Sheaves and Schemes in Algebraic Geometry

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Abstract

We survey the development of scheme theory and demonstrate its use in modern algebraic geometry. This will include a discussion of topics such as sheaf cohomology, commutative algebra and the Riemann Roch Theorem.

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Introduction

Algebraic geometry is often said to be one of the oldest fields in mathematics. This is, however, misleading, as the incarnation of algebraic geometry even a century ago is vastly different to what we would consider "algebraic geometry" today. Indeed, classical algebraic geometry studies objects known as (classical) varieties, which are, roughly speaking, the zeroes of polynomials over an algebraically closed field. The simplest type of variety is an *affine variety*:

Definition 0.0.1. Let k be an algebraically closed field. A set $V \subseteq k^n$ is said to be an *affine variety* if there exists a prime ideal $\mathfrak{p} \subseteq k[x_1, ...x_n]$ such that $V = \{P \in k^n \mid f(P) = 0 \text{ for all } f \in \mathfrak{p}\}$. The ring $A := k[x_1, ..., x_n]/\mathfrak{p}$ is known as the *coordinate ring* of V.

By Hilbert's Nullstellensatz and its corollaries, the points of V have a natural bijection with the maximal ideals of A. In this way, one may recover V up to some form of equivalence (specifically an isomorphism of varieties), knowing A. However, note that A must satisfy certain properties: indeed A must be a finitely-generated algebra over k that is also an integral domain. One may ask then: what if we were to allow analogous constructions for any commutative ring with identity? The answer is that we would end up with a *scheme*.

In fact, arguably the greatest turning point in algebraic geometry is the introduction of schemes by Grothendieck, which replaced varieties as the fundamental object of study. Schemes are a generalisation of classical varieties, but they retain information that would otherwise be lost. Roughly speaking, they are locally ringed spaces which locally look like affine schemes, which are themselves generalisations of affine varieties. A key theorem is the following, which allows us to pass back and forth between ring theory and geometry:

Theorem 0.0.2. *The category of affine schemes is equivalent to the opposite category of rings.*

In this dissertation, we will systematically develop scheme theory, with a view to providing a partial proof to the Riemann Roch Theorem. The standard reference for this is [6], which we will loosely follow.

A key construction in scheme theory is a *sheaf* which, roughly speaking, is a collection of objects parameterised by the open sets of a topological space that keeps track of local data. The first chapter will be devoted to sheaf theory, which we will develop with differential geometry as motivation. A powerful method in algebraic geometry is sheaf cohomology, which gives us tools for defining invariants. The construction of sheaf cohomology through the derived functor approach will be discussed in this chapter.

Chapter 2 then defines and studies the basic properties of schemes and morphisms of schemes. We will see a deep connection between schemes and morphisms of schemes and the rings which define their underlying affine schemes. We will also isolate a class of schemes, known as *abstract varieties* which share similarities with classical varieties.

Finally, Chapter 3 aims to prove the Riemann Roch Theorem. It begins with the theory of divisors, which, in their most basic form, are objects which encode the intrinsic geometry of a scheme or variety, in the form of formal sums of points, subject to an equivalence relation defined by the scheme. This allows us to define invariants of the scheme. Invertible sheaves, which behave like line bundles on a manifold will be discussed too. In particular, we will define an analogous construction to the cotangent bundle on a manifold. We conclude with a partial proof of the Riemann Roch Theorem.

0.1 Novel Elements

Being a very well-developed subject, it is comparatively difficult to prove original results in algebraic geometry. We do not pretend that any new results are proven in this work. However, the author has kept as much of the work original as possible. For example, proofs which were not worked out by the author are kept short or left out altogether, with a reference to the complete proof given. The exceptions to this are proofs which are interesting or enlightening. If a proof does not have a quoted reference, it was worked out by the author, with the possible assistance of peers and advisors.

0.2 Background

The assumed background is a first course in classical algebraic geometry, such as Chapter I of [6] and a first course in commutative algebra such [1]. Well-known results and terminology in commutative algebra and category theory will be used without reference or explanation, though lesser-known results or results which have interesting proofs will be stated and possibly proved in the appendix. We will also assume familiarity with elementary differential geometry, and the terminology of [8] will be used without explanation.

0.3 Conventions

Rings will always be commutative with identity. Unless explicitly stated otherwise, fields are assumed to be algebraically closed.

If A is a ring and $\mathfrak{p}$ is a prime ideal, then if T denotes the multiplicative system $A \setminus \mathfrak{p}$ then we will write $A_{\mathfrak{p}}$ for the localised ring $T^{-1}A$. Functors take both objects and morphisms as inputs. A *classical variety* will mean a variety as defined in [6, p.15]. *Morphism* and *map* will be used interchangeably.

This work will also feature use of sharps ($\sharp$) and flats ($\flat$). Their use is not inherently meaningful, they are simply used to denote objects that are similar in nature. Sharps are generally used to denote a “lifting” of some sort, and flats will be used for the opposite. For example, a presheaf will usually be denoted $\mathcal{F}^{\flat}$ and the associated sheaf will be denoted $\mathcal{F}$, if used in the same context.

Chapter 1

Sheaves

1.1 Definitions and Basic Theory

Sheaves are tools which allow us to keep track of local information on a topological space in a single mathematical object. Their use is ubiquitous throughout algebraic geometry. In this section, we will study their basic theory.

