, and only if, $W^\perp$ is isotropic.*

V) *W is Lagrangian if, and only if, $\omega|_W = 0$ and $\dim W = \frac{1}{2}\dim M$.*

The following theorem is one of the most important results in symplectic geometry. It demonstrates the main difference between Riemannian and symplectic geometry: there are no local invariants in symplectic geometry.

Theorem 2.1.6 (Darboux)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold. For any $p \in M$, there are smooth coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ centred at p such that ω has the coordinate representation*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{2.1.4}$$

Proof. See [Lee12, Chapter 22], or [Sil01, Chapter 8]. □

One of the most important examples of a symplectic manifold is the total space of the cotangent bundle of a smooth manifold M, which carries a canonical symplectic form. First, we define a natural 1-form θ on the total space T^*M called the **tautological**, or **Liouville 1-form**.

A point in T^*M is a 1-form $\eta \in T_p^*M$ for some $p \in M$; we denote such a point by (p, η). The canonical projection $\pi : T^*M \to M$ is then $\pi(p, \eta) = p$. Using the pullback of the projection, we define a 1-form $\theta \in \Omega^1(T^*M)$ by

$$\theta_{(p,\eta)} = \pi^*_{(p,\eta)}\eta. \tag{2.1.5}$$

Its action on a vector field $v \in T_{(p,\eta)}(T^*M)$ is given by

$$\theta_{(p,\eta)}(v) = \big(\pi^*_{(p,\eta)}\eta\big)(v) = \eta\big(d\pi_{(p,\eta)}v\big).$$

Proposition 2.1.7. *Let M be a smooth manifold. The tautological 1-form θ is smooth, and $\omega = -d\theta$ is a symplectic form on the total space T^*M.*

Proof. Let (x^i) be smooth coordinates on M, and let (x^i, ξ_i) denote the corresponding natural coordinates on T^*M. Recall that the coordinates of $(p, \eta) \in T^*M$ are defined to be (x^i, ξ_i), where (x^i) is the coordinate representation of p, and $\sum \xi_i dx^i$ is the coordinate representation of η. This implies that $\pi^*_{(p,\eta)}(dx^i) = (dx^i)_p$, so the coordinate expression for θ is

$$\theta_{(p,\eta)} = \pi^*_{(p,\eta)}\Big(\sum_{i=1}^{n} \xi_i dx^i\Big) = \sum_{i=1}^{n} \xi_i dx^i. \tag{2.1.6}$$

It follows that θ is smooth, as its component functions in these coordinates are linear. Let $\omega = -d\theta \in \Omega^2(T^*M)$. As ω is exact, it is closed. Moreover, in the natural coordinates, (2.1.6) gives

$$\omega = \sum_{i=1}^{n} dx^i \wedge d\xi_i.$$

Under the identification of an open subset of T^*M with an open subset of $\mathbb{R}^{2n}$ (by these coordinates), ω corresponds to the standard symplectic form on $\mathbb{R}^{2n}$. Hence ω is symplectic. □

The cotangent bundle T^*M of a smooth manifold is, arguably, the most important example of a symplectic manifold. This is because symplectic manifolds arose as the mathematical generalisation of classical mechanics in physics. Here T^*M plays the role of phase space for the system, with coordinates on M representing the position and the cotangent coordinates representing momentum.

2.2 Hamiltonian Vector Fields

Definition 2.2.1. Suppose (M, ω) is a symplectic manifold, and f a smooth function on M. Define the **Hamiltonian vector field** of f to be the vector field $X_f = \hat{\omega}^{-1}(df)$. Equivalently, it is the unique vector field such that

$$df = i(X_f)\omega.$$

Note that a Hamiltonian vector field is the symplectic analogue to the gradient from Riemannian geometry. However, its behaviour differs due to the antisymmetry of ω.

Proposition 2.2.2. *Let (M, ω) be a symplectic manifold, and $f \in C^\infty(M)$ a smooth function.*

I) *f is constant along each integral curve of X_f.*

II) *At each regular point of f, the Hamiltonian vector field is tangent to the level set of f.*

Proof. Both claims follow from

$$X_f f = df(X_f) = \omega(X_f, X_f) = 0.$$

For the second assertion, recall that if c is a regular value then $T_p f^{-1}(c) = \ker df_p$. □

In turns out that we can compute the Hamiltonian vector field X_f for some smooth function f explicitly, in Darboux coordinates at least. Let

$$X_f = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i}$$

for some coefficients a^i, b^i. Then

$$i(X_f)\omega = \sum_{i=1}^{n} a^i dy^i - b^i dx^i,$$

while

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^i} dy^i.$$

Equating the two to be equal gives $a^i = \partial f/\partial y^i$ and $b^i = -\partial f/\partial x^i$. Hence, in Darboux coordinates, the Hamiltonian vector field of f is given by

$$X_f = \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i} \Big). \tag{2.2.1}$$

It is natural to ask under what circumstances is an arbitrary vector field $X \in \mathfrak{X}(M)$ a Hamiltonian vector field for some smooth function f?

Definition 2.2.3. A smooth vector field X on a symplectic manifold (M, ω) is said to be **symplectic** if ω is invariant under the flow of X, i.e. $\mathcal{L}_X \omega = 0$. It is said to be **Hamiltonian** (or **globally Hamiltonian**) if there exists a smooth function f such that $X = X_f$. It is **locally Hamiltonian** if at each point $p \in M$, there is a neighbourhood of p on which X is Hamiltonian.

Proposition 2.2.4. *Let (M, ω) be a symplectic manifold.*

I) *A smooth vector field X on M is symplectic if, and only if, it is locally Hamiltonian.*

II) *Every locally Hamiltonian vector field on M is globally Hamiltonian if, and only if, $H^1_{dR}(M) = 0$.*

Proof. I): By Cartan's magic formula,

$$\mathcal{L}_X\omega = d\big(i(X)\omega\big) + i(X)d\omega = d\big(i(X)\omega\big). \tag{2.2.2}$$

Therefore X is symplectic if, and only if, $i(X)\omega$ is closed 1-form. Now if X is locally Hamiltonian, then in a neighbourhood of every point there is a smooth function f such that $i(X)\omega = df$, which is closed. Conversely if X is symplectic, then using the Poincaré lemma each point $p \in M$ has a neighbourhood on which $i(X)\omega$ is exact, i.e. there is a smooth function such that $i(X)\omega = df$. By non-degeneracy of ω, $X = X_f$.

II): Suppose that $H^1_{dR}(M) = 0$. If X is locally Hamiltonian, then it is symplectic by I), and so $i(X)\omega$ is closed by (2.2.2). As $H^1_{dR}(M) = 0$, $i(X)\omega$ is also exact, and so there exists a smooth function f such that $i(X)\omega = df$. Again by non-degeneracy of ω, we conclude $X = X_f$. Conversely, suppose every locally Hamiltonian vector field is globally Hamiltonian. Let η be a closed 1-form, and let $X = \hat{\omega}^{-1}(\eta)$. Then (2.2.2) gives $\mathcal{L}_X\omega = d\eta = 0$. Hence X is symplectic and therefore locally Hamiltonian. By assumption, there is a smooth function f on M such that $X = X_f$. As $\hat{\omega}$ is an isomorphism $\eta = df$. □

In light of Proposition 2.2.4, we can view $H^1_{dR}(M)$ as an obstruction to symplectic vector fields being Hamiltonian.

Proposition 2.2.5. *Let (M, ω) be a symplectic manifold. If X, Y are two symplectic vector fields on M, then $[X, Y]$ is a Hamiltonian vector field with Hamiltonian function $\omega(Y, X)$.*

Proof.

$$\begin{aligned} i([X, Y])\omega &= \mathcal{L}_X i(Y)\omega - i(Y)\mathcal{L}_X\omega \\ &= di(X)i(Y)\omega + i(X)di(Y)\omega - i(Y)di(X)\omega - i(Y)i(X)d\omega \\ &= d\big(\omega(Y, X)\big). \end{aligned}$$

□

Let $\mathfrak{X}^{\mathrm{Symp}}(M)$ be the set of symplectic vector fields on M, and $\mathfrak{X}^{\mathrm{Ham}}(M)$ the set of Hamiltonian vector fields. Then we have the following corollary.

Corollary 2.2.5.1. *Let (M, ω) be a symplectic manifold. Then we have the following inclusions of Lie algebras*

$$(\mathfrak{X}^{\mathrm{Ham}}(M), [\ ,\ ]) \subseteq (\mathfrak{X}^{\mathrm{Symp}}(M), [\ ,\ ]) \subseteq (\mathfrak{X}(M), [\ ,\ ]).$$

2.2.1 Poisson Brackets

Hamiltonian vector fields allow us to define a bracket operation on the algebra $C^\infty(M)$ of smooth function on a symplectic manifold M, which turns $C^\infty(M)$ into a Lie algebra.

Definition 2.2.6. Given $f, g \in C^\infty(M)$, we define their **Poisson bracket** $\{f, g\} \in C^\infty(M)$ by

$$\{f, g\} = -i(X_f)i(X_g)\omega = \omega(X_f, X_g) = X_g f. \tag{2.2.3}$$

Two functions are said to Poisson commute if their Poisson bracket is zero. As vector fields are derivations, it follows from (2.2.3) that the Poisson bracket is also a derivation, i.e.

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Using the equality $\{f, g\} = X_g f$ gives a geometric view of the Poisson bracket: It measures the rate of change of f, along the Hamiltonian flow of g. Moreover, (2.2.1) gives a computation of the Poisson bracket in Darboux coordinates:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i}\frac{\partial g}{\partial x^i} \tag{2.2.4}$$

Proposition 2.2.7 (Properties of the Poisson bracket)**.** *Suppose* (M, ω) *is a symplectic manifold, and* $f, g, h \in C^\infty(M)$*. Then the Poisson bracket satisfies the following properties:*

I) (Bilinearity) $\{f, g\}$ *is linear over* $\mathbb{R}$ *in* f *and* g*.*

II) (Antisymmetry) $\{f, g\} = -\{g, f\}$.

III) (Jacobi Identity) $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$.

IV) $X_{\{f,g\}} = -[X_f, X_g]$.

i.e. $C^\infty(M)$ *is a Lie algebra under the Poisson bracket.*

Proof. I) and II) are clear from the definition of the Poisson bracket. To prove IV), as ω is non-degenerate it suffices to show that

$$\omega\big(X_{\{f,g\}}, Y\big) + \omega\big([X_f, X_g], Y\big) = 0 \tag{2.2.5}$$

for all vector fields Y. First note

$$\omega\big(X_{\{f,g\}}, Y\big) = d\big(\{f, g\}\big)Y = Y\{f, g\} = YX_g f.$$

Now, as Hamiltonian vector fields are symplectic,

$$\begin{aligned} 0 &= \big(\mathcal{L}_{X_g}\omega\big)(X_f, Y) \\ &= X_g\big(\omega(X_f, Y)\big) - \omega\big([X_g, X_f], Y\big) - \omega\big(X_f, [X_g, Y]\big). \end{aligned} \tag{2.2.6}$$

The first and third terms on the right-hand side can be simplified as

$$X_g\big(\omega(X_f, Y)\big) = X_g\big(df(Y)\big) = X_g Y f,$$

and

$$\begin{aligned}\omega\big(X_f, [X_g, Y]\big) &= df\big([X_g, Y]\big) = [X_g, Y]f = X_gYf - YX_gf \\ &= X_gYf - \omega\big(X_{\{f,g\}}, Y\big).\end{aligned}$$

Substituting these into (2.2.6) gives (2.2.5).

Finally, III) follows from II) and IV):

$$\begin{aligned}\{f, \{g, h\}\} &= X_{\{g,h\}}f = -[X_g, X_h]f = -X_gX_hf + X_hX_gf \\ &= -X_g\{f, h\} + X_h\{f, g\} = -\{\{f, h\}, g\} + \{\{f, g\}, h\} \\ &= -\{g, \{h, f\}\} - \{h, \{f, g\}\}.\end{aligned}$$

□

2.3 The Moment Map

Definition 2.3.1. Let (M, ω) be a symplectic manifold, and G a Lie group acting smoothly on M. We say that G acts **symplectically** on M if for all $g \in G$, the map $\mathcal{A}_g : M \to M$ is a symplectomorphism, i.e. $\mathcal{A}_g^*\omega = \omega$.

Fix a smooth manifold (M, ω) and a Lie group G, and suppose G acts symplectically on M.

As $\mathcal{A}_{\exp(tX)}^*\omega = \omega$ for $X \in \mathfrak{g}$, and as $\mathcal{A}_{\exp(tX)}$ is the flow of X_M,

$$\mathcal{L}_{X_M}\omega = \frac{d}{dt}\bigg|_{t=0} \mathcal{A}_{\exp(tX)}^*\omega = \frac{d}{dt}\bigg|_{t=0} \omega = 0. \tag{2.3.1}$$

This shows that the fundamental vector fields are symplectic, and therefore locally Hamiltonian by Proposition 2.2.4. We are interested in the case when the fundamental vector fields are globally Hamiltonian.

Definition 2.3.2. Suppose (M, ω) is a smooth manifold and G a Lie group acting symplectically on M. We say that the G action is **weakly Hamiltonian** if each fundamental vector field on M is Hamiltonian. Equivalently, for every $X \in \mathfrak{g}$ there exists $\mu^X \in C^\infty(M)$ such that

$$d\mu^X = i(X_M)\omega. \tag{2.3.2}$$

It is clear that the collection of functions $\{\mu^X : X \in \mathfrak{g}\}$ are unique up to a constant function. However, these constants can be chosen so that μ^X depends linearly on X; first define μ on a basis for $\mathfrak{g}$, and extend linearly.

The definition of a weakly Hamiltonian action can be restated as follows: Recall that $\mathfrak{X}^{\mathrm{Ham}}(M)$ and $\mathfrak{X}^{\mathrm{Symp}}(M)$ are Lie algebras under the Lie bracket of vector fields, and $C^\infty(M)$ is a Lie algebra under the Poisson bracket, by Proposition 2.2.7. This gives rise to two short exact sequences of Lie algebras

$$0 \longrightarrow \mathfrak{X}^{\text{Ham}}(M) \hookrightarrow \mathfrak{X}^{\text{Symp}}(M) \xrightarrow{X \mapsto [i(X)\omega]} H^1_{dR}(M) \longrightarrow 0, \tag{2.3.3}$$

and

$$0 \longrightarrow \mathbb{R} \hookrightarrow C^\infty(M) \xrightarrow{f \mapsto X_f} \mathfrak{X}^{\text{Ham}}(M) \longrightarrow 0. \tag{2.3.4}$$

From this a weakly Hamiltonian action is equivalent to the existence of a smooth map $\mu^* : \mathfrak{g} \to C^\infty(M)$, called the **comoment map**, such that the following diagram commutes:

$$\begin{array}{ccccccc} C^\infty(M) & \xrightarrow{f \mapsto X_f} & \mathfrak{X}^{\text{Symp}}(M) & \longrightarrow & H^1_{dR}(M) & \longrightarrow & 0. \\ & {\scriptstyle \mu^*} \nwarrow & & \nearrow {\scriptstyle X \mapsto X_M} & & & \\ & & \mathfrak{g} & & & & \end{array} \tag{2.3.5}$$

i.e. the comoment map is a lift of the *infinitesimal action* of $\mathfrak{g}$ on TM.

Definition 2.3.3. A weakly Hamiltonian action is **Hamiltonian** if the comoment map $\mu^* : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism.

Suppose the action of G on M is weakly Hamiltonian with comoment map $\mu^* : \mathfrak{g} \to C^\infty(M)$. Then for all $X \in \mathfrak{g}$, $\mu^*(X)$ is a Hamiltonian function for the fundamental vector field X_M. However, by Proposition 2.2.7(IV) $\{\mu^*(X), \mu^*(Y)\}$ is a Hamiltonian function for $-[X_M, Y_M] = [X, Y]_M$, and so $\{\mu^*(X), \mu^*(Y)\} - \mu^*([X, Y])$ is locally constant:

$$\begin{aligned} d\big(\{\mu^*(X), \mu^*(Y)\} - \mu^*([X,Y])\big) &= d\{\mu^*(X), \mu^*(Y)\} - d\mu^*([X,Y]) \\ &= i\big([X,Y]_M\big)\omega - i\big([X,Y]_M\big)\omega \\ &= 0. \end{aligned} \tag{2.3.6}$$

Hence the obstruction to a weakly Hamiltonian action being Hamiltonian is for this constant to be non-zero.

Given a Hamiltonian group action we can *dualise* the comoment map μ^* via the following procedure: Fix $p \in M$ and consider the function $\mu_p : \mathfrak{g} \to \mathbb{R}$ defined by evaluating $\mu^*(X) \in C^\infty(M)$ at p, i.e. $\mu_p(X) = \big[\mu^*(X)\big](p)$. As μ^* is a Lie algebra homomorphism, it is linear, and so the maps μ_p^* are linear functionals for every $p \in M$. Therefore, as p ranges of M, the maps μ_p can be viewed as a single map $\mu : M \to \mathfrak{g}^*$ defined as $\mu(p) = \mu_p$, called the **moment map**.

Let $\langle\ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ denote the non-degenerate pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$ defined by $\langle \alpha, X \rangle = \alpha(X)$. Unravelling the construction of the moment map, we have $\big[\mu(p)\big](X) = \langle \mu(p), X \rangle$, which implies that we can define $\langle \mu, X \rangle \in C^\infty(M)$ by $\langle \mu, X \rangle (x) = \langle \mu(x), X \rangle$. However, $\langle \mu, X \rangle$ is precisely $\mu^*(X)$, and so (2.3.5) implies

$$d \langle \mu, X \rangle = i(X_M)\omega, \tag{2.3.7}$$

for all $X \in \mathfrak{g}$. We sometimes denote by $\mu^X = \langle \mu, X \rangle$ the X component of μ, and (2.3.7) reads $i(X_M)\omega = d\mu^X$. The relationship (2.3.7) is referred to as the **moment map condition**, and it implies the comoment and moment maps are equivalent.

Note that in (2.3.7) $X \in \mathfrak{g}$ is constant. Hence we can rewrite the left-hand side of (2.3.7) as $\langle d\mu, X \rangle$, where $d\mu$ is the derivative of the moment map. It follows that

$$\langle d\mu_p(v), X \rangle = d\big(\langle \mu, X \rangle \big)_p(v) = \omega(X_M(p), v) \tag{2.3.8}$$

for all $v \in T_pM$.

In the special case that G is a connected Lie group, the fact that the comoment map is a Lie algebra homomorphism can be restated as the moment map being G-equivariant with respect to the action on M and the coadjoint action on $\mathfrak{g}^*$.

Proposition 2.3.4. *Suppose (M, ω) is a symplectic manifold with a weakly Hamiltonian action of a connected Lie group G. Then the comoment map is a Lie algebra homomorphism if, and only if, the moment map is equivariant.*

Proof. First suppose that the comoment map μ^* is a Lie algebra homomorphism. As G is connected, it is generated by elements of the form $\exp(X)$ for $X \in \mathfrak{g}$. Thus, to prove that the moment map is G-equivariant, it is enough to prove it is equivariant on the generators by Corollary $A.3.5.1$. Further, as G is connected, by Proposition $A.3.7$ to show that μ is G-equivariant it suffices to show that it is $\mathfrak{g}$-equivariant relative to the infinitesimal action, i.e.

$$d\mu_p(X_M) = X_{\mathfrak{g}^*}\big(\mu(p)\big), \tag{2.3.9}$$

for all $p \in M$. Now for any $Y \in \mathfrak{g}^*$, we have

$$\big\langle X_{\mathfrak{g}^*}\big(\mu(p)\big), Y \big\rangle = \langle \mu(p), -[X, Y] \rangle = \big[-\mu^*([X, Y])\big](p).$$

However, as μ^* is a homomorphism,

$$\begin{aligned} \langle d\mu_p(X_M), Y \rangle &= \omega_p(Y_M, X_M) \\ &= -\big\{\mu^*(X), \mu^*(Y)\big\}(p) \\ &= \big[-\mu^*([X, Y])\big](p). \end{aligned}$$

So by non-degeneracy of the pairing $\langle\ ,\ \rangle$, (2.3.9) holds and μ is equivariant.

Conversely, suppose that μ is equivariant and let $X, Y \in \mathfrak{g}$. Then

$$\begin{aligned} \big\{\mu^*(X), \mu^*(Y)\big\}(p) = \big[Y_M\big(\mu^*(X)\big)\big](p) &= \frac{d}{dt}\bigg|_{t=0} \langle \mu(\exp(tY) \cdot p), X \rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \Big\langle \mathrm{Ad}^*_{\exp(-tY)} \mu(p), X \Big\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \big\langle \mu(p), \mathrm{Ad}_{\exp(-tY)} X \big\rangle \\ &= \langle \mu(p), [X, Y] \rangle \\ &= \big[\mu^*([X, Y])\big](p), \end{aligned}$$

and μ^* is a Lie algebra homomorphism. $\square$

In light of Proposition 2.3.4, we restate the definition of a Hamiltonian action as follows.

Definition 2.3.5. An action of Lie group G on a symplectic manifold (M, ω) is **Hamiltonian** if there exists a smooth function $\mu : M \to \mathfrak{g}^*$ called the **moment map** such that:

$$d\langle \mu, X\rangle = i(X_M)\omega,$$

for all $X \in \mathfrak{g}$, and μ is equivariant with respect to the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$, i.e.

$$\mu(g \cdot p) = \mathrm{Ad}_g^* \mu(p).$$

We call the quadruple (M, ω, G, μ), for a Hamiltonian action of G with equivariant moment map μ, a **Hamiltonian *G*-space**, or **Hamiltonian *G*-manifold**.

Proposition 2.3.6 (Properties of the moment map)**.**

I) *If $\mu : M \to \mathfrak{g}$ is a moment map, then for all $X \in \mathfrak{g}$*

$$d\mu_p(X_M) = X_{\mathfrak{g}^*}\big(\mu(p)\big).$$

Hence $d\mu_p\big(T_p(G \cdot p)\big) = T_{\mu(p)}\big(G \cdot \mu(p)\big)$, i.e. the image under $d\mu$ of the tangent space to the orbit at p is the tangent space to the coadjoint orbit at $\mu(p)$.

II) *Suppose the action of a connected Lie group G on M is Hamiltonian with moment map $\mu : M \to \mathfrak{g}^*$. Suppose $f : H \to G$ is a Lie group homomorphism of a connected Lie group H, and let $(df)^* : \mathfrak{g}^* \to \mathfrak{h}^*$ denote the dual of the linear map $df : \mathfrak{h} \to \mathfrak{g}$. Then the induced action of H on M given by $h \cdot p = f(h) \cdot p$ is Hamiltonian with moment map $\nu = (df)^* \circ \mu$.*

III) *Suppose G is a Lie group acting in a Hamiltonian way on two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) with moment maps $\mu_1 : M_2 \to \mathfrak{g}^*$ and $\mu_2 : M_2 \to \mathfrak{g}^*$. Then the diagonal action of G on $(M_1 \times M_2, \pi_1^*\omega_1 + \pi_2^*\omega_2)$ is Hamiltonian with moment map*

$$\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2).$$

IV) *Suppose G and H act on M in a Hamiltonian way with moment maps μ_G and μ_H. Suppose the actions of G and H commute, and the moment maps are invariant with respect to the other action. Then there exists a Hamiltonian action of $G \times H$ on M with moment map*

$$\mu : M \to (\mathfrak{g} \times \mathfrak{h})^* = \mathfrak{g}^* \times \mathfrak{h}^*,$$
$$\mu(p) = \big(\mu_G(p), \mu_H(p)\big)$$

Proof. I): We have already shown the first statement in the proof of Proposition 2.3.4. The second statement follows from Corollary $A.4.3.1$,

$$d\mu_p\big(T_p(G \cdot p)\big) = \{d\mu_p(X_M) : X \in \mathfrak{g}\} = \{X_{\mathfrak{g}^*}\big(\mu(p)\big) : X \in \mathfrak{g}\} = T_{\mu(p)}\big(G \cdot \mu(p)\big).$$

II): Let $X \in \mathfrak{h}$, then

$$X_M(p) = \frac{d}{dt}\bigg|_{t=0} f\big(\exp(tX)\big)\cdot p = \frac{d}{dt}\bigg|_{t=0} \exp\big(df(tX)\big)\cdot p = \big(df(X)\big)_M(p).$$

Hence as μ is a moment map for the G-action on M,

$$i(X_M)\omega = i\big(df(X)_M)\omega = d\,\langle \mu, df(X)\rangle = d\,\langle (df)^*\circ\mu, X\rangle\,,$$

and $\nu = (df)^*\circ\mu$ satisfies the moment map condition. As both G and H are connected, to show that ν is equivariant it suffices to show that comoment map ν^* is a Lie algebra homomorphism. Let $X \in \mathfrak{h}$ and $p \in M$, then

$$\big[\nu^*(X)\big](p) = \big[\nu(p)\big](X) = \big[\big((df)^*\circ\mu\big)(p)\big](X) = \big[\mu(p)\big]\big(df(X)\big) = \big[(\mu^*\circ df)(X)\big](p).$$

Thus $\nu^* = \mu^*\circ df$, and ν^* is a Lie algebra homomorphism as both μ^* and df are Lie algebra homomorphisms.

III): Under the identification $T_{(p_1,p_2)}(M_1\times M_2) = T_{p_1}M_1 \times T_{p_2}M_2$, the fundamental vector field associated to $X \in \mathfrak{g}$ is

$$X_{M_1\times M_2}(p_1,p_2) = \frac{d}{dt}\bigg|_{t=0} (\exp(tX)\cdot p_1, \exp(tX)\cdot p_2) = \big(X_{M_1}(p_1), X_{M_2}(p_2)\big).$$

For ease of notation, let $X^\# = X_{M_1\times M_2}$. Hence

$$d(\pi_1)_{(p_1,p_2)}X^\# = X_{M_1}(p_1) \quad\text{and}\quad d(\pi_2)_{(p_1,p_2)}X^\# = X_{M_2}(p_2),$$

and so

$$\begin{aligned} i(X^\#)\omega &= i(X^\#)\big(\pi_1^*\omega_1 + \pi_2^*\omega_2\big)\\ &= i(X^\#)\pi_1^*\omega_1 + i(X^\#)\pi_2^*\omega_2\\ &= \pi_1^*\big(i(X_{M_1})\omega_1\big) + \pi_2^*\big(i(X_{M_2})\omega_2\big)\\ &= \pi_1^*\big(d\,\langle\mu_1, X\rangle\,\big) + \pi_2^*\big(d\,\langle\mu_2, X\rangle\,\big)\\ &= d\big(\pi_1^*\,\langle\mu_1, X\rangle + \pi_2^*\,\langle\mu_2, X\rangle\,\big). \end{aligned}$$

Hence $\mu = \pi_1^*\mu_1 + \pi_2^*\mu_2$ satisfies the moment map condition, and its action on $(p_1,p_2)\in M_1\times M_2$ is

$$\mu(p_1,p_2) = \big(\pi_1^*\mu_1 + \pi_2^*\mu_2\big)(p_1,p_2) = \mu_1(p_1) + \mu_2(p_2).$$

Equivariance of μ is now trivial as the coadjoint action is linear.

IV): The action of $G\times H$ on M is defined by $(g,h)\cdot p = g\cdot(h\cdot p) = h\cdot(g\cdot p)$, where the second equality is the commutativity assumption. Now let $X\in\mathfrak{g}$ and $Y\in\mathfrak{h}$, then

$$\begin{aligned} (X,Y)_M(p) &= \frac{d}{dt}\bigg|_{t=0} \exp(tX)\cdot\exp(tY)\cdot p\\ &= \left(\frac{d}{dt}\bigg|_{t=0}\exp(tX)\right)\big(\exp(0)\cdot p\big) + \left(\frac{d}{dt}\bigg|_{t=0}\exp(tY)\right)\big(\exp(0)\cdot p\big)\\ &= X_M(p) + Y_M(p), \end{aligned}$$

where the second equality uses the chain rule and commutativity of the two actions. Hence

$$i\big((X,Y)_M\big)\omega = i(X_M)\omega + i(Y_M)\omega = d\left\langle \mu_G, X\right\rangle + d\left\langle \mu_H, Y\right\rangle.$$

However, by definition of μ

$$\langle \mu, (X,Y)\rangle = \langle \mu_G, X\rangle + \langle \mu_H, Y\rangle$$

and so μ satisfies the moment map condition. To see that it is equivariant, by direct calculation for all $g \in G$, $h \in H$, and $p \in M$:

$$\begin{aligned}
\mu\big((g,h)\cdot p\big) &= \mu\big(g\cdot(h\cdot p)\big) \\
&= \big(\mu_G\big(h\cdot(g\cdot p)\big), \mu_H\big(g\cdot(h\cdot p)\big)\big) \\
&= (\mu_G(g\cdot p), \mu_H(h\cdot p)) \\
&= \big(\mathrm{Ad}^*_g\,\mu_G(p), \mathrm{Ad}^*_h\,\mu_H(p)\big) \\
&= \big(\,\mathrm{Ad}^*_g \times \mathrm{Ad}^*_h\,\big)\big(\mu_G(p), \mu_H(p)\big) \\
&= \mathrm{Ad}^*_{(g,h)}\,\mu(p).
\end{aligned}$$

□

Let (M,ω,G,μ) be a Hamiltonian G-space. Under the identification $T_\xi\mathfrak{g}^* \cong \mathfrak{g}^*$ for all $\xi \in \mathfrak{g}^*$, we can view the derivative of the moment map as a function $d\mu_p : T_pM \to \mathfrak{g}^*$. The next proposition relates the kernel and image of $d\mu_p$ to well-known subspaces.

Proposition 2.3.7. *Let (M,ω,G,μ) be a Hamiltonian G-space, and view the derivative of the moment map as a function $d\mu_p : T_pM \to \mathfrak{g}^*$. Then*

- **I)** $\ker(d\mu_p) = \big(T_p(G\cdot p)\big)^\omega$.
- **II)** $\mathrm{im}(d\mu_p) = \mathfrak{g}_p^\circ$*, the annihilator of the stabiliser algebra of p.*

Proof. I): By (2.3.8), $\omega\big(X_M(p), v\big) = \langle d\mu_p(v), X\rangle$ for all $v \in T_pM$ and $X \in \mathfrak{g}$. Hence $v \in \ker d\mu_p$ if, and only if, $v \in \big(T_p(G\cdot p)\big)^\omega$.

II): If $X \in \mathfrak{g}_p$ then $\langle d\mu_p(v), X\rangle = \omega\big(X_M(p), v\big) = 0$ for all $v \in T_pM$, implying $\mathrm{im}(d\mu_p) \subseteq \mathfrak{g}_p^\circ$. The reverse inclusion follows via dimension counting:

$$\begin{aligned}
\dim\big(\,\mathrm{im}(d\mu_p)\big) &= \dim T_pM - \dim\big(\,\ker(d\mu_p)\big) \\
&= \dim T_pM - \dim\big(\big(T_p(G\cdot p)\big)^\omega\big) \\
&= \dim\big(T_p(G\cdot p)\big).
\end{aligned}$$

Let τ_p denote the infinitesimal action on T_pM, sending $X \in \mathfrak{g}$ to $X_M(p) \in T_pM$. Then

$$\dim\big(\,\mathrm{im}(d\mu_p)\big) = \dim\big(T_p(G\cdot p)\big) = \dim\big(\,\mathrm{im}(\tau_p)\big),$$

and so

$$\dim\big(\,\mathrm{im}(\tau_p)\big) = \dim\mathfrak{g} - \dim\big(\,\ker(\tau_p)\big) = \dim\mathfrak{g} - \dim\mathfrak{g}_p = \dim\mathfrak{g}_p^\circ.$$

□

Corollary 2.3.7.1. *Let (M, ω, G, μ) be a Hamiltonian G-space. The G action at $p \in M$ is locally free if, and only if, p is a regular point for μ.*

Proof. The action is locally free at $p \in M$ if, and only if, $\mathfrak{g}_p = \{0\}$; which occurs if, and only if, $\mathfrak{g}^* = \mathfrak{g}_p^\circ = \mathrm{im}(d\mu_p)$. Hence $d\mu_p$ is surjective and p is a regular point. □

2.3.1 Examples

In this subsection we provide concrete examples of Hamiltonian G-spaces.

Example 2.3.8. Let S^1 act on $\mathbb{C}$ by multiplication, where $\mathbb{C}$ is viewed as a symplectic manifold with standard symplectic form $\omega = dx \wedge dy$. Recall that we can identify the Lie algebra with $\mathbb{R}$, under which the exponential map is $\exp : \mathbb{R} \to S^1$, $X \mapsto e^{iX}$. Hence the fundamental vector field associated to $X \in \mathbb{R}$ at $z = x + iy \in \mathbb{C}$ is

$$X_{\mathbb{C}}(z) = \left.\frac{d}{dt}\right|_{t=0} e^{itX} z = iXz = -Xy + iXx,$$

or equivalently

$$X_{\mathbb{C}}(z) = -Xy\frac{\partial}{\partial x} + Xx\frac{\partial}{\partial y}.$$

Hence at $z = x + iy$,

$$\begin{aligned} i(X_{\mathbb{C}})\omega &= -Xydy - Xxdx = -X(ydy + xdx) = -\frac{X}{2}(dy^2 + dx^2) = -\frac{X}{2}d(x^2 + y^2) \\ &= d\Big(-\frac{X}{2}|z|^2\Big) \end{aligned}$$

Therefore, we find that $\mu : \mathbb{C} \to \mathbb{R}$ defined by $\mu(z) = -\frac{1}{2}|z|^2$ satisfies the moment map condition. It is trivially equivariant with respect to the S^1 action, and therefore is a moment map for the action.

Suppose instead that we had chosen to identify $\mathrm{Lie}(S^1) = i\mathbb{R}$, and let $i\mathbb{R}$ be identified with its dual via the inner product $(X, Y) = -XY$. Then the previous work shows that S^1-action on $\mathbb{C}$ is Hamiltonian with moment map $\mu(z) = -i/2|z|^2$. (c.f. Example 2.3.9.)

View $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ as n-copies of $\mathbb{C}$. Then the standard symplectic form on $\mathbb{C}^n$ is given by pullback of the standard form on each of the factors. Therefore, using Proposition 2.3.6(III), we have the diagonal action of S^1 on $\mathbb{C}^n$ is Hamiltonian, with moment map $\mu : \mathbb{C}^n \to \mathbb{R}$ defined by

$$\mu(z_1, \ldots, z_n) = -\frac{1}{2}\sum_{i=1}^{n} |z_i|^2.$$

◀

Example 2.3.9. Suppose $\mathrm{U}(n)$ acts on $\mathbb{C}^n$ by matrix multiplication. The Lie algebra $\mathfrak{u}(n)$ consists of anti-self adjoint matrices, and the negative Killing form $(X, Y) = -\operatorname{tr}(XY)$ defines a $\mathrm{U}(n)$-invariant positive-definite inner product on $\mathfrak{u}(n)$. Identify $\mathfrak{u}(n)$ and $\mathfrak{u}(n)^*$ by this inner product. Since $z \in \mathbb{C}^n$ is a column vector, zz^* is a self-adjoint $n \times n$-matrix, so define a function $\mu : \mathbb{C}^n \to \mathfrak{u}(n)^*$ by

$$\mu(z) = \frac{1}{2i} zz^* = -\frac{i}{2} zz^*.$$

We claim that this is a moment map for the $\mathrm{U}(n)$ action on $\mathbb{C}^n$. First note for all $X \in \mathfrak{u}(n)$

$$\langle \mu(z), X \rangle = -\operatorname{tr}(\mu(z)X) = \frac{i}{2} \operatorname{tr}(zz^*X) = \frac{i}{2} z^*Xz.$$

Hence for all $w \in \mathbb{C}^n$,

$$\begin{aligned} d\langle \mu, X \rangle_z (w) &= \frac{d}{dt}\bigg|_{t=0} \langle \mu(z + tw), X \rangle \\ &= \frac{i}{2} \frac{d}{dt}\bigg|_{t=0} \big((z + tw)^* X (z + tw)\big) \\ &= \frac{i}{2}(w^*Xz + z^*Xw). \end{aligned}$$

On the other hand,

$$\begin{aligned} \omega\big(X_{\mathbb{C}^n}(z), w\big) = \omega(Xz, w) &= \operatorname{Im}\big((Xz)^*w\big) \\ &= \frac{1}{2i}\big((Xz)^*w - \overline{(Xz)^*w}\big) \\ &= \frac{1}{2i}\big(z^*X^*w - w^*Xz\big) \\ &= \frac{i}{2}(z^*Xw + w^*Xz), \end{aligned}$$

where we have used the fact that $X^* = -X$. Therefore $d\langle \mu, X \rangle = i(X_{\mathbb{C}^n})\omega$, and μ is a moment map for the action. As $\mathrm{U}(n)$ is a matrix Lie group, $\operatorname{Ad}_g X = gXg^{-1} = gXg^*$ and equivariance follows immediately. ◀

It turns out a large class of Hamiltonian G-spaces can be obtained from certain symplectic manifolds, known as exact symplectic manifolds.

Definition 2.3.10. A symplectic manifold (M, ω) is said to be **exact**, if there exists a 1-form $\theta \in \Omega^1(M)$ such that $\omega = -d\theta$. The form θ is called the **symplectic potential** of M.