To begin: observe that the open sets of a topological space have a natural poset structure under "$\subseteq$". Posets are small categories, and thus one can define a category $\mathsf{Open}(X)$ associated to any topological space X, under this poset.

Definition 1.1.1. Let X be a topological space. A *presheaf* (of abelian groups) $\mathcal{F}^\flat$ on X is a contravariant functor from $\mathsf{Open}(X)$ into Ab, the category of abelian groups. For each open set U, the group $\mathcal{F}^\flat(U)$ will be called the *space of sections over* U and the members of the $\mathcal{F}^\flat(U)$ will be called the *sections* of $\mathcal{F}^\flat$ over U and for every morphism $i : V \to U$, $\mathcal{F}^\flat(i)$ will be called the *restriction map* from U to V. If $f \in \mathcal{F}^\flat(U)$, we will use $f|_V$ to denote $\mathcal{F}^\flat(i)(f)$.

We can replace the category of abelian groups with other categories, such as the category of sets or the category of rings. Unless stated otherwise however, a presheaf will always refer a presheaf of abelian groups.

Example 1.1.2. Let X be a topological space and G an abelian group. Then the functor $\mathcal{G}^\flat$ which maps every open set U to G and every inclusion to the identity is a presheaf, called the constant presheaf, for obvious reasons.

Definition 1.1.3. Let X be a topological space, and let $\mathcal{F}^\flat$ be a presheaf on X. We say $\mathcal{F}^\flat$ is a *sheaf* if it satisfies the following two axioms, called the *identity* and *gluing* axioms, denoted **ID** and **GL** respectively:

ID If $\{V_i\}$ is an open cover of some open set U and if $f, g \in \mathcal{F}^\flat(U)$ are such that $f|_{V_i} = g|_{V_i}$ for all i, then $f = g$.

GL If $\{V_i\}$ is an open cover of some open set U, and for each i we have some $f_i \in \mathcal{F}^\flat(V_i)$, and for every i, j we have that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$, then there exists some $f \in \mathcal{F}^\flat(U)$ such that $f|_{V_i} = f_i$

Note that **ID** implies that the section f obtained by **GL** is unique. If $\mathcal{F}^\flat$ is a sheaf, we will simply denote it as $\mathcal{F}$.

Before we study further examples, we introduce a construction that is extremely important, that is the stalk of a presheaf. The stalk may be thought of as what the presheaf "looks like" near a point.

Definition 1.1.4. Let X be a topological space, and let $\mathcal{F}^\flat$ be a presheaf on X. We define the *stalk* $\mathcal{F}^\flat_P$ of $\mathcal{F}^\flat$ at some point $P \in X$ to be the direct limit of the space of sections taken over all open neighbourhoods of P under the restriction maps.

$$\mathcal{F}^\flat_P := \varinjlim_{V \ni P} \mathcal{F}^\flat(V)$$

An element of the stalk is known as a *germ*.

Example 1.1.5. For a topological space X, we can define the sheaf of continuous real-valued functions $\mathcal{O}_X$, where for each open set $U \subseteq X$, the sections are the continous functions $f : U \to \mathbb{R}$, and the restriction maps are the restriction maps in the usual sense. To check that this is a sheaf, first fix some open set U and an open cover $\{U_i\}$ of U. If two sections $f, g \in \mathcal{O}_X(U)$ satisfy $f|_{U_i} = g|_{U_i}$ for every i, then for every $P \in U$, there will be some $U_i \ni P$, and thus $f(P) = g(P)$, hence $f = g$, and $\mathcal{O}_X$ satisfies **ID**. Next, if we have $f_i \in \mathcal{O}_X(U_i)$ for every i, such that for any i, j we have $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ then we define $f \in \mathcal{O}_X(U)$ as follows: for any $P \in U$ define $f(P) := f_i(P)$, where i is any index where $U_i \ni P$. This is clearly well-defined, and hence **GL** is satisfied. Note that $\mathcal{O}_X$ is a sheaf of rings: the sections inherit the ring structure of $\mathbb{R}$.

Example 1.1.6. Let X be a classical affine variety over some base field k, embedded in $\mathbb{A}(k)^n = k^n$. Recall that a function $\varphi : X \to k$ is *regular* at some point P if for some open neighbourhood U of P, there exist polynomials $f, g \in k[x_1, ..., x_n]$ with $\mathcal{V}(g) \cap U = \emptyset$ such that $\varphi(Q) = \frac{f(Q)}{g(Q)}$ for all $Q \in U$. Then the presheaf $\mathcal{O}$ where $\mathcal{O}(U)$ is the set of regular functions on U is a sheaf. The stalk is the local ring $\mathcal{O}_{P,X}$.

Example 1.1.7. The presheaf of bounded functions on $\mathbb{R}$, which associates to every open set U the set of bounded functions $f : U \to \mathbb{R}$, is not a sheaf, because it does not satisfy **GL**. Indeed, consider the cover of $\{(n-1, n+1)\}_{n \in \mathbb{N}}$ of $\mathbb{R}$, and for each n, take $f_n : (n-1, n+1) \to \mathbb{R}$ to be the restricted identity. Then each f_n is bounded, but clearly the identity, which is what we would get if we glued the f_n together, is not bounded.