Example 2.3.11. The cotangent bundle T^*M is an exact symplectic manifold with symplectic potential θ, the Liouville 1-form defined by (2.1.5). ◀

Remark 2.3.12. Note a closed symplectic manifold (M, ω) can never be exact. If it was, then

$$\omega^n = -d\theta \wedge \omega^{n-1} = -d(\theta \wedge \omega^{n-1}) + \theta \wedge d\omega^{n-1} = -d(\theta \wedge \omega^{n-1})$$

and Stokes' theorem would imply that M has zero volume. ⧫

Theorem 2.3.13. *Suppose (M,ω) is an exact symplectic manifold with symplectic potential θ, and G a Lie group acting symplectically on M. If θ is G invariant, then the action is Hamiltonian with moment map $\langle \mu, X\rangle = i(X_M)\theta$.*

Proof. Since θ is G-invariant $\mathcal{L}_{X_M}\theta = 0$, and Cartan's magic formula implies

$$0 = d\big(i(X_M)\theta\big) + i(X_M)d\theta = d\big(i(X_M)\theta\big) - i(X_M)\omega.$$

Hence, $d\big(i(X_M)\theta\big) = i(X_M)\omega$, and $\langle \mu, X\rangle = i(X_M)\theta$ satisfies the moment map condition for all $X \in \mathfrak{g}$. Thus the action is weakly Hamiltonian, and it remains to show that μ is equivariant. We need to show

$$\langle \mu(g\cdot p), X\rangle = \big\langle \mu(p), \operatorname{Ad}_{g^{-1}} X\big\rangle$$

for all $p \in M$, $g \in G$, and $X \in \mathfrak{g}$. As θ is G-invariant, $\mathcal{A}_g^*\theta = \theta$,

$$\theta_{g\cdot p}\big(d(\mathcal{A}_g)_p Y_p\big) = \theta_p(Y_p)$$

for all $Y_p \in T_pM$. Setting $Y_p = (\operatorname{Ad}_{g^{-1}} X)_M(p) = d(\mathcal{A}_{g^{-1}})_{g\cdot p}\big(X_M(g\cdot p)\big)$, the above equation becomes

$$\begin{aligned}\theta_p\left((\operatorname{Ad}_{g^{-1}} X)_M(p)\right) &= \theta_{g\cdot p}\left(d(\mathcal{A}_g)_p d(\mathcal{A}_{g^{-1}})_{g\cdot p}\big(X_M(g\cdot p)\big)\right)\\ &= \theta_{g\cdot p}\left(d(\mathcal{A}_g\circ\mathcal{A}_{g^{-1}})_{g\cdot p}\big(X_M(g\cdot p)\big)\right)\\ &= \theta_{g\cdot p}\big(X_M(g\cdot p)\big).\end{aligned}$$

However, this implies

$$\left(i\big((\operatorname{Ad}_{g^{-1}} X)_M\big)\theta\right)(p) = \big(i(X_M)\theta\big)(g\cdot p),$$

or equivalently

$$\langle \mu(g\cdot p), X\rangle = \big\langle \mu(p), \operatorname{Ad}_{g^{-1}} X\big\rangle = \big\langle \operatorname{Ad}_g^* \mu(p), X\big\rangle$$

so μ is equivariant. □

Example 2.3.14. Let $G \to \operatorname{Symp}(M,\omega)$ be a symplectic action on (M,ω). We can lift this to an action on T^*M, called the cotangent lift, by defining

$$\begin{aligned}&G\times T^*M \to T^*M,\\ &\big(g,(p,\eta)\big) \mapsto \big(\mathcal{A}_g(p), \mathcal{A}_{g^{-1}}^*\eta\big).\end{aligned}$$

Let $\hat{\mathcal{A}}_g$ denote the lift of $\mathcal{A}_g$ to the action on T^*M. We claim that the Liouville 1-form is invariant under this action. First, note that the canonical projection $\pi : T^*M \to M$ is equivariant with respect to the relevant actions: for $\eta \in T_p^*M$ and $g \in G$

$$\pi\big(g\cdot(p,\eta)\big) = \pi\big((g\cdot p, \mathcal{A}_{g^{-1}}^*\eta)\big) = g\cdot p = g\cdot\pi(p,\eta).$$

Hence $d\pi_\alpha\big(X_{T^*M}(\alpha)\big) = X_M\big(\pi(\alpha)\big)$ for all $\alpha \in T^*M$. Now, for all $v \in T_\alpha(T^*M)$

$$\begin{aligned}\big(\hat{\mathcal{A}}_g^*\theta\big)_\alpha(v) &= \theta_{\hat{\mathcal{A}}_g(\alpha)}\big(d(\hat{\mathcal{A}}_g)_\alpha(v)\big)\\ &= (\mathcal{A}_{g^{-1}}^*\alpha)\big(d(\pi\circ\hat{\mathcal{A}}_g)_\alpha(v)\big)\\ &= (\mathcal{A}_{g^{-1}}^*\alpha)\big(d(\mathcal{A}_g\circ\pi)_\alpha(v)\big)\\ &= \alpha\big(d\pi_\alpha(v)\big)\\ &= \theta_\alpha(v).\end{aligned}$$

Therefore, the induced action on T^*M is symplectic, and Theorem 2.3.13 further implies this action is Hamiltonian with moment map $\mu : T^*M \to \mathfrak{g}^*$ defined by

$$\langle\mu(\alpha), X\rangle = i(X_{T^*M})\theta_\alpha = \alpha\left(d\pi_\alpha\big(X_{T^*M}(\alpha)\big)\right) = \alpha\left(X_M\big(\pi(\alpha)\big)\right).$$

More concretely, if $\alpha = (p,\eta) \in T^*M$ with coordinates (p^i, η_i), then $X_M(p)$ has the coordinate representation $(X_M)^i\partial/\partial p^i \in T_pM$, and

$$\langle\mu(p,\eta), X\rangle = \sum_{i=1}^{n}(X_M)^i\eta_i.$$

◀

Example 2.3.15. View $\mathbb{R}^n$ as a Lie group acting on itself via left-translation. Then the fundamental vector field for $X \in \mathfrak{g} = \mathbb{R}^n$ is

$$X_{\mathbb{R}^n}(p) = \left.\frac{d}{dt}\right|_{t=0}\exp(tX)\cdot p = \left.\frac{d}{dt}\right|_{t=0} tX + p = X,$$

independent of $p \in M$. Now, by Theorem 2.3.13, the moment map for the induced action on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ is $\mu : \mathbb{R}^{2n} \to \mathfrak{g}^* \cong \mathbb{R}^n$ defined by

$$\langle\mu(x,y), X\rangle = \sum_{i=1}^{n} X^i y^i = \langle y, X\rangle$$

where the pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$ is just the standard inner product on $\mathbb{R}^n$. Note this implies $\mu(x,y) = y$, and μ represents linear momentum. ◀

Example 2.3.16. Let G be a subgroup of $\mathrm{GL}(n,\mathbb{R})$ acting on $\mathbb{R}^n$ by matrix multiplication. Then the fundamental vector field associated to $X \in \mathfrak{g}$ is

$$X_{\mathbb{R}^n}(p) = \left.\frac{d}{dt}\right|_{t=0}\exp(tX)\cdot p = \left.\frac{d}{dt}\right|_{t=0}\Big(\sum_{n=0}^{\infty}\frac{1}{n!}(tX)^n\Big)p = Xp.$$

Therefore, by Theorem 2.3.13, the moment map for the induced action on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ is

$$\langle\mu(p,q), X\rangle = \sum_{i=1}^{3}(Xp)^i q^i = (q, Xp)$$

where $(\ , \ )$ is the standard inner product on $\mathbb{R}^n$.

Consider the special case of SO(3) acting on $\mathbb{R}^3$. Recall that

$$\mathfrak{so}(3) = \{X \in M(3, \mathbb{R}) : X = -X^T\},$$

so there is an identification $\mathfrak{so}(3) \cong \mathbb{R}^3$ given by

$$X = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \longleftrightarrow \tilde{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Using the standard inner product on $\mathbb{R}^n$ to identify $\mathbb{R}^n$ with its dual, then

$$\langle \mu(p, q), X \rangle = (q, Xp) = (q, \tilde{X} \times p) = \det(q, \tilde{X}, p) = (p \times q, \tilde{X}).$$

Hence the moment map is given by $\mu(p, q) = p \times q$, and represents angular momentum. ◀

Example 2.3.17. Suppose that a Lie group G acts on itself by left translations. For $X \in \mathfrak{g}$, the fundamental vector field of X is

$$X_G(g) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX)g = \left.\frac{d}{dt}\right|_{t=0} R_g\big(\exp(tX)\big) = d(R_g)_e X,$$

the right-invariant vector field generated by X. Hence by Theorem 2.3.13, the moment map for the induced action on T^*G is

$$\langle \mu(g, \eta), X \rangle = \eta\big(d(R_g)_e X\big),$$

or equivalently, $\mu(g, \eta) = R_g^* \eta$. However, the cotangent bundle T^*G is trivial, as can be seen by the global trivialisation via left translations,

$$\begin{aligned} G \times \mathfrak{g}^* &\to T^*G, \\ (g, \alpha) &\mapsto (g, L_{g^{-1}}^* \alpha) \end{aligned}$$

with inverse $(g, \lambda) \mapsto (g, L_g^* \lambda)$. Under this identification, the moment map $\mu : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is given by the coadjoint action on η,

$$\mu(g, \eta) = \mu(g, L_{g^{-1}}^* \eta) = R_g^*\big(L_{g^{-1}}^* \eta\big) = (L_{g^{-1}} \circ R_g)^* \eta = (\mathrm{Ad}_{g^{-1}})^* \eta = \mathrm{Ad}_g^* \eta.$$

Similarly suppose G acts on itself via right translations, i.e. $g_1 \cdot g_2 = g_2 g_1^{-1}$. Then the fundamental vector field for $X \in \mathfrak{g}$ is

$$X_G(g) = \left.\frac{d}{dt}\right|_{t=0} g \exp(-tX) = \left.\frac{d}{dt}\right|_{t=0} L_g\big(\exp(-tX)\big) = -d(L_g)_e X,$$

the negative left-invariant vector field generated by X. The moment map for the induced action of G on T^*G is $\mu(g, \eta) = -L_g^* \eta$. Using the global trivialisation of T^*G we find that the moment map $\mu : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is negative projection onto the second factor:

$$\mu(g, \eta) = \mu(g, L_{g^{-1}}^* \eta) = -L_g^*\big(L_{g^{-1}}^* \eta\big) = -\eta.$$

◀

2.3.2 Coadjoint Orbits

Let G act on its dual Lie algebra $\mathfrak{g}^*$ via the coadjoint action. In this section, we show the orbits for this action have a natural symplectic form for which the coadjoint action is actually Hamiltonian.

Recall that for adjoint action of G on $\mathfrak{g}$, the fundamental vector field generated by $X \in \mathfrak{g}$ is

$$X_{\mathfrak{g}}(Y) = \left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad}_{\exp(tX)} Y = \operatorname{ad}_X Y = [X, Y],$$

for all $Y \in \mathfrak{g}$. Now consider the coadjoint action of G on $\mathfrak{g}^*$, and let $X \in \mathfrak{g}$. Then for all $\xi \in \mathfrak{g}^*$ and $Y \in \mathfrak{g}$,

$$\begin{aligned} \langle X_{\mathfrak{g}^*}(\xi), Y\rangle &= \left\langle \left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad}^*_{\exp(tX)} \xi, Y \right\rangle \\ &= \left\langle \xi, \left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad}_{\exp(-tX)} Y \right\rangle \\ &= \langle \xi, [Y, X]\rangle . \end{aligned}$$

Thus, let $\xi \in \mathfrak{g}^*$ and define a skew-symmetric bilinear form ω_ξ on $\mathfrak{g}$ by

$$\omega_\xi(X, Y) = \langle \xi, [X, Y]\rangle . \tag{2.3.10}$$

As the adjoint map is a Lie algebra homomorphism, ω_ξ is G-invariant:

$$\begin{aligned} \omega_{g\cdot\xi}(\operatorname{Ad}_g X, \operatorname{Ad}_g Y) &= \left\langle \operatorname{Ad}^*_g \xi, [\operatorname{Ad}_g X, \operatorname{Ad}_g Y]\right\rangle \\ &= \left\langle \operatorname{Ad}^*_g \xi, \operatorname{Ad}_g [X, Y]\right\rangle \\ &= \langle \xi, [X, Y]\rangle \\ &= \omega_\xi(X, Y), \end{aligned}$$

Moreover,

$$\omega_\xi(X, Y) = \langle \xi, [X, Y]\rangle = \langle Y_{\mathfrak{g}^*}(\xi), X\rangle$$

so the kernel of ω is

$$\ker \omega_\xi = \{Y \in \mathfrak{g} : Y_{\mathfrak{g}^*}(\xi) = 0\} = \mathfrak{g}_\xi,$$

the stabiliser algebra of ξ for the coadjoint action. As G acts transitively on the coadjoint orbits

$$T_\xi(G \cdot \xi) = T_e(G/G_\xi) = \mathfrak{g}/\mathfrak{g}_\xi,$$

by Corollary $A.4.3.1$. Thus ω_ξ restricts to a non-degenerate 2-form on the coadjoint orbit $G \cdot \xi$. We now show that ω_ξ is closed. For ease of notation, we denote $X^\# = X_{\mathfrak{g}^*}(\xi)$ for $X \in \mathfrak{g}$. Using the explicit formula for the exterior derivative

$$\begin{aligned} d\omega_\xi(X^\#, Y^\#, Z^\#) = X^\#\omega_\xi(Y^\#, Z^\#) + Z^\#\omega_\xi(X^\#, Y^\#) + Y^\#\omega_\xi(Z^\#, X^\#) \\ - \omega_\xi([Y^\#, Z^\#], X^\#) - \omega_\xi([X^\#, Y^\#], Z^\#) - \omega_\xi([Z^\#, X^\#], Y^\#). \end{aligned} \tag{2.3.11}$$

We first deal with the terms on the first line of (2.3.11). By the properties of the Lie derivative:

$$\begin{aligned}
X^{\#}\omega_\xi(Y^{\#}, Z^{\#}) &= \mathcal{L}_{X^{\#}}\omega_\xi(Y^{\#}, Z^{\#}) \\
&= (\mathcal{L}_{X^{\#}}\omega_\xi)(Y^{\#}, Z^{\#}) + \omega_\xi([X^{\#}, Y^{\#}], Z^{\#}) \\
&\quad + \omega_\xi(Y^{\#}, [X^{\#}, Z^{\#}]) \\
&= \omega_\xi(-[X,Y]^{\#}, Z^{\#}) + \omega_\xi(Y^{\#}, -[X,Z]^{\#}) \\
&= \langle \xi, [-[X,Y], Z] \rangle + \langle \xi, [Y, -[X,Z]] \rangle \\
&= \langle \xi, [Z,[X,Y]] + [Y,[Z,X]] \rangle \\
&= -\langle \xi, [X,[Y,Z]] \rangle,
\end{aligned}$$

where the third equality is because ω is G-invariant, and the last equality is the Jacobi identity. Hence the first line of (2.3.11) becomes

$$\begin{aligned}
X^{\#}\omega_\xi(Y^{\#}, Z^{\#}) &+ Z^{\#}\omega_\xi(X^{\#}, Y^{\#}) + Y^{\#}\omega_\xi(Z^{\#}, X^{\#}) \\
&= -\langle \xi, [X,[Y,Z]] \rangle - \langle \xi, [Z,[X,Y]] \rangle - \langle \xi, [Y,[Y,X]] \rangle \\
&= -\langle \xi, [X,[Y,Z]] + [Z,[X,Y]] + [Y,[Y,X]] \rangle \\
&= 0
\end{aligned}$$

by the Jacobi identity. Dealing with the second line of (2.3.11),

$$\omega_\xi([Y^{\#}, Z^{\#}], X^{\#}) = \omega_\xi(-[Y,Z]^{\#}, X^{\#}) = \langle \xi, [-[Y,Z], X] \rangle = \langle \xi, [X,[Y,Z]] \rangle .$$

Therefore, the second line of (2.3.11) is

$$-\omega([Y^{\#}, Z^{\#}], X^{\#}) - \omega([X^{\#}, Y^{\#}], Z^{\#}) - \omega([Z^{\#}, X^{\#}], Y^{\#}) = 0,$$

again, by the Jacobi identity. Hence $d\omega_\xi = 0$ and ω_ξ is closed.

Altogether, we have proved the following theorem.

Theorem 2.3.18. *The coadjoint orbit $G \cdot \xi$ is a symplectic manifold with symplectic form*

$$\omega_\xi(X,Y) = \langle \xi, [X,Y] \rangle$$

where $X, Y \in \mathfrak{g}$ are viewed as tangent vectors to the coadjoint orbit via $T_\xi(G \cdot \xi) = \mathfrak{g}/\mathfrak{g}_\xi$. The form ω_ξ is called the ***Konstant-Kirillov-Souriau****, or* ***KKS form****.*

In fact, a coadjoint orbit $G \cdot \xi$ is a Hamiltonian G-space, with moment map $\mu : G \cdot \xi \to \mathfrak{g}^*$ the inclusion of the coadjoint orbit in $\mathfrak{g}^*$. Let $\mu : G \cdot \xi \hookrightarrow \mathfrak{g}^*$ be the inclusion map. Then for all $X, Y \in \mathfrak{g}$, $\eta \in G \cdot \xi$:

$$\begin{aligned}
i(Y) d \langle \mu, X \rangle_\eta &= \left.\frac{d}{dt}\right|_{t=0} \left\langle \mu\big(\operatorname{Ad}^*_{\exp(-tY)} \eta\big), X \right\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \left\langle \operatorname{Ad}^*_{\exp(-tY)} \eta, X \right\rangle \\
&= \langle \eta, [X,Y] \rangle \\
&= \omega_\eta(X,Y) \\
&= i(Y)\big(i(X)\omega_\eta\big),
\end{aligned}$$

showing μ satisfies the moment map condition. Moreover, μ is trivially equivariant by definition of a coadjoint orbit.

2.3.3 Existence

Let (M, ω) be a symplectic manifold, and G a Lie group acting symplectically on M. In this section we provide conditions for which the action of G on M is Hamiltonian.

Our first criteria for a symplectic action to be Hamiltonian is a condition on the manifold M.

Theorem 2.3.19. *Suppose (M, ω) is a compact, connected symplectic manifold with a symplectic action of a connected Lie group G. Suppose $H^1_{dR}(M) = 0$, then the action is Hamiltonian.*

Proof. Under the condition $H^1_{dR}(M) = 0$ any symplectic vector field is Hamiltonian by Proposition 2.2.4(II).

Choose a basis $\{e_1, \ldots, e_d\}$ of $\mathfrak{g}$. As every symplectic vector field is Hamiltonian, for each e_i we can find a smooth function $\mu^{e_i} \in C^\infty(M)$ such that $i\big((e_i)_M\big)\omega = d\mu^{e_i}$. The functions μ^{e_i} are unique up to a constant, which we fix by setting

$$\int_M \mu^{e_i}\, \omega^n = 0.$$

Moreover, as $\{e_1, \ldots, e_d\}$ is a basis for $\mathfrak{g}$, every $X \in \mathfrak{g}$ is a unique linear combination $X = \sum_{i=1}^d X^i e_i$, $X^i \in \mathbb{R}$, and set

$$\mu^X = \sum_{i=1}^d X^i \mu^{e_i}.$$

By linearity μ^X satisfies $i(X_M)\omega = d\mu^X$, and so $\mu^* : \mathfrak{g} \to C^\infty(M)$ defined by $\mu^*(X) = \mu^X$ satisfies the moment map condition. Hence the action is weakly Hamiltonian, and to show it is Hamiltonian it suffices to show μ^* is a Lie algebra homomorphism by Proposition 2.3.4. Consider the function $c^{X,Y} \in C^\infty(M)$ defined by

$$c^{X,Y} = \mu^{[X,Y]} - \{\mu^X, \mu^Y\}.$$

By (2.3.6) $c^{X,Y}$ is locally constant on M, and hence constant as M is connected. Now, as $[X, Y] \in \mathfrak{g}$, it is a linear combination of the basis vectors e_i, and therefore $\int_M \mu^{[X,Y]}\, \omega^n = 0$. On the other hand,

$$\begin{aligned}
\{\mu^X, \mu^Y\}\omega^n &= \big(\mathcal{L}_{Y_M}\mu^X\big)\omega^n \\
&= \mathcal{L}_{Y_M}(\mu^X\omega^n) - \mu^X\big(\mathcal{L}_{Y_M}\omega^n\big) \\
&= \mathcal{L}_{Y_M}(\mu^X\omega^n) \\
&= d\big(i(Y_M)(\mu^X\omega^n)\big) + i(Y_M)d(\mu^X\omega^n) \\
&= d\big(\mu^X i(Y_M)\omega^n\big)
\end{aligned}$$

and $\{\mu^X, \mu^Y\}\omega^n$ is exact. Hence Stokes' theorem implies $\int_M \{\mu^X, \mu^Y\}\omega^n = 0$. Therefore $\int_M c^{X,Y}\omega^n = 0$ and as $c^{X,Y}$ is constant, $c^{X,Y} = 0$. Altogether, μ^* is a Lie algebra homomorphism, and the action is Hamiltonian as required. □

The second criterion for a symplectic action to be Hamiltonian is on the Lie group G. To prove it, we need a few results from Lie algebra cohomology.

Let $\mathfrak{g}$ be a Lie algebra. A n-cochain for $\mathfrak{g}$ is an anti-symmetric n-linear map $f : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$, and let $C^n(\mathfrak{g}, \mathbb{R})$ denote the collection of all n-cochains. (Note that $C^n(\mathfrak{g}, \mathbb{R}) = \bigwedge^n \mathfrak{g}^*$.) For each n, define the coboundary operator $\delta_n : C^n(\mathfrak{g}, \mathbb{R}) \to C^{n+1}(\mathfrak{g}, \mathbb{R})$ by

$$\delta_n f(X_0, \ldots, X_n) = \sum_{i<j} (-1)^{i+j} f\big([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n\big),$$

where $\hat{X}_i$ denotes the ommision of X_i. For simplicity we often write δ in place for δ_n, as it is usually clear from context which coboundary map we are using.

The coboundary maps induce a sequence of vector spaces

$$0 \longrightarrow C^1(\mathfrak{g}, \mathbb{R}) \xrightarrow{\delta_1} C^2(\mathfrak{g}, \mathbb{R}) \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-1}} C^n(\mathfrak{g}, \mathbb{R}) \xrightarrow{\delta_n} \cdots, \tag{2.3.12}$$

and we claim that this sequence is a cochain complex.

Proposition 2.3.20. *For all n, $\delta_{n+1}\delta_n = 0$, and* (2.3.12) *is a cochain complex.*

Proof. Let $f \in C^n(\mathfrak{g}, \mathbb{R})$ be a n-cochain. We must show that $\delta_{n+1}\delta_n f = 0$; which will follow from a direct, but tedious, calculation. In the calculation we sequester the ellipses and write $f(X_0, \hat{X}_i, \hat{X}_j, X_n)$ for $f(X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_n)$. Thus

$$\begin{aligned}
\delta^2 f(X_0, \ldots, X_{n+1}) &= \sum_{i<j} (-1)^{i+j} \delta f\big([X_i, X_j], X_0, \hat{X}_i, \hat{X}_j, X_{n+1}\big) \\
&= \sum_{k<i<j} (-1)^{i+j+k} f\big([[X_i, X_j], X_k], X_0, \hat{X}_k, \hat{X}_i, \hat{X}_j, X_{n+1}\big) \\
&\quad + \sum_{i<k<j} (-1)^{i+j+k} f\big([[X_i, X_j], X_k], X_0, \hat{X}_i, \hat{X}_k, \hat{X}_j, X_{n+1}\big) \\
&\quad + \sum_{i<j<k} (-1)^{i+j+k} f\big([[X_i, X_j], X_k], X_0, \hat{X}_i, \hat{X}_j, \hat{X}_k, X_{n+1}\big) \\
&\quad + \sum_{k<l<i<j} (-1)^{i+j+k+l} f\big([X_k, X_l], [X_i, X_j], X_0, \hat{X}_k, \hat{X}_l, \hat{X}_i, \hat{X}_j, X_{n+1}\big) \\
&\quad + \sum_{k<i<l<j} (-1)^{i+j+k+l} f\big([X_k, X_l], [X_i, X_j], X_0, \hat{X}_k, \hat{X}_i, \hat{X}_l, \hat{X}_j, X_{n+1}\big) \\
&\quad + \sum_{k<l<i<j} (-1)^{i+j+k+l} f\big([X_k, X_l], [X_i, X_j], X_0, \hat{X}_k, \hat{X}_l, \hat{X}_i, \hat{X}_j, X_{n+1}\big)
\end{aligned}$$

$$+\sum_{k<i<j<l}(-1)^{i+j+k+l}f\big([X_k,X_l],[X_i,X_j],X_0,\hat{X}_k,\hat{X}_i,\hat{X}_j,\hat{X}_l,X_{n+1}\big)$$
$$+\sum_{i<k<l<j}(-1)^{i+j+k+l}f\big([X_k,X_l],[X_i,X_j],X_0,\hat{X}_i,\hat{X}_k,\hat{X}_l,\hat{X}_j,X_{n+1}\big)$$
$$+\sum_{i<k<j<l}(-1)^{i+j+k+l}f\big([X_k,X_l],[X_i,X_j],X_0,\hat{X}_i,\hat{X}_k,\hat{X}_j,\hat{X}_l,X_{n+1}\big)$$
$$+\sum_{i<j<k<l}(-1)^{i+j+k+l}f\big([X_k,X_l],[X_i,X_j],X_0,\hat{X}_i,\hat{X}_j,\hat{X}_k,\hat{X}_l,X_{n+1}\big).$$

The first three lines cancel each other by the Jacobi identity. The other lines pairwise cancel based on the order of i, j, k, and l. Altogether, $\delta^2 f = 0$ as required. □

Using the coboundary maps, we define two subspaces of $C^n(\mathfrak{g},\mathbb{R})$ by $Z^n(\mathfrak{g},\mathbb{R}) = \ker\delta_n$, and $B^n(\mathfrak{g},\mathbb{R}) = \operatorname{im}\delta_{n-1}$. Elements of $Z^n(\mathfrak{g},\mathbb{R})$ are called **cocycles**, and elements of $B^n(\mathfrak{g},\mathbb{R})$ **coboundaries**. These subspaces are related by the inclusions

$$B^n(\mathfrak{g},\mathbb{R}) \subseteq Z^n(\mathfrak{g},\mathbb{R}) \subseteq C^n(\mathfrak{g},\mathbb{R}),$$

and we define the **Lie algebra cohomology**, or **Chevalley-Eilenberg cohomology groups** as

$$H^n(\mathfrak{g},\mathbb{R}) = Z^n(\mathfrak{g},\mathbb{R})/B^n(\mathfrak{g},\mathbb{R}),$$

the cohomology groups of (2.3.12).

It is the Lie algebra cohomology groups $H^1(\mathfrak{g},\mathbb{R})$ and $H^2(\mathfrak{g},\mathbb{R})$, which provide the second criterion for a symplectic action to be Hamiltonian.

First note $H^1(\mathfrak{g},\mathbb{R}) = \ker\delta_1$, and any 1-cochain $c \in C^1(\mathfrak{g},\mathbb{R})$ is just a linear functional on $\mathfrak{g}$. The condition that $c \in H^1(\mathfrak{g},\mathbb{R})$ is then

$$0 = \delta c(X_0, X_1) = -c\big([X_0, X_1]\big).$$

The **commutator ideal**, or **derived subalgebra** of $\mathfrak{g}$, $[\mathfrak{g},\mathfrak{g}]$, is the ideal generated by the Lie brackets $[X, Y]$ for $X, Y \in \mathfrak{g}$. Thus $\delta c = 0$ is equivalent to c vanishing on the derived subalgebra. Hence $H^1(\mathfrak{g},\mathbb{R}) = [\mathfrak{g},\mathfrak{g}]^\circ$, where $[\mathfrak{g},\mathfrak{g}]^\circ \subseteq \mathfrak{g}^*$ is the annihilator of $[\mathfrak{g},\mathfrak{g}]$.

If $c \in C^2(\mathfrak{g},\mathbb{R})$, then c is bilinear map from $\mathfrak{g}\times\mathfrak{g}$ to $\mathbb{R}$ with

$$\delta c(X_0, X_1, X_2) = -c\big([X_0, X_1], X_2\big) + c\big([X_0, X_2], X_1\big) - c\big([X_1, X_2], X_0\big).$$

Further, if $c \in B^2(\mathfrak{g},\mathbb{R})$, then $c = \delta b$ for some 1-cochain b and

$$c(X_0, X_1) = \delta b(X_0, X_1) = -b\big([X_0, X_1]\big).$$

Theorem 2.3.21. *Let G be a connected Lie group with $H^1(\mathfrak{g},\mathbb{R}) = H^2(\mathfrak{g},\mathbb{R}) = 0$. Then every symplectic G-action on a connected manifold M is Hamiltonian.*

Proof. If $H^1(\mathfrak{g},\mathbb{R}) = [\mathfrak{g},\mathfrak{g}]^\circ = 0$, then $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Hence any fundamental vector field X_M can be written as a linear combination of Lie brackets $[Y_M, Z_M]$, which is Hamiltonian by Proposition 2.2.5.

Repeating the proof of Theorem 2.3.19 take a basis $\{e_1,\dots,e_n\}$ for $\mathfrak{g}$, lift them to Hamiltonian vector fields with functions $\mu_0^{e_i}$, and extend linearly to define μ_0^X for all $X \in \mathfrak{g}$. However, unlike in Theorem 2.3.19, the map $\mu_0^* : \mathfrak{g} \to C^\infty(M)$ defined by $\mu_0^*(X) = \mu_0^X$ may not be a Lie algebra homomorphism.

As

$$c^{X,Y} = \mu_0^{[X,Y]} - \{\mu_0^X, \mu_0^Y\}$$

is constant on M, the function $c : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ defined by $c(X,Y) = c^{X,Y}$ is bilinear and anti-symmetric, and defines a 2-cochain $c \in C^2(\mathfrak{g},\mathbb{R})$. Using the Jacobi identity for the Lie and Poisson brackets,

$$\begin{aligned}
\delta c(X_0,X_1,X_2) &= -c\big([X,Y],Z\big) + c\big([X,Z],Y\big) - c\big([Y,Z],X\big)\\
&= -c\big([X,Y],Z\big) - c\big([Z,X],Y\big) - c\big([Y,Z],X\big)\\
&= -\mu_0^{[[X,Y],Z]} - \mu_0^{[[Z,X],Y]} - \mu_0^{[[Y,Z],X]} + \{\mu_0^{[X,Y]}, \mu_0^Z\}\\
&\quad + \{\mu_0^{[Z,X]}, \mu_0^Y\} + \{\mu_0^{[Y,Z]}, \mu_0^X\}\\
&= -\mu_0^{[[X,Y],Z]} - \mu_0^{[[Z,X],Y]} - \mu_0^{[[Y,Z],X]} + \big\{\{\mu_0^X, \mu_0^Y\}, \mu_0^Z\big\}\\
&\quad + \big\{\{\mu_0^Z, \mu_0^X\}, \mu_0^Y\big\} + \big\{\{\mu_0^Y, \mu_0^Z\}, \mu_0^X\big\}\\
&= 0,
\end{aligned}$$

which implies $c \in Z^2(\mathfrak{g},\mathbb{R})$ is a 2-cocycle. Since $H^2(\mathfrak{g},\mathbb{R}) = 0$, c is also a coboundary, which implies there is $b \in C^1(\mathfrak{g},\mathbb{R}) = \mathfrak{g}^*$ such that $c = \delta b$, and

$$c(X,Y) = \delta b(X,Y) = -b\big([X,Y]\big).$$

Define $\mu^* : \mathfrak{g} \to C^\infty(M)$ by $\mu^*(X) = \mu_0^X + b(X)$. Then μ^* is linear, and a Lie algebra homomorphism:

$$\mu^*\big([X,Y]\big) = \mu_0^{[X,Y]} + b\big([X,Y]\big) = \mu_0^{[X,Y]} - c(X,Y) = \{\mu_0^X, \mu_0^Y\} = \{\mu^*(X), \mu^*(Y)\}. \tag{2.3.13}$$

The last equality follows as $b(X) \in \mathbb{R}$ is a constant function on M. It remains to show that μ^* satisfies the moment map condition. As $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$, by linearity it suffices to show $i\big([X,Y]_M\big) = d\mu^*\big([X,Y]\big)$. However, (2.3.13) shows $\mu^*\big([X,Y]\big) = \{\mu_0^X, \mu_0^Y\}$, and

$$d\mu^*\big([X,Y]\big) = d\{\mu_0^X, \mu_0^Y\} = i\big(-[X_M, Y_M]\big)\omega = i\big([X,Y]_M\big)\omega.$$

□

The question now is: which Lie groups have $H^1(\mathfrak{g},\mathbb{R}) = H^2(\mathfrak{g},\mathbb{R}) = 0$?

Definition 2.3.22. A Lie group G is **semisimple** if $\mathfrak{g} = \mathrm{Lie}(G)$ has no non-trivial abelian ideals. When G is compact, this is equivalent to $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Example 2.3.23. Any abelian Lie group is not semisimple. ◀

Example 2.3.24. $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$ are examples of compact semisimple Lie groups for $n > 1$. The unitary group $\mathrm{U}(n)$ is not semisimple. It has a non-trivial centre given by $S^1 \cdot I$, where I is the identity matrix, which at the Lie algebra level is the 1-dimensional abelian ideal $\mathbb{R} \cdot I$. ◀

Theorem 2.3.25 (Whitehead)**.** *Let G be a compact Lie group. Then G is semisimple if, and only if, $H^1(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$.*

A proof of this result can be found in [Jac62, pages 93-95]. As a corollary, a symplectic action of a compact semisimple Lie group G is Hamiltonian.

2.3.4 Uniqueness

Suppose (M, ω, G, μ) is a Hamiltonian G-space with M and G connected. In this section we answer whether the moment map μ is unique. That is, if μ_1 and μ_2 are two moment maps for the Hamiltonian action, what is their difference $\mu_1 - \mu_2$?

By definition, for all $X \in \mathfrak{g}$, μ_1^X and μ_2^X are both Hamiltonian functions for X_M. Hence

$$0 = d\mu_1^X - d\mu_2^X = d(\mu_1^X - \mu_2^X),$$

and the difference $\mu_1 - \mu_2$ is locally constant. As M is connected, the difference is constant so $\mu_1^X - \mu_2^X = c(X)$ is constant. It is clear that $c(X)$ depends linearly on X, and so $c \in \mathfrak{g}^*$. Thus $\mu_1 = \mu_2 + c$, and the moment maps differ by a constant in $\mathfrak{g}^*$. However, recall both μ_1^* and μ_2^* are Lie algebra homomorphisms, so for any $X, Y \in \mathfrak{g}$:

$$\begin{aligned} c\big([X,Y]\big) &= \mu_1^{[X,Y]} - \mu_2^{[X,Y]} \\ &= \{\mu_1^X, \mu_1^Y\} - \{\mu_2^X, \mu_2^Y\} \\ &= \{\mu_2^X + c(X), \mu_2^Y + c(Y)\} - \{\mu_2^X, \mu_2^Y\} \\ &= \{\mu_2^X, \mu_2^Y\} + \{\mu_2^X, c(Y)\} + \{c(X), \mu_2^Y\} + \{c(X), c(Y)\} - \{\mu_2^X, \mu_2^Y\} \\ &= 0. \end{aligned}$$

The last equality follows from the bracket involving $c(X)$ or $c(Y)$ being zero, as both functions are constant. Thus, it follows that c lies in the annihilator of $[\mathfrak{g}, \mathfrak{g}]$. Conversely, for any $c \in [\mathfrak{g}, \mathfrak{g}]^\circ = H^1(\mathfrak{g}, \mathbb{R})$, and any moment map μ, then $\mu + c$ is another moment map for the action. To show that $\mu + c$ is equivariant, we have to show that $\mathrm{Ad}_g^* c = c$ for all $g \in G$. To see this, as G is connected it suffices to show that $\mathrm{Ad}_{\exp(X)}^* c = c$ for all $X \in \mathfrak{g}$. Let $Y \in \mathfrak{g}$, then

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tX)}^* c(Y) = c\left(\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(-tX)} Y \right) = c\big(-[X,Y]\big) = 0$$

as c annihilates $[\mathfrak{g}, \mathfrak{g}]$. Hence $\mathrm{Ad}^*_{\exp(tX)} c$ is locally constant, and as G is connected, constant for all $t \in \mathbb{R}$. Therefore

$$\mathrm{Ad}^*_{\exp(X)} c = \mathrm{Ad}^*_{\exp(0)} c = c$$

as required. (Note if G is also compact, this shows that $\mathfrak{z}(\mathfrak{g})^* = [\mathfrak{g}, \mathfrak{g}]^\circ$ is the fixed point set for the coadjoint action of G on $\mathfrak{g}^*$.)

Therefore, we have proved

Theorem 2.3.26 (Uniqueness)**.** *Any two moment maps for the same Hamiltonian action differ by a constant in* $H^1(\mathfrak{g}, \mathbb{R}) = [\mathfrak{g}, \mathfrak{g}]^\circ \subseteq \mathfrak{g}^*$.

Corollary 2.3.26.1.

I) *Let G be a compact Lie group with $H^1(\mathfrak{g}, \mathbb{R}) = 0$, then the moment map for a Hamiltonian G-action is unique.*

II) *Let G be an abelian Lie group. If μ is a moment map for a Hamiltonian G-action, then $\mu + c$ is another moment map for all $c \in \mathfrak{g}$.*

Proof. As $H^1(\mathfrak{g}, \mathbb{R}) = [\mathfrak{g}, \mathfrak{g}]^\circ$, I) is clear. For II) if G is abelian then $[\mathfrak{g}, \mathfrak{g}] = 0$ so $[\mathfrak{g}, \mathfrak{g}]^\circ = \mathfrak{g}^*$. □

Example 2.3.27. Any moment map for a Hamiltonian action of a compact semisimple Lie group is unique. ◀

Example 2.3.28. Let S^1 act on $\mathbb{C}^n$ by the diagonal action. In Example 2.3.8, we showed this action is Hamiltonian with moment map

$$\mu(z_1, \ldots, z_n) = -\frac{1}{2}(|z_1|^2 + \cdots + |z_n|^2).$$

As S^1 is an abelian Lie group, Corollary 2.3.26.1(II) shows that

$$\nu(z_1, \ldots, z_n) = -\frac{1}{2}(|z_1|^2 + \cdots + |z_n|^2) + c$$

is a moment map for all $c \in \mathbb{R}$. ◀

Chapter 3

Symplectic Reduction

Symplectic reduction is the extension of the quotient manifold theorem to symplectic manifolds, endowing certain subspaces of the orbit space with a canonical symplectic structure.

The quotient manifold theorem itself is not sufficient in the symplectic category. To see why, recall that a symplectic manifold is necessarily even dimensional. Hence if M is a symplectic manifold equipped with a smooth S^1-action, then the orbit space M/S^1 has dimension $\dim M - 1$ which is odd. Thus, there is no way to endow M/S^1 with a symplectic structure.

Many of the results and proofs of this chapter rely on [Dwi+19, Chapter 6], and [Sil01, Chapters 23-24]. Moreover, unless stated otherwise, every Lie group G in this chapter is assumed to be connected.

3.1 Symplectic Reduction

Definition 3.1.1. A smooth map $f : M \to N$ between two manifolds M and N is **transverse** to a submanifold $S \subseteq N$ if for every $x \in f^{-1}(S)$ we have $T_{f(x)}N = T_{f(x)}S + df_x(T_xM)$.

Theorem 3.1.2 (Transversality). *If $f : M \to N$ is a smooth map transverse to an embedded submanifold $S \subseteq N$, then $f^{-1}(S)$ is an embedded submanifold of M with codimension the codimension of S in N.*

Proof. [Lee12, Theorem 6.30]. □

Corollary 3.1.2.1. *If $y \in N$ is a regular value for $f : M \to N$, then $f^{-1}(y)$ is an embedded submanifold of M with codimension* $\dim N$.

Thus the regular level sets of a smooth function provide a large class of embedded submanifold. In our situation, we hope that the regular level sets of smooth functions on (M, ω) inherit a symplectic structure. In particular, if (M, ω) is endowed with a Hamiltonian G-action, we ask whether the regular level sets of the moment map are symplectic submanifolds. This is not the case.

The restriction of ω to a regular level set will be degenerate for any tangent vector also tangent to the orbit. To see why, as ξ is a regular value $T_p\mu^{-1}(\xi) = \ker(d\mu_p) = \big(T_p(G\cdot p)\big)^{\omega}$ for all $p \in \mu^{-1}(\xi)$ by Proposition 2.3.7(I). Hence

$$\ker(\iota^*\omega) = T_p\mu^{-1}(\xi) \cap \big(T_p\mu^{-1}(\xi)\big)^{\omega} = T_p\mu^{-1}(\xi) \cap T_p(G\cdot p).$$

However, on each tangent space $T_p\mu^{-1}(\xi)$ for a regular value ξ, we can quotient the degeneracies of $\iota^*\omega$. We then ask whether $\iota^*\omega$ pushes forward to a non-degenerate form on this quotient, and also ask whether this process extends from a pointwise notion to a global notion on the whole submanifold $\mu^{-1}(\xi)$. To answer these questions, we first need some definitions from principal G-bundles.

For a fibre bundle $\pi : P \to B$ of smooth manifolds, recall that there is a short exact sequence of tangent spaces

$$0 \longrightarrow \ker d\pi \longrightarrow TP \longrightarrow TB \longrightarrow 0,$$

where elements of $\ker d\pi$ are called vertical vector fields.

Definition 3.1.3. A **horizontal k-form** on a fibre bundle $\pi : P \to B$ is a form $\alpha \in \Omega^k(P)$ which annihilates the vertical vector fields, i.e. $i(X)\alpha = 0$ for all $X \in \ker d\pi$. The vertical vector fields for a principal G-bundle are precisely the fundamental vector fields on the total space P (see [Tu17, Corollary 27.19])

If $\pi : P \to B$ is a principal G-bundle, then a **basic k-form** on P is a horizontal G-invariant form.

The reason for considering basic forms is they are the forms on the total space P which are pulled back from the base B.

Theorem 3.1.4. *Every basic k-form $\alpha \in \Omega^k(P)$ on a principal G-bundle $\pi : P \to B$ determines a unique k-form $\beta \in \Omega^k(B)$ such that $\pi^*\beta = \alpha$. Moreover, if α is closed, then so β.*

For a proof, see [Tu17, Theorem 31.12].

We are now ready to start our work on symplectic reduction; and we start by defining it at the vector space level.

Proposition 3.1.5 (Linear reduction)**.** *Suppose S is a subspace of a vector space V equipped with a symplectic form ω. Let $\iota : S \to V$ denote the inclusion map, and $\pi : S \to S/(S \cap S^{\omega})$ the canonical projection. Then there exists a unique symplectic form ω_S on $S/(S \cap S^{\omega})$ satisfying $\pi^*\omega_S = \iota^*\omega$.*

Proof. The relation $\pi^*\omega_S = \iota^*\omega$ forces us to define $\omega_S\big([u],[v]\big) = \omega(u,v)$ for all $u, v \in S$; from which uniqueness of ω_S is clear. To see that ω_S is well-defined, consider $u + x$ and $v + y$ for

$x, y \in S \cap S^\omega$. Then $[u + x] = [u]$, $[v + y] = [v]$ and

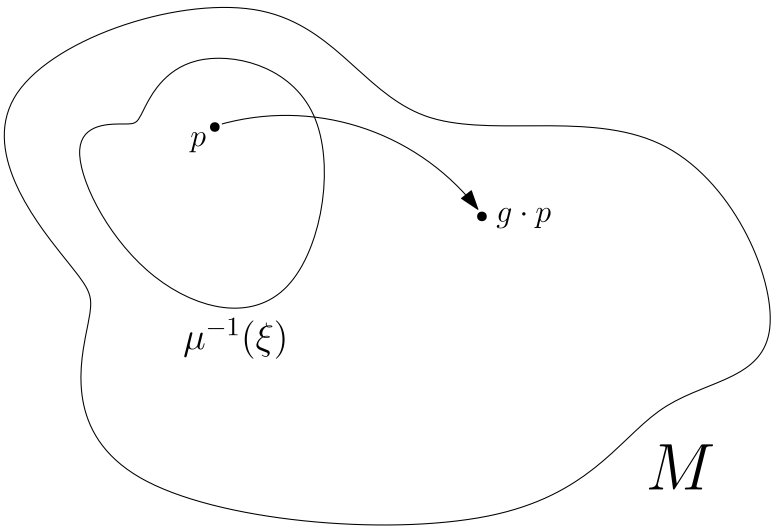
$$\begin{aligned}\omega(u + x, v + y) &= \omega(u, v) + \omega(u, y) + \omega(x, v) + \omega(x, y)\\ &= \omega(u, v)\end{aligned}$$

showing ω_S is independent of the choice of representative. It is immediate that ω_S is bilinear. To see that ω_S is symplectic; if $[u] \in S/(S \cap S^\omega)$ and

$$0 = \omega_S\big([u], [v]\big) = \omega(u, v)$$

for all $[v] \in S/(S \cap S^\omega)$, then $u \in S^\omega$ implying $[u] = 0$. □

Returning to our original situation of a regular level set $\mu^{-1}(\xi)$, the action of G may not preserve $\mu^{-1}(\xi)$, and so $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G$ may not be a principal G-bundle.



Thus, we only consider the action of subgroups which do fix the level set. As the moment map is equivariant, the elements fix $\mu^{-1}(\xi)$ are those which fix ξ under the coadjoint action, and so the largest such subgroup is the coadjoint stabiliser group G_ξ, of ξ.

However, this means we can only remove the degenerate directions which are tangent to the orbit of G_ξ. The next lemma shows that this is sufficient.

Lemma 3.1.6. *Let (M, ω, G, μ) be a Hamiltonian G-space, and suppose $\xi \in \mathfrak{g}^*$ a regular value of μ. Then for any $p \in \mu^{-1}(\xi)$,*

$$T_p(G_\xi \cdot p) = T_p\mu^{-1}(\xi) \cap T_p(G \cdot p).$$

It follows that there is a family of unique symplectic forms ω_p^ξ defined on the quotients $T_p\mu^{-1}(\xi)/T_p(G_\xi \cdot p)$.

Proof. First note that as ξ is a regular value for $p \in \mu^{-1}(\xi)$, $T_p\mu^{-1}(\xi) = \ker(d\mu_p)$. Hence

$$\begin{aligned} T_p\mu^{-1}(\xi) \cap T_p(G \cdot p) &= \big\{X_M(p) : X \in \mathfrak{g} \text{ and } 0 = d\mu_p(X_M) = X_{\mathfrak{g}^*}\big(\mu(p)\big)\big\} \\ &= \{X_M(p) : X \in \mathfrak{g}_\xi\} \\ &= T_p(G_\xi \cdot p). \end{aligned}$$

Let $\iota : \mu^{-1}(\xi) \hookrightarrow M$ be the inclusion map. Then

$$\begin{aligned} \ker(\iota^*\omega) &= T_p\mu^{-1}(\xi) \cap \big(T_p\mu^{-1}(\xi)\big)^\omega \\ &= T_p\mu^{-1}(\xi) \cap T_p(G \cdot p) \\ &= T_p(G_\xi \cdot p), \end{aligned}$$

where the second equality is Proposition 2.3.7(I). The last statement of the lemma now follows from linear reduction applied to the vector space $T_p\mu^{-1}(\xi)$. □

Theorem 3.1.7 (Symplectic Reduction)**.** *Let (M, ω, G, μ) be a Hamiltonian G-space with G and M connected, and suppose the action of G on M is proper. For a point $\xi \in \mathfrak{g}^*$ let $\iota : \mu^{-1}(\xi) \hookrightarrow M$ denote the inclusion map, $M_\xi = \mu^{-1}(\xi)/G_\xi$ the orbit space, and $\pi : \mu^{-1}(\xi) \to M_\xi$ the canonical projection. If ξ is a regular value of μ and G_ξ acts freely and properly on $\mu^{-1}(\xi)$, then there is a unique symplectic structure ω^ξ on M_ξ satisfying $\pi^*\omega^\xi = \iota^*\omega$.*

Proof. Since ξ is a regular value, the level set $\mu^{-1}(\xi)$ is a smooth submanifold of codimension $\dim G$, and as the G_ξ action is free and proper it is a principal G_ξ-bundle over M_ξ. Now consider $p \in \mu^{-1}(\xi)$. By Lemma 3.1.6, the pullback $\iota^*\omega$ is a smooth closed 2-form on $\mu^{-1}(\xi)$ whose kernel is $T_p(G_\xi \cdot p)$, the vertical bundle. Note $\iota^*\omega$ is also G_ξ-equivariant as both ι and ω are. Hence $\iota^*\omega$ is a basic 2-form on $\mu^{-1}(\xi)$, and pushes forward to a unique closed 2-form ω^ξ on the base M_ξ satisfying $\pi^*\omega^\xi = \iota^*\omega$. It remains to show that ω^ξ is symplectic, or equivalently, ω^ξ is non-degenerate. This follows from linear reduction as $\omega^\xi_{\pi(p)}$ is the pushforward of $\iota^*\omega_p$ to the quotient of $T_p\mu^{-1}(\xi)$ by its kernel. □

Theorem 3.1.7 is also referred to as the **Marsden-Weinstein-Meyer** theorem, eponymous after the authors who originally discovered this result: Marsden and Weinstein in [MW74], and Meyer in [Mey73].

The symplectic manifold (M_ξ, ω^ξ) constructed in Theorem 3.1.7 is called the **symplectic quotient** of M by G at ξ, and is also denoted by $M/\!\!/_\xi G$. We often drop the subscript 0 and write $M/\!\!/G$ for the reduction of M by G at $0 \in \mathfrak{g}^*$.

Example 3.1.8. Consider the symplectic manifold $(\mathbb{C}^n, \omega_0)$ with the diagonal action of S^1. By Example 2.3.8 this action is Hamiltonian with moment map

$$\mu(z_1, \dots, z_n) = -\frac{1}{2}\sum_{i=1}^n |z_i|^2,$$

where $c \in \mathbb{R}$. Now for all $r \neq 0$, r is a regular value of μ, and so

$$\mu^{-1}(-1/2) = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 = 1 \right\} = S^{2n-1}$$

is a regular level set. It is clear that the induced action S^1 on $\mu^{-1}(-1/2)$ is free and preserves the level set. Hence by Theorem 3.1.7 there is a unique symplectic form ω_{red} on the quotient space

$$\mu^{-1}(-1/2)/S^1 = S^{2n-1}/S^1 = \mathbb{C}P^{n-1}.$$

The form ω_{red} on $\mathbb{C}P^{n-1}$ is called the **Fubini-Study form**. ◀

Example 3.1.9. Consider the action of a connected Lie group G on itself via right translations and lift this action to the cotangent bundle T^*G. By the working in Example 2.3.17 this action is Hamiltonian, and under the global trivialisation $T^*G \cong G \times \mathfrak{g}^*$ via left translations, the moment map μ for this action is just negative the projection onto $\mathfrak{g}^*$, $\mu(g, \xi) = -\xi$. Hence every $\xi \in \mathfrak{g}^*$ is a regular value for μ and $\mu^{-1}(\xi) = G \times \{-\xi\}$. Thus $\mu^{-1}(\xi)/G_\xi \cong G/G_\xi$ the orbit space for left-translations by G_ξ, and so $\mu^{-1}(\xi)/G_\xi \cong G \cdot (-\xi)$ the coadjoint orbit of $-\xi$. As the action of G_ξ on $\mu^{-1}(\xi)$ is free (being left-translation), it remains to show it is proper. For this let (g_n) and (h_n) be sequences in G such that (g_n) and $(g_n h_n)$ converge. We must show that a subsequence of (h_n) converges by Proposition $A.1.11$. But this is immediate as

$$\lim h_n = \lim g_n^{-1} g_n h_n = \lim g_n^{-1} \lim g_n h_n = \big(\lim g_n\big)^{-1} \lim g_n h_n$$

where we have used that both multiplication and inversion are smooth (and hence continuous) on a Lie group. Therefore by Theorem 3.1.7 $T^*G /\!\!/_\xi G \cong G \cdot (-\xi)$ inherits a symplectic structure. In fact one can show that the induced symplectic structure on $G \cdot (-\xi)$ is precisely the KKS symplectic form on a coadjoint orbit [Kir04, p. 9]. ◀

As the requirements of Theorem 3.1.7 are quite stringent, it is natural to ask whether there exists any generalisations. First if $\xi \in \mathfrak{g}^*$ is a regular value for μ, then the G-action on $\mu^{-1}(\xi)$ is at worst locally free by Corollary 2.3.7.1. Hence M_ξ is an orbifold which inherits symplectic structure turning M_ξ into a symplectic orbifold. (See [LT97] for more details.)

Another generalisation relaxes the requirement for $\xi \in \mathfrak{g}^*$ to be a regular value of μ. If G is compact, then by [SL91] M_ξ is a so called *stratified symplectic space*. This theory of *singular reduction* is the focus of section 3.4. (We note that this situation has been further generalised to proper group actions in [BL97].)

3.2 The Shifting Trick

In Theorem 3.1.7 we considered the action of the coadjoint stabiliser group on the regular level set. We would like to extend this to the action of the whole group, to alleviate the need to calculate coadjoint stabilisers. One way to do this is by enlarging the regular level set.

By equivariance of the moment map, the preimage of a coadjoint orbit is stable under the action of G. Thus if $\mathcal{O}_\xi$ denotes the coadjoint orbit of $\xi \in \mathfrak{g}^*$, then $\pi : \mu^{-1}(\mathcal{O}_\xi) \to M_{\mathcal{O}_\xi} = \mu^{-1}(\mathcal{O}_\xi)/G$ is a principal G-bundle. This subsection deals with trying find an analogue to Theorem 3.1.7 for $\mu^{-1}(\mathcal{O}_\xi)$.

Proposition 3.2.1. *Let (M, ω, G, μ) be a Hamiltonian G-space with G connected. If the coadjoint orbit $\mathcal{O}_\xi$ for $\xi \in \mathfrak{g}^*$ contains a single regular value of μ, then every point in $\mathcal{O}_\xi$ is a regular value of μ. In this case, μ is transverse to the coadjoint orbit and $\mu^{-1}(\mathcal{O}_\xi)$ is a smooth submanifold with codimension* $\dim G_\xi$.

Proof. Let $\xi \in \mathcal{O}_\xi$ be a regular value for μ. Then by Corollary 2.3.7.1 the G-action at $p \in \mu^{-1}(\xi)$ is locally free, i.e. $\mathfrak{g}_p = \{0\}$. Let $\eta \in \mathcal{O}_\xi$, then there exist $g \in G$ such that $\eta = \mathrm{Ad}_g^* \xi$. As for $q \in \mu^{-1}(\eta)$, $g^{-1} \cdot q \in \mu^{-1}(\xi)$, so it follows that the stabiliser groups for q and $g^{-1} \cdot q$ are conjugate. However, the stabiliser group for $g^{-1} \cdot q$ is discrete by the previous work, and so the stabiliser group for $q \in \mu^{-1}(\eta)$ is also discrete. Hence $\mathfrak{g}_q = \{0\}$ for all $q \in \mu^{-1}(\eta)$ and η is a regular value. As η is an arbitrary point in the coadjoint orbit, the whole orbit consists of regular values.

As the whole coadjoint orbit consists of regular values, $d\mu_p$ is surjective for all $p \in \mu^{-1}(\mathcal{O}_\xi)$, so

$$T_{\mu(p)}\mathfrak{g}^* = d\mu_p(T_pM) = d\mu_p(T_pM) + T_{\mu(p)}\mathcal{O}_\xi.$$

Thus μ is transverse to the coadjoint orbit, and $\mu^{-1}(\mathcal{O}_\xi)$ is a submanifold of M with codimension $\dim G - \dim \mathcal{O}_\xi$ by Theorem 3.1.2. The final statement of the lemma now follows as the orbit $\mathcal{O}_\xi$ is diffeomorphic to homogeneous space G/G_ξ by Proposition $A.4.3$ which has dimension $\dim G - \dim G_\xi$. □

Lemma 3.2.2. *Let (M, ω, G, μ) be a Hamiltonian G-space with G compact, and let $\xi \in \mathfrak{g}^*$. The action of G_ξ on $\mu^{-1}(\xi)$ is free if, and only if, the action of G on $\mu^{-1}(\mathcal{O}_\xi)$ is free.*

Proof. If the G-action on $\mu^{-1}(\mathcal{O}_\xi)$ is free, then the G_ξ-action on $\mu^{-1}(\xi)$ is free as $\xi \in \mathcal{O}_\xi$ and $G_\xi \subseteq G$. Conversely, suppose the G_ξ-action on $\mu^{-1}(\xi)$ is free. Let $p \in \mu^{-1}(\mathcal{O}_\xi)$. Then $\mu(p) = \eta$, and there exists $g \in G$ such that $\eta = \mathrm{Ad}_g^* \xi$. If $h \in G_p$, then by equivariance of μ, $h \in G_\eta$ which is conjugate to G_ξ. Thus we have $h = gh_0g^{-1}$ for some $h_0 \in G_\xi$ which fixes $g^{-1} \cdot p$. However, $g^{-1} \cdot p \in \mu^{-1}(\xi)$ and the G_ξ-action on $\mu^{-1}(\xi)$ is free which implies that h_0 is the identity element. Thus $h = gh_0g^{-1}$ is also the identity element, which shows that G_p is trivial and the G-action is free. □

Hence under the assumptions of Theorem 3.1.7, the action of G on $\mu^{-1}(\mathcal{O}_\xi)$ defines a principal G-bundle which we denote by

$$\pi_{\mathcal{O}_\xi} : \mu^{-1}(\mathcal{O}_\xi) \to \mu^{-1}(\mathcal{O}_\xi)/G = M_{\mathcal{O}_\xi}.$$

We further denote by $\iota_{\mathcal{O}_\xi} : \mu^{-1}(\mathcal{O}_\xi) \hookrightarrow M$ the inclusion map.

One would hope that $\iota_{\mathcal{O}_\xi}^*\omega$ is basic, so that it pushes forward to a symplectic form on the base $M_{\mathcal{O}_\xi}$, as in Theorem 3.1.7. Unfortunately, $\iota_{\mathcal{O}_\xi}\omega$ is not basic as it is not horizontal. However, for any point $p \in \mu^{-1}(\mathcal{O}_\xi)$, and vertical tangent vector $w \in \ker d\big(\pi_{\mathcal{O}_\xi}\big)_p$, we have

$$\begin{aligned} T_p\mu^{-1}(\mathcal{O}_\xi) &= (d\mu_p)^{-1}(T_{\mu(p)}\mathcal{O}_\xi) \\ &= (d\mu_p)^{-1}\left\{X_{\mathfrak{g}}\big(\mu(p)\big) : X \in \mathfrak{g}\right\} \\ &= \left\{v \in T_pM : \text{there exists } X \in \mathfrak{g} \text{ such that } d\mu_p(v) = X_{\mathfrak{g}^*}\big(\mu(p)\big)\right\} \end{aligned}$$

and

$$\ker\big(d(\pi_{\mathcal{O}_\xi})_p\big) = T_p(G\cdot p) = \{w \in T_pM : \text{there exists } X \in \mathfrak{g} \text{ such that } w = X_M(p)\}.$$

Hence $i(w)\omega$, restricted to $T_p\mu^{-1}(\mathcal{O}_\xi)$, can be described by the KKS form ω^{KKS} on the coadjoint orbit:

$$\begin{aligned} \omega_p(w,v) &= \langle d\mu_p(v), X\rangle = \big\langle Y_{\mathfrak{g}^*}\big(\mu(p)\big), X\big\rangle = \langle \mu(p), [X,Y]\rangle = \omega^{KKS}_{\mu(p)}\big(X_{\mathfrak{g}^*}, Y_{\mathfrak{g}^*}\big) \\ &= (\mu^*\omega^{KKS})_p(w,v), \end{aligned}$$

for all $v \in T_p\mu^{-1}(\mathcal{O}_\xi)$. As μ is equivariant and ω^{KKS} is G-invariant, it follows that $\mu^*\omega^{KKS}$ is G-invariant. Hence, the difference $\omega - \mu^*\omega^{KKS}$ is a closed invariant 2-form, and its restriction to $\mu^{-1}(\mathcal{O}_\xi)$, $\iota_{\mathcal{O}_\xi}^*\big(\omega - \mu^*\omega^{KKS}\big)$, is basic. Define $\omega^{\mathcal{O}_\xi}$ to be the pushforward of this basic form to the quotient space $M_{\mathcal{O}_\xi}$.

The form $\omega^{\mathcal{O}_\xi}$ vanishes on vectors of the form $d\pi_{\mathcal{O}_\xi}(v)$ for which $i(v)\omega = i(v)\big(\mu^*\omega^{KKS}\big)$. This holds only if v is also tangent to the G-orbit, i.e. v is a vertical vector, and so $d\pi_{\mathcal{O}_\xi}(v) = 0$. Thus $\omega^{\mathcal{O}_\xi}$ is non-degenerate and defines a symplectic structure on $M_{\mathcal{O}_\xi}$.

We claim that there exists a diffeomorphism between M_ξ and $M_{\mathcal{O}_\xi}$. To see this let $\iota_\xi : \mu^{-1}(\xi) \hookrightarrow \mu^{-1}(\mathcal{O}_\xi)$ denote the inclusion of $\mu^{-1}(\xi)$ into $\mu^{-1}(\mathcal{O}_\xi)$, and also let $\pi_\xi : \mu^{-1}(\xi) \to M_\xi$ be the canonical projection. As ι_ξ is G_ξ-equivariant, $\pi_{\mathcal{O}_\xi} \circ \iota_\xi$ is a smooth map constant on the fibres of π_ξ. As π_ξ is a smooth submersion, there exists a smooth function $F : M_\xi \to M_{\mathcal{O}_\xi}$ such that following diagram commutes

$$\begin{array}{ccc} \mu^{-1}(\xi) & \xrightarrow{\iota_\xi} & \mu^{-1}(\mathcal{O}_\xi) \\ \Big\downarrow{\scriptstyle \pi_\xi} & & \Big\downarrow{\scriptstyle \pi_{\mathcal{O}_\xi}} \\ M_\xi & \underset{F}{\dashrightarrow} & M_{\mathcal{O}_\xi}. \end{array} \tag{3.2.1}$$

To show that F is a diffeomorphism, we first show it is bijective. Suppose that $F\big([p]\big) = F\big([q]\big)$ for $p, q \in \mu^{-1}(\xi)$. Then $\pi_{\mathcal{O}_\xi}(p) = \pi_{\mathcal{O}_\xi}(q)$ and there exists $g \in G$ such that $q = g \cdot p$. By equivariance of μ,

$$\xi = \mu(q) = \mu(g\cdot p) = \operatorname{Ad}_g^*\mu(p) = \operatorname{Ad}_g^*\xi,$$

and $g \in G_\xi$. Hence $[p] = [q]$ and F is injective. Now let $p \in \mu^{-1}(\mathcal{O}_\xi)$ and consider $\pi_{\mathcal{O}_\xi}(p) \in M_{\mathcal{O}_\xi}$. As $\mu(p) \in \mathcal{O}_\xi$, there exists $g \in G$ such that $\mathrm{Ad}_g^* \mu(p) = \xi$, and so equivariance of μ implies $g \cdot p \in \mu^{-1}(\xi)$. Thus

$$F\big([g \cdot p]\big) = \pi_{\mathcal{O}_\xi}\big(\iota_\xi(g \cdot p)\big) = \pi_{\mathcal{O}_\xi}(g \cdot p) = \pi_{\mathcal{O}_\xi}(p),$$

and F is surjective. It is clear that F is a diffeomorphism as both π_ξ, $\pi_{\mathcal{O}_\xi}$ are submersions, ι_ξ is an immersion, and (3.2.1) commutes.

We ask whether M_ξ is actually symplectomorphic to $M_{\mathcal{O}_\xi}$. The answer is yes; but before we show this, we consider reduction from yet another view point.

Let $-\mathcal{O}_\xi$ denote the coadjoint orbit with symplectic form minus the KKS form, $-\omega^{KKS}$. By Proposition 2.3.6(III), the diagonal action of G on $\big(M \times -\mathcal{O}_\xi, \nu = \pi_1^*\omega + \pi_2^*(-\omega^{KKS})\big)$ is Hamiltonian with moment map $\Phi(p, \eta) = \mu(p) - \eta$. It follows that the zero level set of Φ

$$\Phi^{-1}(0) = \{(p, \eta) \in M \times -\mathcal{O}_\xi : \mu(p) = \eta\}$$

is equivariantly diffeomorphic to $\mu^{-1}(\mathcal{O}_\xi)$, which can be seen by taking the graph of the moment map. The next proposition shows that the assumptions of Theorem 3.1.7 apply so one can form the symplectic quotient of $M \times -\mathcal{O}_\xi$ by G at 0.

Proposition 3.2.3. *If $\xi \in \mathfrak{g}^*$ is a regular value for μ and G_ξ acts freely and properly on $\mu^{-1}(\xi)$, then $0 \in \mathfrak{g}^*$ is a regular value for Φ and G acts freely and properly on $\Phi^{-1}(0)$.*

Proof. If G_ξ acts freely on $\mu^{-1}(\xi)$, then G_η acts freely on $\mu^{-1}(\eta)$ for any $\eta \in \mathcal{O}_\xi$ as the stabilisers are conjugate. Now for $x \in \Phi^{-1}(0)$ we can write $x = \big(p, \eta\big)$ where $\eta = \mu(p) \in \mathcal{O}_\xi$. Suppose $g \in G$ stabilises x, then

$$\big(g \cdot p, \mathrm{Ad}_g^* \eta\big) = g \cdot x = x = (p, \eta).$$

Hence $\mathrm{Ad}_g^* \eta = \eta$ and $g \in G_\eta$. However, the action of G_η on $\mu^{-1}(\eta)$ is free it follows that g is the identity element. Thus G acts freely on $\Phi^{-1}(0)$, and hence 0 is also a regular value for Φ by Corollary 2.3.7.1. □

Hence if the assumptions of Theorem 3.1.7 are satisfied, we have three different symplectic quotients

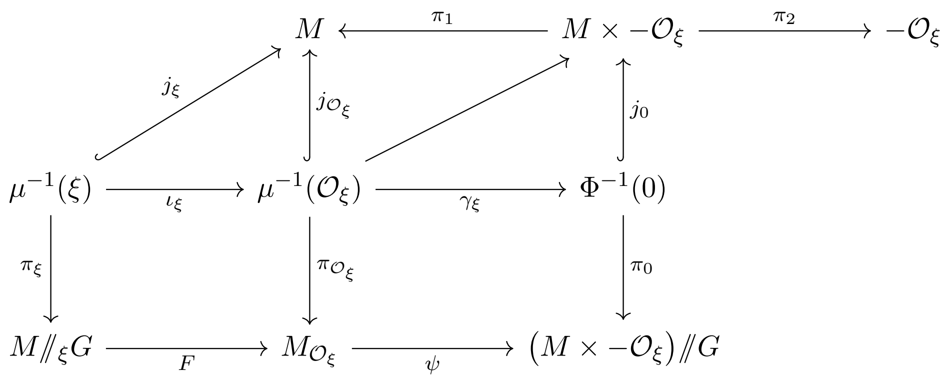
$$\big(M/\!\!/_\xi G, \omega^\xi\big) \quad (M_{\mathcal{O}_\xi}, \omega^{\mathcal{O}_\xi}) \quad \big((M \times -\mathcal{O}_\xi)/\!\!/G, \nu^0\big), \tag{3.2.2}$$

which are all diffeomorphic to each other. The next theorem shows that they are in fact symplectomorphic.

Theorem 3.2.4 (Shifting Trick)**.** *Under the assumptions of Theorem* 3.1.7*, the three symplectic quotients in* (3.2.2) *are symplectomorphic to each other.*

Proof. We have already seen that the inclusion $\iota_\xi : \mu^{-1}(\xi) \hookrightarrow \mu^{-1}(\mathcal{O}_\xi)$ descends to a diffeomorphism between $M_\xi = M/\!\!/_\xi G$ and $M_{\mathcal{O}_\xi}$, and that the graph of the moment map provides an equivariant diffeomorphism between $\mu^{-1}(\mathcal{O}_\xi)$ and $\Phi^{-1}(0)$. The graph of μ restricted to $\mu^{-1}(\mathcal{O}_\xi)$

factors through an equivariant diffeomorphism γ_ξ between $\mu^{-1}(\mathcal{O}_\xi)$ and $\Phi^{-1}(0)$. As an equivariant diffeomorphism of principal G-bundles, γ_ξ descends to a diffeomorphism ψ on the orbit spaces. All this can be combined with the previous results to obtain the following commutative diagram



where F is a diffeomorphism by previous work. To see that F is a symplectomorphism, we have

$$\begin{aligned}\pi_\xi^* F^* \omega^{\mathcal{O}_\xi} &= \iota_\xi^* \pi_{\mathcal{O}_\xi}^* \omega^{\mathcal{O}_\xi} \\ &= \iota_\xi^* \iota_{\mathcal{O}_\xi}^* \big(\omega - \mu^* \omega^{KKS}\big) \\ &= j_\xi^* \omega - (\mu \circ j_\xi)^* \omega^{KKS}\end{aligned}$$

where the first and third equalities follows from commutativity of the diagram, and the second equality is the defining property of $\omega^{\mathcal{O}_\xi}$. However, as μ is constant on $\mu^{-1}(\xi)$, $(\mu \circ j_\xi)^* \omega^{KKS} = 0$ and $\pi_\xi^* F^* \omega^{\mathcal{O}_\xi} = j_\xi^* \omega$. Hence by the uniqueness of Theorem 3.1.7 $F^* \omega^{\mathcal{O}_\xi} = \omega^\xi$ and $M/\!\!/_\xi G$ and $M_{\mathcal{O}_\xi}$ are symplectomorphic. Similarly we see that

$$\begin{aligned}\pi_{\mathcal{O}_\xi}^* \psi^* \nu^0 &= \gamma_\xi^* \pi_0^* \nu^0 \\ &= \gamma_\xi^* j_0^* \nu \\ &= \gamma_\xi^* j_0^* \big(\pi_1^* \omega + \pi_2^*(-\omega^{KKS})\big) \\ &= j_{\mathcal{O}_\xi}^* \omega - (\mu \circ j_{\mathcal{O}_\xi})^* \omega^{KKS} \\ &= j_{\mathcal{O}_\xi}^* (\omega - \mu^* \omega^{KKS}).\end{aligned}$$

Hence $\psi^* \nu^0 = \omega^{\mathcal{O}_\xi}$, and $M_{\mathcal{O}_\xi}$ is symplectomorphic to $\big(M \times -\mathcal{O}_\xi\big)/\!\!/G$. □

Hence the shifting trick allows us to talk about reduction exclusively at the zero level set.

Corollary 3.2.4.1. *For any two points ξ and η in the same coadjoint orbit, there is a symplectomorphism between $M/\!\!/_\xi G$ and $M/\!\!/_\eta G$.*

Proof. As ξ and η lie in the same coadjoint orbit $\mathcal{O}_\xi = \mathcal{O}_\eta$, and so both symplectic quotients are symplectomorphic to $M_{\mathcal{O}_\xi}$. □

3.3 Reduction in Stages

Suppose that (M, ω, G, μ) is a Hamiltonian G-space. Further suppose that K is another Lie group acting in a Hamiltonian fashion on M with moment map ϕ. If the G and K-actions commute, and the moment maps are invariant under the other's action, then we have already shown in Proposition 2.3.6 that M is Hamiltonian $G \times K$-space with moment map $\mu \times \phi$. However, this begs the question: if we reduce M by the G-action, what happens to the residual K-action? Does it descend? If so, is the induced action Hamiltonian on the quotient space? The goal of this section is to answer these questions, which leads to the theory of reduction in stages.

Proposition 3.3.1. *Let (M, ω, G, μ) be a Hamiltonian G-space with G a connected Lie group, and suppose K is another connected Lie group acting on M in a Hamiltonian way with moment map ϕ. Suppose the actions of K and G on M commute, and that ϕ is G-invariant. Then*

- **I)** *μ is K-invariant.*
- **II)** *If the symplectic quotient $(M/\!\!/_\xi G, \omega^\xi)$ is well defined at $\xi \in \mathfrak{g}^*$, then there is an induced Hamiltonian action of K on $M/\!\!/_\xi G$ with the moment map being induced from ϕ.*

Proof. I): G-invariance of ϕ implies that $d\phi^X(Y_M) = 0$ for all $X \in \mathfrak{k}$ and $Y \in \mathfrak{g}$. As K is connected, to show μ is K-invariant it suffices to show that $d\mu^Y(X_M) = 0$ for all $Y \in \mathfrak{g}$ and $X \in \mathfrak{k}$ by Proposition $A.3.7$. However, using the moment map condition

$$d\mu^Y(X_M) = \omega(Y_M, X_M) = -\omega(X_M, Y_M) = -d\phi^X(Y_M) = 0$$

proving μ is K-invariant.

II): Denote the G and K actions on M by $\mathcal{A}^G$ and $\mathcal{A}^K$, respectively. By I), $\mu^{-1}(\xi)$ is invariant under $\mathcal{A}^K$ for all $\xi \in \mathfrak{g}^*$. As the actions of G and K commute, we obtain a well defined action $\tilde{\mathcal{A}}^K$ on $M/\!\!/_\xi G$ such that $\tilde{\mathcal{A}}^K_k \pi(p) = \pi\big(\mathcal{A}^K_k(p)\big)$ for all $k \in K$, where $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G_\xi = M/\!\!/_\xi G$ is the canonical projection. Now for all $k \in K$,

$$\begin{aligned}
\pi^*(\tilde{\mathcal{A}}^K_k)^*\omega^\xi &= \big(\tilde{\mathcal{A}}^K_k \circ \pi\big)^*\omega^\xi \\
&= (\pi \circ \mathcal{A}^K_k)^*\omega \\
&= (\mathcal{A}^K_k)^*\pi^*\omega^\xi \\
&= (\mathcal{A}^K_k)^*\iota_\xi^*\omega \\
&= \iota_\xi^*(\mathcal{A}^K_k)^*\omega \\
&= \iota_\xi^*\omega,
\end{aligned}$$

where $\iota_\xi : \mu^{-1}(\xi) \hookrightarrow M$ is the inclusion map. Thus by the uniqueness of the symplectic form ω^ξ, we have $(\tilde{\mathcal{A}}^K_k)^*\omega^\xi = \omega^\xi$ and the action is symplectic.

As ϕ is G-invariant it is constant on the fibres of submersion π. Hence there is a unique smooth map $\tilde{\phi} : M/\!\!/_\xi \to \mathfrak{k}^*$ such that $\tilde{\phi} \circ \pi = \phi$. We claim that ϕ is a moment map for the induced K

action on $M/\!\!/_\xi G$. Indeed, for all $X \in \mathfrak{k}$ the vector fields X_M and $X_{M/\!\!/_\xi G}$ are π-related, and so

$$\pi^*\big(i(X_{M/\!\!/_\xi G})\omega^\xi\big) = i(X_M)\iota^*\omega = \iota^*\big(i(X_M)\omega\big) = \iota^*(d\phi^X) = \pi^*(d\tilde{\phi}^X).$$

As π is a surjective submersion, π^* is injective and so $i(X_{M/\!\!/_\xi G})\omega^\xi = d\tilde{\phi}^X$. As $\tilde{\phi} \circ \pi = \phi$ and both π and ϕ are equivariant, it follows that $\tilde{\phi}$ is equivariant on elements of the form $\pi(p)$. As π is surjective every element of $M/\!\!/_\xi G$ is of this form, and $\tilde{\phi}$ is equivariant. □

Hence we see that if M is a Hamiltonian $G \times K$-space, under certain conditions the residual K-action descends to the reduced space $M/\!\!/G$. Thus we could consider the iterated quotient $\big(M/\!\!/G\big)/\!\!/K$. In an ideal world, we should be able to relate this quotient to the quotients $M/\!\!/(G \times K)$, and $\big(M/\!\!/K\big)/\!\!/G$. Fortunately for us we can, leading to the following theorem.

Theorem 3.3.2 (Commuting Reduction in Stages)**.** *Suppose* (M, ω, G, μ) *and* (M, ω, K, ϕ) *are two Hamiltonian spaces with* K *and* G *compact, connected Lie groups. Suppose that the actions of* K *and* G *commute, and* ϕ *is* G*-invariant. Then* M *is a Hamiltonian* $G \times K$*-space with moment map* $\mu \times \phi$*. Suppose that for* $\xi \in \mathfrak{g}^*$ *and* $\eta \in \mathfrak{k}^*$*,* G_ξ *acts freely on* $\mu^{-1}(\xi)$ *and* K_η *acts freely on* $\phi^{-1}(\eta)$*. Then* $G_\xi \times K_\eta$ *acts freely on* $(\mu, \phi)^{-1}(\xi, \eta) = \mu^{-1}(\xi) \cap \phi^{-1}(\eta)$ *and the symplectic quotients* $\big(M/\!\!/_{(\xi,\eta)}(G \times K), \omega^{(\xi,\eta)}\big)$ *and* $\big((M/\!\!/_\xi G)/\!\!/_\eta K, \omega^{\xi,\eta}\big)$ *are well defined and are symplectomorphic.*

Proof. Under the given assumptions, the fact M is a Hamiltonian $G \times K$-space with moment map (μ, ϕ) is given by Proposition 2.3.6(IV). Further, by Proposition 3.3.1 there is an induced Hamiltonian K-action on $M/\!\!/_\xi G$ with moment map $\tilde{\phi}$ induced from ϕ. The fact that the K_η action on $\phi^{-1}(\eta)$ is free implies that the induced K_η on $\tilde{\phi}^{-1}(\eta)$ is also free. As the respective actions are free, it follows that ξ and η are regular values for their respective moment maps by Corollary 2.3.7.1. Hence by Theorem 3.1.7 we can form the symplectic quotients $\big((M/\!\!/_\xi G)/\!\!/_\eta K, \omega^{\xi,\eta}\big)$ and $\big(M/\!\!/_{(\xi,\eta)}(G \times K), \omega^{(\xi,\eta)}\big)$.

Let

$$j : (\mu, \phi)^{-1}(\xi, \eta) = \mu^{-1}(\xi) \cap \phi^{-1}(\eta) \hookrightarrow \mu^{-1}(\xi)$$

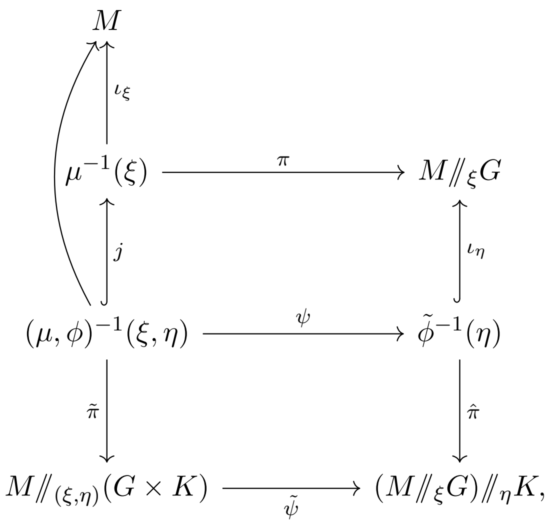
be the inclusion map. Composing with the canonical projection $\pi : \mu^{-1}(\xi) \to M/\!\!/_\xi G$ gives a map $\pi \circ j : (\mu, \phi)^{-1}(\xi, \eta) \to M/\!\!/_\xi G$. As $\tilde{\phi} \circ \pi = \phi$ it follows that $\pi \circ j$ actually maps to $\tilde{\phi}^{-1}(\eta)$. So we have a map

$$\psi : (\mu, \phi)^{-1}(\xi, \eta) \to \tilde{\phi}^{-1}(\eta).$$

As π is G_ξ-invariant and K_η-equivariant, it follows that ψ is equivariant with respect to the $G_\xi \times K_\eta$ action on $(\mu, \phi)^{-1}(\xi, \eta)$ and the K_η action on $\tilde{\phi}^{-1}(\eta)$. Thus ψ descends to a map

$$\tilde{\psi} : M/\!\!/_{(\xi,\eta)}(G \times K) \to (M/\!\!/_\xi G)/\!\!/_\eta K,$$

which fits into the following commutative diagram



where the other maps are either canonical inclusions, or canonical projections. From commutativity, it follows that

$$\tilde{\pi}^*\tilde{\psi}^*\omega^{\xi,\eta} = \psi^*\hat{\pi}^*\omega^{\xi,\eta} = \psi^*\iota_\eta^*\omega^\xi = j^*\pi^*\omega^\xi = j^*\iota_\xi^*\omega = (\iota_\xi\circ j)^*\omega,$$

and $\tilde{\psi}^*\omega^{\xi,\eta} = \omega^{(\xi,\eta)}$ by the unique property of $\omega^{(\xi,\eta)}$. Thus $\tilde{\psi}$ is symplectic.

It remains to show that $\tilde{\psi}$ is a diffeomorphism, and we do this by calculating its inverse. Let

$$\varphi : \tilde{\phi}^{-1}(\eta) \to M/\!\!/_{(\xi,\eta)}(G\times K)$$

be defined as follows. Choose an equivalence class $[p]_\xi \in \tilde{\phi}^{-1}(\eta) \subseteq M/\!\!/_\xi G$, where $p\in\mu^{-1}(\xi)$ and the equivalence relation is for the G_ξ action. Note that for each such point

$$\phi(p) = \tilde{\phi}\big([p]_\xi\big) = \eta,$$

and so $p\in(\mu,\phi)^{-1}(\xi,\eta)$. Hence we can consider the equivalence class $[p]_{(\xi,\eta)}$ of p relative to the $G_\xi\times K_\eta$ action, and set

$$\varphi\big([p]_\xi\big) = [p]_{(\xi,\eta)}.$$

It follows that φ is well-defined as any other choice of representative for $[p]_\xi$ is of the form $\mathcal{A}_g^G(p)$ for $g\in G_\xi$, and so defines the same class in $M/\!\!/_{(\xi,\eta)}(G\times K)$. Further note that φ is K_η-invariant, and so there is a unique smooth map

$$\tilde{\varphi} : (M/\!\!/_\xi G)/\!\!/_\eta K \to M/\!\!/_{(\xi,\eta)}(G\times K)$$

on the quotient. It is clear from the definition of φ that $\tilde{\varphi}$ is the inverse for $\tilde{\psi}$. □

Now suppose that H is a Lie subgroup of G, and that $G = H \times G/H$. Then by Proposition 2.3.6 we know that a Hamiltonian G-space can be considered as a Hamiltonian H-space. Moreover, as G splits we can apply Theorem 3.3.2 to obtain the iterated quotient $(M/\!\!/H)/\!\!/(G/H) \cong M/\!\!/G$. This gives a subgroup version of Theorem 3.3.2.

However, not every Lie group and subgroup splits in this way, so extra work must be done to obtain such a result.

Consider the inclusion $j : H \hookrightarrow G$ of a normal subgroup H of G. Then we have a short exact sequence of Lie groups

$$0 \longrightarrow H \xhookrightarrow{j} G \xrightarrow{q} G/H \longrightarrow 0,$$

and by differentiating and taking duals induces another two short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathfrak{h} \xrightarrow{dj} \mathfrak{g} \xrightarrow{dq} \mathfrak{g}/\mathfrak{h} \longrightarrow 0, \\ 0 \longrightarrow (\mathfrak{g}/\mathfrak{h})^* \xrightarrow{(dq)^*} \mathfrak{g}^* \xrightarrow{(dj)^*} \mathfrak{h}^* \longrightarrow 0. \end{gathered} \tag{3.3.1}$$

The moment map for the Hamiltonian H-action is given by $\mu_H = (dj)^* \circ \mu$ by Proposition 2.3.6, and assuming $0 \in \mathfrak{h}^*$ is a regular value for μ_H and H acts freely on $\mu_H^{-1}(0)$ then we can take the symplectic quotient $M/\!\!/H$, whose symplectic form we denote by ω^H. The kernel of $(dj)^*$ is given by

$$\ker(dj)^* = \{\xi \in \mathfrak{g}^* : \langle (dj)^*\xi, X\rangle = \langle \xi, dj(X)\rangle = 0 \text{ for all } X \in \mathfrak{h}\},$$

and by using dj to identify $\mathfrak{h}$ as a subspace of $\mathfrak{g}$, we have $\ker(dj)^* = \mathfrak{h}^\circ$ the annihilator of $\mathfrak{h}$.