Definition 1.1.8. Let $\mathcal{F}^\flat, \mathcal{G}^\flat$ be presheaves on X. A *morphism* of presheaves $\varphi : \mathcal{F}^\flat \to \mathcal{G}^\flat$ is a natural transformation from $\mathcal{F}^\flat$ to $\mathcal{G}^\flat$. In other words, it is a collection of morphisms $\varphi_U : \mathcal{F}^\flat(U) \to \mathcal{G}^\flat(U)$ for each open set U such that for any $V \subseteq U$ open, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^\flat(U) & \xrightarrow{\varphi_U} & \mathcal{G}^\flat(U) \\ \downarrow & & \downarrow \\ \mathcal{F}^\flat(V) & \xrightarrow{\varphi_V} & \mathcal{G}^\flat(V) \end{array}$$

By abuse of notation, we will sometimes write φ for the induced map on sections too, if there is no ambiguity. An *isomorphism* of presheaves is a morphism with a left and right inverse. For an open set U, we define the *sections functor* Γ_U which maps $\mathcal{F}^\flat$ to its section $\mathcal{F}^\flat(U)$ over U.

A morphism $\varphi : \mathcal{F}^\flat \to \mathcal{G}^\flat$ of presheaves also induces a morphism of stalks φ_P. Indeed, given a point $P \in X$, for every open neighbourhood U we have a map $\mathcal{F}^\flat(U) \to \mathcal{G}^\flat(U) \to \mathcal{G}^\flat_P$. Thus by the universal property of direct limits, we have a map $\varphi_P : \mathcal{F}^\flat_P \to \mathcal{G}^\flat_P$. The following result is very useful; it will be our main tool for showing that a morphism of sheaves is an isomorphism.

Proposition 1.1.9. *Let $\mathcal{F}, \mathcal{G}$ be sheaves on X. Then a map $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if at every point the induced map of stalks is an isomorphism.*

Proof. [6, p.63] □

Remark 1.1.10. However, it is not always true that two sheaves that have isomorphic stalks are isomorphic. Indeed, the above proposition requires that the isomorphism of stalks be induced by the same morphism of sheaves. For an explicit example, see Proposition 3.2.8.

We now present a very important lemma:

Lemma 1.1.11. *Let $\mathscr{B}$ be a base of the topology of X, and $\mathcal{F}$ a sheaf. Then for any open set U, we have*

$$\mathcal{F}(U) \cong \varprojlim_{\substack{V \subseteq U \\ V \in \mathscr{B}}} \mathcal{F}(V)$$

Proof. Observe that an element of $\varprojlim \mathcal{F}(V)$ defines a section on each base open set V, and this commutes with restriction. Thus by the sheaf axioms, this corresponds to a unique section in U, since the base open sets V clearly cover U, and thus we have a unique map $\varprojlim \mathcal{F}(V) \to \mathcal{F}(U)$ such that for every base open W contained in U the following diagram commutes:

$$\begin{tikzcd} \varprojlim \mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[dl] \\ \mathcal{F}(W) & \end{tikzcd}$$

Since any object with morphisms to each V factors uniquely through $\varprojlim \mathcal{F}(V)$, this means that $\mathcal{F}(U)$ satisfies the same universal property. This concludes the proof. □

Corollary 1.1.12. *A sheaf $\mathcal{F}$ on X is uniquely determined by the values it takes on a basis for the topology on X. A morphism $\mathcal{F} \to \mathcal{G}$ is also determined by the morphisms at each base open set.*

Proof. This follows from the above lemma. □

Remark 1.1.13. In fact, in order to define a sheaf it suffices [2, Proposition I-12] to give the sections over the *base* open sets and restriction maps between them, as long as the sheaf axioms are satisfied for the base open sets. We will make heavy use of this reduction.

We will now need suitable definitions of familiar concepts such as kernels, images, and cokernels for our morphisms. One may be tempted to simply define them as $U \mapsto \ker(\varphi_U), U \mapsto \operatorname{im}(\varphi_U), U \mapsto \operatorname{coker}(\varphi_U)$, however, this does not work since the presheaves $U \mapsto \operatorname{im}(\varphi_U), U \mapsto \operatorname{coker}(\varphi_U)$, referred to as the *presheaf image* and the *presheaf cokernel* of φ is not necessarily a sheaf (though the presheaf kernel, defined similarly, is a sheaf). See Example 3.1.12 for an example. This motivates the following question: how do we make a sheaf which best approximates a presheaf? This is answered below:

Theorem 1.1.14. *Let $\mathcal{F}^\flat$ be a presheaf. Then there exists a sheaf $\mathcal{F}$ along with a morphism $\mathcal{F}^\flat \to \mathcal{F}$ that satisfies the following universal property: if $\mathcal{G}$ is a sheaf and $\varphi : \mathcal{F}^\flat \to \mathcal{G}$ a morphism of presheaves, there exists a unique $\psi : \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{F}^\flat \arrow[r] \arrow[d, "\varphi"] & \mathcal{F} \arrow[dl, "\psi"] \\ \mathcal{G} & \end{tikzcd}$$

Moreover, φ_P is an isomorphism for every P.