As H is a normal subgroup, it is closed under conjugation by G. Hence $\mathfrak{h}$ is closed under the adjoint action of G. Thus the coadjoint action of G fixes the annihilator of $\mathfrak{h}$, and by equivariance of μ, G also fixes $\mu_H^{-1}(0)$ as

$$\mu_H^{-1}(0) = \mu^{-1}\left(\left((dj)^*\right)^{-1}(0)\right) = \mu^{-1}\big(\ker(dj)^*\big) = \mu^{-1}(\mathfrak{h}^\circ).$$

Suppose $p \in \mu_H^{-1}(0)$, and let $h \cdot p$ denote a point in $H \cdot p$, the H-orbit of p. The action of G on $h \cdot p$ is given by

$$g \cdot (h \cdot p) = (gh) \cdot p = (ghg^{-1}g) \cdot p = \underbrace{ghg^{-1}}_{\in H} \cdot (g \cdot p),$$

which shows that $g \cdot (h \cdot p)$ lies in the H-orbit of $g \cdot p$. Thus we can define a G-action on $M/\!\!/H$ by $g \cdot [p] = [g \cdot p]$, and $\pi_H : \mu_H^{-1}(0) \to M/\!\!/H$ is equivariant with respect to this action. This action descends to an action of G/H by $gH \cdot [p] = [g \cdot p]$, and the next proposition shows that this action is Hamiltonian.

Proposition 3.3.3. *Let $(M/\!\!/H, \omega^H)$ be the symplectic quotient with the principal H-bundle structure denoted by $\pi_H : \mu_H^{-1}(0) \to M/\!\!/H$, and let inclusion of the zero level set be denoted by $\iota_H : \mu_H^{-1}(0) \hookrightarrow M$. Then there is a Hamiltonian action of G/H on $M/\!\!/H$ with moment map $\mu_{G/H}$ satisfying $(dq)^* \circ \big(\pi_H^* \mu_{G/H}\big) = \iota_H^* \mu$.*

Proof. We have already shown that G/H acts on $M /\!/ H$ by $gH \cdot [p] = [g \cdot p]$, so it remains to prove that the action is Hamiltonian, and the associated moment map $\mu_{G/H}$ satisfies the given relation. As H acts trivially on $\mathfrak{h}^\circ$, equivariance of μ implies its restriction to $\mu_H^{-1}(0)$, $\iota_H^*\mu$, is constant on H-orbits, and so descends to a smooth map on the orbit space $M /\!/ H \to \mathfrak{h}^\circ$. Further, as $\mathfrak{h}^\circ = \ker(dj)^* = \operatorname{im}(dq)^*$ by exactness of (3.3.1), the induced map factors uniquely through the quotient which we denote by $\mu_{G/H}$; i.e. we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathfrak{g}/\mathfrak{h})^* & \xrightarrow{(dq)^*} & \mathfrak{g}^* & \xrightarrow{(dj)^*} & \mathfrak{h}^* & \longrightarrow & 0 \\
 & & \Big\uparrow{\scriptstyle \mu_{G/H}} & & \Big\uparrow{\scriptstyle \iota_H^*\mu} & & & & \\
 & & M /\!/ H & \xleftarrow[\pi_H]{} & \mu_H^{-1}(0) & & & &
\end{array}$$

We claim that $\mu_{G/H}$ is equivariant with respect to the G/H actions on $M /\!/ H$ and $(\mathfrak{g}/\mathfrak{h})^*$, so let $gH \in G/H$ and $H \cdot p \in M /\!/ H$. To show that

$$\mu_{G/H}\big(gH \cdot [p]\big) = \operatorname{Ad}^*_{gH}\, \mu_{G/H}\big([p]\big)$$

it suffices to show this holds when paired with elements of $\mathfrak{g}/\mathfrak{h}$, which can be written as $dq(X)$ for $X \in \mathfrak{g}$ as q is a submersion. Now, by commutativity of the diagram and equivariance of μ and dq,

$$\begin{aligned}
\big\langle \mu_{G/H}\big(gH \cdot [p]\big), dq(X)\big\rangle &= \big\langle (dq)^*\mu_{G/H}\big([p]\big), X\big\rangle \\
&= \langle \mu(g \cdot p), X\rangle \\
&= \big\langle \operatorname{Ad}^*_g\, \mu(p), X\big\rangle \\
&= \big\langle \operatorname{Ad}^*_g (dq)^*\mu_{G/H}\big([p]\big), X\big\rangle \\
&= \big\langle \mu_{G/H}\big([p]\big), dq\big(\operatorname{Ad}_{g^{-1}}(X)\big)\big\rangle \\
&= \big\langle \mu_{G/H}\big([p]\big), \operatorname{Ad}_{g^{-1}}\big(dq(X)\big)\big\rangle \\
&= \big\langle \operatorname{Ad}^*_g\, \mu_{G/H}\big([p]\big), dq(X)\big\rangle \\
&= \big\langle \operatorname{Ad}^*_{gH}\, \mu_{G/H}\big([p]\big), dq(X)\big\rangle .
\end{aligned}$$

Thus $\mu_{G/H}$ is equivariant, and it remains to show that it satisfies the moment map condition, i.e. we must show that

$$\Big(i(v_{M/\!/H})\omega^H_{[p]}\Big)(w) = \big\langle d(\mu_{G/H})_{[p]}(w), v\big\rangle$$

for all $[p] \in M /\!/ H$, $v \in \mathfrak{g}/\mathfrak{h}$, and $w \in T_{[p]}\big(M /\!/ H\big)$. Choose $X \in \mathfrak{g}$ and $Y \in T_p\mu_H^{-1}(0)$ so that $dq(X) = v$ and $d\pi_H(Y) = w$. As π_H is equivariant, we have $v_{M/\!/H} = \big(dq(X)\big)_{M/\!/H} =$

$d\pi_H\big(X_{\mu_H^{-1}(0)}\big)$. Hence

$$\begin{aligned} i(v_{M/\!\!/H})\omega^H_{H\cdot p}(w) &= \omega^H_{\pi_H(p)}\big(d(\pi_H)_p(X_{\mu_H^{-1}(0)}), d(\pi_H)_p(Y)\big) \\ &= (\pi_H^*\omega^H)_p(X_{\mu_H^{-1}(0)}, Y) \\ &= (\iota_H^*\omega)_p(X_{\mu_H^{-1}(0)}, Y) \\ &= \langle d\mu_p(Y), X\rangle \\ &= \big\langle (dq)^* d(\mu_{G/H})_{\pi_H(p)} d(\pi_H)_p(Y), X\big\rangle \\ &= \big\langle d(\mu_{G/H})_{\pi_H(p)}(w), v\big\rangle\,. \end{aligned}$$

□

It is routine to check that if the actions of H on $\mu_H^{-1}(0)$ and of G on $\mu^{-1}(0)$ are both free and proper, then the action of G/H on $\mu_{G/H}^{-1}(0)$ is free and proper. Thus by Proposition 3.3.3 we can form the iterated symplectic quotient $\mu_{G/H}^{-1}(0)/(G/H) = (M/\!\!/H)/\!\!/G/H$, which we claim is diffeomorphic to $M/\!\!/G$.

As $\mu_{G/H}^{-1}(0) = \pi_H\big(\mu^{-1}(0)\big) = \mu^{-1}(0)/H$, we consider the map $\pi_{G,G/H} : \mu^{-1}(0) \to \mu_{G/H}^{-1}(0)$ sending p to its H orbit $[p]$. This map is equivariant with respect to the G action on $\mu^{-1}(0)$ and the G/H action on $\mu_{G/H}^{-1}(0)$,

$$\pi_{G,G/H}(g\cdot p) = gH\cdot \pi_{G,G/H}(p).$$

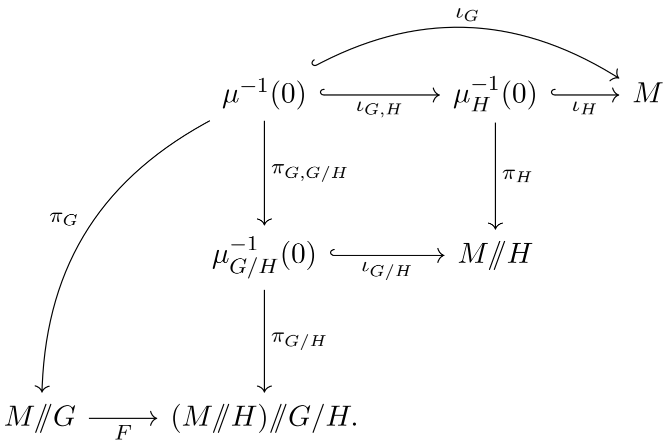
Hence $\pi_{G,G/H}$ induces a smooth map $F : M/\!\!/G \to (M/\!\!/H)/\!\!/G/H$ such that the following diagram commutes

$$\begin{array}{ccc} \mu^{-1}(0) & \xrightarrow{\pi_{G,G/H}} & \mu_{G/H}^{-1}(0) \\ \big\downarrow & & \big\downarrow \\ M/\!\!/G & \overset{}{\underset{F}{\dashrightarrow}} & (M/\!\!/H)/\!\!/G/H. \end{array}$$

Again, it is routine to check that F is a diffeomorphism; and the next theorem shows that F is a symplectomorphism between $M/\!\!/G$ and $(M/\!\!/H)/\!\!/G/H$.

Theorem 3.3.4 (Reduction in stages)**.** *Under the conditions for which the iterated symplectic quotient* $\big((M/\!\!/H)/\!\!/G/H, \omega^{G/H}\big)$ *can be formed, then it is symplectomorphic to* $(M/\!\!/G, \omega^G)$.

Proof. The fact that $F : M/\!\!/G \to (M/\!\!/H)/\!\!/G/H$ is a symplectomorphism will follow from the following commutative diagram



From commutativity of the diagram, and the universal properties of $\omega^{G/H}$ and ω^H:

$$\begin{aligned}\pi_G^*(F^*\omega^{G/H}) &= \pi_{G,G/H}^*(\pi_{G/H}^*\omega^{G/H})\\ &= \pi_{G,G/H}^*(\iota_{G/H}^*\omega^H)\\ &= \iota_{G,H}^*(\pi_H^*\omega^H)\\ &= \iota_{G,H}^*(\iota_H^*\omega)\\ &= \iota_G^*\omega.\end{aligned}$$

Thus by uniqueness property of ω^G, it follows that $F^*\omega^{G/H} = \omega^G$. □

3.4 Singular Reduction

Let (M, ω, G, μ) be a Hamiltonian G-space. Recall from Theorem 3.1.7 that in order for the reduction of M at the level $\xi \in \mathfrak{g}^*$ to be well defined, ξ needs to be a regular value for the moment map μ, and the coadjoint stabiliser group G_ξ needs to act freely and properly on the level set $\mu^{-1}(\xi)$. These conditions are rather restrictive, and so it is natural to ask what, if any, conditions in Theorem 3.1.7 can be loosened? The condition that the stabiliser group G_ξ acts properly is easy to deal with; if G is a compact lie group, then the stabiliser groups are compact and therefore act properly by Corollary A.1.11.1. For the second, if G_ξ does not act freely on $\mu^{-1}(\xi)$ where $\xi \in \mathfrak{g}^*$ is a regular value, then by Corollary 2.3.7.1 the G_ξ-action is locally free on $\mu^{-1}(\xi)$. Hence the reduced space $M/\!\!/_\xi G$ is a symplectic orbifold. Thus, the main condition we would like to loosen is for $\xi \in \mathfrak{g}^*$ to be a regular value.

Removing the condition for ξ to be a regular value of the moment map is rather problematic, as the resulting level set may not be a submanifold of M. However, much work has been done over the years to alleviate this assumption, which culminated in the 1991 paper of Sjamaar and Lerman [SL91], and is the focus of this section.

However, before we can state Sjamaar and Lerman's result, we need to recall some advanced topics from group actions on manifolds which are not contained in Appendix A. A good supplementary source for the following material is [Bre72].

3.4.1 The Slice Theorem

Let G be a Lie group acting smoothly and properly on a manifold M, and denote by G_p the stabiliser group of $p \in M$. Then for all $g \in G_p$ the differential of $\mathcal{A}_g : M \to M$ is a function $d\mathcal{A}_g : T_pM \to T_pM$. Thus G_p acts linearly on T_pM, and this action is called the **isotropy**, or **stabiliser representation** of G_p on T_pM. The tangent space to the orbit $G \cdot p$ is invariant under the isotropy representation. Indeed, by Corollary $A.4.3.1$ $T_p(G \cdot p) = \{X_M(p) : X \in \mathfrak{g}\}$, and so for all $X \in \mathfrak{g}$ and $g \in G_p$,

$$d(\mathcal{A}_g)_p\big(X_M(p)\big) = (\operatorname{Ad}_g X)_M(g \cdot p) = (\operatorname{Ad}_g X)_M(p) \in T_p(G \cdot p)$$

by Proposition $A.3.1$. Since the action is proper, the stabiliser group G_p is compact, and so there exists a G_p-invariant inner product on T_pM by Proposition $A.1.9$. Let W be the orthogonal complement of $T_p(G \cdot p)$ in T_pM relative to this inner product. Then W is G_p-invariant, and the tangent space T_pM splits into the G_p-invariant decomposition

$$T_pM = T_p(G \cdot p) \oplus W. \tag{3.4.1}$$

By (3.4.1) we can identify W with the quotient vector space $T_pM/T_p(G \cdot p)$. Since G_p acts on W, we can further form the associated bundle $G \times_{G_p} W$, which is a vector bundle over G/G_p with fibre W. The associated bundle has an induced G-action given by $g_1 \cdot [g_2, v] = [g_1g_2, v]$, and the manifold G/G_p can be identified with the zero section of $G \times_{G_p} W$,

$$G/G_p \cong \{[g, 0] \in G \times_{G_p} W : g \in G\} \subseteq G \times_{G_p} W.$$

However, as the action is proper, by Proposition $A.4.3$ G/G_p is equivariantly diffeomorphic to the orbit $G \cdot p$ of p. It is therefore natural to ask whether we can extend the identification of $G/G_p \cong G \cdot p$ with the zero section of $G \times_{G_p} W$ to a G-equivariant diffeomorphism between $G \times_{G_p} W$ and a neighbourhood of $G \cdot p$ in M? The answer is yes, and is a result known as the **slice theorem**.

To prove the slice theorem we will use the following **local linearisation**, or **Bochner linearisation theorem**. It is a special case of the slice theorem when $p \in M$ is a fix point for the G-action.

Theorem 3.4.1 (Local linearisation theorem). *Let G be a compact Lie group acting on a manifold M. Suppose that $p \in M$ is a fixed point for the G-action, i.e. $G_p = G$. Then there exists a G-equivariant diffeomorphism from a neighbourhood of $0 \in T_pM$ to a neighbourhood of $p \in M$.*

Proof. Let U be a G-invariant neighbourhood of $p \in M$, i.e. $g \cdot p \in U$ for all $g \in G$. Suppose $f : U \to T_pM$ is any smooth map such that $f(p) = 0$, and whose differential df at p is the identity on T_pM. Let $F : U \to T_pM$ be the average of f, i.e. for all $q \in U$

$$F(q) = \int_G d\mathcal{A}_g\big(f(g^{-1}\cdot q)\big)dm(g),$$

where m is the Haar measure on G, see Theorem $A.1.7$. Then F is smooth with $F(p) = 0$, and we claim that F is G-equivariant. Suppose $q \in U$ and $g \in G$, then

$$\begin{aligned} d\mathcal{A}_g\big(F(g^{-1}\cdot q)\big) &= d\mathcal{A}_g\left(\int_G d\mathcal{A}_h\big(f(h^{-1}\cdot g^{-1}\cdot q)\big)dm(h)\right) \\ &= \int_G d\mathcal{A}_g d\mathcal{A}_h\big(f(h^{-1}\cdot g^{-1}\cdot q)\big)dm(h) \\ &= \int_G d\mathcal{A}_{gh} f\big((gh)^{-1}\cdot q\big)dm(h) \\ &= \int_G d\mathcal{A}_h\big(f(h^{-1}\cdot q)\big)dm(h), \\ &= F(q) \end{aligned}$$

where the second equality follows as $d\mathcal{A}_g$ is continuous, and the second to last equality follows as m is left invariant. Hence

$$d\mathcal{A}_g \circ F \circ \mathcal{A}_{g^{-1}} = F,$$

which implies that $d\mathcal{A}_g \circ F = F \circ \mathcal{A}_g$, and F is G-equivariant. Furthermore, as $df_p = \mathrm{Id}_{T_pM}$, for all $v \in T_pM$ we find

$$\begin{aligned} dF_p(v) &= d\left(\int_G d\mathcal{A}_g\big(f(\mathcal{A}_{g^{-1}}q)\big)dm(g)\right)_p(v) \\ &= \int_G d\left(d\mathcal{A}_g\big(f(\mathcal{A}_{g^{-1}}\cdot q)\big)\right)_p(v)dm(g) \\ &= \int_G \big(d(\mathcal{A}_g)_p \circ df_p \circ d(\mathcal{A}_{g^{-1}})_p\big)(v)dm(g) \\ &= \int_G v\; dm(g) \\ &= v, \end{aligned}$$

where the last equality follows as $\int_G dm(g) = 1$. Hence $dF_p = \mathrm{Id}_{T_pM}$, and so the implicit function theorem implies that there exists a neighbourhood U of $p \in M$, and a neighbourhood V of $0 \in T_pM$, such that $F : U \to V$ is a diffeomorphism. □

Theorem 3.4.2 (Slice theorem). *Let G be a Lie group acting properly on a smooth manifold M, and let $p \in M$. Then there exists a small ball B in W about 0 with respect to some G_p-invariant metric, and a G-equivariant diffeomorphism from the associated bundle $G \times_{G_p} B$ to a neighbourhood of the orbit $G\cdot p$ in M; whose restriction to the zero section $G \times_{G_p} \{0\} \cong G/G_p$ is the map F^p defined in Proposition $A.4.3$.*

Under the assumptions of Theorem 3.4.1, $G \cdot p = \{p\}$ so that $T_pM = W$ by (3.4.1) and $G \times_G W \cong T_pM$. Hence Theorem 3.4.1 is indeed a specific case of the slice theorem.

Proof. As the action is proper, the stabiliser group G_p is compact, and so by Theorem 3.4.1 there exists a G_p-equivariant diffeomorphism E from a neighbourhood of 0 in T_pM to a neighbourhood of p in M. Fix any G_p-invariant inner product on W, where W is the subspace defined by (3.4.1). Further, let B be a ball in W relative to the inner product which is small enough to be contained in domain of E. Define

$$\psi : G \times_{G_p} B \to M,$$
$$[g, v] \mapsto g \cdot E(v).$$

To see that ψ is well defined, suppose $[g_1, v_1] = [g_2, v_2]$. Then there exists $h \in G_p$ such that $(g_1h^{-1}, h \cdot v_1) = (g_2, v_2)$ and

$$\psi\big([g_2, v_2]\big) = g_2 \cdot E(v_2) = g_1h^{-1} \cdot E(h \cdot v_1) = g_1h^{-1}h \cdot E(v_1) = g_1 \cdot E(v_1) = \psi\big([g_1, v_1]\big),$$

as E is G_p-equivariant. It is clear that ψ is also G-equivariant with respect to the G-action on $G \times_{G_p} B$ defined by $g_1 \cdot [g_2, v] = [g_1g_2, v]$. We also see that ψ is a local diffeomorphism at $[e, 0]$ because E is local diffeomorphism, and so equivariance gives ψ is a local diffeomorphism at the points $[g, 0]$ which is the zero section of the bundle $G \times_{G_p} B$. It remains to show that ψ is injective if B is a sufficiently small ball.

Assume, for a contradiction, that ψ is not injective on any neighbourhood of 0 in W. Then we have two sequences (u_n) and (v_n) in W converging to 0, and two sequences (g_n), (h_n) in G such that $[g_n, u_n] \neq [h_n, v_n]$ but

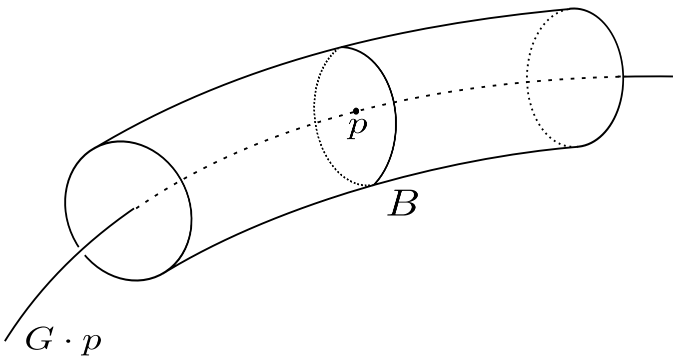
$$g_n \cdot E(u_n) = \psi\big([g_n, u_n]\big) = \psi\big([h_n, v_n]\big) = h_n \cdot E(v_n).$$

By taking the action with h_n^{-1}, without a loss of generality we may assume that $h_n = e$ for all n. Then $g_n \cdot E(u_n) = E(v_n)$ converges to p by continuity of E. This, however, does not imply that g_n converges to e in G. Under the proper map $G \times M \to M \times M$ the sequence $\big(g_n, E(u_n)\big)$ maps to the convergent sequence $\big(E(v_n), E(u_n)\big)$. Since the action is proper, the sequence $\big(g_n, E(u_n)\big)$ is contained a compact subset of $G \times M$, and so the sequence (g_n) is contained in a compact subset of G. Therefore the sequence (g_n) has a convergent subsequence (g_{n_j}) converging to some $g_0 \in G$. However, this implies we have sequences such that $[g_{n_j}, u_{n_j}] \neq [e, v_{n_j}]$ but

$$\lim_{j\to\infty} \psi\big([g_{n_j}, u_{n_j}]\big) = \lim_{j\to\infty} g_{n_j} \cdot E(u_{n_j}) = \lim_{j\to\infty} E(v_{n_j}) = p = \psi\big([e, 0]\big),$$

which is a contradiction to ψ being a local diffeomorphism at $[e, 0]$. □

From the Theorem 3.4.2, we obtain the following picture of how a neighbourhood of the orbit $G \cdot p$ *looks like*, at least locally:



Here a neighbourhood of $G \cdot p$ looks like a tube in the normal direction, centred on the orbit $G \cdot p$. It is for this reason the slice theorem, Theorem 3.4.2, is also referred to as the **equivariant tubular neighbourhood theorem**.

Further, as a G_p invariant ball B in W is equivariantly diffeomorphic to W itself, we obtain the following corollary.

Corollary 3.4.2.1. *Let G be a Lie group acting properly on a smooth manifold M, $p \in M$, and W be a G_p-invariant complementary subspace to $T_p(G \cdot p)$. Then there exists a G-equivariant diffeomorphism from the associated bundle $G \times_{G_p} W$ onto a neighbourhood of the orbit $G \cdot p$, whose restriction to the zero section $G \times_{G_p} \{0\} \cong G/G_p$ is the map F^p defined in Proposition A.4.3.*

The ball B constructed in Theorem 3.4.2 embeds in M by sending $v \in B$ to $\psi\big([e, v]\big)$, and this embedded submanifold has nice properties as we now show.

Proposition 3.4.3. *Let $p \in M$, and consider and neighbourhood U_p of its orbit $G \cdot p$ in M. By Theorem 3.4.2, there is a G-equivariant diffeomorphism $\psi : G \times_{G_p} W \to U_p$ where W is normal to the orbit. Then W embeds in M by mapping $v \mapsto \psi\big([e, v]\big)$, and let S be the image of W under this map. Then S is an embedded submanifold which satisfies the following conditions:*

- **I)** *S is G_p-invariant.*
- **II)** *S is transverse to $G \cdot p$ at p.*
- **III)** *For $s \in S$, $(G \cdot s) \cap S = G_p \cdot S$.*

Proof. Denote the embedding of W in M by φ. At p we have $T_pS = T_{[e,0]}W = W$, and as W is normal to $G \cdot p$ at p, we have

$$T_pM = T_p(G \cdot p) \oplus W.$$

Hence S and $G \cdot p$ intersect transversely at p, proving II). To see that S is G_p-invariant, let $\psi\big([e, v]\big) \in S$ and $h \in G_p$. Then as ψ is G-equivariant,

$$h \cdot \psi\big([e, v]\big) = \psi\big(h \cdot [e, v]\big) = \psi\big([h, v]\big) = \psi\big([e, h \cdot v]\big) \in S,$$

as W is G_p-invariant which proves I). Suppose that $s \in S$ and $g \cdot s \in S$ for some $g \in G$. Then we can write $s = \psi\big([e,v]\big)$, and $g \cdot s = \psi\big([e,u]\big)$. However, as ψ is G-equivariant,

$$\psi\big([e,u]\big) = g \cdot \psi\big([e,v]\big) = \psi\big(g \cdot [e,v]\big) = \psi\big([g,v]\big).$$

Hence there exists $h \in G_p$ such that $gh^{-1} = e$, and $h \cdot v = u$. However, the first condition immediately implies $h = g$ and so $g \in G_p$ proving the last statement. □

In light of Proposition 3.4.3, we have the following definition.

Definition 3.4.4. Let G be a Lie group acting smoothly on M. A submanifold $S \subseteq M$ is a **slice** for the group action at $p \in M$ if S is

i) S is G_p-invariant.

ii) S is transverse to $G \cdot p$ at p.

iii) For $s \in S$, $(G \cdot s) \cap S = G_p \cdot S$.

Corollary 3.4.4.1. *Let G be a Lie group acting smoothly and properly on M. Then for all $p \in M$ there exists a slice for the group action at p.*

3.4.2 Orbit Types

Suppose G is a Lie group acting smoothly and properly on a manifold M. Let $p \in M$, $g \in G$ and consider the stabiliser subgroups G_p and $G_{g \cdot p}$. How do the stabiliser subgroups G_p and $G_{g \cdot p}$ relate?

Note for $h \in G_{g \cdot p}$, we have the chain of equivalences

$$h \cdot (g \cdot p) = g \cdot p \iff (gg^{-1}hg) \cdot p = g \cdot p \iff (g^{-1}hg) \cdot p = p$$

and $g^{-1}hg \in G_p$. Thus it follows that $h \in G_{g \cdot p}$ if, and only if, $h \in gG_pg^{-1}$ and the stabiliser groups are related by conjugation. Hence each orbit $G \cdot p$ for a group action corresponds to a unique conjugacy class of stabilisers.

Following this idea, we define a relation on the subgroups of G by defining two subgroups H, K of G are related, written $H \sim K$, if, and only if, H is G-conjugate to K, $H = gKg^{-1}$ for some $g \in G$. It is clear that $\sim$ is an equivalence relation, and we denote by (H) the equivalence class of H. We further define a partial ordering on the set of equivalence classes by saying $(H_1) < (H_2)$ if H_2 is G-conjugate to a subgroup of H_1. Note that under this relation, the minimal class is (G), while the maximal class is $\big(\{e\}\big)$.

Definition 3.4.5. For a subgroup H of G we define

$$M^H = \{p \in M : H \subseteq G_p\}$$

to be the **fixed point set** of the H-action, and

$$M_{(H)} = \{p \in M : (H) = (G_p)\}$$

to be the set of **orbit type** (H).

If $p \in M_{(H)}$, then $G \cdot p \subseteq M_{(H)}$, as the stabiliser groups of points in an orbit are related by conjugation. Hence to each orbit corresponds a unique orbit type. Further note that we have the disjoint decomposition of M into orbit types

$$M = \coprod_H M_{(H)}$$

where H is a subgroup of G.

This is not only a decomposition as sets, but a decomposition by submanifolds.

Proposition 3.4.6. *For each subgroup H of a Lie group G acting smoothly and properly on M, $M_{(H)}$ is an embedded submanifold.*

Proof. Let $p \in M$ be a point whose stabiliser is H. Then $G \cdot p \subseteq M_{(H)}$, and to show $M_{(H)}$ is a submanifold it suffices to show that $G \cdot p$ has a neighbourhood whose intersection with $M_{(H)}$ is a submanifold. By the slice theorem, a neighbourhood U of the orbit $G \cdot p$ is modelled $G \times_H W$ with the G-action on $G \times_H W$ given by $g_1 \cdot [g, v] = [g_1 g, v]$. Further it follows that $U \cap M_{(H)}$ maps to $(G \times_H W)_{(H)}$ by equivariance. Thus it suffices to show $(G \times_H W)_{(H)}$ is a subbundle of $G \times_H W$.

Let $g_1 \in G_{[g,v]}$. By the definition of the associated bundle $[g_1 g, v] = [g, v]$ if, and only if, there exists $h \in H$ such that $(g_1 g, v) = (gh^{-1}, h \cdot v)$. Thus $h \in H_v$ and $g_1 = gh^{-1}g^{-1}$, and the stabiliser group of $[g, v]$ is given by

$$G_{[g,v]} = gH_v g^{-1},$$

where H_v is the stabiliser group of v relative to H. Moreover, the stabiliser groups of points in $G \times_H W$ are conjugate to a subgroup of H, and are conjugate to H itself if, and only if, $H_v = H$. (Note this means the orbit types in the model look like $(G \times_H W)_{(K)} = G \times_H W_{(K)}$, where $(H) \leq (K)$.) Therefore

$$(G \times_H W)_{(H)} = \{[g, v] : (G_{[g,v]}) = (H)\} = G \times_H W^H = G/H \times W^H,$$

which is a vector subbundle of $G \times_H W$ containing the zero section. Mapping this subbundle back to M, we obtain the required embedded submanifold. □

Corollary 3.4.6.1. *Let G be a Lie group acting smoothly and properly on M. Then the fixed point set M^G is a submanifold of M.*

Proof. The fixed point set M^G corresponds to the orbit type (G). □

Proposition 3.4.7. *Let G be a Lie group acting smoothly and properly on M. Then the number of orbit types is locally finite, i.e. every point $p \in M$ has a neighbourhood which meets only finitely many of the sets $M_{(H)}$.*

Proof. We proceed by induction on the dimension of M. If $\dim M = 0$, then the result holds as M is a countable union of points. Suppose now the proposition holds for all manifolds with dimension at most n, and suppose $\dim M = n$. As M is locally compact, to prove the proposition, it suffices to show the local model $G \times_H W$, with H compact, given by the slice theorem has finitely many orbit types.

Choose a H-invariant metric on the vector space W and let $S(W)$ be the sphere for the metric. Then $G \times_H S(W)$ is a subbundle whose corresponding embedded submanifold is of dimension $n-1$. Thus the induction hypothesis implies that there are only finitely many orbit types in $G \times_H S(W)$. As the H action on W is linear, it follows that $G_{[g,v]} = G_{[g,tv]}$ for all $t \neq 0$ and so the orbit type of $[g, tv] \in G \times_H W$ is the same as the orbit type of $[g, v] \in G \times_H S(W)$. Therefore there are only finitely many orbit types in $G \times_H W$: the orbit types of $G \times_H S(W)$, and the orbit type for the zero section G/H.

Hence by induction the number of orbit types is locally finite for any Lie group acting smoothly and properly on M. □

Corollary 3.4.7.1. *Suppose G is a Lie group acting smoothly and properly on a compact manifold M. Then the number of orbit types is finite.*

3.4.3 Stratifications

Proposition 3.4.6 shows that if G is a Lie group acting smoothly and properly on M, then M decomposes into embedded submanifolds given by the orbit types for the action. A natural question to ask is how these submanifolds piece together inside M. This subsection is dedicated to answering this question.

Definition 3.4.8. Let X be a paracompact, Hausdorff topological space and let $\mathcal{I}$ be a partially ordered set with order relation $\leq$. An **$\mathcal{I}$-decomposition** of X is a locally finite collection of disjoint, locally closed manifolds $S_i \subseteq X$ called **pieces** such that

i) $X = \coprod_{i \in I} S_i$,

ii) $S_i \cap S_j \neq \emptyset$ if, and only if, $S_i \subseteq \overline{S}_j$ which occurs if, and only if, $i \leq j$.

The second condition ii) is called the **frontier condition**. If $S_i \subseteq \overline{S}_j$ we write $S_i \leq S_j$; and if $S_i \leq S_j$ with $S_i \neq S_j$ we write $S_i < S_j$. Note that there is no requirement for the pieces to be connected.

We also define the **dimension** of X to be $\dim X = \sup_{i \in I} \dim S_i$, and for this thesis we only care about finite-dimensional decomposed spaces.

Example 3.4.9. Let G be a Lie group acting smoothly and properly on M. We will show later that M is a decomposed space with pieces the points of orbit type (H), for H a subgroup of G. The indexing set is given by reverse conjugation. ◀

Example 3.4.9 illustrates an issue with the definition of a decomposed space, the pieces may not have a well-defined dimension. For example let S^1 act on $\mathbb{C}P^2$ by $e^{i\theta} \cdot [z_0 : z_1 : z_2] = [e^{i\theta} z_0 : z_1 : z_2]$.

The fixed point set of this action, or equivalently the set of orbit type (G), consists of the point $[1:0:0]$ and the line $\{[0:z_1:z_2]:z_1,z_2\in\mathbb{C}\}$. The solution is to either allow the pieces to have connected components of differing dimension; or to further refine the decomposition into connected components. We choose the latter.

Definition 3.4.10. Let X be a $\mathcal{I}$-decomposed space with pieces $\{S_i\}_{i\in\mathcal{I}}$. The **depth** of a piece S in X is

$$\operatorname{depth}_X S = \sup\{n\in\mathbb{N} : \text{there exists pieces } S=S^0<S^1<\cdots<S^n\}.$$

The depth of S is bounded above by the codimension of S, $\dim X - \dim S$, and so is a non-negative integer. We define the depth of X to be

$$\operatorname{depth} X = \sup_{i\in\mathcal{I}}\{\operatorname{depth}_X S_i\}.$$

Note that $\operatorname{depth} X \le \dim X$.

If M is a manifold, then M is a decomposed space containing a single piece M, and $\operatorname{depth} M = 0$. Recall that if Y is a manifold, then the cone over Y, denoted by $\mathring{C}Y$, is the space constructed by collapsing the boundary $Y\times\{0\}$ of the half-open cylinder $Y\times[0,\infty)$ to a point. If $\mathring{C}X$ is a cone over a manifold X, then $\mathring{C}X$ decomposes into two pieces $X\times(0,\infty)$ and the vertex of the cone. Hence $\operatorname{depth}\mathring{C}X = 1$. In general, if X is a decomposed space with pieces $\{S_i\}_{i\in\mathcal{I}}$, then $\mathring{C}X$ is a decomposed space with pieces of the form $S_i\times(0,\infty)$ and the vertex of the cone. Thus

$$\operatorname{depth}\mathring{C}X = \operatorname{depth} X + 1.$$

Definition 3.4.11. A decomposed space X is a **stratified space of depth** n if the pieces of X, called **strata**, satisfy the following condition:

Given a point x in a piece S there exists an open neighbourhood U of x in X, an open (topological) ball B around x in S, a stratified space L of depth at most $n-1$ called the **link** of x, and a homeomorphism

$$\varphi : B\times\mathring{C}L\to U,$$

which preserves the decomposition.

The decomposition itself is referred to as a **stratification** of X.

Remark 3.4.12. Note that there are many notions of topological stratified spaces. For example one of the more common definitions is a **Whitney stratification** for submanifolds of $\mathbb{R}^n$, see [GM88, p. 37]. However, we follow the definition presented in [SL91, Definition 1.1]. ♦

We can extend the notion of a stratification to the smooth category as follows. If the strata of the space X are smooth manifolds, consider the subalgebra $C^\infty(X)$ of continuous functions on X, defined by $f\in C^\infty(X)$ if the restriction $f|_S$ to each stratum S is smooth. Call the subalgebra $C^\infty(X)$ the **smooth structure** on X. Given two stratified spaces X and Y, a continuous map $\varphi: X\to Y$ is **smooth** if for any $f\in C^\infty(Y)$ we have $f\circ\varphi\in C^\infty(Y)$. For example, the inclusion of a stratum S into X is smooth. In particular, we say that two stratified spaces X and Y are **diffeomorphic** if there exists a bijective smooth map $\varphi : X \to Y$ with a smooth inverse.

Let G be a Lie group acting smoothly and properly on a smooth manifold M, and let $X = M/G$ be the orbit space. Then M has a decomposition into orbit types $M = \coprod_{(H)} M_{(H)}$, which induces a decomposition of the orbit space $X = \coprod_{(H)} X_{(H)}$ where $X_{(H)} = M_{(H)}/G$. Further decompose X into the connected components of $X_{(H)}$, i.e.

$$X = \coprod_i X_i,$$

where X_i is a connected component of some $X_{(H_i)}$. Let M_i be the preimage of X_i under the quotient map, then we have a decomposition

$$M = \coprod_i M_i.$$

In an abuse of notation, we say these refinements are the **orbit type decomposition** of X and M, respectively.

Theorem 3.4.13 (Orbit Type Stratification)**.** *Let G be a Lie group acting smoothly and properly on M with orbit space $X = M/G$, then the decompositions $M = \coprod_i M_i$ and $X = \coprod X_i$ are stratifications.*

Proof. As always, by Theorem 3.4.2 it suffices to prove the theorem in the local model near any orbit $G \cdot p \subseteq M_{(H)}$ where $G_p = H$. We have already shown in Proposition 3.4.6 and Proposition 3.4.7 that each $M_{(H)}$ is an embedded submanifold, and the collection of orbit types is locally finite. In particular, this implies that the collection of components M_i is smooth and locally finite. To see that each X_i is smooth, recall that in the local model we have

$$(G \times_H W)_{(H)} = G \times_H W^H = G/H \times W^H,$$

where W is normal to the orbit, and so $(G \times_H W^H)_{(H)}/G = W^H$ which shows that the X_i are also smooth. The collection of X_i is also locally finite, as the collection M_i is locally finite and the quotient map is open.

To see that each $M_{(K)}$ is locally closed in M, take $p \in \overline{M_{(K)}}$. Then $(G_p) \leq (K)$. If equality holds, any orbit in $\overline{M_{(K)}}$ has orbit type greater than or equal to (G_p), and hence equals (G_p). Hence we have $M_{(K)}$ is open in $\overline{M_{(K)}}$, and therefore locally closed in M.

To see that the frontier condition holds suppose $M_{(H)} \cap \overline{M_{(K)}} \neq \emptyset$. We showed in Proposition 3.4.7 that $G_{[g,v]} = G_{[g,tv]}$ for all $t \neq 0$, so the orbit type stratum $(G \times_H W)_{(K)} = G \times_H W_{(K)}$ where $(K) \leq (H)$ is invariant under the scaling $(g, v) \mapsto (g, tv)$. Now if we view G/H as the zero section in $G \times_H W$, then it follows that G/H is contained in the closure of each orbit type stratum $(M \times_H W)_{(K)}$. Thus we find the closure of each orbit type stratum also contains $(G \times_H W)_{(H)} = G \times_H W^H$, however at the manifold level this implies $M_{(H)} \subseteq \overline{M_{(K)}}$.

To show the strata fit together in the required way, we again use induction on the dimension of M. If $\dim M = 0$, then the result holds as M is a disjoint union of points. Otherwise suppose the result holds for all manifolds with dimension less than n, and suppose M has dimension n. In the local model, choose an H-invariant inner product on W and let V be the orthogonal complement of W^H in W. Then

$$G \times_H W = G \times_H \left(W^H \oplus V\right) = W^H \times \left(G \times_H V\right) = W^H \times \left(G \times_H \mathring{C}\big(S(V)\big)\right),$$

where $S(V) \subseteq V$ is the sphere bundle. By induction, the H-action on $S(V)$ gives a stratification $S(V) = \coprod_j S(V)_j$. Then the orbit type decomposition for $G \times_H W$ is

$$(G \times_H W)_j = W^H \times \left((0, \infty) \times \left(G \times_H S(V)_j\right)\right),$$

together with the vertex $G/H \times W^H$. This shows that the orbit type decomposition is a stratification.

As the orbit space is modelled locally by

$$(G \times_H W)/G = W/H = W^H \times V/H = W^H \times \mathring{C}\big(S(V)/H\big),$$

a similar induction as before shows that this forms a stratification. □

Theorem 3.4.14. *Suppose G is a Lie group acting smoothly and properly on a manifold M such that orbit space M/G is connected. Among the conjugacy classes of stabiliser groups, there is a unique conjugacy class (K_{prin}) such that $(G_p) \leq (K_{\text{prin}})$ for any other stabiliser group. The corresponding orbit type stratum $M_{(K_{\text{prin}})}$ is an open dense subset of M, and its quotient $X_{(K_{\text{prin}})} = M_{(K_{\text{prin}})}/G$ is an open, dense, connected subset of M/G.*

Definition 3.4.15. The orbit type (K_{prin}) whose existence is guaranteed in Theorem 3.4.14 is called the **principal orbit type** for the action. Its orbits are the **principal orbits**.

Proof. We show that there exists a unique orbit type stratum (K_{prin}) which is open and dense in M. Such a stratum necessarily has depth zero and satisfies $(G_p) \leq (K_{\text{prin}})$: if $p \in M = \overline{M_{(K_{\text{prin}})}}$, then $(G_p) \leq (K_{\text{prin}})$.

We first show the result holds if both M and G are connected, and then show we can reduce to this case.

Once more, we prove this using induction on $\dim M$. The case when $\dim M = 0$ is trivial, so suppose the result holds manifolds up to $\dim M = n$. As always, it suffices to prove the result in the local model $G \times_H W$. Let $S(W)$ be the unit sphere relative to some H-invariant inner product on W. If $G \times_H S(W)$ is connected, then we can apply the induction hypothesis to conclude there exists a unique orbit type (K_{prin}) such that $\big(G \times_H S(W)\big)_{(K_{\text{prin}})} = G \times_H S(W)_{(K_{\text{prin}})}$ is open and dense. The only way $G \times_H S(W)$ is disconnected is when $\dim W = 1$, so that $S(W) = \{1, -1\}$. In this case we see that $G \times_H S(W) \to G/H$ is a two-sheeted covering. If it is non-trivial, then $G \times_H S(W)$ is connected and we are done. If the covering is trivial, then H acts trivially on W.

Hence all orbits of $G \times_H W$ are of the same type and the result holds. In either case it follows that there exists a unique orbit type (K_{prin}) such that $\big(G \times_H S(W)\big)_{(K_{\text{prin}})} = G \times_H S(W)_{(K_{\text{prin}})}$ is open and dense in $G \times_H S(W)$. As the orbit types for $G \times_H W$ consist of the orbit types of $G \times_H S(W)$ and the orbit type for the zero section, the result follows.

Suppose now that $M = \bigcup M_\alpha$ is disconnected. Let $G_\alpha = \{g \in G : g \cdot M_\alpha = M_\alpha\}$. Let $\tilde{M}_\alpha$ be the union of the G_α-principal orbits in M_α. For $\alpha \neq \beta$, there exists $g \in G$ such that $g \cdot M_\alpha = M_\beta$, which implies $g \cdot \tilde{M}_\alpha = \tilde{M}_\beta$ as both are open and dense. Thus the principal orbits in $\tilde{M}_\alpha$ and $\tilde{M}_\beta$ are conjugated by $g \in G$, and so define the same principal orbit type.

Now suppose that M is connected, and G is disconnected. Then $G = \bigcup g_\alpha G_0$, where G_0 is the identity component of G. Let $\tilde{M}_0$ be the union of the principal orbits for the G_0-action on M. Then $\tilde{M} = \bigcup g_\alpha \tilde{M}_0$ is an open dense subset of M, and all orbits in $\tilde{M}_0$ have the same orbit type. □

Suppose G is a Lie group acting smoothly and properly on M. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be Lie subalgebras of $\mathfrak{g}$. Define a relation on the subalgebras by setting $\mathfrak{h}_1 \sim \mathfrak{h}_2$ if there exists $g \in G$ such that $\mathrm{Ad}_g(\mathfrak{h}_1) = \mathfrak{h}_2$. This relation is subalgebra conjugation by G, and it is clear that it is an equivalence relation. Using this we can define the decomposition of M into infinitesimal orbit types by defining

$$M_{(\mathfrak{h})} = \{p \in M : \mathfrak{g}_p \sim \mathfrak{h}\},$$

where $\mathfrak{g}_p$ is the stabiliser algebra of p. Note that $M_{(\mathfrak{h})}$ is the union of all $M_{(H)}$ where H is a subgroup of G such that $\mathrm{Lie}(H) = \mathfrak{h}$.

We have a stratification of M obtained by decomposing M into the infinitesimal orbit types $M_{(\mathfrak{h})}$, and then further into connected components M_i. The strata of the stratification are called the **infinitesimal orbit type strata**. It follows that the previous results on orbit types and stratifications extend to the infinitesimal orbit types. However, we make explicit the following theorem.

Theorem 3.4.16. *Suppose G is a Lie group acting smoothly and properly on a connected manifold M. Then there exists a unique subalgebra $\mathfrak{k}$ such that $M_{(\mathfrak{k})}$ is an open, dense, and connected submanifold of M.*

3.4.4 The Result

We are finally ready to state the singular reduction result of Sjamaar and Lerman.

Theorem 3.4.17 (Singular Reduction [SL91]). *Let (M, ω, G, μ) be a Hamiltonian G-space with G a compact Lie group. The intersection of zero level set of the moment map $\mu^{-1}(0)$ and the points of orbit type (H) $M_{(H)}$ is a G-stable subset of $\mu^{-1}(0)$ and the orbit space*

$$(M/\!\!/G)_{(H)} = \big(\mu^{-1}(0) \cap M_{(H)}\big)/G$$

is naturally a symplectic manifold, whose symplectic form is induced by the restriction of ω to $\mu^{-1}(0) \cap M_{(H)}$. Thus the orbit type decomposition of M induces a decomposition of the reduced *space $M/\!\!/G$ into a disjoint union of symplectic manifolds*

$$M/\!\!/G = \coprod_{(H)} (M/\!\!/G)_{(H)}.$$

The proof of this result requires a symplectic version of the slice theorem, Theorem 3.4.2. This leads to a **local normal form** for the moment map; due to Guillemin and Sternberg [GS84a], and Marle [Mar85]. It provides a description of the moment map in terms of the local model defined by the symplectic slice. This allowed Sjamaar and Lerman to prove the statements in this local model (just like the case for properties of the orbit type decomposition).

As in the case of orbit type decompositions the pieces $(M/\!\!/G)_{(H)}$ may not be manifolds in the sense they could have connected components of different dimensions. As in the orbit type decomposition, we rectify this by taking a further refinement of $(M/\!\!/G)_{(H)}$ into its connected components.

Moreover, we know that the orbit type decomposition of M is in fact a stratification in the sense of Definition 3.4.11, so we could ask whether the decomposition of the reduced space given in Theorem 3.4.17 is also a stratification? The answer is yes, and is proven in [SL91, Theorem 6.11]. As a very rough outline of the proof, Sjamaar and Lerman show that each piece embeds inside $\mathbb{R}^n$. The embedded pieces in $\mathbb{R}^n$ are Whitney stratified, which are also stratified in our sense.

As such we call each piece $(M/\!\!/G)_{(H)}$ a **symplectic stratum** of $M/\!\!/G$. Furthermore, one can show that the smooth structure on $M/\!\!/G$ is a Poisson algebra, and the embeddings $(M/\!\!/G)_{(H)} \hookrightarrow M/\!\!/G$ are Poisson maps. All this leads to the following definition.

Definition 3.4.18. A **stratified symplectic space** X is a stratified space with a smooth structure, $C^\infty(X)$, such that:

i) each stratum X_i is a symplectic manifold.

ii) $C^\infty(X)$ is a Poisson algebra.

iii) The embeddings $X_i \hookrightarrow X$ are Poisson maps.

Hence Theorem 3.4.17 can be restated as the reduced space of Hamiltonian G-manifold, for G a compact Lie group, is a symplectic stratified space. Moreover, it is shown in [SL91] that a lot of the standard results of symplectic reduction apply in the case of singular reduction. For example, it follows that both forms of reduction in stages, Theorem 3.3.2 and Theorem 3.3.4, have singular analogues.

Chapter 4

Symplectic Implosion

Symplectic Implosion is a procedure introduced by Guillemin, Jeffrey, and Sjamaar in [GJS02], which abelianises reduction. The idea is that given a Hamiltonian G-space (M, ω, G, μ) with G a compact, connected Lie group; we seek a Hamiltonian T-space M_{impl}, where $T \subseteq G$ is a maximal torus of G, such that $M /\!\!/_{\xi} G \cong M_{\text{impl}} /\!\!/_{\xi} T$ for suitable values ξ. However, this abelianisation introduces singularities into the imploded space M_{impl}, and so M_{impl} is not a Hamiltonian T-space in the sense of Definition 2.3.5, but rather a stratified Hamiltonian T-space.

The goal of this chapter is provide an introduction to symplectic implosion, following the original paper [GJS02]. However, we expand on some of the proofs and results which were terse in said paper.

4.1 The Fundamental Weyl Chamber

Let (M, ω, G, μ) be a Hamiltonian G-space. Recall by the shifting trick, Theorem 3.2.4, the reduced space $M /\!\!/_{\xi} G$ at the level $\xi \in \mathfrak{g}^*$ is symplectomorphic to the quotient $M_{G \cdot \xi} = \mu^{-1}(G \cdot \xi)/G$. Hence to understand reduction, we only have to compute at choice of representative for a coadjoint orbit. So the question is: what is the orbit space $\mathfrak{g}^*/G$ for the coadjoint action? It is well known that the space of orbits for the coadjoint action is given by the product of a vector space (the fixed point set of the coadjoint action), and a proper closed convex polytope, see [Kir04, p. 148]. We set $\mathfrak{t}^*_+ = \mathfrak{g}^*/G$, and call it the **fundamental Weyl chamber**.

However, the fundamental Weyl chamber has a more insightful description in terms of the roots of G, which we provide a brief outline of. The proof of the following statements can be found in any book on Lie theory: for example [Hal15], [Kna96], or [Sep07].

Consider the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$. Let $\mathfrak{t} = \text{Lie}(T)$ be the Lie algebra of T, and $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$ its complexification. It is well known that $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, a maximal abelian subalgebra consisting of ad-semisimple elements. Hence we can form the root space decomposition of $\mathfrak{g}_\mathbb{C}$

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where Δ is a finite subset of $\mathfrak{t}^*_\mathbb{C} \setminus \{0\}$, and

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_\mathbb{C} : \mathrm{ad}_H X = \alpha(H)X, \text{ for all } H \in \mathfrak{t}_\mathbb{C}\}.$$

We refer to $\alpha \in \Delta$ as **root** for $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{t}_\mathbb{C}$, and Δ is a **root system** for $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{t}_\mathbb{C}$. It is known that the roots are valued in $i\mathfrak{t}^*$, and so we identify them as lying inside $\mathfrak{t}^*$. Moreover the roots span $\mathfrak{t}^*_1 = \mathfrak{t}^* \cap [\mathfrak{g},\mathfrak{g}]^*$, the semisimple part of $\mathfrak{t}^*$. A **base** for the root system is a subset $B \subseteq \Delta$ such that B spans $\mathfrak{t}^*_1$ and every root $\alpha \in \Delta$ can be written as a linear combination of elements of B with either non-negative or non-positive integer coefficients. An element of the base is called a **simple root**. Note that a base B is a basis for the subspace $\mathfrak{t}_1$ so $|B| = \dim \mathfrak{t}_1$. We define the **rank** of G as $\dim \mathfrak{t}_1$, which is well-defined as maximal tori are related by inner automorphisms.

Now as G is compact, there exists an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}_\mathbb{C}$, which induces an inner product on $\mathfrak{t}_\mathbb{C}$. This inner product further induces an inner product on the dual $\mathfrak{t}^*_\mathbb{C}$, and therefore defines an inner product on the root system which we denote by $(\, , \,)$. Consider the set

$$\mathfrak{t}^*_\mathbb{C} \setminus \bigcup_{\alpha\in\Delta} \{\alpha\}^\perp = \{H \in \mathfrak{t}^*_\mathbb{C} : (H,\alpha) \neq 0 \text{ for all } \alpha \in \Delta\}, \tag{4.1.1}$$

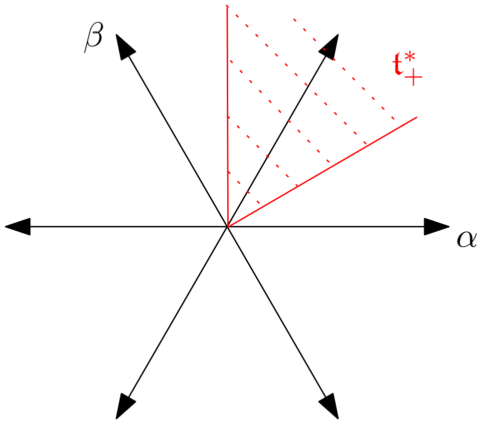
where $\{\alpha\}^\perp$ denotes the hyperplane orthogonal the the root α. The connected components of (4.1.1) are referred to as **open Weyl Chambers**. Given a base B for the root system, we have a canonical closed Weyl chamber called the **fundamental Weyl chamber** given by

$$\mathfrak{t}^*_+ = \overline{\{H \in \mathfrak{t}^*_\mathbb{C} : (H,\alpha) > 0 \text{ for all } \alpha \in B\}}.$$

Again viewing the root system as lying in $\mathfrak{t}$, it follows that $\mathfrak{t}^*_+$ is the same set parameterising the coadjoint orbit which was defined earlier.

It is clear that $\mathfrak{t}^*_+$ is a convex polytope, and thus we can consider the disjoint decomposition of $\mathfrak{t}^*_+$ into the open faces of said polytope. These are given by intersection of the elements $\mathfrak{t}_1$ which are orthogonal to a subset of the simple roots. Thus $\mathfrak{t}^*_+$ decomposes into 2^r disjoint open faces, where r is the rank of G, as this is the number of subsets of B. As an example, we find that the lowest dimensional face of $\mathfrak{t}^*_+$ is the origin, being the only element orthogonal to all the simple roots; while the highest dimensional face is the interior of $\mathfrak{t}^*_+$, consisting of elements not orthogonal to any simple root. From now on, let Σ denote the collection of open faces of $\mathfrak{t}^*_+$, and $\sigma \in \Sigma$ an open face. Further we define a partial ordering $\leq$ on Σ by setting $\sigma \leq \sigma'$ if, and only if, $\sigma \subseteq \overline{\sigma'}$.

Example 4.1.1. Consider $G = \mathrm{SU}(3)$. Then $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(3,\mathbb{C})$ which has the well known A_2 root system. This root system is two dimensional consisting of six roots, all of equal length, and all spaced apart at an angle of $2\pi/3$. A base of the system consists of two roots which are spaced apart at an angle of $4\pi/3$. The following figure is a pictorial representation of the root system for $\mathrm{SU}(3)$. A base of the system is $B = \{\alpha,\beta\}$ and the corresponding fundamental Weyl chamber $\mathfrak{t}^*_+$ is shown in red.



◀

One final note is the following: given a point $\xi \in \mathfrak{t}^*_+$, let σ be the open face of $\mathfrak{t}^*_+$ containing ξ. If G_ξ denotes the coadjoint stabiliser group of ξ, it follows that $G_\eta = G_\xi$ for all points $\eta \in \sigma$ (see [DK00, Chapter 3]). Hence we can refer to G_σ as the coadjoint stabiliser for the open face σ of $\mathfrak{t}^*_+$.

4.1.1 The Kirwan Polytope

Returning to the realm of symplectic geometry, as $\mathfrak{t}^*_+$ parameterises the coadjoint orbits, to understand reduction we only have to consider levels $\xi \in \mathfrak{t}^*_+$. Thus if (M, ω, G, μ) is a Hamiltonian G-space, the set $\mu(M) \cap \mathfrak{t}^*_+$ is an important object to consider. As $\mathfrak{t}^*_+$ is a convex polytope, one could ask whether $\mu(M) \cap \mathfrak{t}^*_+$ is also a convex polytope?

The case where G is abelian, i.e. a torus, is true: $\mu(M) \cap \mathfrak{t}^*_+$ is a convex polytope. This result is the celebrated **Atiyah-Guillimen-Sternberg convexity theorem**, and was shown independently by Atiyah [Ati82], and Guillemin-Sternberg [GS82]. In [GS82], Guillemin and Sternberg conjectured that this result should hold for non-abelian Lie groups as well, however they were unable to prove this. This conjecture was proved two years later by Kirwan in [Kir84], with the additional assumption that M is compact connected. It was later shown by Hilgert, Neeb, and Plank in [HNP94], that the assumption that M is compact could be replaced with the weaker condition that the moment map μ is proper.

In any case, we refer to the set $\Delta(M) = \mu(M) \cap \mathfrak{t}^*_+$ as the **moment set**, or **Kirwan polytope** of M, and it plays a pivotal role in the rest of this chapter.

4.2 Symplectic Cross-Sections

Given a symplectic manifold (M, ω) a common problem is trying to construct symplectic submanifolds of M. One such construction is through the use of symplectic cross-sections, which is originally due to Guillemin and Sternberg [GS84b, Theorem 26.7]. However, we will use a refinement of Guillemin and Sternberg's result, which is due to Lerman et al [Ler+98].

Suppose (M, ω, G, μ) is a Hamiltonian G-space with G a compact, connected Lie group, and consider the coadjoint action of G on $\mathfrak{g}^*$. Recalling the definition of a slice for a group action given in Definition 3.4.4, we have the following.

Lemma 4.2.1. *The open subset $S_\lambda \subseteq \mathfrak{g}_\lambda^* \subseteq \mathfrak{g}^*$ defined by*

$$S_\lambda = G_\lambda \cdot \{\xi \in \mathfrak{t}_+^* : G_\xi \subseteq G_\lambda\}$$

is a slice at $\lambda \in \mathfrak{t}_+^*$ for the coadjoint action of G. Moreover, S_λ is the largest subset of $\mathfrak{g}_\lambda^*$ that is a slice at λ.

Proof. As G is connected the stabiliser groups for the coadjoint orbits are connected as well, see [GLS96]. Hence the inclusion $G_\xi \subseteq G_\lambda$ of stabiliser groups is equivalent to the inclusion of the stabiliser algebras $\mathfrak{g}_\xi \subseteq \mathfrak{g}_\lambda$. Thus S_λ is a G_λ-invariant open neighbourhood of λ. Also recall that $T_\lambda(G \cdot \lambda) = \mathfrak{g}/\mathfrak{g}_\lambda$, so $T_\lambda(G \cdot \lambda) = \mathfrak{g}_\lambda^\perp$ relative to some G_λ-invariant inner product, which implies that S_λ is transverse to $G \cdot \lambda$ at λ.

Now suppose $\xi \in S_\lambda$ and $g \cdot \xi \in S_\lambda$ for $g \in G$. By definition of S_λ, there exists $h, k \in G_\lambda$ such that $h \cdot \xi$, and $kg \cdot \xi$ are in the closed fundamental Weyl Chamber $\mathfrak{t}_+^*$. However, the fundamental Weyl chamber parameterises the coadjoint orbits, and so it follows that $h \cdot \xi = kg \cdot \xi$. Thus $\xi = h^{-1}kg \cdot \xi$ and $h^{-1}kg \in G_\xi \subseteq G_\lambda$, and so $g \in G_\lambda$ as required. □

The slice S_λ constructed in Lemma 4.2.1 is referred to as the **natural slice** at λ.

Corollary 4.2.1.1. *The coadjoint stabiliser group is the same for every point in an open face of the closed fundamental Weyl chamber. Therefore if $\lambda \in \mathfrak{t}_+^*$ and σ is the open face containing λ, then the natural slice at λ is equal to*

$$S_\sigma = G_\sigma \cdot \{\xi \in \mathfrak{t}_+^* : G_\xi \subseteq G_\lambda\} = G_\sigma \cdot \operatorname{star} \sigma = G_\sigma \cdot \bigcup_{\sigma \leq \tau} \tau,$$

where G_σ is the stabiliser group of σ and τ is an open face of the fundamental Weyl chamber.

Using Corollary 4.2.1.1 we have the following **cross-section** theorem.

Theorem 4.2.2 ([Ler+98], Theorem 3.8)**.** *Let (M, ω, G, μ) be a connected Hamiltonian G-space with G a compact, connected Lie group. Let $\lambda \in \mathfrak{g}^*$ and S_λ be the natural slice at λ. Then the **cross-section** $M_\lambda = \mu^{-1}(S_\lambda)$ is a G_λ-invariant symplectic submanifold of M. Further the G_λ-action is Hamiltonian with moment map given by the restriction of μ to M_λ.*

As $G_\lambda = G_\sigma$ for the open face of $\mathfrak{t}_+^*$ containing λ, Theorem 4.2.2 extends to show that $M_\sigma = \mu^{-1}(S_\sigma)$ is a symplectic submanifold of M.

Note that the cross-section M_λ may not be a slice for the action of G on M, since G_λ is not necessarily the stabiliser group for a point in M_λ. However, as S_λ is a slice for the coadjoint action, and as the moment map μ is equivariant, we find that the saturation $G \cdot M_\lambda$ is an open subset of M equivariantly diffeomorphic to the associated bundle $G \times_{G_\lambda} M_\lambda$ over $G \cdot \lambda$.

One of the main applications of Theorem 4.2.2 is to reduced statements about G orbits in M to the case when the orbit is contained in the zero level set of a moment map for the action. To see this let $p \in \mu^{-1}(\mathfrak{t}_+^*)$, and let σ be the open face of $\mathfrak{t}_+^*$ containing $\mu(p) = \lambda \in \mathfrak{t}_+^*$. Let S_σ be the natural slice at σ and M_σ the corresponding cross-section of M. By Theorem 4.2.2 M_σ is a Hamiltonian G_σ-space. Moreover, there is a unique G_σ invariant decomposition

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}_\sigma) \oplus [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] \oplus \mathfrak{m}_\sigma,$$

where $\mathfrak{z}(\mathfrak{g}_\sigma)$ is the centre of $\mathfrak{g}_\sigma$, $[\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]$ is the semi-simple part of $\mathfrak{g}_\sigma$, and $\mathfrak{m}_\sigma$ is the complement of $\mathfrak{g}_\sigma$. As $\mathfrak{z}(\mathfrak{g}_\sigma)$ can be characterised as the fixed point set for the G_σ-action on $\mathfrak{g}$, it follows that the linear span of σ is $\mathfrak{z}(\mathfrak{g}_\sigma)^*$. Since $\lambda = \mu|_{M_\sigma}(p) \in \sigma \subseteq \mathfrak{z}(\mathfrak{g}_\sigma)^* = [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]^\circ$, we can shift the moment map $\mu|_{M_\sigma}$ by λ to obtain a new moment map $\tilde{\mu}$ such that $p \in \tilde{\mu}^{-1}(0)$.

4.2.1 The Principal Cross-Section

Suppose (M, ω, G, μ) is a connected Hamiltonian G-space with G a compact, connected Lie group. The main result of this section is that there exists a unique face σ_{prin} of $\mathfrak{t}_+^*$ such that $\mu(M) \cap \sigma$ is dense in $\mu(M) \cap \mathfrak{t}_+^*$, and the cross-section $M_{\sigma_{\text{prin}}} = \mu^{-1}(\sigma_{\text{prin}})$ is a connected Hamiltonian T-space.

Recall from Theorem 3.4.16 that there exists a unique infinitesimal orbit type $(\mathfrak{k})$ such that $M_{(\mathfrak{k})}$ is open, dense, and connected in M. Note that we also call $M_{(\mathfrak{k})}$ the **principal stratum**, or **principal orbit type**. Suppose now that $p \in M_{(\mathfrak{k})}$, then as $\operatorname{im} d\mu_p = \mathfrak{g}_p^\circ$ by Proposition 2.3.7, and as $\mathfrak{k} = \operatorname{Ad}_g(\mathfrak{g}_p)$ for some $g \in G$, it follows that $\dim \mathfrak{k} = \dim \mathfrak{g}_p$ and the restriction of the moment map to $M_{(\mathfrak{k})}$ has constant rank.

Theorem 4.2.3 ([Ler+98], Theorem 3.1)**.** *Let (M, ω, G, μ) be a connected Hamiltonian G-space with G a compact, connected Lie group. Then*

I) *There exists a unique open face σ_{prin} of $\mathfrak{t}_+^*$ such that $\mu(M) \cap \sigma_{\text{prin}}$ is dense in $\mu(M) \cap \mathfrak{t}_+^*$.*

II) *The commutator subgroup $[G_{\sigma_{\text{prin}}}, G_{\sigma_{\text{prin}}}]$ acts trivially on the preimage $M_{\sigma_{\text{prin}}} = \mu^{-1}(S_{\sigma_{\text{prin}}})$. Hence $M_{\sigma_{\text{prin}}}$ is a connected Hamiltonian T-manifold, with moment map given by the restriction of μ to $M_{\sigma_{\text{prin}}}$.*

III) *The saturation of $M_{\sigma_{\text{prin}}}$, $G \cdot M_{\sigma_{\text{prin}}} = \{g \cdot p : g \in G, p \in M_{\sigma_{\text{prin}}}\}$ is dense in M.*

The principal face is constructed out of the image of the principal orbit type stratum under the moment map μ, justifying the name. It is also shown that σ_{prin} is the lowest dimensional face of $\mathfrak{t}_+^*$ such that the moment polytope $\mu(M) \cap \mathfrak{t}_+^*$ is contained in its closure.

By the definition of the natural slice, we have the following corollary.

Corollary 4.2.3.1. *Let σ_{prin} be the unique open face constructed in Theorem* 4.2.3*, then $M_{\sigma_{\text{prin}}} = \mu^{-1}(\sigma)$. Thus if the cross-section M_σ for a face σ is non-empty, then its saturation $G \cdot M_\sigma$ is dense.*

Proof. The principal face σ_{prin} is the lowest dimensional face of $\mathfrak{t}_+^*$ such that $\mu(M) \cap \mathfrak{t}_+^* \subseteq \overline{\sigma_{\text{prin}}}$. Hence $\mu^{-1}(\sigma_{\text{prin}}) = \mu^{-1}(S_{\sigma_{\text{prin}}}) = M_{\sigma_{\text{prin}}}$. For the second statement, by Corollary 4.2.1.1 the slice for a face σ, is given by $S_\sigma = G_\sigma \cdot \operatorname{star} \sigma$. Hence if M_σ is non-empty, then $\sigma_{\text{prin}} \subseteq \operatorname{star} \sigma$ and so $G \cdot M_\sigma$ is dense as $G \cdot M_{\sigma_{\text{prin}}} \subseteq G \cdot M_\sigma$. □

Therefore, we refer to $M_{\sigma_{\text{prin}}}$ as the **principal cross-section** of M. For most situations σ_{prin} is the interior of the fundamental Weyl chamber $(\mathfrak{t}_+^*)^\circ$; for example if G is semisimple. The issue lies in that $\mathfrak{t}_+^*$ is the product $\mathfrak{z}(\mathfrak{g})^* \times \big(\mathfrak{t}_+^* \cap [\mathfrak{g}, \mathfrak{g}]^*\big)$, so if $\mathfrak{z}(\mathfrak{g})^* \neq 0$, then $\mu(M)$ may only hit a boundary face of $\mathfrak{t}_+^*$.

4.3 Imploded Cross-Sections

Let (M, ω, G, μ) be a connected Hamiltonian G-space with G a compact, connected Lie group. Fix a maximal torus $T \subseteq G$ which defines a fundamental Weyl chamber in $\mathfrak{t}^* = \operatorname{Lie}(T)$. We wish to construct a Hamiltonian T-space M_{impl}, called the **imploded cross-section**, or **imploded space**, of M such that $M /\!\!/_\xi G \cong M_{\text{impl}} /\!\!/_\xi T$ for all $\xi \in \mathfrak{t}_+^*$. This implies that M_{impl} *abelianises* the reduction procedure.

Now by Theorem 4.2.3 we know that the preimage of the principal face σ_{prin}, $M_{\sigma_{\text{prin}}}$ is a connected Hamiltonian T-space whose saturation is dense in M, and hence provides a suitable basis for the construction of M_{impl}. However, we need to *complete* $M_{\sigma_{\text{prin}}}$ to cover all of M. A natural choice is then to take $\mu^{-1}(\mathfrak{t}_+^*) = \mu^{-1}(\overline{\sigma_{\text{prin}}})$; but there is no guarantee that the preimages of the boundary faces are smooth manifolds, let alone symplectic. To rectify this situation we will contract, or *implode*, certain parts of the boundary faces.

Definition 4.3.1. Define a relation $\sim$ on $\mu^{-1}(\mathfrak{t}_+^*)$ by setting $p \sim q$ if, and only if, there exists $g \in [G_{\mu(p)}, G_{\mu(p)}]$ such that $q = g \cdot p$.

By equivariance of the moment map μ, if $p \sim q$ then $\mu(p) = \mu(q)$ and $\sim$ is an equivalence relation on $\mu^{-1}(\mathfrak{t}_+^*)$.

Definition 4.3.2. The **imploded cross-section**, or **imploded space**, of M is the quotient space $M_{\text{impl}} = \mu^{-1}(\mathfrak{t}_+^*)/{\sim}$ endowed with the quotient topology. The quotient map $\mu^{-1}(\mathfrak{t}_+^*) \to M_{\text{impl}}$ will be denoted by π. The **imploded moment map** μ_{impl} is the continuous map $M_{\text{impl}} \to \mathfrak{t}_+^*$ induced by μ, i.e. so that the following diagram commutes

$$\begin{array}{ccc} \mu^{-1}(\mathfrak{t}_+^*) & & \\ \downarrow{\scriptstyle \pi} & \searrow{\scriptstyle \mu} & \\ M_{\text{impl}} & \xrightarrow[\mu_{\text{impl}}]{} & \mathfrak{t}_+^*. \end{array}$$

Note $\mu_{\text{impl}}(M_{\text{impl}}) = \mu(M) \cap \mathfrak{t}_+^*$. Moreover, as all points in an open face σ of $\mathfrak{t}_+^*$ have the same stabiliser G_σ, we have

$$M_{\text{impl}} = \coprod_{\sigma\in\Sigma} \mu^{-1}(\sigma)/[G_\sigma, G_\sigma],$$

where Σ is the set of faces of $\mathfrak{t}^*_+$.

Lemma 4.3.3 ([GJS02], Lemma 2.3)**.** *The projection π is proper and M_{impl} is locally compact, Hausdorff, and second countable. If M is compact, then so is M_{impl}. Moreover, $\mu^{-1}(\sigma)/[G_\sigma, G_\sigma]$ is locally closed in M_{impl} for every face σ.*

Recall that by Theorem 2.3.26 the moment map is unique up to addition by a constant in $[\mathfrak{g},\mathfrak{g}]^\circ$. However, as G is compact $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g},\mathfrak{g}]$, and we identify $\mathfrak{z}(\mathfrak{g})^*$ with the annihilator of $[\mathfrak{g},\mathfrak{g}]$. Thus the moment map is unique up to a constant from $\mathfrak{z}(\mathfrak{g})^*$. As $\mathfrak{t}^*_+ = \mathfrak{z}(\mathfrak{g})^* \times \big(\mathfrak{t}^*_+ \cap [\mathfrak{g},\mathfrak{g}]^*\big)$, we find the imploded cross-section is independent of the choice of moment map.

Using Theorem 4.2.2, we can recover more information about the subsets $\mu^{-1}(\sigma)/[G_\sigma, G_\sigma]$. In particular, we claim that each of the sets $\mu^{-1}(\sigma)/[G_\sigma, G_\sigma]$ is a symplectic quotient of some submanifold of M. By Theorem 4.2.2, $M_\sigma = \mu^{-1}(S_\sigma)$ is a connected Hamiltonian G_σ-space for every face σ, whose moment map μ_σ, is the restriction of μ to M_σ. We turn M_σ into a Hamiltonian $[G_\sigma, G_\sigma]$-space by composing μ_σ with the projection $\pi_\sigma : \mathfrak{g}^*_\sigma \to [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]^*$ to get the moment map $\tilde{\mu}_\sigma$ (see Proposition 2.3.6). Its zero level set is given by

$$\tilde{\mu}_\sigma^{-1}(0) = \mu_\sigma^{-1}\big(\mathfrak{z}(\mathfrak{g}_\sigma)^*\big) = \mu^{-1}\big(\mathfrak{z}(\mathfrak{g}_\sigma)^*\big) \cap M_\sigma = \mu^{-1}\big(\mathfrak{z}(\mathfrak{g}_\sigma)^* \cap S_\sigma\big).$$

However, $\mathfrak{z}(\mathfrak{g}_\sigma)^*$ can be identified with the linear span of σ, and so $\mathfrak{z}(\mathfrak{g}_\sigma)^* \cap S_\sigma = \sigma$. Hence $\tilde{\mu}_\sigma^{-1}(0) = \mu^{-1}(\sigma)$ for every face σ of $\mathfrak{t}^*_+$, and we have a decomposition

$$M_{\text{impl}} = \coprod_{\sigma\in\Sigma} \tilde{\mu}_\sigma^{-1}(0)/[G_\sigma, G_\sigma] = \coprod_{\sigma\in\Sigma} M_\sigma /\!\!/ [G_\sigma, G_\sigma]. \tag{4.3.1}$$

Note that the piece corresponding to the smallest face, the origin, is $\mu^{-1}\big(\mathfrak{z}(\mathfrak{g})^*\big)/[G,G] = M/\!\!/[G,G]$.

On the other end of the spectrum, as $[G_{\sigma_{\text{prin}}}, G_{\sigma_{\text{prin}}}]$ acts trivially on $M_{\sigma_{\text{prin}}}$, if $\mu(p) \in \sigma_{\text{prin}}$ then $q \sim p$ implies that $p = q$. Hence the restriction of π to $M_{\sigma_{\text{prin}}}$ is a homeomorphism onto its image. As $M_{\sigma_{\text{prin}}}$ is connected and open, $\pi(M_{\sigma_{\text{prin}}})$ is connected and open. As $\mu^{-1}(M_{\sigma_{\text{prin}}})$ is dense in $\mu^{-1}(\mathfrak{t}^*_+)$, $\pi(M_{\sigma_{\text{prin}}})$ is dense in M_{impl} and M_{impl} is connected. We recap the last result in the following proposition.

Proposition 4.3.4. *The restriction of π to $M_{\sigma_{\text{prin}}} = \mu^{-1}(\sigma_{\text{prin}})$ is a homeomorphism onto its image, and the image is connected, open, and dense in M_{impl}. Thus M_{impl} is a connected space.*

While M_{impl} can be decomposed into a collection of symplectic quotients by (4.3.1), not all pieces may be symplectic manifolds; as there is no guarantee that μ is transverse to every face σ of $\mathfrak{t}^*_+$, and the action of $[G_\sigma, G_\sigma]$ may not be free. However, by Theorem 3.4.17 we know that further decomposing by orbit types, then quotienting is symplectic.

Let σ be a face of $\mathfrak{t}^*_+$, and for ease of notation let $G' = [G_\sigma, G_\sigma]$. For any closed subgroup H of G', set

$$M_{\sigma,(H)} = \{p \in M_\sigma : (G'_p) = H \text{ in } G'\}$$

to be the points of orbit type (H) in M_σ. By Theorem 3.4.17,

$$\tilde{\mu}_\sigma^{-1}(0) \cap M_{\sigma,(H)} = \mu^{-1}(\sigma) \cap M_{\sigma,(H)}$$

is a G'-stable smooth submanifold of M_σ, and the quotient

$$\big(\mu^{-1}(\sigma) \cap M_{\sigma,(H)}\big)/G' \tag{4.3.2}$$

is symplectic manifold. As in the regular singular reduction theory we elect to use the connected components rather than the submanifolds directly. Hence let $\{X_i\}_{i\in I}$ be the collection of connected components of all manifolds of the form (4.3.2), where σ is a face of $\mathfrak{t}^*_+$ and (H) is a conjugacy class of the subgroup $[G_\sigma, G_\sigma]$. There is a partial ordering on the index set I defined by $i \leq j$ if, and only if, $X_i \subseteq \overline{X_j}$. Since the orbit type decomposition is locally finite by Proposition 3.4.7, and as the quotient map π is proper by Lemma 4.3.3, it follows that the collection $\{X_i\}_{i\in I}$ is locally finite. Moreover, Proposition 4.3.4 implies that I has a maximal element. Thus we have the following theorem.

Theorem 4.3.5. *The imploded cross-section M_{impl} is a locally finite disjoint union of locally closed connected subspaces, each of which is a symplectic manifold*

$$M_{\text{impl}} = \coprod_{i\in I} X_i. \tag{4.3.3}$$

There exists a unique open piece which is dense in M_{impl} and symplectomorphic to $M_{\sigma_{\text{prin}}}$, the principal cross-section of M.

It is shown in [GJS02, Section 5] that decomposition (4.3.3) forms a stratification of M_{impl} in the sense of Definition 3.4.11. Hence we call the pieces, X_i, of the decomposition (4.3.3) **strata**, and by Theorem 4.3.5 each strata is a symplectic manifold. However, M_{impl} is not a symplectic stratified space in the sense of Definition 3.4.18, as the strata do not have a natural algebra of functions equipped with a Poisson bracket. Nevertheless, in an abuse of notation, we still refer to M_{impl} as a symplectic stratified space.

4.4 Abelianisation

We previously claimed that the imploded cross-section M_{impl} of a Hamiltonian G-space is in some sense a Hamiltonian T-space that abelianises reduction. The goal of this section is to prove and expand on this idea.

Suppose X is a topological space with a decomposition $X = \coprod_{i\in I} X_i$ of connected subspaces, each of which is a symplectic manifold with symplectic form ω_i. A continuous action of a Lie group H on X is **Hamiltonian** if:

i) The action preserves the decomposition, i.e. $H \cdot X_i \subseteq X_i$.

ii) The action is smooth on each X_i.

iii) There exists a continuous coadjoint equivariant map $\mu_X : X \to \mathfrak{h}^*$, called the **moment map**, such that $\mu_X|_{X_i}$ is a moment map in the sense of Definition 2.3.5 for the H-action on X_i.

The tuple $\big(X, \{(X_i, \omega_i)\}_{i\in I}, H, \mu_X\big)$ is referred to as a **Hamiltonian H-space**. If the decomposition $X = \coprod_{i\in I} X_i$ is a symplectic stratification, we sometimes referred to $\big(X, \{(X_i, \omega_i)\}_{i\in I}, H, \mu_X\big)$ as a **stratified Hamiltonian H-space** to distinguish it from an regular Hamiltonian H-space.

An **isomorphism** of two Hamiltonian H-spaces $\big(X, \{(X_i, \omega_i)\}_{i\in I}, H, \mu_X\big)$ and $\big(Y, \{(Y_j, \omega_j)\}_{j\in J}, H, \mu_Y\big)$ is a pair (F, f), where $F : X \to Y$ is a homeomorphism, and $f : I \to J$ is a bijection such that:

i) F is equivariant.

ii) $\mu_X = \mu_Y \circ F$.

iii) F is a symplectomorphism from X_i to $Y_{f(i)}$ for all $i \in I$.

Now let (M, ω, G, μ) be an ordinary Hamiltonian G-space. We claim that M_{impl} with the decomposition given in (4.3.3) is a stratified Hamiltonian T-space. To prove this, we need the following lemma.

Lemma 4.4.1. *For every face σ of $\mathfrak{t}^*_+$, we have $T \subseteq G_\sigma$. Furthermore, T normalises the commutator subgroups $[G_\sigma, G_\sigma]$ for every face σ.*

Proof. As T is abelian, and the coadjoint action is induced from conjugation, it follows that T acts trivially on $\mathfrak{t}^*$ via the coadjoint action. Hence $T \subseteq G_\sigma$ for every face σ of $\mathfrak{t}^*_+$.

To see that T normalises $[G_\sigma, G_\sigma]$ for every face σ, by induction and the definition of a commutator subgroup, it suffices to show T normalises the generators of $[G_\sigma, G_\sigma]$. So let $t \in T$, and $g_1 g_2 g_1^{-1} g_2^{-1} \in [G_\sigma, G_\sigma]$. Then

$$\begin{aligned} t g_1 g_2 g_1^{-1} g_2^{-1} t^{-1} &= t g_1 t^{-1} t g_2 t^{-1} t g_1^{-1} t^{-1} t g_2^{-1} t^{-1} \\ &= (t g_1 t^{-1})(t g_2 t^{-1})(t g_1 t^{-1})^{-1}(t g_2 t^{-1})^{-1}. \end{aligned}$$

As $T \subseteq G_\sigma$, $t g_i t^{-1} \in G_\sigma$. Thus $t g_1 g_2 g_1^{-1} g_2^{-1} t^{-1} \in [G_\sigma, G_\sigma]$, and $t[G_\sigma, G_\sigma]t^{-1} \subseteq [G_\sigma, G_\sigma]$. Conversely, if $g_1 g_2 g_1^{-1} g_2^{-1} \in [G_\sigma, G_\sigma]$, then for all $t \in T$

$$\begin{aligned} g_1 g_2 g_1^{-1} g_2^{-1} &= t t^{-1} g_1 t t^{-1} g_2 t t^{-1} g_1^{-1} t t^{-1} g_2^{-1} t t^{-1} \\ &= t(t^{-1} g_1 t)(t^{-1} g_2 t)(t^{-1} g_1 t)^{-1}(t^{-1} g_2 t)^{-1} t^{-1}. \end{aligned}$$

Again, as $t^{-1} g_i t \in G_\sigma$, it follows $g_1 g_2 g_1^{-1} g_2^{-1} \in t[G_\sigma, G_\sigma]t^{-1}$. Therefore, for every $t \in T$, $t[G_\sigma, G_\sigma]t^{-1} = [G_\sigma, G_\sigma]$, and T normalises $[G_\sigma, G_\sigma]$. $\square$

To see that M_{impl} is a stratified Hamiltonian T-space, first note by equivariance of μ and as T acts trivially on $\mathfrak{t}_+^*$, $\mu^{-1}(\mathfrak{t}_+^*)$ is a T-stable subset of M. Moreover, suppose that $p, q \in \mu^{-1}(\mathfrak{t}_+^*)$ and $p \sim q$. Then there exists $g \in [G_\sigma, G_\sigma]$, where σ is the open face containing $\mu(p)$, such that $q = g \cdot p$. As T normalises $[G_\sigma, G_\sigma]$ by Lemma 4.4.1, for all $t \in T$ we have

$$t \cdot q = (tg) \cdot p = tgt^{-1} \cdot (t \cdot p),$$

and $t \cdot q \sim t \cdot p$. Hence the action of T preserves $\sim$, and so descends to a continuous action on $M_{\text{impl}} = \mu^{-1}(\mathfrak{t}_+^*)/\sim$ given by

$$t \cdot [p] = [t \cdot p],$$

and we claim that this action is Hamiltonian with moment map given by μ_{impl}.