Proof Sketch. Taken from [6, p.64]. For each open U, we define $\mathcal{F}(U)$ to be the set of functions $f : U \to \bigsqcup_{P \in U} \mathcal{F}_P$ such that the following hold true:

1. For each $f \in \mathcal{F}(U)$ we have $f(P) \in \mathcal{F}_P$
2. For each $f \in \mathcal{F}(U)$ and $P \in U$, there exists an open neighbourhood V of P and $f^\flat \in \mathcal{F}^\flat(V)$ such that $f(Q) = f^\flat_Q$ for all $Q \in V$

The restriction maps are then the restriction maps in the usual sense. Details can be found in the above reference. $\square$

Definition 1.1.15. Let $\mathcal{F}^\flat$ be a presheaf on X. We define the *sheaf associated to* $\mathcal{F}^\flat$ to be the sheaf $\mathcal{F}$ defined in the above theorem. The process of constructing $\mathcal{F}$ from $\mathcal{F}^\flat$ is known as *sheafification*.

Note that sheafification is left-adjoint to the forgetful functor that maps a sheaf to its presheaf.

Example 1.1.16. Let $X, G, \mathcal{G}^\flat$ be as in Example 1.1.2. The sheaf $\mathcal{G}$ associated to $\mathcal{G}^\flat$ is the sheaf defined as follows: Give G the discrete topology. Then the elements of $\mathcal{G}(U)$ are the continuous functions from U to G, and the restriction maps are the restriction maps of functions. Then the map $\mathcal{G}^\flat \to \mathcal{G}$ maps $g \in \mathcal{G}^\flat(U) = G$ to the constant function g, where U is any open set.

To see that $\mathcal{G}$ is indeed the sheaf associated to $\mathcal{G}^\flat$, we check that $\mathcal{G}$ satisfies the required universal property. So let $\mathcal{F}$ be a sheaf, and $\varphi : \mathcal{G}^\flat \to \mathcal{F}$ be a morphism of presheaves. This is equivalent to giving a map $G \to \mathcal{F}(X)$. Now note that if U is a connected open subset of X, then $\mathcal{G}(U) \cong G$, since the only G-valued continuous functions on U are the constant functions. Since the connected open sets of any topological space form a base, we have a map $\mathcal{G}(U) \to \mathcal{F}(U)$ for any connected open set U. Lemma 1.1.11 implies there is a unique morphism of sheaves $\mathcal{G} \to \mathcal{F}$ such commutes with the map $\mathcal{G}^\flat \to \mathcal{G}$.

Definition 1.1.17. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then we define the *kernel, image* and *cokernel* of φ to be the sheaf associated with their respective presheaf counterparts. We say φ is *injective* if $\ker \varphi = 0$ and *surjective* if $\operatorname{im} \varphi = \mathcal{G}$.

While it is true that a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if for every open set U the induced map of sections is injective, φ being surjective does not necessarily mean that the induced map of sections is surjective for every open set (see Example 3.1.12). However, it is true that a morphism is surjective if and only if it is surjective at the stalks:

Lemma 1.1.18. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then φ is injective (resp surjective) if and only if for every point P, the map of stalks $\varphi_P : \mathcal{F}_P \to G_P$ is injective (resp surjective)*

Proof. The map φ is injective if and only if for every open set U, the induced map of sections is injective. Since every section induces a germ and every germ is induced by a section, the result is immediate.

To prove the statement about surjectivity, we observe that by the construction of sheafification, for every P, the stalk of the image $(\operatorname{im}\varphi)_P$ and the stalk of the presheaf image are equal. Moreover, it is obvious that the stalk of the presheaf image at P is equal to the image of the induced map of stalks $\operatorname{im}(\varphi_P)$, and thus $\operatorname{im}(\varphi_P) = (\operatorname{im}\varphi)_P$. Now φ is surjective if and only if $\operatorname{im}\varphi = \mathcal{G}$, and since we have a canonical injection $\operatorname{im}\varphi \to \mathcal{G}$, this happens if and only if the stalks are equal, that is $G_P = (\operatorname{im}\varphi)_P$ and since we have established $\operatorname{im}(\varphi_P) = (\operatorname{im}\varphi)_P$, the result follows. □

Lemma 1.1.19. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then φ is surjective if and only if for every open set U and $s \in \mathcal{G}(U)$, there exists an open cover $\{U_i\}$ of U and sections $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$.*

Proof. Suppose φ is surjective. Then by Lemma 1.1.18, the induced map of stalks is surjective at every point $P_i \in U$, so that there exists $t_{P_i} \in \mathcal{F}_{P_i}$ such that $\varphi_{P_i}(t_i) = s_i$. We now construct our open cover as follows: for each i, choose a pair (U_i, t_i) so that $P_i \in U_i$ and t_i induces the germ t_{P_i} and $\varphi(t_i)$ and s agree on U_i, which exists because their germs agree. This is an open cover and satisifes our conditions.