As $[G_\sigma, G_\sigma]$ is a subgroup of G_σ, by Proposition 3.3.3 the G_σ action on $M_\sigma = \mu^{-1}(S_\sigma)$ descends to an action of $G_\sigma/[G_\sigma, G_\sigma]$ on $M_\sigma /\!\!/ [G_\sigma, G_\sigma]$. Further, as $T \subseteq G_\sigma$ and T normalises $[G_\sigma, G_\sigma]$, there exists a canonical surjection $T \to G_\sigma/[G_\sigma, G_\sigma]$ sending t to its coset.

We claim that the continuous T-action on M_{impl} preserves strata. Indeed as the coadjoint action of T on $\mathfrak{t}_+^*$ is trivial, and μ is G-equivariant, we find that $\mu^{-1}(\sigma)$ is T-stable for every face σ. Further, as T normalises $G' = [G_\sigma, G_\sigma]$, we have the equality of conjugacy classes of stabilisers,

$$(G'_{t \cdot p}) = (tG'_p t^{-1}) = (G'_p).$$

Therefore, the T-action also preserves the G'-orbit types $M_{\sigma,(H)}$, which implies that the T-action preserves the strata of M_{impl}. Moreover, by Proposition 3.3.3, the T-action on

$$\big(\mu^{-1}(\sigma) \cap M_{\sigma,(H)}\big)/[G_\sigma, G_\sigma]$$

is Hamiltonian whose moment map is induced from the restriction of the moment map to $\mu^{-1}(\sigma) \cap M_{\sigma,(H)}$. Thus, we find that the moment map on each strata is just the restriction of μ_{impl} to each strata, which proves that M_{impl} is a stratified Hamiltonian T-space.

We are now ready to state and prove the main theorem concerning imploded cross-sections. Let $\lambda \in \mathfrak{t}_+^*$, and σ the open face containing λ. Then by definition of the imploded moment map

$$\mu^{-1}(\lambda) = (\mu_{\text{impl}} \circ \pi)^{-1}(\lambda) = \pi^{-1}\big(\mu_{\text{impl}}^{-1}(\lambda)\big).$$

However, as π is surjective we have

$$\mu^{-1}(\lambda)/[G_\sigma, G_\sigma] = \pi\big(\mu^{-1}(\lambda)\big) = \mu_{\text{impl}}^{-1}(\lambda),$$

and there exists a quotient map $\mu^{-1}(\lambda) \to \mu_{\text{impl}}^{-1}(\lambda)$.

Theorem 4.4.2 (Abelianisation)**.** *For all $\lambda \in \mathfrak{t}_+^*$, the quotient map $\mu^{-1}(\lambda) \to \mu_{\text{impl}}^{-1}(\lambda)$ induces a symplectomorphism*

$$M /\!\!/_\lambda G \cong M_{\text{impl}} /\!\!/_\lambda T.$$

Proof. Let σ be the open face containing λ. First assume that all points in $\mu^{-1}(\lambda)$ are of the same orbit type for $G_\sigma = G_\lambda$, so that $M/\!\!/_\lambda G$ is symplectic by Theorem 3.4.17. By reduction in stages, Theorem 3.3.4, it is symplectomorphic to the iterated quotient

$$\big(M_\sigma/\!\!/[G_\sigma, G_\sigma]\big)/\!\!/_\lambda T. \tag{4.4.1}$$

Since

$$\mu_{\mathrm{impl}}^{-1}(\lambda) = \mu^{-1}(\lambda)/[G_\sigma, G_\sigma] \subseteq M_\sigma/\!\!/[G_\sigma, G_\sigma],$$

and the restriction of μ_{impl} to $M_\sigma/\!\!/[G_\sigma, G_\sigma]$ is the moment map for the T-action on $M_\sigma/\!\!/[G_\sigma, G_\sigma]$, it follows that (4.4.1) is equal to $M_{\mathrm{impl}}/\!\!/_\lambda T$.

If $\mu^{-1}(\lambda)$ consists of more than one stratum; the same argument, using stratified reduction in stages [SL91, Section 4], shows the quotient map $\mu^{-1}(\lambda) \to \mu_{\mathrm{impl}}^{-1}(\lambda)$ induces a homeomorphism $M/\!\!/_\lambda G \to M_{\mathrm{impl}}/\!\!/_\lambda T$ which restricts to a symplectomorphism on each strata. □

4.5 The Universal Imploded Cross-Section

In this section we investigate the cross-section and imploded cross-sections of the cotangent bundle T^*G, where G is a compact, connected Lie group. As it follows that the imploded cross-section $(T^*G)_{\mathrm{impl}}$ acts an *universal object* for implosion.

Consider the action of G on itself via left and right translations

$$L_g(h) = gh \qquad \text{and} \qquad R_g(h) = hg^{-1},$$

respectively. Trivialising $T^*G \cong G \times \mathfrak{g}^*$ by left translations, the cotangent lifts of the left and right translations are given by

$$\hat{L}_g(h, \lambda) = (gh, \lambda) \qquad \hat{R}_g(h, \lambda) = (hg^{-1}, \mathrm{Ad}_g^* \lambda).$$

Relative to the symplectic form $\omega = d\theta$, where θ is the tautological 1-form, by Example 2.3.17 these actions are Hamiltonian with moment maps

$$\mu_L(g, \lambda) = -\mathrm{Ad}_g^* \lambda \qquad \mu_R(g, \lambda) = \lambda,$$

respectively. (Note here that ω is negative the canonical symplectic structure $-d\theta$ on T^*G.)

The inversion map $\mathrm{inv}(g) = g^{-1}$ intertwines the left and right actions on G:

$$R_g\big(\mathrm{inv}(h)\big) = R_g(h^{-1}) = h^{-1}g^{-1} = (gh)^{-1} = \mathrm{inv}\,\big(L_g(h)\big),$$

and similarly for L_g. Its cotangent lift is

$$\hat{\mathrm{inv}}(g, \lambda) = (g^{-1}, -\mathrm{Ad}_g^* \lambda),$$

and defines a symplectic involution of T^*G. Further, as

$$\mu_L\big(\hat{\mathrm{inv}}(g, \lambda)\big) = \mu_L(g^{-1}, -\mathrm{Ad}_g^* \lambda) = -\mathrm{Ad}_{g^{-1}}^*(-\mathrm{Ad}_g^* \lambda) = \lambda = \mu_R(g, \lambda),$$

and

$$\mu_R\big(\hat{\text{inv}}(g,\lambda)\big) = \mu_R(g^{-1}, -\operatorname{Ad}_g^*\lambda) = -\operatorname{Ad}_g^*\lambda = \mu_L(g,\lambda),$$

$\hat{\text{inv}}$ intertwines the moments μ_R and μ_L on T^*G. Thus the cross-sections for μ_L and μ_R are symplectomorphic, and for simplicity we choose to use μ_R. Let $M = T^*G$. For every face σ of $\mathfrak{t}_+^*$

$$M_\sigma = \mu_R^{-1}(S_\sigma) = G \times S_\sigma,$$

and so

$$\begin{aligned}(T^*G)_{\text{impl}} &= \coprod_{\sigma\in\Sigma} (G\times S_\sigma)/\!\!/[G_\sigma, G_\sigma] \\ &= \coprod_{\sigma\in\Sigma} \mu_R^{-1}(\sigma)/[G_\sigma, G_\sigma] \\ &= \coprod_{\sigma\in\Sigma} \big(G\times\sigma\big)/[G_\sigma, G_\sigma] \\ &= \coprod_{\sigma\in\Sigma} \frac{G}{[G_\sigma, G_\sigma]}\times\sigma.\end{aligned} \tag{4.5.1}$$

Thus as G is connected, in this situation the decompositions (4.3.1) and (4.3.3) are equal.

Let $\pi_R : \mu_R^{-1}(\mathfrak{t}_+^*) \to (T^*G)_{\text{impl}}$ denote the projection onto the imploded cross-section generated by the right action on T^*G. The left and right actions on T^*G commute

$$\hat{L}_g\big(\hat{R}_h(k,\lambda)\big) = \hat{L}_g(kh^{-1}, \operatorname{Ad}_h^*\lambda) = (gkh^{-1}, \operatorname{Ad}^*\lambda) = \hat{R}_h(gk,\lambda) = \hat{R}_h\big(\hat{L}_g(k,\lambda)\big),$$

and the respective moment maps are invariant under the other action:

$$\mu_R\big(\hat{L}_g(h,\lambda)\big) = \mu_R(gh,\lambda) = \lambda = \mu_R(h,\lambda),$$

and

$$\mu_L\big(\hat{R}_g(h,\lambda)\big) = \mu_L(hg^{-1}, \operatorname{Ad}_g^*\lambda) = -\operatorname{Ad}_{hg^{-1}}^*\big(\operatorname{Ad}_g^*\lambda\big) = -\operatorname{Ad}_h^*\lambda = \mu_L(h,\lambda).$$

Hence by Proposition 2.3.6(IV), T^*G is a Hamiltonian $G\times G$-space. Hence if we imploded T^*G with respect to the right action, then the left action descends to a Hamiltonian G-action on $(T^*G)_{\text{impl}}$ with moment map $\tilde{\mu}_L$, induced from μ_L. As the G-action on $(T^*G)_{\text{impl}}$ is given by

$$g\cdot\pi_R(h,\lambda) = \pi_R\big(g\cdot(h,\lambda)\big) = \pi_R(gh,\lambda),$$

it follows that the T and G-actions on $(T^*G)_{\text{impl}}$ commute because the left and right actions on T^*G commute. As both $\mu_{\text{impl}}\circ\pi_R = \mu_R$ and $\tilde{\mu}_L\circ\pi_R = \mu_L$, the moment maps are invariant under the others action. Therefore $(T^*G)_{\text{impl}}$ is a Hamiltonian $G\times T$-space, and one may ask what happens when $(T^*G)_{\text{impl}}$ is reduced with respect to the residual G-action?

Lemma 4.5.1. *Let (M,ω,G,μ) be a Hamiltonian G-space and define $f : M\to M\times T^*G$ by $f(p) = \big(p, e, \mu(p)\big)$. Then f is a symplectic embedding and induces an isomorphism of Hamiltonian G-spaces*

$$\bar{f} : M \to (M\times T^*G)/\!\!/G.$$

The right hand side is the quotient with respect to the diagonal G-action, where G acts on the left on T^*G. The G-action on $(M \times T^*G)/\!\!/G$ is the action induced by the diagonal G-action on $M \times T^*G$ where G acts trivially on M and as the right action on T^*G.

Proof. Note the map $p \mapsto \big(e, \mu(p)\big)$ sends M to the Lagrangian $\mathfrak{g}^* \subseteq T^*G$, and therefore $f : M \to M \times T^*G$ defined by $f(p) = \big(p, e, \mu(p)\big)$ is a symplectic embedding. As both M and T^*G are Hamiltonian G-spaces, the product $M \times T^*G$ is a Hamiltonian G-space for the diagonal G-action with moment map

$$\psi(p, g, \lambda) = \mu(p) - \operatorname{Ad}^*_g \lambda.$$

Thus we find f maps M to $\psi^{-1}(0)$. Let $(p, g, \lambda) \in \psi^{-1}(0)$. Then $\mu(p) = \operatorname{Ad}^*_g \lambda$ implying $\lambda = \operatorname{Ad}^*_{g^{-1}} \mu(p) = \mu(g^{-1} \cdot p)$. Hence

$$\begin{aligned} g^{-1} \cdot (p, g, \lambda) &= (g^{-1} \cdot p, g^{-1}g, \lambda) \\ &= (g^{-1} \cdot p, e, \lambda) \\ &= \big(g^{-1} \cdot p, e, \mu(g^{-1} \cdot p)\big) \\ &= f(g^{-1} \cdot p). \end{aligned}$$

Thus every element of $\psi^{-1}(0)$ is a G-translate of the image of f, and so f descends to the smooth surjective map $\bar{f} : M \to \psi^{-1}(0)/G = (M \times T^*G)/\!\!/G$ such that $\bar{f} = \pi \circ f$ where $\pi : \psi^{-1}(0) \to \psi^{-1}(0)/G$ is the canonical projection. We claim that $\bar{f}$ is also injective. Suppose $p, q \in M$ such that

$$\bar{f}(p) = \big[p, e, \mu(p)\big] = \big[q, e, \mu(q)\big] = \bar{f}(q).$$

Then there exists $g \in G$ such that

$$\big(q, e, \mu(q)\big) = g \cdot \big(p, e, \mu(p)\big) = \big(g \cdot p, g, \mu(p)\big),$$

which implies that $g = e$, and so $p = q$. Hence we see that $\bar{f} : M \to (M \times T^*G)/\!\!/G$ is a smooth bijection. Its inverse, being the map $\big[p, g, \lambda\big] \mapsto p$, is smooth and so $\bar{f}$ is a diffeomorphism. As f is a symplectic embedding, by the uniqueness property of the symplectic form on the quotient, we see that $\bar{f}$ is a symplectomorphism.

Let $\hat{L}$ and $\hat{R}$ denote the actions on $M \times T^*G$ induced by the left and right actions on T^*G, respectively, as stated in the theorem. Further, let $\tilde{R}$ denote the action on $(M \times T^*G)/\!\!/G$ induced from the right action $\hat{R}$. Since $\hat{L}$ and $\hat{R}$ commute,

$$\begin{aligned} f(g \cdot p) &= \big(g \cdot p, e, \mu(g \cdot p)\big) \\ &= \big(g \cdot p, gg^{-1}, \operatorname{Ad}^*_g \mu(p)\big) \\ &= \hat{R}_g\big(g \cdot p, g, \mu(p)\big) \\ &= \hat{R}_g \hat{L}_g\big(p, e, \mu(p)\big) \\ &= \hat{R}_g \hat{L}_g\big(f(p)\big). \end{aligned}$$

Thus

$$\begin{aligned}\bar{f}(g\cdot p) &= (\pi\circ f)(g\cdot p)\\ &= \pi\left(\hat{R}_g\hat{L}_g\big(f(p)\big)\right)\\ &= \pi\left(\hat{R}_g\big(f(p)\big)\right)\\ &= \tilde{R}_g\pi\big(f(p)\big)\\ &= \tilde{R}_g\big(\bar{f}(p)\big)\end{aligned}$$

showing that $\bar{f}$ is G-equivariant. Moreover, let μ_R denotes the moment map for the action on $M\times T^*G$ given by the right action on T^*G, and $\tilde{\mu}_R$ the moment map for the induced action on $(M\times T^*G)/\!\!/G$. As $\mu_R\big(f(p)\big)=\mu(p)$, it follows

$$\tilde{\mu}_R\big(\bar{f}(p)\big)=\tilde{\mu}_R\left(\pi\big(f(p)\big)\right)=[\tilde{\mu}_R\circ\pi]\big(f(p)\big)=\mu_R\big(f(p)\big)=\mu(p).$$

Therefore, $\bar{f}$ intertwines the moments map on M and $(M\times T^*G)/\!\!/G$, and defines an isomorphism of Hamiltonian G-spaces. □

Corollary 4.5.1.1. *For every face σ of $\mathfrak{t}^*_+$, f maps M_σ to $M\times(G\times S_\sigma)\subseteq M\times T^*G$, and induces an isomorphism of Hamiltonian G_σ-manifolds*

$$\bar{f}_\sigma: M_\sigma\to(M\times G\times S_\sigma)/\!\!/G.$$

Here the quotient is taken as in Lemma 4.5.1.

Proof. Proved in the same way as Lemma 4.5.1. □

Recall in Example 3.1.9 that for all $\xi\in\mathfrak{g}^*$, the reduction of $(T^*G,-d\theta)$ (again we note the difference of symplectic form here) relative to the right action of G on T^*G is coadjoint orbit through ξ with symplectic form negative the KKS form, which we denote by $-\mathcal{O}_\xi$. Further recall that the shifting trick, Theorem 3.2.4, states that given a Hamiltonian G-space M then

$$M/\!\!/_\xi G\cong(M\times-\mathcal{O}_\xi)/\!\!/G\cong\big(M\times(T^*G/\!\!/_\xi G\big)/\!\!/G$$

for all $\xi\in\mathfrak{g}^*$. However, by Theorem 4.4.2 we know that

$$-\mathcal{O}_\xi\cong T^*G/\!\!/_\xi G\cong(T^*G)_{\mathrm{impl}}/\!\!/_\xi T.$$

Hence we ask whether Lemma 4.5.1 descends to a statement about the imploded cross-sections. Taking into account the sign difference of the symplectic forms on T^*G.

First we note that if f is the map defined in Lemma 4.5.1, then f maps $\mu^{-1}(\mathfrak{t}^*_+)$ to $\coprod_\sigma M\times G\times\sigma$, and hence induces a continuous map $\tilde{f}:\mu^{-1}(\mathfrak{t}^*_+)\to M\times(T^*G)_{\mathrm{impl}}$ defined by $\tilde{f}(p)=\big(p,\pi_R\big(e,\mu(p)\big)\big)$.

Theorem 4.5.2 (Universal Property). *The map f in Lemma 4.5.1 induces an isomorphism of Hamiltonian T-spaces*

$$\bar{f}_{\mathrm{impl}} : M_{\mathrm{impl}} \to \big(M \times (T^*G)_{\mathrm{impl}}\big)/\!\!/G,$$

where the quotient is taken with respect to the diagonal G-action. The T-action on $\big(M \times (T^*G)_{\mathrm{impl}}\big)/\!\!/G$ is induced from the trivial T-action on M and T-action on $(T^*G)_{\mathrm{impl}}$.

Proof. Recall that $M \times (T^*G)_{\mathrm{impl}}$ is a Hamiltonian G-space for the diagonal action of G with moment map

$$\Psi\big(p, \pi_R(g, \lambda)\big) = \mu(p) - \mathrm{Ad}_g^* \lambda,$$

so that $\tilde{f}$ maps $\mu^{-1}(\mathfrak{t}_+^*)$ to $\Psi^{-1}(0)$. Let $p \in \mu^{-1}(\mathfrak{t}_+^*)$ and $g \in [G_{\mu(p)}, G_{\mu(p)}]$. Then

$$\begin{aligned} \tilde{f}(g \cdot p) &= \big(g \cdot p, \pi_R\big(e, \mu(g \cdot p)\big)\big) \\ &= \big(g \cdot p, \pi_R\big(gg^{-1}, \mathrm{Ad}_g^* \mu(p)\big)\big) \\ &= \big(g \cdot p, \pi_R\big(g, \mu(p)\big)\big) \\ &= \big(g \cdot p, g \cdot \pi_R\big(e, \mu(p)\big)\big) \\ &= g \cdot \big(p, \pi_R\big(e, \mu(p)\big)\big) \\ &= g \cdot \tilde{f}(p), \end{aligned} \tag{4.5.2}$$

and $\tilde{f}$ is equivariant with respect to group elements in $[G_{\mu(p)}, G_{\mu(p)}]$. Therefore define a function $\bar{f}_{\mathrm{impl}} : M_{\mathrm{impl}} \to \big(M \times (T^*G)_{\mathrm{impl}}\big)/\!\!/G$ by $\bar{f}_{\mathrm{impl}}[p] = \big[\tilde{f}(p)\big]$, i.e. so that the following diagram commutes

$$\begin{array}{ccc} \mu^{-1}(\mathfrak{t}_+^*) & \xrightarrow{\quad \tilde{f} \quad} & M \times (T^*G)_{\mathrm{impl}} \\ \Big\downarrow & & \Big\downarrow \\ M_{\mathrm{impl}} & \xrightarrow[\quad \bar{f}_{\mathrm{impl}} \quad]{} & \big(M \times (T^*G)_{\mathrm{impl}}\big)/\!\!/G. \end{array}$$

To see that $\bar{f}_{\mathrm{impl}}$ is well defined, suppose that $[p], [q] \in M_{\mathrm{impl}}$ and $[q] = [p]$. Then there exists $g \in [G_{\mu(p)}, G_{\mu(p)}]$ such that $q = g \cdot p$, hence (4.5.2) implies

$$\bar{f}_{\mathrm{impl}}[q] = \big[\tilde{f}(g \cdot p)\big] = \big[g \cdot \tilde{f}(p)\big] = \big[\tilde{f}(p)\big] = \bar{f}_{\mathrm{impl}}[p].$$

Therefore $\bar{f}_{\mathrm{impl}}$ is a well defined continuous function. We claim that $\bar{f}_{\mathrm{impl}}$ is actually a homeomorphism. Surjectivity is proved similarly to Lemma 4.5.1. Take $\big(p, \pi_R(g, \lambda)\big) \in \Psi^{-1}(0)$, which implies that $\lambda \in \mathfrak{t}_+^*$. Then $\lambda = \mathrm{Ad}_{g^{-1}}^* \mu(p) = \mu(g^{-1} \cdot p)$, so $g^{-1} \cdot p \in \mu^{-1}(\mathfrak{t}_+^*)$. Moreover,

$$\begin{aligned} \big(p, \pi_R(g, \lambda)\big) &= \big(p, \pi_R\big(g, \mu(g^{-1} \cdot p)\big)\big) \\ &= \big(gg^{-1} \cdot p, g \cdot \pi_R\big(e, \mu(g^{-1} \cdot p)\big)\big) \\ &= g \cdot \big(g^{-1} \cdot p, \pi_R\big(e, \mu(g^{-1} \cdot p)\big)\big)\,, \end{aligned}$$

which implies $\tilde{f}_{\text{impl}}[g^{-1} \cdot p] = \big[\big(p, \pi_R\big(g,\lambda\big)\big)\big]$. To see that $\bar{f}_{\text{impl}}$ is injective, suppose that $\bar{f}_{\text{impl}}[p] = \bar{f}_{\text{impl}}[q]$. Then $\big[\tilde{f}(p)\big] = \big[\tilde{f}(q)\big]$ and there exists $g \in G$ such that

$$\begin{aligned}\big(p, \pi_R\big(e,\mu(p)\big)\big) &= \tilde{f}(p) = g \cdot \tilde{f}(q) = g \cdot \big(q, \pi_R\big(e,\mu(q)\big)\big)\\ &= \big(g\cdot q, \pi_R\big(g,\mu(q)\big)\big)\,.\end{aligned}$$

The condition

$$\pi_R\big(e,\mu(p)\big) = \pi_R\big(g,\mu(q)\big),$$

forces $\mu(p) = \mu(q)$ and $e = ga^{-1}$ for $a \in [G_{\mu(q)}, G_{\mu(q)}]$. However, this implies that $g = a$, and hence $[p] = [q]$. Thus $\bar{f}_{\text{impl}}$ is a continuous bijection, and it is easy to see that its inverse is continuous implying it is a homeomorphism.

Recall that $M \times (T^*G)_{\text{impl}}$ also has a diagonal T-action given by the trivial action on M and the standard T-action on $(T^*G)_{\text{impl}}$. The moment map for this T-action is given by

$$\tilde{\mu}_R\big(p, \pi_R(g,\lambda)\big) = \lambda,$$

and this induces a T-action on $\big(M \times (T^*G)_{\text{impl}}\big)/\!\!/G$ whose moment map is

$$\bar{\mu}_R\left[\big(p, \pi_R(g,\lambda)\big)\right] = \tilde{\mu}_R\big(p, \pi_R(g,\lambda)\big) = \lambda.$$

We wish to show that $\bar{f}_{\text{impl}}$ is an isomorphism of Hamiltonian T-spaces. Indeed, recall that $\tilde{f}$ is equivariant on group elements of the form $[G_{\mu(p)}, G_{\mu(p)}]$ by (4.5.2). As $T \subseteq [G_{\mu(p)}, G_{\mu(p)}]$ for every $p \in \mu^{-1}(\mathfrak{t}^*_+)$, we see that $\tilde{f}$ is T-equivariant. Hence for $t \in T$,

$$\bar{f}_{\text{impl}}\big(t \cdot [p]\big) = \big[\tilde{f}(t\cdot p)\big] = \big[t \cdot \tilde{f}(p)\big] = t \cdot \big[\tilde{f}(p)\big] = t \cdot \bar{f}_{\text{impl}}[p],$$

and $\bar{f}_{\text{impl}}$ is T-equivariant. Similarly, $\bar{f}_{\text{impl}}$ intertwines the moment maps μ_{impl} and $\bar{\mu}_R$, as

$$\begin{aligned}\bar{\mu}_R\big(\bar{f}_{\text{impl}}[p]\big) &= \bar{\mu}_R\big[\tilde{f}(p)\big]\\ &= \bar{\mu}_R\left[\big(p, \pi_R\big(e,\mu(p)\big)\big)\right]\\ &= \tilde{\mu}_R\left(p, \pi_R\big(e,\mu(p)\big)\right)\\ &= \mu(p)\\ &= \mu_{\text{impl}}[p].\end{aligned}$$

Hence it remains to show that f_{impl} does not mix strata. Consider $p \in \mu^{-1}(\mathfrak{t}^*_+)$. Then by (4.5.1) $\tilde{f}(p) \in M \times (G \times S_\sigma)/\!\!/[G_\sigma, G_\sigma]$ for σ the open face of $\mathfrak{t}^*_+$ containing $\mu(p)$. Thus we find that $\bar{f}_{\text{impl}}$ restricts to the map

$$M_\sigma/\!\!/[G_\sigma, G_\sigma] \to \big(M \times (G \times S_\sigma)/\!\!/[G_\sigma, G_\sigma]\big)/\!\!/G. \tag{4.5.3}$$

However, this is precisely the map induced from the map $\bar{f}_\sigma$ defined in Corollary 4.5.1.1. As $\bar{f}_\sigma$ is an isomorphism of Hamiltonian G_σ-spaces, it preserves the $[G_\sigma, G_\sigma]$ orbit types by equivariance. Thus (4.5.3) maps strata to strata, and is symplectic on each strata. □

Due to Theorem 4.5.2 we refer to $(T^*G)_{\text{impl}}$ as the **universal imploded cross-section**. Its use allows one to calculate the imploded cross-sections for various Hamiltonian actions. One can also consult [GJS02, Section 6] for explicit calculations of $(T^*G)_{\text{impl}}$ for various Lie groups G. These calculations often involve techniques from algebraic geometry, via the use of the Kempf-Ness theorem, and are beyond the scope of this thesis. However, in the remainder of this section we will compute the imploded cross-section for $T^*\,\mathrm{SU}(2)$.

4.5.1 The Imploded Cross-Section of $T^*\,\mathrm{SU}(2)$

Recall that we have the decomposition

$$(T^*G)_{\text{impl}} = \coprod_{\sigma} (G \times S_\sigma) /\!\!/ [G_\sigma, G_\sigma],$$

and $G \times S_\sigma$ has a symplectic form induced from T^*G by Theorem 4.2.2. We give an another characterisation of this form.

Let σ be a face of $\mathfrak{t}^*_+$, and let G_σ be the coadjoint stabiliser subgroup of σ. As G_σ is closed, we can view G as a principal G_σ-bundle:

$$\begin{array}{ccc} G_\sigma & \longrightarrow & G \\ & & \downarrow \\ & & G/G_\sigma. \end{array} \tag{4.5.4}$$

Moreover, as G is a compact, connected Lie group, there exists a $\mathrm{Ad}(G_\sigma)$-invariant decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{g}_\sigma \oplus \mathfrak{m}$, where $\mathfrak{g}_\sigma = \mathrm{Lie}(G_\sigma)$. Using this we can define a canonical connection 1-form $\beta \in \Omega^1(G, \mathfrak{g}_\sigma)$ as the projection of the Maurer–Cartan form for G onto the subspace $\mathfrak{g}_\sigma$.

Definition 4.5.3. A connection $\alpha \in \Omega^1(\mathcal{P}, \mathfrak{k})$ on a principal K-bundle $\mathcal{P}$ is **fat** at $\lambda \in \mathfrak{k}^*$ if the 2-form $d\langle \lambda, \alpha \rangle = \langle \lambda, d\alpha \rangle$ is non-degenerate on the horizontal subspaces of $\mathcal{P}$.

Let $\mathrm{pr}_2 : P \times \mathfrak{k}^* \to \mathfrak{k}^*$ be projection onto the second factor. Then a connection α on $\mathcal{P}$ being fat at $\lambda \in \mathfrak{k}^*$ is equivalent to the closed 2-form $d\langle \mathrm{pr}_2, \alpha \rangle$ being non-degenerate on $\mathcal{P} \times \{\lambda\}$.

Lemma 4.5.4 ([GLS96], Corollary 2.3.8)**.** *The canonical 1-form β on the bundle* (4.5.4) *is fat at $\lambda \in \mathfrak{g}^*_\sigma$ if, and only if, $\lambda \in S_\sigma$.*

Hence the form $d\langle \mathrm{pr}_2, \beta \rangle$ on $G \times \mathfrak{g}^*_\sigma$ is symplectic on $G \times S_\sigma$. In fact, we show that $\langle \mathrm{pr}_2, \beta \rangle$ is equal to the restriction of the tautological 1-form θ to $G \times S_\sigma$. To see this, by G-invariance we only need to check equality at points of the form $(e, \lambda) \in G \times S_\sigma$. Take $(X, \xi) \in T_{(e,\lambda)}(G \times S_\sigma) = \mathfrak{g} \times \mathfrak{g}^*_\sigma$. Then by the definition of the tautological 1-form θ, $\theta_{(e,\lambda)}(X, \xi) = \lambda(X)$. However, on the other hand,

$$\langle \mathrm{pr}_2, \beta \rangle_{(e,\lambda)} (X, \xi) = \langle \lambda, \mathrm{Id} \rangle (X, \xi) = \langle \lambda, X \rangle = \lambda(X).$$

Therefore $\theta|_{G\times S_\sigma} = \langle \mathrm{pr}_2, \beta\rangle$, and so $\omega = d\theta = d\langle \mathrm{pr}_2, \beta\rangle$. Hence the symplectic form on the stratum $(G\times S_\sigma)/\!\!/[G_\sigma, G_\sigma]$ be interpreted as the form obtained from reducing $\big(G\times S_\sigma, d\langle \mathrm{pr}_2, \beta\rangle\big)$.

We now consider the case when $G = \mathrm{SU}(2)$, which we identify with the unit quaternions $S^3 \subseteq \mathbb{H}$. The Lie algebra $\mathfrak{su}(2)$ can be viewed as either skew-Hermitian matrices with trace zero, or the set of purely imaginary quaternions. We also identify $\mathfrak{su}(2)$ with its dual $\mathfrak{su}(2)^*$ via the Killing form. Let $T = S^1 \subseteq \mathbb{C}$ denote a maximal torus in SU(2), where S^1 embeds as

$$t \mapsto \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$

The fundamental Weyl chamber $\mathfrak{t}^*_+$ is given by the closed ray $[0,\infty)$, which decomposes into two faces $\{0\}$ and $(0,\infty)$ with $(\mathfrak{t}^*_+)^\circ = (0,\infty)$ the principal face.

The fibration in (4.5.4) for the principal face corresponds to the Hopf fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \big\downarrow \\ & & S^2. \end{array}$$

The symplectic form on $\mathrm{SU}(2)\times(\mathfrak{t}^*_+)^\circ = S^3\times(0,\infty)$ is $(d\beta)x = d(x\beta)$, where $d\beta$ is the curvature of the connection of the Hopf fibration, and x the standard coordinate on $(0,\infty)$. Now, relative to the standard symplectic structure on $\mathbb{H}\cong\mathbb{R}^4$, it is clear that the map

$$\begin{aligned} \phi : S^3\times(0,\infty) &\to \mathbb{H}\setminus\{0\}, \\ (z,x) &\mapsto \sqrt{2x}z, \end{aligned}$$

is a symplectomorphism. The other stratum, corresponding to $\{0\}$, is a single point.

We claim that the continuous map

$$\begin{aligned} \psi : T^*\,\mathrm{SU}(2) &\to \mathbb{H}\cong\mathbb{C}^2, \\ (g,\lambda) &\mapsto \sqrt{2\,\|\lambda\|}g \end{aligned}$$

induces a homeomorphism $\tilde{\psi} : \big(T^*\,\mathrm{SU}(2)\big)_{\mathrm{impl}} \to \mathbb{C}^2$. (Note that $\|\lambda\|$ is the norm of λ viewed as a purely imaginary quaternion.) Suppose that $\pi_R(g,\lambda) = \pi_R(h,\xi)$, so that $\lambda = \xi$. If $\lambda \in (\mathfrak{t}^*_+)^\circ$, then the action of $[G_\lambda, G_\lambda]$ is trivial and $(g,\lambda) = (h,\xi)$ which implies $\psi(g,\lambda) = \psi(h,\xi)$. Alternatively if $\lambda = 0$, then $\psi(g,0) = 0$ for all $g\in\mathrm{SU}(2)$. In either case, we find that ψ is constant of the equivalence classes of $\sim$, and thus descends uniquely to a continuous map $\tilde{\psi}$ on the quotient $\big(T^*\,\mathrm{SU}(2)\big)_{\mathrm{impl}}$. It is a bijection as its restriction to each stratum is bijective. Its inverse is clearly continuous, and so $\tilde{\psi}$ is a homeomorphism. Hence we find that $\big(T^*\,\mathrm{SU}(2)\big)_{\mathrm{impl}}$ is actually symplectic and isomorphic to $\mathbb{C}^2$.

However, we know that $\big(T^* \mathrm{SU}(2)\big)_{\mathrm{impl}}$ is a Hamiltonian $G \times T$-space and we now look at the corresponding actions under the isomorphism $\tilde{\psi}$. Let $\pi_R(g, \lambda) \in \big(T^* \mathrm{SU}(2)\big)_{\mathrm{impl}}$, $h \in \mathrm{SU}(2)$, and $t \in T$. Then

$$\begin{aligned}\tilde{\psi}\big(h \cdot \pi_R(g, \lambda)\big) &= \tilde{\psi}\big(\pi_R(hg, \lambda)\big) \\ &= \sqrt{2\,\|\lambda\|}hg \\ &= h\big(\sqrt{2\,\|\lambda\|}g\big),\end{aligned}$$

i.e. the induced left action on $\big(T^* \mathrm{SU}(2)\big)_{\mathrm{impl}}$ is the standard representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$. Similarly,

$$\begin{aligned}\tilde{\psi}\big(t \cdot \pi_R(g, \lambda)\big) &= \tilde{\psi}\big(\pi_R(gt^{-1}, \lambda)\big) \\ &= \sqrt{2\,\|\lambda\|}gt^{-1} \\ &= t^{-1}\sqrt{2\,\|\lambda\|}g\end{aligned}$$

and the corresponding T-action on $\mathbb{C}^2$ is given by $t \cdot z = t^{-1}z$.

Note that in this example $\big(T^* \mathrm{SU}(2)\big)_{\mathrm{impl}}$ is the smooth symplectic manifold $\mathbb{C}^2$, and leads to the question of when is $(T^*G)_{\mathrm{impl}}$ smooth for other Lie groups G? The answer is given in [GJS02, Proposition 6.15]; if the commutator subgroup $[G, G]$ is a product of $\mathrm{SU}(2)$, then the universal imploded cross-section $(T^*G)_{\mathrm{impl}}$ is smooth.

Chapter 5

Real Structures on Imploded Spaces

In this chapter we introduce the definition of real structures on symplectic manifolds and Hamiltonian G-spaces, giving rise to real symplectic manifolds and real Hamiltonian G-spaces, respectively. A motivation for considering real structures is that their fixed point sets are either empty or Lagrangian submanifolds; an important class of submanifolds generalising the notion of conjugate momenta in classical mechanics.

We then investigate under what conditions the imploded cross-section of a real Hamiltonian G-space, a stratified Hamiltonian T-space for T a maximal torus, inherits a real Hamiltonian structure. The resulting space would be a **real imploded cross-section**, or a **real imploded space**, and the overall procedure **real symplectic implosion**.

5.1 Real Structures

We begin with the definition of a real structure on a symplectic manifold.

Definition 5.1.1. Suppose (M, ω) is a symplectic manifold. A **real structure** on M is a smooth map $f : M \to M$ such that

i) f is an involution, i.e. $f^2 = \mathrm{Id}_M$;

ii) and f is anti-symplectic $f^*\omega = -\omega$.

We call (M, ω) equipped with a real structure a **real symplectic manifold**. We also define the fixed point set of the real structure f as

$$M^f = \{p \in M : f(p) = p\}.$$

We also refer to the fixed point set M^f as the **real locus** of M.

The following example provides an explanation as to why the involution and fixed point set have the names real structure and real locus, respectively.

Example 5.1.2. Consider the vector space $\mathbb{C}^n$ as $2n$-dimensional real vector space with symplectic form

$$\omega_0 = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j,$$

or equivalently,

$$\omega_0(z,w) = \mathrm{Im}(z^* w),$$

as in Example 2.1.3. Define an involution f_0 on $\mathbb{C}^n$ by component-wise complex conjugation, $f_0(z) = \overline{z}$. Then

$$f_0^*\omega_0 = \frac{i}{2}\sum_{j=1}^{n} f_0^*\big(dz_j \wedge d\bar{z}_j\big) = \frac{i}{2}\sum_{j=1}^{n} f_0^*(dz_j) \wedge f_0^*(d\bar{z}_j) = \frac{i}{2}\sum_{j=1}^{n} d\bar{z}_j \wedge dz_j = -\omega_0$$

The real locus is $(\mathbb{C}^n)^f = \mathbb{R}^n$, and we call f_0 the **standard real structure** on $\mathbb{C}^n$. Further note that from $\omega_0(z,w) = \mathrm{Im}(z^*w)$ it follows that $\omega_0|_{(\mathbb{C}^n)^{f_0}} = 0$, and that $\dim(\mathbb{C}^n)^{f_0} = \dim \mathbb{R}^n = n = \frac{1}{2}\dim \mathbb{C}^n$. Hence $(\mathbb{C}^n)^{f_0}$ is a Lagrangian subspace of $\mathbb{C}^n$. ◀

The fact that the real locus of $\mathbb{C}^n$ in Example 5.1.2 is Lagrangian is no coincidence; the real locus of a real symplectic manifold is either empty or a Lagrangian submanifold. Before we prove this result we will need the following theorem, which is an equivariant version of Darboux's theorem Theorem 2.1.6. This result was proven by Weinstein in [Wei71], however we follow the proof presented in [Dui83].

Theorem 5.1.3 (Equivariant Darboux)**.** *Let G be a compact Lie group acting smoothly on a symplectic manifold (M,ω) such that $\mathcal{A}_g^*\omega = \varepsilon(g)\omega$ for $\varepsilon : G \to \{1,-1\}$ a continuous homomorphism. Let $p \in M^G$ be a fixed point for the action. Suppose $\tilde{\omega}$ is another symplectic form defined on a neighbourhood U of p, such that $\mathcal{A}_g^*\tilde{\omega} = \varepsilon(g)\tilde{\omega}$, and $\omega_p = \tilde{\omega}_p$. Then there exists a G-equivariant local diffeomorphism ϕ around p such that $\phi(p) = p$ and $\phi^*\tilde{\omega} = \omega$.*

Proof. As in the proof of the ordinary Darboux theorem, we make use of Moser's trick. Set $\eta = \tilde{\omega} - \omega$. Then η is a closed 2-form, and by the Poincaré lemma it is locally exact, i.e. there exists a smooth 1-form α on U such that $d\alpha = -\eta$. Without a loss of generality, we may assume that $\alpha_p = 0$.

For $t \in [0,1]$ write

$$\omega_t = \omega + t\eta = \omega + t(\tilde{\omega} - \omega)$$

so that $\omega_0 = \omega$ and $\omega_1 = \tilde{\omega}$. Also note that $\mathcal{A}_g^*\omega_t = \varepsilon(g)\omega_t$ for all t. As $(\omega_t)_p = \omega_p$ for all t is non-degenerate, there exists a neighbourhood U_1 of p contained in U such that ω_t is non-degenerate on U_1. Hence we have an isomorphism $TU_1 \to T^*U_1$ induced by ω_t for all t. Hence define a time-dependent vector field V_t by

$$i(V_t)\omega_t = \alpha,$$

which implies that

$$d\big(i(V_t)\omega_t\big) = -(\tilde{\omega} - \omega). \tag{5.1.1}$$

As $\alpha|_p = 0$ it follows that $(V_t)_p = 0$ for all t. Note that replacing V_t by $d\mathcal{A}_g(V_t)$ still satisfies (5.1.1), for

$$\begin{aligned}
\varepsilon(g)(\tilde{\omega} - \omega) &= \mathcal{A}_g^*(\tilde{\omega} - \omega) \\
&= \mathcal{A}_g^*\left(-d\big(i(V_t)\omega_t\big)\right) \\
&= -d\left(\mathcal{A}_g^*\big(i(V_t)\omega_t\big)\right) \\
&= -d\left(i\big(d\mathcal{A}_g(V_t)\big)\mathcal{A}_g^*\omega_t\right) \\
&= -\varepsilon(g)\ d\left(i\big(d\mathcal{A}_g(V_t)\big)\omega_t\right).
\end{aligned}$$

Hence by averaging over G, we may assume without a loss of generality that V_t is G-invariant. Let ϕ_t denote the flow of V_t, and by restricting U_1 if necessary we may assume that the flow exists for all $t \in [0,1]$. As V_t is G-invariant it follows that ϕ_t is G-equivariant, and satisfies

$$\begin{aligned}
\frac{d}{dt}\phi_t^*\omega_t &= \phi_t^*\left(\mathcal{L}_{V_t}\omega_t + \frac{d}{dt}\omega_t\right) \\
&= \phi_t^*\left(d\big(i(V_t)\omega_t\big) + \eta\right) \\
&= \phi_t^*(d\alpha + \eta) \\
&= 0.
\end{aligned}$$

Hence by integration it follows that $\phi_t^*\omega_t = \phi_0^*\omega_0 = \omega$. Taking $\phi = \phi_1$ gives the result. □

Let (M, ω) be a real symplectic manifold with real structure denoted by f. We define a smooth $\mathbb{Z}_2$-action on M by setting $\mathcal{A}_1 = \mathrm{Id}_M$, and $\mathcal{A}_{-1} = f$. Hence $\mathcal{A}_g^*\omega = \varepsilon(g)\omega$ for all $g \in \{1, -1\}$, where $\varepsilon : \mathbb{Z}_2 \to \mathbb{Z}_2$ is the identity homomorphism. It follows that the real locus M^f coincides with the fixed point set for this $\mathbb{Z}_2$-action, $M^f = M^{\mathbb{Z}_2}$. Hence, using Theorem 5.1.3 we obtain the following.

Proposition 5.1.4. *Let (M, ω) be a real symplectic manifold with real structure given by f. Suppose that the real locus M^f of M is non-empty, then it is a Lagrangian submanifold of M.*

Proof. Set $n = \frac{1}{2}\dim M$, and consider $\mathbb{C}^n$ with the standard symplectic structure ω_0 and standard real structure f_0 defined in Example 5.1.2. Let $p \in M^f$, then by Theorem 5.1.3 there exists a neighbourhood U of p preserved by f, and a chart $\phi : U \to \mathbb{C}^n$ centred at p such that $\phi^*\omega = \omega_0$, and $\phi \circ f = f_0 \circ \phi$. Moreover,

$$M^f \cap U = \phi^{-1}\big((\mathbb{C}^n)^{f_0}\big) = \phi^{-1}(\mathbb{R}^n)$$

which shows that M^f is a Lagrangian submanifold. □

We want to extend the definition of a real symplectic manifold to the case of a Hamiltonian G-space. As a Hamiltonian G-space is already a symplectic manifold, we could just take the same definition of an anti-symplectic involution. However, we would ideally like the real structure to interact with the action of G, and with the moment map on M. To do this we need the Lie group

G to be equipped with a Lie group automorphism $\tau : G \to G$ which is also an involution. (We will refer to such a τ as an involutive automorphism from now on.)

As τ is an homomorphism, τ maps the identity of G to itself. Hence the derivative of τ is a involution on the Lie algebra of G. As τ is a Lie group homomorphism, it follows that $d\tau$ is a Lie algebra homomorphism. This also induces an involution on the dual Lie algebra, where τ acts on the dual Lie algebra by pullbacks:

$$\big(\tau^*\lambda\big)(X) = \lambda\big(d\tau(X)\big)$$

for all $\lambda \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

This leads to the following definition.

Definition 5.1.5. Let (M, ω, G, μ) be a Hamiltonian G-space. A **real structure** on (M, ω, G, μ) is a pair of smooth maps

$$f : M \to M,$$
$$\tau : G \to G,$$

such that τ is an involutive automorphism, and f is a real structure on (M, ω). We also require that f and τ are compatible in the following sense:

$$\mu\big(f(p)\big) = -\tau^*\mu(p), \tag{5.1.2}$$
$$f(g \cdot p) = \tau(g) \cdot f(p). \tag{5.1.3}$$

If (M, ω, G, μ) is a Hamiltonian G-space with real structure given by involutions f on M and τ on G, then we say that (M, ω, G, μ) is a **real Hamiltonian** (G, τ)**-space** with real structure f on M. As before, we define the fixed point set of τ to be

$$G^\tau = \{g \in G : \tau(g) = g\}.$$

These spaces were first studied by Duistermaat in [Dui83], who considered the case when $G = T$ is a torus acting on M with $\tau : T \to T$ being the inversion involution $\tau(t) = t^{-1}$. Duistermaat used this structure to prove a real version of Kirwan's convexity theorem; the moment polytope of the real locus $\Delta(M^f)$ is equal to the moment polytope of M. Duistermaat's result was generalised to non-abelian groups by O'Shea and Sjamaar in [OS00]. It was here where O'Shea and Sjamaar presented the defintion of a real Hamiltonian (G, τ)-space, and proved the moment polytope of the real locus is equal to a subpolytope of the Kirwan polytope of M [OS00, Theorem 3.1].

The conditions (5.1.2) and (5.1.3) turn out to actually be related, as the next proposition shows.

Proposition 5.1.6. *Let (M, ω, G, μ) be a real Hamiltonian (G, τ)-space with anti-symplectic involution f on M. If G is connected then* (5.1.2) *implies* (5.1.3)*. Conversely, if* (5.1.2) *holds then we can shift the moment μ by a constant so that* (5.1.2) *is satisfied.*

Proof. Suppose that (5.1.2) holds and G is connected. Using the moment map condition $\langle d\mu, X\rangle = d\langle \mu, X\rangle = i(X_M)\omega$ for $X \in \mathfrak{g}$, we have

$$\begin{aligned} i\big(df(X_M)\big)\omega &= -i\big(df(X_M)\big)f^*\omega \\ &= -f^*\big(i(X_M)\omega\big) \\ &= -f^*d\langle \mu, X\rangle \\ &= -d\big(f^*\langle \mu, X\rangle\big) \\ &= -d\langle \mu \circ f, X\rangle \\ &= d\langle \tau^* \circ \mu, X\rangle \\ &= d\langle \mu, d\tau(X)\rangle \\ &= i\big(d\tau(X)_M\big)\omega. \end{aligned}$$

Hence $df(X_M) = d\tau(X)_M$ by the non-degeneracy of ω. As G is connected, this implies (5.1.3) by Proposition A.3.7.

Conversely, suppose that (5.1.3) holds (regardless of whether G is connected). Then as f is τ-equivariant, it follows that $df(X_M) = d\tau(X)_M$ which implies $X_M = df\big(d\tau(X)_M\big)$ as df is an involution. Thus

$$\begin{aligned} \langle d\mu, X\rangle = i(X_M)\omega &= i\left(df\big(d\tau(X)_M\big)\right)\omega \\ &= -i\left(df\big(d\tau(X)_M\big)\right)f^*\omega \\ &= -f^*\left(i\big(d\tau(X)_M\big)\omega\right) \\ &= -f^*\langle d\mu, d\tau(X)\rangle \\ &= -\langle d(\mu\circ f), d\tau(X)\rangle \\ &= -\langle d(\tau^*\circ\mu\circ f), X\rangle, \end{aligned}$$

for all $X \in \mathfrak{g}$, so that $d(\tau^*\circ\mu\circ f) = -d\mu$ and $\tau^*\circ\mu\circ f$ satisfies the moment map condition. We wish to show that it is also G-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$. By direct calculation we see that for all $p \in M$ and $g \in G$,

$$\begin{aligned} \big(\tau^*\circ\mu\circ f\big)(g\cdot p) &= \tau^*\left(\mu\big(f(g\cdot p)\big)\right) \\ &= \tau^*\left(\mu\big(\tau(g)\cdot f(p)\big)\right) \\ &= \tau^*\left(\mathrm{Ad}^*_{\tau(g)}\,\mu\big(f(p)\big)\right) \\ &= \mathrm{Ad}^*_g\,\tau^*\left(\mu\big(f(p)\big)\right) \\ &= \mathrm{Ad}^*_g\left(\tau^*\circ\mu\circ f\right)(p). \end{aligned}$$

Where the second equality used (5.1.3), and the fourth equality used Lemma 5.1.7 (proved below). Hence $\tau^*\circ\mu\circ f$ is a moment map for the G-action on M, and so by uniqueness of the moment map Theorem 2.3.26, there exists an $\mathrm{Ad}^*(G)$-invariant $c \in \mathfrak{g}^*$ such that

$$\tau^*\circ\mu\circ f = -\mu + c.$$

Rearranging, we have $\mu = -\tau^* \circ \mu \circ f + \tau^*(c)$ which implies that $\tau^*(c) = c$. Setting $\tilde{\mu} = \mu - c/2$, it follows that $\tilde{\mu}$ is a G-equivariant moment map satisfying

$$\tau^* \circ \tilde{\mu} \circ f = \tau^* \circ (\mu - c/2) \circ f = \tau^* \circ \mu \circ f - c/2 = -\mu + c - c/2 = -\tilde{\mu},$$

which implies $\tilde{\mu} \circ f = -\tau^* \circ \tilde{\mu}$. □

Lemma 5.1.7. *Let τ be an involutive automorphism of a Lie group G, and consider its action on the dual Lie algebra $\mathfrak{g}^*$. Then for all $\lambda \in \mathfrak{g}^*$, we have*

$$\tau^*\big(\operatorname{Ad}_g^* \lambda\big) = \operatorname{Ad}^*_{\tau(g)}(\tau^*\lambda).$$

Proof. Recall that the adjoint action is given by differential at the identity of the conjugation map $c_g(h) = ghg^{-1}$. Hence as τ is a homomorphism we have

$$(c_g \circ \tau)(h) = g\tau(h)g^{-1} = \tau\big(\tau(g)\big)\tau(h)\tau\big(\tau(g)^{-1}\big) = \tau\big(\tau(g)h\tau(g)^{-1}\big) = (\tau \circ c_{\tau(g)})(h)$$

for all $g, h \in G$, which implies $c_g \circ \tau = \tau \circ c_{\tau(g)}$. So let $g \in G$, $X \in \mathfrak{g}$, and $\lambda \in \mathfrak{g}^*$. Then

$$\begin{aligned}
\left[\tau^*\big(\operatorname{Ad}_g^* \lambda\big)\right](X) &= \big(\operatorname{Ad}_g^* \lambda\big)\big(d\tau(X)\big) \\
&= \lambda\left(\operatorname{Ad}_{g^{-1}}\big(d\tau(X)\big)\right) \\
&= \lambda\left(d\big(c_{g^{-1}} \circ \tau\big)(X)\right) \\
&= \lambda\left(d\big(\tau \circ c_{\tau(g)^{-1}}\big)(X)\right) \\
&= \lambda\left(d\tau\big(\operatorname{Ad}_{\tau(g)^{-1}} X\big)\right) \\
&= \left[\operatorname{Ad}^*_{\tau(g)}\left(\tau^*\lambda\right)\right](X).
\end{aligned}$$

□

We now return to the real locus M^f of a real Hamiltonian (G, τ)-space for G a connected Lie group. Let K denote the identity component of the fixed point group G^τ, and let $\mathfrak{q}$ denote the -1-eigenspace of $d\tau$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, where $\mathfrak{k}$ is the Lie algebra of K. This induces a decomposition $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{q}^*$, where we identify $\mathfrak{k}^*$ with the annihilator of $\mathfrak{q}$ and vice versa. Note that we can also view $\mathfrak{q}^*$ as the -1-eigenspace of τ^*.

Proposition 5.1.8. *Suppose (M, ω, G, μ) is a real Hamiltonian (G, τ)-space, with G a connected Lie group, and real structure f on M. Then if the real locus M^f is non-empty, it is a G^τ-stable Lagrangian submanifold of M, and $\mu(M^f)$ is contained in $\mathfrak{q}^*$.*

Proof. To see that M^f is G^τ-stable, let $p \in M^f$ and $g \in G^\tau$. Then by (5.1.3)

$$f(g \cdot p) = \tau(g) \cdot f(p) = g \cdot p$$

and $g \cdot p \in M^f$. To see that $\mu(M^f) \subseteq \mathfrak{q}^*$, again take $p \in M^f$, and by (5.1.2)

$$\mu(p) = \mu\big(f(p)\big) = -\tau^*\mu(p)$$

so $\mu(p) \in \mathfrak{q}^*$. The fact that M^f is Lagrangian was already shown in Proposition 5.1.4.

□

5.1.1 Examples

In this subsection we provide a bank of examples of real Hamiltonian (G, τ)-spaces.

Example 5.1.9 (Complex Space)**.** Consider the action of $\mathrm{U}(n)$ on $\mathbb{C}^n$ by matrix multiplication. As shown in Example 2.3.9 this action is Hamiltonian with moment map

$$\mu(z) = \frac{1}{2i} z z^*,$$

relative to the standard symplectic structure ω_0 on $\mathbb{C}^n$. Consider the standard real structure f_0 on $\mathbb{C}^n$, and let $\tau : \mathrm{U}(n) \to \mathrm{U}(n)$ denote the involutive automorphism given by matrix conjugation $\tau(g) = \bar{g} = (g^{-1})^T$. We claim that $\mathbb{C}^n$ is a real Hamiltonian $\big(\mathrm{U}(n), \tau\big)$-space. Condition (5.1.3) is immediate:

$$f(g \cdot z) = \overline{gz} = \bar{g}\bar{z} = \tau(g) \cdot f(z).$$

For (5.1.2), we note that the action of $d\tau$ on $\mathfrak{u}(n)$ is also given by conjugation, i.e. for $X \in \mathfrak{u}(n)$ (an anti-self dual matrix) we have $d\tau(X) = \bar{X}$. Moreover, we also note that $z^* X z$ is purely imaginary for all $X \in \mathfrak{u}(n)$ and $z \in \mathbb{C}^n$. Thus

$$\begin{aligned} \big\langle \mu\big(f_0(z)\big), X \big\rangle &= \frac{i}{2} \bar{z}^* X \bar{z} = \frac{i}{2} \overline{z^* \overline{X} z} = \frac{i}{2} \overline{z^* d\tau(X) z} = -\frac{i}{2} z^* d\tau(X) z = -\left\langle \mu(z), d\tau(X) \right\rangle \\ &= \big\langle -\tau^*\big(\mu(z)\big), X \big\rangle, \end{aligned}$$

and $\mu \circ f = -\tau^* \circ \mu$ by non-degeneracy of the pairing.

Therefore, it follows that $\mathbb{C}^n$ with the standard real structure is a real Hamiltonian $\big(\mathrm{U}(n), \tau\big)$-space. Its real locus $(\mathbb{C}^n)^{f_0}$ is $\mathbb{R}^n$, and is invariant under the action of $G^\tau = \mathrm{O}(n)$, the n-dimensional orthogonal group. ◀

Example 5.1.10 (Cotangent Bundles)**.** Let Q be a smooth manifold, and $\bar{f}$ a smooth involution on Q. Let T^*Q be the cotangent bundle of Q equipped with the canonical symplectic form $\omega = -d\theta$ for θ the tautological-form. Let $f : T^*Q \to T^*Q$ defined by $f = (d\bar{f}^{-1})^*$ be the **cotangent lift** of $\bar{f}$. For all $\alpha \in \Omega^1(Q)$, which we can view as a function $Q \to T^*Q$, f fits into the following commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\ \bar{f}\ } & Q \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle (\bar{f}^{-1})^*\alpha} \\ T^*Q & \xrightarrow[\ f\ ]{} & T^*Q. \end{array}$$

We claim that f is a symplectomorphism of T^*Q. To prove this we show that f preserves the tautological-form. Recall that the tautological-form θ is the unique 1-form on T^*Q satisfying $\alpha^*\theta = \alpha$ for all $\alpha \in \Omega^1(Q)$. Hence we have

$$\alpha^*\big(f^*\theta\big) = (f \circ \alpha)^*\theta = \big((\bar{f}^{-1})^*\alpha \circ \bar{f}\big)^*\theta = \bar{f}^*\big((\bar{f}^{-1})^*\alpha\big)^*\theta = \bar{f}^*(\bar{f}^{-1})^*\alpha = \alpha,$$

and by uniqueness $f^*\theta = \theta$. Hence $f^*\omega = \omega$ for the symplectic form $\omega = -d\theta$, and f is a symplectomorphism. We can turn f into an anti-symplectic involution by multiplying the cotangent directions by -1. If G is a Lie group acting on T^*Q such that $f(g \cdot p) = \tau(g) \cdot f(p)$ for τ an involutive automorphism of G, it follows that T^*Q is a real Hamiltonian (G, τ)-space. This follows as the moment map for the G-action on T^*Q is given by $\langle \mu, X \rangle = i(X_M)\theta$, as shown in Theorem 2.3.13.

An important example to consider is the cotangent bundle T^*G of a Lie group G. Let τ be an involutive automorphism of G. Trivialising $T^*G \cong G \times \mathfrak{g}^*$ via left-translations, we define an involution f on T^*G by setting $f(g, \lambda) = \big(\tau(g), -\tau^*\lambda\big)$. This is an anti-symplectic involution by the previous work, and the real locus of T^*G is $(T^*G)^f = G^\tau \times \mathfrak{q}^*$. Further, as the cotangent lifts of the left and right multiplication on T^*G are Hamiltonian by Example 2.3.17, it follows that T^*G is a real Hamiltonian (G, τ)-space in two different ways. ◀

Example 5.1.11 (Coadjoint Orbits)**.** Let G be a connected Lie group and τ an involutive automorphism of G. Take $\lambda \in \mathfrak{g}^*$, and consider the coadjoint orbit $G \cdot \lambda$ through λ endowed with the KKS form. Assume that $-\tau^*\lambda \in G \cdot \lambda$, so that $-\tau^*(G \cdot \lambda) = G \cdot \lambda$ by Lemma 5.1.7. Define an involution f on $G \cdot \lambda$ by setting $f(\xi) = -\tau^*\xi$. As τ is a homomorphism, τ^* is a Poisson automorphism of $\mathfrak{g}^*$. Hence as $G \cdot \lambda$ is a symplectic leaf of $\mathfrak{g}^*$, it follows that $f = -\tau^*$ is anti-symplectic.

As the moment map for the coadjoint action on $G \cdot \lambda$ is simply inclusion, (5.1.2) holds. Further (5.1.3) is satisfied by Lemma 5.1.7. Hence $G \cdot \lambda$ is a real Hamiltonian (G, τ)-space. Its real locus is $(G \cdot \lambda)^f = G \cdot \lambda \cap \mathfrak{q}^*$; and if it is non-empty, we may assume that $\lambda \in \mathfrak{q}^*$. We call such a coadjoint orbit **symmetric**. ◀

Example 5.1.12 (Products)**.** Suppose M_1 and M_2 are two real Hamiltonian (G, τ)-spaces, and let f_1, f_2 denote their respective real structures. We claim that $M_1 \times M_2$ is a real Hamiltonian (G, τ)-space with respect to the diagonal G-action. Define an involution $f : M_1 \times M_2 \to M_1 \times M_2$ by $f(p_1, p_2) = \big(f_1(p_1), f_2(p_2)\big)$. This is anti-symplectic because f_1 and f_2 are anti-symplectic. To be more specific, let $\pi_i : M_1 \times M_2 \to M_i$ denote the canonical projection. Then

$$\begin{aligned}
f^*\omega &= f^*(\pi_1^*\omega_1 + \pi_2^*\omega) \\
&= (\pi_1 \circ f)^*\omega_1 + (\pi_2 \circ f)^*\omega_2 \\
&= (f_1 \circ \pi_1)^*\omega_1 + (f_2 \circ \pi_2)^*\omega_2 \\
&= \pi_1^*\big(f_1^*\omega_1\big) + \pi_2^*\big(f_2^*\omega_2\big) \\
&= -\pi^*\omega_1 - \pi_2^*\omega_2 \\
&= -\omega.
\end{aligned}$$

Moreover,

$$\begin{aligned}
f\big(g \cdot (p_1, p_2)\big) = f(g \cdot p_1, g \cdot p_2) = \big(f_1(g \cdot p_1), f_2(g \cdot p_2)\big) &= \big(\tau(g) \cdot f_1(p_1), \tau(g) \cdot f_2(p_2)\big) \\
= \tau(g) \cdot \big(f_1(p_1), f_2(p_2)\big)& \\
= \tau(g) \cdot f(p_1, p_2),&
\end{aligned}$$

for all $g \in G$, $p_1 \in M_1$, and $p_2 \in M_2$ so that (5.1.3) is satisfied. Let μ_1 and μ_2 denote the moment maps on M_1 and M_2, respectively. Then the moment map for the diagonal G-action on $M_1 \times M_2$ is given by $\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2)$. Thus

$$\begin{aligned}\mu\big(f(p_1, p_2)\big) &= \mu\big(f_1(p_1), f_2(p_2)\big)\\ &= \mu_1\big(f_1(p_1)\big) + \mu_2\big(f_2(p_2)\big)\\ &= -\tau^*\mu(p_1) - \tau^*\mu_2(p_2)\\ &= -\tau^*\mu(p_1, p_2),\end{aligned}$$

showing that (5.1.2) holds. Therefore $M_1 \times M_2$ is a real Hamiltonian (G, τ)-space as claimed. ◀

5.2 Real Reduction

In this section we investigate whether the reduced space $M/\!\!/G$ of a real Hamiltonian (G, τ)-space M is a real Hamiltonian (K, ϕ)-space for some Lie group K and involutive automorphism ϕ of K. To answer this question, we first find conditions for a real structure to descend to the reduced space.

Note that most of this section is contained in [OS00], but is stated without proof. In this section, we expand on exposition and fill in the details.

For ease of notation, let M_0 denote the reduced space $M/\!\!/G$ of a Hamiltonian G-space M.

Proposition 5.2.1. *Suppose (M, ω, G, μ) is a Hamiltonian G-space, and f is a real structure on M. Further suppose that 0 is a regular value of μ and that G acts freely and properly on $\mu^{-1}(0)$, so that the reduced space M_0 is a symplectic manifold by Theorem 3.1.7 with symplectic form ω_0. If f preserves the zero level set $\mu^{-1}(0)$ and sends G-orbits to G-orbits, then f descends to a real structure $\tilde{f}$ on M_0.*

Proof. As f preserves the level set $\mu^{-1}(0)$, f restricts to an involution $f : \mu^{-1}(0) \to \mu^{-1}(0)$. Further, as f sends G-orbits to G-orbits, f descends to a smooth map $\tilde{f}$ on $\mu^{-1}(0)/G = M_0$ such that the following diagram commutes:

$$\begin{array}{ccc}\mu^{-1}(0) & \xrightarrow{\;f\;} & \mu^{-1}(0)\\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi}\\ M_0 & \xrightarrow[\;\tilde{f}\;]{} & M_0,\end{array}$$

i.e. $\tilde{f}\big([p]\big) = [f(p)]$. It is clear that $\tilde{f}$ is an involution as f is an involution. Further, as $\tilde{f}$ is surjective, it is a diffeomorphism. To see that $\tilde{f}$ is anti-symplectic,

$$\begin{aligned} \pi^*(-\tilde{f}^*\omega_0) &= -(\tilde{f}\circ\pi)^*\omega_0 \\ &= -(\pi\circ f)^*\omega_0 \\ &= -f^*(\pi^*\omega_0) \\ &= -f^*(\iota^*\omega) \\ &= -\iota^*(f^*\omega) \\ &= -\iota^*(-\omega) \\ &= \iota^*\omega, \end{aligned} \tag{5.2.1}$$

where $\iota : \mu^{-1}(0) \hookrightarrow M$ is the inclusion map. Hence using the uniqueness property of ω_0, it follows that $-\tilde{f}^*\omega_0 = \omega_0$ and $\tilde{f}^*$ is anti-symplectic. ☐

Corollary 5.2.1.1. *Suppose (M, ω, G, μ) is real Hamiltonian (G, τ)-space, and G acts freely and properly on $\mu^{-1}(0)$ with 0 a regular value for μ. Then the reduced space (M_0, ω_0) is a real symplectic manifold with real structure induced from the real structure on M.*

Proof. If f denotes the real structure on M, then for all $p \in \mu^{-1}(0)$

$$\mu\big(f(p)\big) = -\tau^*\mu(p) = -\tau^*0 = 0$$

by linearity of τ^*. Hence f preserves the zero level set of μ. Further (5.1.3) shows that f sends G-orbits to G-orbits, and Proposition 5.2.1 implies that M_0 is a real symplectic manifold. ☐

Recall that in certain situations we can endow the reduced space M_0 with a Hamiltonian structure. Explicitly, suppose that (M, ω, G, μ) is a Hamiltonian manifold, and K is another Lie group acting on M in a Hamiltonian fashion with moment map ϕ. If the actions of G and K commute and μ is K-invariant, then ϕ is G-invariant and M is a Hamiltonian $G \times K$-manifold by Proposition 2.3.6(IV). Moreover, if the reduced space $M/\!\!/G$ is a well-defined symplectic manifold, then the induced action of K on $M/\!\!/G$ is Hamiltonian with moment map induced by ϕ (see Proposition 3.3.1). (This is basis of commuting reduction in stages, Theorem 3.3.2.)

Extending this to the real Hamiltonian setting, we obtain the following theorem.

Theorem 5.2.2. *Let (M, ω, G, μ) be a real Hamiltonian (G, τ)-space with real structure f. Suppose K is another Lie group acting on M in a Hamiltonian fashion with moment map ϕ. Assume the actions of K and G commute and that $\phi^{-1}(0)$ is G-stable. Further suppose that f maps $\phi^{-1}(0)$ to $\phi^{-1}(0)$ and sends K-orbits to K-orbits. If the reduced space $M/\!\!/K$ is a well-defined symplectic manifold, then $M/\!\!/K$ is a real Hamiltonian (G, τ)-space with real structure induced by f.*

Proof. As f sends K-orbits to K-orbits, and $\phi^{-1}(0)$ is f-stable, it follows that $M/\!\!/K$ is a real symplectic manifold whose real structure, $\tilde{f}$, is induced from f by Proposition 5.2.1. However,

$M/\!\!/K$ is a Hamiltonian G-space with moment map $\tilde{\mu}$ induced from μ, and we claim that this turns $M/\!\!/K$ into a real Hamiltonian (G,τ)-space with real structure $\tilde{f}$. Indeed,

$$\tilde{\mu}\big(\tilde{f}[p]\big) = \tilde{\mu}\left(\big[f(p)\big]\right) = \mu\big(f(p)\big) = -\tau^*\mu(p) = -\tau^*\left(\tilde{\mu}[p]\right)$$

giving (5.1.2). For (5.1.3), we find

$$\tilde{f}\big(g \cdot [p]\big) = \tilde{f}[g \cdot p] = \big[f(g \cdot p)\big] = \big[\tau(g) \cdot f(p)\big] = \tau(g) \cdot \big[f(p)\big] = \tau(g) \cdot \tilde{f}[p].$$

Hence $M/\!\!/K$ is a real Hamiltonian (G,τ)-space as claimed. □

5.2.1 Singular Reduction

Recall by Theorem 3.4.17 that the requirement of 0 to be a regular value of the moment map in symplectic reduction can be dropped. The resulting quotient space is no longer a symplectic manifold, but a stratified symplectic space, with stratification induced by the orbit type stratification. The goal of this section is to generalise Theorem 5.2.2 to singular reduction. Intuition suggests that this should be possible, we just have to check the real Hamiltonian structure does not mix the various strata.

Let (M,ω,G,μ) be a real Hamiltonian (G,τ)-space for a compact Lie group G, with real structure given by f. By Corollary 5.2.1.1 we know that f already preserves $\mu^{-1}(0)$. Hence to have a singular analogue of Theorem 5.2.2, we also require f to preserve the orbit type decomposition i.e. $f\big(M_{(H)}\big) = M_{(H)}$ for all subgroups H of G. So let $p \in M$, and $g \in G_p$. Then by (5.1.3),

$$g \cdot p = p \implies f(g \cdot p) = f(p) \implies \tau(g) \cdot f(p) = f(p), \tag{5.2.2}$$

so $\tau(g) \in G_{f(p)}$. Further, as τ is an involution, each implication in (5.2.2) is reversible. Therefore G_p and $G_{f(p)}$ are isomorphic subgroups of G for every $p \in M$, and so define the same conjugacy class in G. Hence if $p \in M_{(H)}$ then

$$\big(G_{f(p)}\big) = (G_p) = (H),$$

implying $f(p) \in M_{(H)}$. Altogether, we have proved the following lemma.

Lemma 5.2.3. *Suppose (M,ω,G,μ) is a real Hamiltonian (G,τ)-space for a compact Lie group G, with real structure f. Then f preserves the orbit type decomposition of M.*

Using Lemma 5.2.3 we obtain the following, singular analogue, of Proposition 5.2.1.

Proposition 5.2.4. *Suppose (M,ω,G,μ) is a real Hamiltonian (G,τ)-space for a compact Lie group G, with real structure f. Then f descends to a continuous involution on the stratified symplectic space M_0, which restricts to a real structure on each strata.*

Proof. Analogous to the proof of Proposition 5.2.1, as f preserves the zero level set $\mu^{-1}(0)$ and sends G-orbits to G-orbits, it descends to a continuous involution $\tilde{f}$ on M. To see that f preserves the individual strata, as f is surjective and preserves orbit types by Lemma 5.2.3, it follows that

$$f\big(\mu^{-1}(0) \cap M_{(H)}\big) = f\big(\mu^{-1}(0)\big) \cap f\big(M_{(H)}\big) = \mu^{-1}(0) \cap M_{(H)}$$

for all subgroups H of G. Hence $\tilde{f}$ preserves the strata, and (5.2.1) shows that $\tilde{f}$ is a real structure on the strata. □

Proposition 5.2.4 leads to the following definition.

Definition 5.2.5. A **stratified real symplectic space** M is a stratified symplectic space with a continuous involution $f : M \to M$ which preserve the strata, and whose restriction to said strata is a real structure. The involution f is still referred to as a **real structure**.

Hence Proposition 5.2.4 can be restated as the reduction of a real Hamiltonian (G, τ)-space is a stratified real symplectic space. Moreover, we can extend this definition to the case of stratified Hamiltonian spaces as follows.

Definition 5.2.6. A **stratified real Hamiltonian** (G, τ)**-space** M is a real stratified symplectic space with a continuous action of G such that the action preserves each symplectic strata, and each strata is a real Hamiltonian (G, τ)-space with real structure given by the restriction of the real structure on M.

Similarly, we see that we have a singular analogue of Theorem 5.2.2.

Theorem 5.2.7. *Suppose (M, ω, G, μ) is a real Hamiltonian (G, τ)-space for a compact Lie group G, with real structure f. Suppose K is another compact Lie group acting on M in a Hamiltonian fashion with moment map ϕ. Suppose that the actions of G and K commute, and that f preserves the zero level set $\phi^{-1}(0)$ and sends K-orbits to K-orbits. Further suppose that both f and the G-action preserve the K-orbit types. Then the reduced space $M/\!\!/K$ is a stratified real Hamiltonian (G, τ)-space with real structure induced from f.*

Proof. By Proposition 5.2.4 f descends to a real structure on $M/\!\!/K$. As the G-action preserves the K-orbit type strata, we may apply Theorem 5.2.2 strata by strata to conclude that $M/\!\!/K$ is a stratified real Hamiltonian (G, τ)-space. □

Corollary 5.2.7.1. *Suppose the assumptions of Theorem* 5.2.7 *hold, and let $\tilde{f}$ denote the real structure on $M/\!\!/K$. Then if the fixed point set*

$$(M/\!\!/K)^{\tilde{f}} = \{[p] \in M/\!\!/K : \tilde{f}[p] = [p]\}$$

is non-empty, it is a Lagrangian submanifold in each strata of $M/\!\!/K$.

Hence we have a notion of a **Lagrangian submanifold** in a stratified symplectic space: it is a subset whose intersection with each strata is a Lagrangian submanifold. Moreover, Theorem 5.2.7 gives a way to construct such subsets.

5.3 Real Implosion

In this section we answer the main question of this thesis: under what conditions does the imploded cross-section of a real Hamiltonian (G, τ)-space inherit a real Hamiltonian (T, τ) structure?

Let (M, ω, G, μ) be a connected Hamiltonian G-space for G a compact, connected Lie group. Let $T \subseteq G$ be a maximal torus. Recall by (4.3.1) we have a decomposition of the imploded cross-section M_{impl}

$$M_{\text{impl}} = \coprod_{\sigma} M_\sigma /\!\!/ [G_\sigma, G_\sigma] \tag{5.3.1}$$

where σ is a face of the fundamental Weyl chamber $\mathfrak{t}^*_+$ in $\mathfrak{t}^*$. Using (5.3.1) to determine if the imploded cross-section inherits a real Hamiltonian structure, it is sufficient to determine when the real structure restricts to each term in (5.3.1).

Suppose M is also a real Hamiltonian (G, τ)-space with real structure given by f. We want the real Hamiltonian structure on M to restrict to a real Hamiltonian structure on each cross-section $M_\sigma = \mu^{-1}(S_\sigma)$. Now, each cross-section M_σ is a Hamiltonian G_σ-space with moment map $\mu_\sigma = \mu|_{M_\sigma}$. Hence for f to descend to a real Hamiltonian (G_σ, τ)-structure, by (5.1.2) we need $-\tau^*(S_\sigma) = S_\sigma$ for every face σ. Similarly, for (5.1.3) to hold we need τ to map G_σ to itself for every face. Under these conditions, it is clear that M_σ is a real Hamiltonian (G_σ, τ)-space.

However, in implosion we consider M_σ as a Hamiltonian $[G_\sigma, G_\sigma]$-space with moment map $\tilde{\mu}_\sigma$ given by μ_σ composed with the projection $\pi_\sigma : \mathfrak{g}^*_\sigma \to [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma]^*$. As τ is an automorphism which preserves the stabiliser groups G_σ, it preserves the commutator subgroups $[G_\sigma, G_\sigma]$ so that (5.1.3) holds. For (5.1.2), we have

$$\tilde{\mu}_\sigma\big(f(p)\big) = \pi_\sigma\left(\mu_\sigma\big(f(p)\big)\right) = -\pi_\sigma\big(\tau^*\mu(p)\big) = -\tau^*\left(\pi_\sigma\big(\mu(p)\big)\right) = -\tau^*\tilde{\mu}_\sigma(p),$$

and M_σ is a real Hamiltonian $\big([G_\sigma, G_\sigma], \tau\big)$-space with real structure f. Therefore, for every face σ of $\mathfrak{t}^*_+$, by Proposition 5.2.4 $M_\sigma /\!\!/ [G_\sigma, G_\sigma]$ is a stratified real symplectic space with real structure induced by f.

However, the imploded cross-section M_{impl} comes equipped with a continuous T-action which restricts to a Hamiltonian action on each strata with moment map μ_{impl}, the continuous map induced from μ. We claim that this T-action turns M_{impl} into a real Hamiltonian (T, τ)-space with the real structure constructed previously.

Theorem 5.3.1 (Real Symplectic Implosion). *Let (M, ω, G, μ) be a connected real Hamiltonian (G, τ)-space for G a compact, connected Lie group with real structure f. Fix a maximal torus T in G. Suppose that*

- **I)** $\tau^*(\operatorname{star}\sigma) = -\operatorname{star}\sigma$ *for every face σ.*
- **II)** τ *fixes the coadjoint stabiliser groups G_σ for every face σ.*
- **III)** τ *fixes the maximal torus T.*

Then f descends to a real structure f_{impl} on the imploded cross-section M_{impl}, and M_{impl} is a real Hamiltonian (T, τ)-space with moment map μ_{impl}.

Proof. Recall that $S_\sigma = G_\sigma \cdot \operatorname{star}\sigma$. Hence by conditions I), II), and Lemma 5.1.7, it is clear that $-\tau^*(S_\sigma) = S_\sigma$. Thus the argument in the preceding paragraphs implies that $M_\sigma = \mu^{-1}(S_\sigma)$

is a real Hamiltonian $\big([G_\sigma, G_\sigma], \tau\big)$-space with real structure given by f. As τ preserves the commutator subgroups $[G_\sigma, G_\sigma]$ for every face σ, it follows that f sends $[G_\sigma, G_\sigma]$-orbits to $[G_\sigma, G_\sigma]$-orbits by (5.1.3). Therefore, f descends to a continuous involution f_{impl} on M_{impl} defined by $f_{\text{impl}}[p] = \big[f(p)\big]$. By Proposition 5.2.4 f_{impl} preserves strata and is a real structure on M_{impl}.

To see that M_{impl} is a stratified real Hamiltonian (T, τ)-space, let $p \in M_{\text{impl}}$ and $\tau \in T$. Then

$$\mu_{\text{impl}}\big(f_{\text{impl}}[p]\big) = \mu_{\text{impl}}\big[f(p)\big] = \mu\big(f(p)\big) = -\tau^*\mu(p) = -\tau^*\big(\mu_{\text{impl}}[p]\big),$$

and

$$f_{\text{impl}}\big(t\cdot[p]\big) = f_{\text{impl}}[t\cdot p] = \big[f(t\cdot p)\big] = \big[\tau(t)\cdot f(p)\big] = \tau(t)\cdot\big[f(p)\big] = \tau(t)\cdot f_{\text{impl}}[p],$$

where the second to last equality holds as $\tau(t) \in T$. Therefore (5.1.2) and (5.1.3) hold and M_{impl} is a stratified real Hamiltonian (T, τ)-space as required. □

Corollary 5.3.1.1. *The fixed point set*

$$M_{\text{impl}}^{f_{\text{impl}}} = \{[p] \in M_{\text{impl}} : f_{\text{impl}}[p] = [p]\},$$

is either empty or a Lagrangian submanifold in each strata, i.e. if $M_{\text{impl}} = \amalg_{i\in I} X_i$ *is the decomposition into strata, then*

$$M_{\text{impl}}^{f_{\text{impl}}} = \coprod_{i\in I} X_i^{f_{\text{impl}}}$$

where $X_i^{f_{\text{impl}}}$ *is either empty or a Lagrangian.*

Theorem 5.3.1 provides a real analogue of symplectic implosion. Hence we refer to the imploded cross-section M_{impl}, viewed as a stratified real Hamiltonian (T, τ)-space, as the **real imploded cross-section**, or **real imploded space** of M.

An important corollary is the following, which is a real version of the abelianisation theorem, Theorem 4.4.2.

Corollary 5.3.1.2. *Suppose M is a connected real Hamiltonian (G, τ)-space, where τ is an involutive automorphism satisfying conditions I)-III) in Theorem* 5.3.1. *Then f induces a real structure f_0 on $M/\!\!/G$, and real structure f_{impl} on M_{impl} (where M is imploded with respect to the maximal torus τ fixes). Furthermore, f_{impl} induces a real structure $\tilde{f}_{\text{impl}}$ on $M_{\text{impl}}/\!\!/T$ which corresponds to the real structure f_0.*

Proof. The real structure f_0 on $M/\!\!/G$ is given by $f_0[p]_G = \big[f(p)\big]_G$, while the real structure f_{impl} is given by $f_{\text{impl}}[p]_{\text{impl}} = \big[f(p)\big]_{\text{impl}}$. Now, f_{impl} further induces a real structure on $M_{\text{impl}}/\!\!/T$ defined by $\tilde{f}_{\text{impl}}\big[[p]_{\text{impl}}\big]_T = \Big[\big[f(p)\big]_{\text{impl}}\Big]_T$. Thus it is clear that, under the isomorphism $M/\!\!/G \cong M_{\text{impl}}/\!\!/T$, we have the following commutative diagram

$$\begin{array}{ccc} M_{\mathrm{impl}}/\!\!/T & \xrightarrow{\tilde{f}_{\mathrm{impl}}} & M_{\mathrm{impl}}/\!\!/T \\ \downarrow & & \downarrow \\ M/\!\!/G & \xrightarrow[f_0]{} & M/\!\!/G. \end{array}$$

□

While the conclusion of Theorem 5.3.1 is certainly interesting, the requirements I)-III) on the involutive automorphism τ are restrictive and, a priori, such a non-trivial involution may not exist. However, for compact, connected Lie groups there exists such an automorphism τ, called the **Chevalley involution**, which satisfies conditions I)-III) relative to a certain maximal torus.

Definition 5.3.2. Let G be a compact, connected Lie group. A **Chevalley involution** is an involutive automorphism τ of G satisfying $\tau(t) = t^{-1}$ for all t in some maximal torus T of G, and $\tau^*\alpha = -\alpha$ for all roots α of G relative to $\mathfrak{t} = \mathrm{Lie}(T)$.

Example 5.3.3. The Chevalley involution for $\mathrm{SU}(n)$ is matrix conjugation $\tau(g) = \overline{g} = (g^{-1})^T$. It fixes the maximal torus T defined by

$$T = \left\{ \begin{bmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{bmatrix} : \sum_{j=1}^{n} \theta_j = 0 \right\}.$$

◀

It shown in [Sam90, Section 2.10] that a Chevalley involution on a compact, connected Lie group G exists, and is unique up to conjugation. By definition of the Chevalley involution, τ satisfies condition III) of Theorem 5.3.1 for the maximal torus T that τ fixes. The fact that τ satisfies conditions I) and II) of Theorem 5.3.1 is shown in the following proposition.