Conversely, suppose the latter condition is satisifed. By Lemma 1.1.18, it suffices to show that the map of stalks is surjective at every point. Let P be a point and suppose $s_P \in \mathcal{G}_P$. Then s_P is the germ of some $s \in \mathcal{G}(U)$ for some open neighbourhood U of P, and moreover we can assume U is small enough so that there exists some $t \in \mathcal{F}(U)$ such that $\varphi(t) = s$. Then $\varphi_P(t_P) = s_P$ as required. □

We will now explore the homological behaviour of sheaves. We begin by exploring the exactness of the sections functor:

Proposition 1.1.20. *The sections functor Γ_U is left exact for every open U.*

Proof. Suppose we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0$$

and an open set U. We consider the following sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\iota_U} \mathcal{G}(U) \xrightarrow{\pi_U} \mathcal{H}(U)$$

Now ι_U is still injective, so the sequence is exact at $\mathcal{F}(U)$. Now suppose we have $s \in \ker \pi$. Since ι surjects onto the kernel of π, by Lemma 1.1.19 there exists an open cover U_i of U such that for each i there exists $t_i \in \mathcal{F}(U_i)$ such that $\iota(t_i) = s|_{U_i}$. Now considering t_i and t_j for some i and j, we have that $\iota(t_i|_{U_i \cap U_j}) = s|_{U_i \cap U_j} = \iota(t_j|_{U_i \cap U_j})$. But since ι is injective, that means $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$. This means there exists $t \in \mathcal{F}(U)$ such that $t|_{U_i} = t_i$ for every i, so that $\iota(t) = s$. This completes the proof. $\square$

However, the sections functor could be exact. As an example, we introduce the concept of a flasque sheaf.

Definition 1.1.21. Let $\mathcal{F}$ be a sheaf. We say $\mathcal{F}$ is *flasque* or *flabby* if for every inclusion $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Example 1.1.22. The sheaf of functions on any topological space is flasque. The sheaf of continuous real-valued functions on $\mathbb{R}$ is not (for example the function $f : x \mapsto 1/x$ defined on $\mathbb{R} \setminus \{0\}$ cannot be extended to the whole of $\mathbb{R}$). The structure sheaf on a scheme, which we will define later, is also not flasque.

Example 1.1.23. Let X be a topological space, A an abelian group and $P \in X$ a point. We define the *skyscraper sheaf* associated to A at P, denoted $A(P)$ to be the sheaf defined as follows: if $P \in U$ then $A(P)(U) = A$, if $P \notin U$ then $A(P)(U) = 0$. It is easily seen to be a sheaf. This is flasque, since a section is always an element of A, which is also the space of global sections.

Proposition 1.1.24. *Consider the following short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0$$

Suppose that $\mathcal{F}$ is flasque. Then the sections functor Γ_U is exact for every open set U.

Proof. We know that the functor is left exact, so it only remains to check that π_U is surjective. Suppose $s \in \mathcal{H}(U)$. We consider the set Ω of pairs (U_i, s_i) where $s_i \in \mathcal{G}(U_i)$ and $s|U_i = \pi(t_i)$. Lemma 1.1.19 guarantees Ω is not empty. Ordering by inclusion, it is easy to see that each chain has an upper bound, since we can take the union of the U_i, and the gluing axiom guarantees that such a section exists. Thus by Zorn's Lemma, there exists some maximal element (U_0, s_0).

Supposing for contradiction that $U_0 \neq U$, then there exists some $V \subseteq U$ which is not a subset of U_0 and some corresponding $s' \in \mathcal{G}(V)$ such that $s|V = \pi(s')$. Now if $V \cap U_0 = \emptyset$, then s and s' clearly agree on their intersection (which is the trivial group), and thus we may glue them together. Otherwise, $W := V \cap U_0$ is nonempty. Now since $\pi(s_0|_W) = s_W = \pi(s'|_W)$, so that $s_0|_W - s'|_W \in \ker \pi_W = \operatorname{im} \iota_W$, so that there exists some $t_0 \in \mathcal{F}(W)$ such that $\iota(t_0) = s_0|_W - s'|_W$. By the flasque property, we may extend t_0 to some $t \in \mathcal{F}(V)$. But we observe that $s' + \iota(t) \in \mathcal{G}(V)$ and $\pi(s' + \iota(t)) = \pi(s') = s|V$ and moreover $s_0|_W - (s' + \iota(t))_W = s_0|_W - s'_W - \iota(t_0) = 0$, so that we may glue s_0 and $s' + \iota(t)$ together on $U_0 \cup V$, contradicting the maximality of U_0. $\square$

The converse is not true; for example the global sections functor on the structure sheaf $\mathcal{O}_X$ of an affine scheme, as we will see later, satisfies the exactness property, but $\mathcal{O}_X$ is not flasque.

Corollary 1.1.25. *Consider the following short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0$$

If $\mathcal{F}$ and $\mathcal{G}$ are flasque, then $\mathcal{H}$ is also flasque.

Proof. Suppose we have an inclusion $V \subseteq U$ and some $s \in \mathcal{H}(V)$. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(U) \arrow[r, "\iota_U"] \arrow[d] & \mathcal{G}(U) \arrow[r, "\pi_U"] \arrow[d] & \mathcal{H}(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}(V) \arrow[r, "\iota_V"] & \mathcal{G}(V) \arrow[r, "\pi_V"] & \mathcal{H}(V) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathcal{F}$ is flasque, we know π_V is surjective by Proposition 1.1.24. Similarly, since $\mathcal{G}$ is flasque, the map $\mathcal{G}(U) \to \mathcal{G}(V)$ is also surjective. Therefore there exists some $t \in \mathcal{G}(U)$ such that $\pi_V(t|_V) = s$. Then $\pi_U(t)|_V = s$ as required. □

We conclude this section with a discussion about the pushforwards and pullbacks of sheaves along continuous maps. These are functors which allow sheaves to be passed along continuous maps. To be precise:

Definition 1.1.26. Let $f : X \to Y$ be a continous map, $\mathcal{F}$ a sheaf on X, and $\mathcal{G}$ a sheaf on Y. Then we define the *pushforward sheaf* $f_*\mathcal{F}$ on Y as $U \mapsto \mathcal{F}(f^{-1}(U))$. It is easy to see that this is a sheaf. Dually, we define the *pullback sheaf* $f^{-1}\mathcal{G}$ on X as the sheaf associated to the presheaf defined by the following direct limit

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

It is clear that the pushforward and the pullback are functorial. Note that while the pullback is more complicated than the pushforward, it is very well-behaved; for example at a point $P \in X$, the stalk $f^{-1}\mathcal{G}_P$ is isomorphic to $\mathcal{G}_{f(P)}$, whereas the analogous result does not hold for the pushforward.

Theorem 1.1.27 (The Adjoint Property of f^{-1})**.** *There exists a natural bijection of sets*

$$\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$$

Proof. Let $\{\varphi_{U,V}\}$ be a collection of maps $\varphi_{U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$, running across all open sets $U \subseteq X$, $V \subseteq Y$ where V contains $f(U)$. We say this collection is *compatible* if it commutes with restriction. We claim that there is a bijection from $\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$ to the set of all collections of compatible maps.

Suppose firstly that we have a morphism of sheaves $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$. Then given open subsets $U \subseteq X$ and $V \subseteq Y$ where V contains $f(U)$, we have a natural map $\mathcal{G}(V) \to \varinjlim_{W \supseteq f(U)} \mathcal{G}(W)$ which then induces a map $\mathcal{G}(V) \to \mathcal{F}(U)$ through composition. We now check that this map is compatible.

Let $V' \subseteq V$ and $U' \subseteq U$ be open subsets where $V' \supseteq f(U')$. Since all maps are compatible with restriction, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[r] \arrow[d] & \varinjlim_{W \supseteq f(U)} \mathcal{G}(W) \arrow[r] \arrow[d] & f^{-1}\mathcal{G}(U) \arrow[r, "\varphi"] \arrow[d] & \mathcal{F}(U) \arrow[d] \\
\mathcal{G}(V') \arrow[r] & \varinjlim_{W \supseteq f(U')} \mathcal{G}(W) \arrow[r] & f^{-1}\mathcal{G}(U') \arrow[r, "\pi_V"] & \mathcal{F}(U')
\end{tikzcd}$$

Conversely, given a compatible collection of maps $\{\varphi_{U,V}\}$, we construct a morphism $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ as follows: suppose we have U, V as above. Now we observe that since every $f(U) \subseteq W$ comes with a map $\mathcal{G}(W) \to \mathcal{F}(U)$ which commutes with restriction, by the universal property of direct limits we get a map $\varinjlim_{W \supseteq f(U)} \mathcal{G}(W) \to \mathcal{F}(U)$, which, by the universal property of sheafification induces a natural map $\varphi_U : f^{-1}\mathcal{G}(U) \to \mathcal{F}(U)$. By construction, these maps are compatible with restriction, and thus define a morphism of sheaves. Moreover, it is easily checked through "diagram-chasing" that the compatible collection generated by φ is $\{\varphi_{U,V}\}$. We omit the details.

To establish the other bijection, first suppose we have a morphism $\varphi : \mathcal{G} \to f_*\mathcal{F}$. Then for every (U, V) pair as above, we define the map $\varphi_{U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$ to be the composition of φ with the natural map $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$. Once again, the collection $\{\varphi_{U,V}\}$ is easily seen to be compatible.

Conversely, suppose we have a compatible collection $\{\varphi_{U,V}\}$. Then we have a map $\varphi : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ for every $V \subseteq Y$ which commutes with restriction, thus giving us a morphism of sheaves. Moreover, the compatible maps generated by φ are exactly $\{\varphi_{U,V}\}$, and thus we have a bijection. The naturality of the bijection is easy to check. $\square$

1.2 Ringed Spaces and Sheaves of Modules

Ringed spaces play a big role in modern geometry. Indeed, many familiar geometric objects such as varieties and manifolds are all ringed spaces. Schemes, as we will see later, are also ringed spaces. In this section, we will develop the theory of ringed spaces in preparation for scheme theory, but prove the results in the more general setting.

In particular, we are interested in sheaves of modules, as they can encode the information of an underlying space in very useful ways. Two particularly important classes of sheaves, known as coherent and quasicoherent sheaves, are a generalisation of vector bundles, and their role in algebraic geometry cannot be overstated. We will motivate them by first studying vector bundles, before giving a few properties.

We begin with the definition of a ringed space, and a few examples.

Definition 1.2.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$, called the *structure sheaf.* A *morphism* of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$ consisting of a continuous map $f : X \to Y$ and a morphism of sheaves $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that the stalk at every point is a local ring. A *morphism* of locally ringed spaces is a morphism of ringed spaces $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_X)$ such that the induced morphism $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism for every $P \in X$; that is, the preimage of the unique maximal ideal of $\mathcal{O}_{X,P}$ is the unique maximal ideal of $\mathcal{O}_{Y,f(P)}$ for every $P \in X$.

Of course, not every ringed space is a locally ringed space, for example any topological space equipped with the constant sheaf $\mathbb{Z}$ is a ringed space but not a locally ringed space. However, many interesting familiar geometric examples are in fact locally ringed spaces. We will study some.