Proposition 5.3.4. *Let G be a compact, connected Lie group and τ a Chevalley involution of G. Then $\tau^*(\mathfrak{t}_+^*) = -\mathfrak{t}_+^*$ for the fundamental Weyl chamber $\mathfrak{t}_+^*$ relative to the maximal torus τ fixes. Moreover, τ preserves the coadjoint stabiliser groups G_σ for every face σ of $\mathfrak{t}_+^*$.*

Proof. Recall that $\tau^*\alpha = -\alpha$ for every root of G relative to $\mathfrak{t}$. As the roots of G span the fundamental Weyl chamber $\mathfrak{t}_+^*$, it follows that $\tau^*|_{\mathfrak{t}_+^*} = -\mathrm{Id}$ and so $\tau^*(\mathfrak{t}_+^*) = -\mathfrak{t}_+^*$. To see the second statement, let $\lambda \in \mathfrak{t}_+^*$, σ the face of $\mathfrak{t}_+^*$ containing λ, and $g \in G_\sigma$. Then by the first statement,

$$\mathrm{Ad}^*_{\tau(g)} \lambda = \mathrm{Ad}^*_{\tau(g)}(-\tau^*\lambda) = -\tau^*\big(\mathrm{Ad}^*_g \lambda\big) = -\tau^*\lambda = \lambda,$$

showing $\tau(g) \in G_\sigma$. □

Hence if (M, ω, G, μ) is a real Hamiltonian (G, τ)-space for τ a Chevalley involution of G, then the imploded cross-section M_{impl} inherits a real Hamiltonian (T, τ)-structure if we imploded with respect to the maximal torus T that τ fixes.

5.3.1 The Universal Imploded Cross-Section

From the previous section, we know that if M is a connected real Hamiltonian (G, τ)-space for τ a Chevalley involution of G, then the imploded cross-section M_{impl}, imploded relative to the maximal torus τ fixes, is a stratified real Hamiltonian (T, τ)-space. However, we still do not have an example of a real Hamiltonian space for the action of the Chevalley involution. Luckily for us, the universal object of implosion $(T^*G)_{\text{impl}}$ is an example of such a space.

To recall, let G be a compact, connected Lie group and consider the symplectic manifold T^*G with symplectic form $\omega = d\theta$. Trivialising $T^*G \cong G \times \mathfrak{g}^*$ via left translations, the cotangent lift of the right action of G on G is given by

$$h \cdot (g, \lambda) = (gh^{-1}, \operatorname{Ad}_h^* \lambda).$$

This action is Hamiltonian with moment map projection onto the second factor $\mu_R(g, \lambda) = \lambda$ (as we have seen so many times before). Imploding T^*G relative to this action gives the universal imploded cross-section $(T^*G)_{\text{impl}}$, in the sense of Theorem 4.5.2.

Suppose now that τ is a Chevalley involution of G. Then by Example 5.1.10 the cotangent lift of τ to $(T^*G, d\theta)$ can be modified to be a real structure of f. Under the trivialisation $T^*G \cong G \times \mathfrak{g}^*$, the real structure is given $f(g, \lambda) = \big(\tau(g), -\tau^*(\lambda)\big)$. Moreover the cotangent lift of the right action makes T^*G a real Hamiltonian (G, τ)-manifold. Hence, if T is the maximal torus fixed by τ, the imploded cross-section $(T^*G)_{\text{impl}}$ is a stratified Hamiltonian (T, τ)-space by Theorem 5.3.1 with real structure induced from f.

We give an explicit description of the induced real structure f_{impl} as follows. Let $\pi_R(g, \lambda) \in (T^*G)_{\text{impl}}$, i.e. $g \in G$ and $\lambda \in \mathfrak{t}_+^*$. Then

$$f_{\text{impl}}\big(\pi_R(g, \lambda)\big) = \pi_R\big(f(g, \lambda)\big) = \pi_R\big(\tau(g), -\tau^*\lambda\big) = \pi_R\big(\tau(g), \lambda\big),$$

where the last equality holds as $-\tau^*$ acts as the identity on $\mathfrak{t}_+^*$. From this we also obtain compatibility with the T-action:

$$\begin{aligned} f_{\text{impl}}\big(t \cdot \pi_R(g, \lambda)\big) &= f_{\text{impl}}\big(\pi_R(gt^{-1}, \operatorname{Ad}_t^* \lambda)\big) \\ &= \pi_R\big(\tau(g)\tau(t)^{-1}, -\tau^*(\operatorname{Ad}_t^* \lambda)\big) \\ &= \pi_R\big(\tau(g)\tau(t)^{-1}, \operatorname{Ad}_{\tau(t)}^* \lambda\big) \\ &= \tau(t) \cdot \pi_R\big(\tau(g), \lambda\big) \\ &= \tau(t) \cdot f_{\text{impl}}\big(\pi_R(g, \lambda)\big). \end{aligned}$$

However, recall we also have a residual Hamiltonian G-action on $(T^*G)_{\text{impl}}$ induced from left-action on T^*G. We claim that f_{impl} with the induced left-action is also a real Hamiltonian (G, τ)-space.

As the moment map $\tilde{\mu}_L$ for the left action on $(T^*G)_{\text{impl}}$ is induced from the moment map μ_L for the left action on T^*G, we have

$$\begin{aligned}\tilde{\mu}_L\left(f_{\text{impl}}\big(\pi_R(g,\lambda)\big)\right) &= \tilde{\mu}_L\left(\pi_R\big(\tau(g),\lambda\big)\right)\\ &= \operatorname{Ad}^*_{\tau(g)}\lambda\\ &= -\operatorname{Ad}^*_{\tau(g)}\tau^*\lambda\\ &= -\tau^*\big(\operatorname{Ad}^*_g\lambda\big)\\ &= -\tau^*\left(\tilde{\mu}_L\big(\pi_R(g,\lambda)\big)\right).\end{aligned}$$

To see that τ is compatible with the left action,

$$f_{\text{impl}}\big(h\cdot\pi_R(g,\lambda)\big) = f_{\text{impl}}\big(\pi_R(hg,\lambda)\big) = \pi_R\big(\tau(h)\tau(g),\lambda\big) = \tau(h)\cdot f_{\text{impl}}\big(\pi_R(g,\lambda)\big)$$

for all $h\in G$. Hence it follows that $(T^*G)_{\text{impl}}$ is a real Hamiltonian $(G\times T,\tau)$-space.

This gives the following real analogue of Theorem 4.5.2.

Theorem 5.3.5. *Suppose M is a connected real Hamiltonian (G,τ)-space. Then the isomorphism in Theorem* 4.5.2

$$M_{\text{impl}}\to\big(M\times(T^*G)_{\text{impl}}\big)/\!\!/G$$

is an isomorphism of real Hamiltonian (T,τ)-spaces, where T is the maximal torus τ fixes.

Proof. Let f denote the real structure on M, and let $\tilde{\phi}$ denote the real structure on $(T^*G)_{\text{impl}}$ induced from the real structure $\phi(g,\lambda) = \big(\tau(g),-\tau^*\lambda\big)$ on T^*G. By previous work, we can view $M\times(T^*G)_{\text{impl}}$ as a real Hamiltonian $(G\times T,\tau)$-space where the $G\times T$-action is defined as in Theorem 4.5.2. The real structure on $M\times T^*G$ is given by

$$\tilde{f}\big(p,\pi_R(g,\lambda)\big) = \big(f(p),\pi_R\big(\tau(g),\lambda\big)\big)$$

and it induces a real structure $\tilde{f}_0$ on the quotient $\big(M\times(T^*G)_{\text{impl}}\big)/\!\!/G$. As the isomorphism $M_{\text{impl}}\to\big(M\times(T^*G)_{\text{impl}}\big)/\!\!/G$ is given by $[p]\mapsto\big[p,\pi_R\big(e,\mu(p)\big)\big]$, it is clear that the following diagram commutes

$$\begin{array}{ccc} M_{\text{impl}} & \xrightarrow{\;f_{\text{impl}}\;} & M_{\text{impl}}\\ \downarrow & & \downarrow\\ \big(M\times(T^*G)_{\text{impl}}\big)/\!\!/G & \xrightarrow[\tilde{f}_0]{} & \big(M\times(T^*G)_{\text{impl}}\big)/\!\!/G,\end{array}$$

where f_{impl} is the real structure on M_{impl} induced from f. ☐

To end we compute the induced real structure on the imploded cross-section of $T^*\operatorname{SU}(2)$ with the Chevalley involution. The Chevalley involution on $\operatorname{SU}(2)$ is given by complex conjugation which clearly fixes the maximal torus $S^1 \subseteq \mathbb{C}$. Consider the imploded cross-section $\big(T^* \mathrm{SU}(2)\big)_{\text{impl}}$ relative to this maximal torus. We have shown already in Section 4.5.1 that $\big(T^* \mathrm{SU}(2)\big)_{\text{impl}}$ is isomorphic to $\mathbb{C}^2$, with the isomorphism given by

$$\begin{aligned}\tilde{\psi} : \big(T^* \mathrm{SU}(2)\big)_{\text{impl}} &\to \mathbb{C}^2,\\ \pi_R(g,\lambda) &\mapsto \sqrt{2\,\|\lambda\|}g.\end{aligned}$$

Under this isomorphism, we see that the real structure f_{impl} is given by

$$\tilde{\psi}\left(f_{\text{impl}}\big(\pi_R(g,\lambda)\big)\right) = \tilde{\psi}\left(\pi_R\big(\tau(g),\lambda\big)\right) = \sqrt{2\,\|\lambda\|}\tau(g) = \sqrt{2\,\|\lambda\|}\bar{g},$$

and so f_{impl} corresponds to the standard involution f_0 on $\mathbb{C}^2$ given by complex conjugation. Compatibility with action of S^1 is given by

$$\begin{aligned}\tilde{\psi}\left(f_{\text{impl}}\big(t\cdot\pi_R(g,\lambda)\big)\right) &= \tilde{\psi}\left(f_{\text{impl}}\big(\pi_R(gt^{-1},\lambda)\big)\right)\\ &= \tilde{\psi}\left(\pi_R\big(\tau(g)\tau(t)^{-1},\lambda\big)\right)\\ &= \sqrt{2\,\|\lambda\|}\tau(g)\tau(t)^{-1},\end{aligned}$$

or equivalently

$$f_0(t\cdot z) = f_0(t^{-1}z) = \bar{t}^{-1}\bar{z} = \tau(t)\cdot f_0(z).$$

Hence $\big(T^* \mathrm{SU}(2)\big)_{\text{impl}}$ with the induced real structure and $\mathbb{C}^2$ with the standard real structure are isomorphic real Hamiltonian (S^1,τ)-spaces. Therefore the real locus $\big(T^* \mathrm{SU}(2)\big)_{\text{impl}}^{f_{\text{impl}}}$ is isomorphic to $\mathbb{R}$; the real locus of $\mathbb{C}^2$ under the standard real structure.

5.4 Further Directions

In this section we propose some further directions of research.

The first potential direction is applying this theory to the case of quasi-Hamiltonian implosion. Quasi-Hamiltonian G-spaces where introduced by Alekseev et al. in [AMM98], they differ from Hamiltonian spaces in a few ways, most notably in that the moment map is valued in the Lie group rather than the dual Lie algebra. Implosion of such spaces was introduced by Hurtubise, Jeffrey, and Sjamaar in [HJS06], and has interesting links to moduli problems in geometry.

We expect that real implosion will work in the setting of quasi-Hamiltonian spaces. This is because Hurtubise et al. show that the imploded cross-section has a decomposition into pieces which are the reduction of some quasi-Hamiltonian submanifold. Moreover, Schaffhauser [Sch07] has shown when an anti-symplectic involution on a quasi-Hamiltonian manifold descends to the quotient. The use of these results should allow one to develop an analogous theory of real quasi-Hamiltonian implosion.

Another avenue of research is to develop a real theory for hyperkähler implosion, which was introduced by Dancer et al. in a series of papers [DKS13a; DKS13b; DKS14; Dan+16]. In these papers, they show that the imploded cross-section could be constructed through quiver diagrams, so a first step would be to investigate the induced real structure in the quiver model. Moreover, one could investigate the situation where the real structure preserves additional structures. For example, we could also require the real structure to be anti-holomorphic.

Appendix A

Group Actions

A.1 Preliminaries

The goal of this appendix is to provide the requisite background material for a Lie group acting on a smooth manifold needed for this thesis. It also doubles in stating what sign conventions this thesis uses. The material in this appendix is taken from [GGK02, Appendix B], and [Aud04, Chapter 1].

Definition A.1.1. An **(left) action** of a group G on a set M is a collection of maps $\mathcal{A}_g : M \to M$ for $g \in G$ such that $\mathcal{A}_{gh} = \mathcal{A}_g \circ \mathcal{A}_h$, and $\mathcal{A}_e = \mathrm{Id}_M$ for e the identity of G. For ease of notation, we will often write $g \cdot p$ for $\mathcal{A}_g(p)$.

Remark A.1.2. A right action of G on M is defined similarly, except $\mathcal{A}_{gh} = \mathcal{A}_h \circ \mathcal{A}_g$. In a right action, we write $\mathcal{A}_g(p) = p \cdot g$. ♦

If M is a vector space and the maps $\mathcal{A}_g$ are linear maps for all $g \in G$, the action is called a **linear representation of** G.

Definition A.1.3. In the case that G is a Lie group and M is a smooth manifold, we say that an action of G on M is **smooth** if the maps $\mathcal{A}_g$ are **smooth** for all $g \in G$. We sometimes denote a smooth Lie group action by the group homomorphism $G \to \mathrm{Diff}(M)$ it defines.

A.1.1 Adjoint Actions

Consider now the situation where G is a Lie group acting on itself. There are many ways that it can do this, for example G may act on itself via left or right translations. However, we consider the situation where G acts on itself via conjugation, i.e. $C_g(h) = g \cdot h = ghg^{-1}$. It follows that C_g is a Lie group automorphism for all $g \in G$, and so the conjugation action of G on itself is smooth.

As C_g is a Lie group homomorphism, its derivative at the identity $d(C_g)_e : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism. This Lie algebra homomorphism plays such a pivotal role in Lie theory that it is given its own name, the **adjoint map**, and is denoted by $\mathrm{Ad}_g = d(C_g)_e$. Viewing the adjoint map as a function $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, defined by $\mathrm{Ad}(g) = \mathrm{Ad}_g$, we have the following proposition.

Proposition A.1.4. *Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$. The adjoint map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a Lie group representation, called the **adjoint representation** of G.*

Proof. [Lee12, Proposition 20.24]. □

There also exists an adjoint representation for Lie algebras. Given a Lie algebra $\mathfrak{g}$, for every $X \in \mathfrak{g}$, the map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}_X Y = [X, Y]$. The adjoint maps for a Lie group and Lie algebra are related in the following way.

Theorem A.1.5. *Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$. If $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation of G, then the induced Lie algebra representation $d(\mathrm{Ad})_e : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is $d(\mathrm{Ad})_e = \mathrm{ad}$.*

Proof. [Lee12, Theorem 20.27]. □

Moreover, using the adjoint action of G on $\mathfrak{g}$, we can define an action of G on $\mathfrak{g}^*$ by

$$\left[\,\mathrm{Ad}_g^* \,\xi\right](X) = \xi\left(\,\mathrm{Ad}_{g^{-1}} X\right)$$

for all $g \in G$, $X \in \mathfrak{g}$, and $\xi \in \mathfrak{g}^*$. This action is the **coadjoint action** of G on $\mathfrak{g}^*$. Note that the change from g to g^{-1} in the definition of the coadjoint action; this is required to ensure it satisfies the properties of a left action.

A.1.2 Averaging with a Compact Lie Group

Lemma A.1.6. *Suppose G is a compact Lie group. Then there exists a unique G-invariant volume form ω^n on G such that $\int_G \omega^n = 1$.*

Proof. Let $\omega^n_e \in \bigwedge^{\dim G} \mathfrak{g}^*$, and define a top degree left invariant form $\omega^n \in \Omega^{\dim G}(G)$ by $\omega_g = L_g^* \omega$, where $L_g : G \to G$ is left-translation by $g \in G$. We claim that ω^n is also right invariant, i.e. $R_h^* \omega^n = \omega^n$ where $R_h : G \to G$ is right-translation by $h \in G$.

First note that as R_h and L_g commute for all $g, h \in G$, it follows that $R_h^* \omega^n$ is also left-invariant. As $\dim\left(\bigwedge^{\dim G} \mathfrak{g}^*\right) = 1$, there exists a function $f : G \to \mathbb{R}^\times$ such that $R_h^* \omega^n = f(h)\omega^n$. By the properties of a right action, f is a group homomorphism. Moreover, as

$$f(g) = R_g^* \omega = R_g^*\left(L_{g^{-1}}^* \omega^n\right) = \left(L_{g^{-1}} \circ R_g\right)^* \omega^n$$

it follows that $f(g) = \det\left(\,\mathrm{Ad}_{g^{-1}}\,\right)$ which is smooth as both the determinant and adjoint maps are smooth.

Assume first that G is also connected. Now as f is a group homomorphism, $f(G)$ is a compact connected subgroup of $\mathbb{R}^\times$ as G is compact connected. As $1 \in f(G)$, we have $f(G) = [\alpha, \beta] \subseteq (0, \infty)$. We claim that $f(G) = \{1\}$. Suppose not, and take $f(x) \in f(G)$ with $x > 1$. Then there exists n such that $f(x^n) = f(x)^n \notin f(G)$, a contradiction to $f(G)$ being a subgroup of $\mathbb{R}^\times$. Therefore $f(G) = \{1\}$, which implies that $R_g^*\omega^n = \omega^n$ for all $g \in G$. If G is not connected then the previous work generalises to show that $f(G) = \{1, -1\}$.

In either case, as ω^n is a G-invariant volume it follows that $(\int_G \omega^n)^{-1}\omega^n$ is also a G-invariant volume form whose integral over G is 1 and this form is unique. □

The top form ω^n constructed in Lemma $A.1.6$ gives a canonical G-invariant measure on G.

Theorem A.1.7. *Let G be a compact Lie group. Then there exists a unique G-invariant measure m of G defined on a σ-algebra of G which contains the Borel σ-algebra.*

Proof. Let ω^n be the G-invariant top form constructed in Lemma $A.1.6$. Define a linear functional $\Lambda : C(G) \to \mathbb{R}$ on the space of continuous functions on G by

$$\Lambda(f) = \int_G f\omega^n.$$

Then Λ is a positive linear functional in the sense that $f(G) \subseteq [0, \infty)$ implies $\big[\Lambda(f)\big](G) \subseteq [0, \infty)$. Therefore by the Riesz representation theorem, [Rud87, Theorem 2.14], there exists a unique measure m defined on a σ-algebra of G which contains the Borel σ-algebra such that

$$\int_G f dm = \Lambda(f) = \int_G f\omega^n.$$

The measure m is G-invariant because ω^n is. For example, left invariance is given by

$$\int_G f(hg)dm(g) = \int_G (L_h^* f)\omega^n = \int_G (L_h^* f)(L_h^*\omega^n) = \int_G L_h^*(f\omega^n) = \int_G f\omega^n = \int_G f(g)dm(g).$$

□

Definition A.1.8. The unique measure defined in Theorem $A.1.7$ is called the **Haar Measure** of G.

The Haar measure is extremely useful because it allows *averaging* over a compact Lie group. This will allow us to construct G-invariant inner products, which play a major in the construction of slices for a group action.

Proposition A.1.9. *Let G be a compact Lie group acting linearly on a vector space V. Then there exists a G-invariant inner product on V.*

Proof. Let $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ be any inner product on V. Define $(\ ,\ ) : V \times V \to \mathbb{R}$ by *averaging* $\langle\ ,\ \rangle$ with respect to G, i.e. for all $u, v \in V$

$$(u, v) = \int_G \langle g \cdot u, g \cdot v\rangle\, dm(g)$$

where m is the Haar measure on G. It is clear that $(\ , \ )$ is symmetric and bilinear. It is also non-degenerate as

$$(u,u) = \int_G \langle g \cdot u, g \cdot u \rangle \, d\mu(g) > 0$$

for $u \neq 0$ as $\langle g \cdot u, g \cdot u \rangle > 0$. Further, the fact that $(\ , \ )$ is G-invariant follows immediately from the Haar measure being G-invariant. □

Corollary A.1.9.1. *Let G be a compact Lie group acting smoothly on a manifold M. Then there exists a G-invariant Riemannian metric on M. In this situation G acts by isometries on M.*

Proof. Let G act on M by the functions $\mathcal{A}_g : M \to M$ for all $g \in G$. This induces an action of G on the tangent bundle TM with the functions $d\mathcal{A}_g$. Now let g' be any Riemannian metric on M. Then by Proposition $A.1.9$

$$g(X,Y) = \int_G g'\big(d\mathcal{A}_h(X), d\mathcal{A}_h(Y)\big) dm(h)$$

is a G-invariant Riemannian metric on M. □

A.1.3 Proper Actions

In this thesis, we are mainly interested in compact Lie groups. However, a few statements hold more generally, and compactness can be replaced by the following condition.

Definition A.1.10. An action of G on M is **proper** if the map

$$\begin{aligned} G \times M &\to M \times M, \\ (g,p) &\mapsto (g \cdot p, p) \end{aligned}$$

is proper, i.e. the preimage of a compact set is compact.

The next proposition gives conditions under which an action is proper.

Proposition A.1.11 (Characterisation of proper actions)**.** *Let G be a Lie group acting smoothly on M. Then the following are equivalent*

I) *The action is proper.*

II) *If (p_n) is a sequence in M and (g_n) is a sequence in G such that both (p_n) and $(g_n \cdot p_n)$ converge, then a subsequence of (g_n) converges.*

III) *For every compact subset $K \subseteq M$, the set $G_K = \{g \in G : (g \cdot K) \cap K \neq \emptyset\}$ is compact.*

Proof. [Lee12, Proposition 21.5] □

Corollary A.1.11.1. *Every smooth action of a compact Lie group G on a manifold M is proper.*

A.2 Orbits and Stabilisers

Suppose G is a Lie group acting smoothly on M.

Definition A.2.1. The **stabiliser** of $p \in M$ is

$$G_p = \{g \in G : g \cdot p = p\}.$$

From the definition of a group action, it is clear that G_p is a subgroup of G. We claim that it is also closed. Consider a sequence (g_n) in G_p. Then as the group action is smooth, it is continuous, and

$$p = \lim_{n\to\infty} g_n \cdot p = \big(\lim_{n\to\infty} g_n\big) \cdot p$$

and so $\lim_{n\to\infty} g_n \in G_p$, showing G_p is closed. Thus by the closed subgroup theorem [Lee12, Theorem 20.12] G_p is a Lie subgroup. If the action is proper, then the stabiliser group G_p is compact. This follows easily as G_p is homeomorphic to the preimage of (p, p), a compact subset of $M \times M$.

Definition A.2.2. An action is **free** if $G_p = \{e\}$ for all $p \in M$. The action is **locally free** if the stabiliser are discrete, i.e. $\mathrm{Lie}(G_p) = \{0\}$. The action is **effective** if the group homomorphism $G \to \mathrm{Diff}(M)$ defining the action is injective, this is equivalent to $\bigcap_{p\in M} G_p = \{e\}$.

Definition A.2.3. The **orbit** of $p \in M$ is

$$G \cdot p = \{g \cdot p : g \in G\} \subseteq M.$$

The map

$$\begin{aligned} \mathcal{A}^p : G &\to M, \\ g &\mapsto g \cdot p, \end{aligned}$$

is called the **orbit map**, as $\mathcal{A}^p(G) = G \cdot p$.

Define a relation on M by stating that p and q are related if, and only if, they lie in the same orbit. It is clear that this is an equivalence relation and the equivalence classes are the orbits. Hence we denote the quotient space M/G to be the set of orbits, and endow it with the quotient topology induced from M.

Proposition A.2.4. *Given a proper group action of a Lie group G on M, then every orbit is a closed subset of M and the orbit space M/G is Hausdorff.*

Proof. Let $K \subseteq M$ be compact. Then by continuity $(\mathcal{A}^p)^{-1}(K)$ is closed in G. However, $(\mathcal{A}^p)^{-1}(K) \subseteq G_{K\cup\{p\}}$ which is compact as the action is proper, and so $(\mathcal{A}^p)^{-1}(K)$ is compact in G, and $\mathcal{A}^p$ is proper. As M is a smooth manifold, it is locally compact and Hausdorff, and so $\mathcal{A}^p$ is also a closed map. Thus the orbits are closed as $\mathcal{A}^p(G) = G \cdot p$.

Let $\mathcal{R} = \{(g \cdot p, p) \in M \times M : p \in M, g \in G\}$ be the orbit relation, and $\pi : M \to M/G$ be the canonical projection. Then π is an open quotient map, and to show M/G is Hausdorff it suffices to show $\mathcal{R}$ is closed in $M \times M$. This follows immediately as the G-action is proper and G, and M are smooth manifold and so locally compact, Hausdorff spaces. □

A.3 Fundamental Vector Fields

A smooth action of a Lie group G on M induces a linear map $\mathfrak{g} \to \mathfrak{X}(M)$ by $X \mapsto X_M$, where X_M is the vector field on M defined by

$$X_M(p) = \frac{d}{dt}\Big|_{t=0} \exp(tX) \cdot p,$$

where $\exp : \mathfrak{g} \to G$ is the Lie group exponential. The vector field X_M is called the **fundamental vector field** on M associated to $X \in \mathfrak{g}$, and it is the vector field whose flow is $\mathcal{A}_{\exp(tX)}$. It has another description due to the orbit map. Computing the differential using curves, we find

$$X_M(p) = \frac{d}{dt}\Big|_{t=0} \mathcal{A}^p\big(\exp(tX)\big) = d(\mathcal{A}^p)_e(X).$$

Proposition A.3.1. *Let G be a Lie group acting smoothly on M. Then for all $g \in G$, $X, Y \in \mathfrak{g}$*

I) $\big(\operatorname{Ad}_g X\big)_M = d\mathcal{A}_g(X_M)$.

II) $[X_M, Y_M] = -[X, Y]_M$.

Proof. I): For all $p \in M$, we have

$$\begin{aligned}
\big(\operatorname{Ad}_g X\big)_M(p) &= \frac{d}{dt}\Big|_{t=0} \mathcal{A}_{\exp(t \operatorname{Ad}_g X)}(p) \\
&= \frac{d}{dt}\Big|_{t=0} \mathcal{A}_{g\exp(tX)g^{-1}}(p) \\
&= \frac{d}{dt}\Big|_{t=0} \mathcal{A}_g \mathcal{A}_{\exp(tX)} \mathcal{A}_{g^{-1}}(p) \\
&= d\mathcal{A}_g\Big(\frac{d}{dt}\Big|_{t=0} \mathcal{A}_{\exp(tX)}(g^{-1} \cdot p)\Big) \\
&= d\mathcal{A}_g\big(X_M(g^{-1} \cdot p)\big) \\
&= \big(d\mathcal{A}_g(X_M)\big)(p).
\end{aligned}$$

II): As the Lie bracket of vector fields is given by the Lie derivative, using I) we have for all

$p \in M$:

$$\begin{aligned}[X_M, Y_M](p) = \big(\mathcal{L}_{X_M} Y_M\big)(p) &= \frac{d}{dt}\bigg|_{t=0} \big[d\mathcal{A}_{\exp(-tX)}(Y_M)\big](p) \\ &= \frac{d}{dt}\bigg|_{t=0} \big[\operatorname{Ad}_{\exp(-tX)} Y\big]_M(p) \\ &= \big(-\operatorname{ad}_X Y\big)_M(p) \\ &= -[X,Y]_M(p).\end{aligned}$$

□

Thus it follows that the map $\mathfrak{g} \to \mathfrak{X}(M)$ sending X to its fundamental vector field is a Lie algebra *anti-homomorphism.*

Definition A.3.2. For all $p \in M$ define the **stabiliser algebra** of p to be the set

$$\mathfrak{g}_p = \{X \in \mathfrak{g} : X_M(p) = 0\} \subseteq \mathfrak{g}.$$

From the properties of fundamental vector fields, it is clear that $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ (a subspace closed under the Lie bracket).

The notational similarities between the stabiliser algebra $\mathfrak{g}_p$ and the stabiliser group G_p is not a coincidence, as the next proposition shows.

Proposition A.3.3. *For a smooth action of a Lie group G on M, the Lie algebra of the stabiliser group G_p is the stabiliser algebra $\mathfrak{g}_p$ for all $p \in M$.*

Proof. Since G_p is a Lie subgroup of G, its Lie algebra is

$$\operatorname{Lie}(G_p) = \{X \in \mathfrak{g} : \exp(tX) \in G_p, \text{ for all } t \in \mathbb{R}\}.$$

If $X \in \operatorname{Lie}(G_p)$, then $\exp(tX) \in G_p$ for all t and $\gamma(t) = \exp(tX) \cdot p = p$ is a constant curve for all t. Differentiating γ and setting $t = 0$ gives $X_M(p) = 0$ and so $X \in \mathfrak{g}_p$.

Conversely, if $X \in \mathfrak{g}_p$ then to show $X \in \operatorname{Lie}(G_p)$, it is enough to show that $\exp(tX) \cdot p$ is a constant curve. This follows because

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=s} \exp(tX) \cdot p &= \frac{d}{dt}\bigg|_{t=0} \exp\big((t+s)X\big) \cdot p \\ &= \frac{d}{dt}\bigg|_{t=0} \exp(sX)\exp(tX) \cdot p \\ &= d\mathcal{A}_{\exp(sX)}\left(\frac{d}{dt}\bigg|_{t=0} \exp(tX) \cdot p\right) \\ &= d\mathcal{A}_{\exp(sX)}\big(X_M(p)\big) \\ &= 0.\end{aligned}$$

□

A.3.1 Infinitesimal Actions

Suppose that G is a Lie group and $\mathfrak{g}$ its Lie algebra.

Definition A.3.4. An action of the Lie algebra $\mathfrak{g}$ on M is a Lie algebra anti-homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, $X \mapsto \hat{X}$ such that the map

$$\begin{aligned}\mathfrak{g} \times M &\to TM, \\ (X, p) &\mapsto \big(p, \hat{X}(p)\big),\end{aligned}$$

is smooth. A Lie algebra action is also called an **infinitesimal action** on M.

We have already seen that every smooth action of G on M induces a Lie algebra action by considering the fundamental vector fields. We claim that if G is connected, then the converse holds. This will follow from the following lemma.

Lemma A.3.5. *Suppose G is a connected Lie group, and $U \subseteq G$ a neighbourhood of the identity $e \in G$. Then every $g \in G$ can be written as a finite product of elements in U, i.e. $g = g_1 \cdots g_n$ where $g_i \in U$.*

Proof. Without a loss of generality we may assume that U is closed under inversions. For each n define $U^n = \{g_1 \cdot g_n : g_i \in U\}$, we have to show that $\bigcup_{n=0}^{\infty} U^n = G$. We first show that U^n is open for every n. Note that for all $g \in G$ $gU = (L_{g^{-1}})^{-1}(U)$, where $L_{g^{-1}}$ is left translation. Thus gU is open by continuity, so U^n is open by induction and therefore $\bigcup_{n=0}^{\infty} U^n$ is open. It is clear that $\bigcup_{n=0}^{\infty} U^n$ is actually a subgroup of G, and so writing $H = \bigcup_{n=0}^{\infty} U^n$ it follows that the cosets gH are disjoint, and open by the previous work. Hence $G \setminus H = \bigcup_{g \in G \setminus H} gH$ is open, and H is therefore closed. As G is connected and $H = \bigcup_{n=0}^{\infty} U^n$ is non-empty, it follows that $G = \bigcup_{n=0}^{\infty}$. □

Corollary A.3.5.1. *If G is a connected Lie group then every element of G is product of exponentials.*

Proof. The exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism about the identities in $\mathfrak{g}$ and G. This gives the whole group by Lemma A.3.5. □

Many definitions relating to group actions have infinitesimal counterparts.

Definition A.3.6. A smooth map $f : M \to N$ between two manifolds M, and N, both with a $\mathfrak{g}$-action, is **$\mathfrak{g}$-equivariant** if $df(X_M) = X_N$ for all $X \in \mathfrak{g}$.

Proposition A.3.7. *Let M and N be smooth manifolds both equipped with a smooth action of a Lie group G. If $f : M \to N$ is G-equivariant, then it is $\mathfrak{g}$-equivariant with respect to the induced Lie algebra action. If G is connected then the converse holds.*

Proof. If f is G-equivariant then $f\big(\exp(tX) \cdot p\big) = \exp(tX) \cdot f(p)$, so taking the derivative with respect to t and setting $t = 0$ gives $df\big(X_M(p)\big) = X_N\big(f(p)\big)$.

Conversely, suppose G is connected and $df(X_M) = X_N$ for all $X \in \mathfrak{g}$. Then by the naturality of vector flows, [Lee12, Proposition 9.13], f preserves the flows of X_M and X_N, i.e. $f\big(\exp(tX)\cdot p\big) = \exp(tX)\cdot f(p)$. Thus f is equivariant on elements of the form $\exp(X)$ for $X \in \mathfrak{g}$. As G is connected, these elements generate G by Corollary $A.3.5.1$ and f is equivariant. $\square$

A.4 Principal G-Bundles

Definition A.4.1. A **principal G-bundle** over a manifold M is a manifold P with a free right action of G on P, together with a map $\pi : P \to M$ whose level sets are the G orbits in P, and every point in M has a neighbourhood U together with a diffeomorphism

$$\phi_U : \pi^{-1}(U) \to U \times G$$

such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times G \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & U & \end{array}$$

where π_1 is just projection onto the first factor. We further require ϕ_U to be G-equivariant with respect to the induce action on $\pi^{-1}(U)$ and the right action $(p,h)\cdot g = (p,hg)$ on $U \times G$.

Proposition A.4.2. *Let G be a Lie group and H a closed subgroup. Then the quotient G/H is a manifold, its tangent space at eH is $\mathfrak{g}/\mathfrak{h}$, and the quotient map $\pi : G \to G/H$ is a principal H-bundle.*

Proof. Let H act on G by right translations, then $g_1, g_2 \in G$ are in the same H-orbit if, and only if, $g_1 h = g_2$ for $h \in H$, which is equivalent to g_1 and g_2 lying in the same H coset. Thus the orbit space determined by the right action is just the left coset space G/H. As H is properly embedded by the closed subgroup theorem, the action of H on G is smooth being the restriction of the multiplication map. It is also free as $gh = g$ implies $h = e$. To see that it is proper, let (g_n) be a sequence in G and (h_n) a sequence in H such that (g_n) and $(g_n h_n)$ converge in G. Then by continuity $h_n = g_n^{-1}(g_n h_n)$ converges in G, and as H is closed it follows that (h_n) converges in H.

As the action is proper, it follows that the quotient is Hausdorff. We will show that every coset of H in G has a neighbourhood which is equivariantly diffeomorphic to $U \times H$, where U is a subset of $\mathbb{R}^k$ and H acts on $U \times H$ by $(p,h_1)\cdot h_2 = (p, h_1 h_2)$. This will prove G/H is a manifold, its atlas is induced by U and it is second countable because G is second countable. It will also show that $\pi : G \to G/H$ is a principal H-bundle.

Let $\mathfrak{n}$ be a complementary subspace to $\mathfrak{h} = \mathrm{Lie}(H)$ in $\mathfrak{g}$; $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$. Define the map ψ by

$$\begin{aligned} \Psi : \mathfrak{n} \times H &\to G, \\ (X,h) &\mapsto \exp(X)h. \end{aligned}$$

The differential $d\Psi|_{(0,e)} : N \times \mathfrak{h} \to \mathfrak{g}$ is the identity map. By H-equivariance, $d\Psi|_{(0,h)}$ is a bijection for all $h \in H$, and by continuity there exists a neighbourhood U of 0 in N such that $d\Psi|_{(u,h)}$ is a bijection for all $u \in U$. By the inverse function theorem, $\Psi : U \times H \to G$ is a local diffeomorphism. Restricting U if necessary, we claim that Ψ is actually a diffeomorphism. To show this, it suffices to show that Ψ is injective. So suppose, for a contradiction, that Ψ is not injective. Then there is a sequence of pairs $(X_n, a_n) \neq (Y_n, b_n)$ with $\lim_{n\to\infty} X_n = \lim_{n\to\infty} Y_n = 0$, and $\exp(X_n)a_n = \exp(Y_n)b_n$. Replacing a_n with $a_n b_n^{-1}$ and b_n with e, we obtain the sequence $(X_n, a_n b_n^{-1}) \neq (Y_n, e)$ with

$$\lim_{n\to\infty} \exp(X_n)a_n b_n^{-1} = \lim_{n\to\infty} \exp(Y_n) = e.$$

However, this is a contradiction as Ψ is injective on a neighbourhood of $(0, e)$. □

Proposition A.4.3. *Let G be a Lie group acting smoothly on M. Then for all $p \in M$, the map*

$$F^p : G/G_p \to M,$$
$$gG_p \mapsto g \cdot p,$$

is an injective immersion. If the action is proper, the orbit $G \cdot p$ is an embedded submanifold in M, and the map is a equivariant diffeomorphism between G/G_p and M.

Proof. We first show that F^p is well defined. Assume that $g_1 G_p = g_2 G_p$, so that $g_2 = g_1 h$ for some $h \in G_p$. Then

$$F^p(g_2 G_p) = g_2 \cdot p = g_1 h \cdot p = g_1 \cdot p = F^p(g_1 G_p).$$

It is clear that F^p is G-equivariant. Furthermore, F^p is smooth as it is obtained via the orbit map $\mathcal{A}^p$ by passing to the quotient: $F^p \circ \pi = \mathcal{A}^p$. Finally, F^p is injective, as $g_1 \cdot p = g_2 \cdot p$ implies $g_2^{-1} g_1 \in G_p$.

Now, the differential at the identity of the orbit map $\mathcal{A}^p$ is the induced $\mathfrak{g}$-action on M, $X \mapsto X_M(p)$. Its kernel is $\mathfrak{g}_p$ by Proposition $A.3.3$, hence F is an immersion at identity eH as $T_{eG_p}(G/G_p) = \mathfrak{g}/\mathfrak{g}_p$. By equivariance it follows that F is an immersion everywhere.

If the action is proper, then the map F is proper as the orbit map $\mathcal{A}^p$ is proper. As F is a proper injective immersion, it is an embedding. □

Corollary A.4.3.1. *Suppose G is a Lie group acting smoothly and properly on M. Then the tangent space to the orbit $G \cdot p$ for all $p \in M$ is*

$$T_p(G \cdot p) = \mathfrak{g}_M(p) = \{X_M : X \in \mathfrak{g}\}.$$

Proof. Since $G \cdot p$ is a submanifold by Proposition $A.4.3$, and $F^p : G/G_p \to G \cdot p$ is a diffeomorphism, we have the tangent space to $G \cdot p$ at p is given by

$$T_p(G \cdot p) = d(F^p)_{eG_p}(G/G_p).$$

However, recall that $\pi : G \to G/G_p$ is a submersion, and so $d\pi_e : \mathfrak{g} \to T_{eG_p}(G/G_p)$ is surjective. Therefore, by the chain rule

$$T_p(G \cdot p) = d(F^p)_{eG_p}\big(d\pi_e(\mathfrak{g})\big) = d(F^p \circ \pi)_e(\mathfrak{g}) = d(\mathcal{A}^p)_e(\mathfrak{g}) = \mathfrak{g}_M(p).$$

□

Definition A.4.4. A rank k **vector bundle** over a manifold M is a manifold E with a map $\pi : P \to M$ whose level sets $\pi^{-1}(p)$ are k-dimensional vector spaces, and every point in M has a neighbourhood U and a vector space V together with a linear isomorphism

$$\phi_U : \pi^{-1}(U) \to U \times V$$

such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times V \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & U & \end{array}$$

where π_1 is just projection onto the first factor.

Let $P \to M$ be a principal H-bundle, and let H act linearly on a vector space V. The **associated bundle** is

$$P \times_H V = (P \times V)/H$$

where the action of H on $P \times V$ is

$$(p, v) \cdot h = (ph^{-1}, h \cdot v).$$

We denote by $[p, v]$ the equivalence class of $(p, v) \in P \times V$ in $P \times_H V$. It follows that $[ph, v] = [p, h \cdot v]$ for all $h \in H$, and the projection $\pi : P \to M$ induces a projection $\beta : P \times_H V \to M$ by $\beta\big([p, v]\big) = \pi(p)$. Further it is well known that the associated bundle $P \times_H V$ is a vector bundle with fibre V under the operations

$$[p, v_1] + [p, v_2] = [p, v_1 + v_2], \quad \lambda[p, v] = [p, \lambda v]$$

for all $v_1, v_2 \in V$ and $\lambda \in \mathbb{R}$. For a proof see [Tu17, Sections 31.1-2].

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