Example 1.2.2. A classical variety X over a field k with its sheaf of regular functions $\mathcal{O}_X$ is an example of a locally ringed space. Recall that a morphism of varieties is a continuous function $\varphi : X \to Y$ such that given an open subset $V \subseteq Y$ and a regular function $f : V \to k$, the composition $f \circ \varphi : f^{-1}(V) \to k$ is regular [6, p.15]. It is easily checked that the induced map on sheaves, and by extension stalks is a local homomorphism, making a morphism of varieties a morphism of locally ringed spaces.

Example 1.2.3. For a smooth manifold M, the sheaf $\mathcal{O}_M$ of C^∞ real-valued functions forms a structure sheaf, turning the pair $(M, \mathcal{O}_M)$ into a ringed space. In fact, it is also a locally ringed space; the unique maximal ideal of the stalk $\mathcal{O}_{M,P}$ consists of the functions which vanish at P. A smooth map $f : M \to N$ induces a morphism of sheaves $\mathcal{O}_N, \to f_* \mathcal{O}_M$ where each section $s \in \mathcal{O}_N(U)$ is mapped to the section $s \circ f$. In fact, this is a local homomorphism, since $s \circ f$ vanishes at P if and only if s vanishes at $f(P)$.

We now develop the theory of sheaves of modules. We will begin with a few basic definitions and results, and then we will look at smooth vector bundles from a sheaf theoretic point of view. In particular, we will show that smooth vector bundles are the same thing as locally free sheaves (Definition 1.2.15). This will motivate the definition of coherent and quasicoherent sheaves, which, instead of locally free, are locally the cokernel of free sheaves.

Definition 1.2.4. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of* $\mathcal{O}_X$*-modules*, or simply an $\mathcal{O}_X$*-module*, is a sheaf of abelian groups $\mathcal{F}$ such that for every open subset $U \subseteq X$, the sections $\mathcal{F}(U)$ over U form an $\mathcal{O}_X(U)$-module, and for every inclusion of open sets $V \subseteq U$, the map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. More precisely, given $a \in \mathcal{O}_X(U), x \in \mathcal{F}(U)$, we have $(ax)|_V = a|_V x|_V$. A *morphism* of $\mathcal{O}_X$-modules is a morphism of sheaves that is compatible with the module structure.

A special case would be a *sheaf of ideals*, in which the module is in fact an ideal.

Proposition 1.2.5. *Let* $(X, \mathcal{O}_X)$ *be a ringed space and* $\mathcal{F}$ *a presheaf of* $\mathcal{O}_X$*-modules (that is, each* $\mathcal{F}(U)$ *is a module over* $\mathcal{O}_X(U)$*, and the restriction maps of* $\mathcal{F}$ *are compatible with the retriction maps of* $\mathcal{O}_X$*). Then* $\mathcal{F}^\sharp$*, the sheaf associated to* $\mathcal{F}$ *is an* $\mathcal{O}_X$*-module.*

Proof. This follows from the construction of sheafification. Indeed, each section of $\mathcal{F}^\sharp$ is a function that maps a point to a germ, which is locally induced. We can impose an $\mathcal{O}_X$-module structure on $\mathcal{F}^\sharp$ by defining multiplication pointwise on the functions. $\square$

Corollary 1.2.6. *The kernel, cokernel, image of a morphism of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module. Any quotient sheaf of an $\mathcal{O}_X$-module is an $\mathcal{O}_X$-module.*

Definition 1.2.7. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups. We define their *direct sum*, denoted $\mathcal{F} \oplus \mathcal{G}$ to be the sheaf $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$. It is easily seen to be a sheaf. If they are $\mathcal{O}_X$-modules, we define their *tensor product over* $\mathcal{O}_X$, denoted $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

By the above corollary, it is also an $\mathcal{O}_X$-module.

We will now define the pullback and pushforward of an $\mathcal{O}_X$-module with respect to a continuous map.

Definition 1.2.8. Let $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{G}$ an $\mathcal{O}_Y$-module. Then the *pushforward* of $\mathcal{F}$ is $f_*\mathcal{F}$, the same as the pushforward of $\mathcal{F}$ as a sheaf of abelian groups. Note that $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module and hence an $\mathcal{O}_Y$-module via the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. However, while $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$ module, it is not an $\mathcal{O}_X$-module. However, we do have a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ through the adjunction, and thus we can define the *pullback* of $\mathcal{F}$ to be

$$f^*(\mathcal{G}) := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

which is then an $\mathcal{O}_X$-module. As in Theorem 1.1.27, one can show that f^* is left adjoint to f_*.

Example 1.2.9. The structure sheaf on any ringed space $(X, \mathcal{O}_X)$ is an $\mathcal{O}_X$-module.

Example 1.2.10. Let M be a smooth manifold and $\mathcal{O}_M$ be its sheaf of smooth functions. Then the tangent bundle is an $\mathcal{O}_M$-module. Indeed, given any section s over an open subset U, we have a natural module structure through pointwise multiplication by a smooth function.

Example 1.2.10 paves the way to a very important phenomenon, which we will now study in greater generality. We will take a brief detour into the world of differential geometry to explore this. Note that the tangent bundle in Example 1.2.10 is a special case of a more general object known as a smooth vector bundle. We recall its definition:

Definition 1.2.11. A *smooth vector bundle*, or just *bundle* over a smooth manifold M, is a smooth manifold E equipped with a surjective map $\pi : E \to M$ such that the following hold

1. For every $p \in M$, the preimage $E_p := \pi^{-1}(p)$, known as the *fibre* over p is an n-dimensional real vector space for some fixed n, which we call the *rank* of E.

2. For every $p \in M$, there exists an open neighbourhood U and a diffeomorphism (known as the *local trivialisation*) $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, leftrightarrow, "\Phi"] \arrow[d, "\pi"] & U \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ U & \end{tikzcd}$$

and for every $q \in U$, the restriction of Φ to E_q is a linear isomorphism from E_q to $\{q\} \times \mathbb{R}^n \cong \mathbb{R}^n$.

A *section* of the bundle over an open subset $U \subseteq M$ is a smooth function $\varphi : U \to E$ such that $\pi \circ \varphi = \mathrm{id}$.

Example 1.2.12. A cylinder (of infinite height) and a Möbius strip are examples of a bundle over the circle. The details will take us too far afield, but they can be found in [8, p.251].

Proposition 1.2.13. *Let M be a smooth manifold, $\mathcal{O}_M$ its sheaf of smooth real-valued functions and E a smooth vector bundle. Then the sections of E on each open subset form an $\mathcal{O}_M$-module.*

Proof. The module structure follows from pointwise multiplication by a smooth function. Clearly, the product of a smooth function and a smooth section (of the bundle) is a smooth section. The compatibility with restriction maps is obvious. Since smoothness is a local property, the sheaf axioms are true. $\square$

Definition 1.2.14. We call the sheaf in the above proposition the *sheaf associated to E*.

What is particularly interesting about smooth vector bundles from a sheaf-theoretic point of view is the local trivialisation. Indeed, this is saying that, locally, a smooth vector bundle looks like copies of the underlying space. The sheaf analogue is known as a *locally free sheaf*, which we will define below. In fact, we will show that locally free sheaves and smooth vector bundles are the “same thing”.

Definition 1.2.15. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *free* if $\mathcal{F} \cong \bigoplus_{i \in I} \mathcal{O}_X$ for some (possibly infinite) indexing set I. It is *locally free* if X can be covered by open subsets $\{U_i\}$ such that for each U_i there exists some indexing set J such that $\mathcal{F}|_{U_i} \cong \bigoplus_{j \in J} \mathcal{O}_X|_{U_i}$. For each U_i, we say that the *rank* of $\mathcal{F}$ is the cardinality of J if it is finite, and infinite otherwise. If the rank is the same for all such open subsets (for example if X is connected), we will simply refer to the rank of $\mathcal{F}$.

Example 1.2.16. Let M be any A-module. For any $r \in \mathbb{N}$ we define the *r-th exterior power* of M, denoted $\bigwedge^r M$ to be the free module generated by the symbols $x_1 \wedge ... \wedge x_r$ for $(x_1, ..., x_r) \in M^r$, subject to the following relations:

- $a(x_1 \wedge ... \wedge x_r) = (ax_1) \wedge ... \wedge x_r = ... = x_1 \wedge ... \wedge (ax_r)$
- $x_1 \wedge ... \wedge x_{i-1} \wedge (x_i + y_i) \wedge x_{i+1} \wedge ... \wedge x_r = (x_1 \wedge ... \wedge x_{i-1} \wedge x_i \wedge x_{i+1} \wedge ... \wedge x_r) + (x_1 \wedge ... \wedge x_{i-1} \wedge y_i \wedge x_{i+1} \wedge ... \wedge x_r)$

- $x_1 \wedge ... \wedge x_r = 0$ if $x_i = x_j$ and $i \neq j$.

A counting argument will show that if M is free of rank n, then $\bigwedge^r M$ is also free of rank $\binom{n}{r}$.

Now if $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module, then we define $\bigwedge^r \mathcal{F}$ to be the sheaf associated to the presheaf $U \mapsto \bigwedge^r \mathcal{F}(U)$. If $\mathcal{F}$ is locally free of rank r, then we $\bigwedge^r \mathcal{F}$ is free of rank $\binom{n}{r}$.

Theorem 1.2.17. *Let E be a smooth vector bundle of rank n over a smooth manifold M, and $\mathcal{F}$ the sheaf associated to E. Then $\mathcal{F}$ is locally free of rank n. Conversely, if $\mathcal{F}$ is a locally free sheaf of finite rank n, there exists a bundle E whose associated sheaf is isomorphic to $\mathcal{F}$.*

Proof. Let M be covered by $\{U_i\}$ such that E is locally trivial on each U_i. Now on each U_i, every section $s : U_i \to \pi^{-1}(U_i)$ can be identified with a map $s' : U_i \to \mathbb{R}^n$ through the local trivialisation and the identification of fibres $E_p \cong \{p\} \times \mathbb{R}^n$. Now since s is smooth, the component functions of s' are also smooth, and thus we can identify s with n smooth functions $s_j : U_i \to \mathbb{R}$ for $1 \leq j \leq n$, giving us an element of $\bigoplus_{j=1}^n \mathcal{O}_M(U)$. Conversely, an element of $\bigoplus_{j=1}^n \mathcal{O}_M(U)$ is simply a tuple of n smooth functions $s_j : U_i \to \mathbb{R}$, which gives us a smooth function $s' : U_i \to \mathbb{R}^n$, which we can identify with a unique $s : U_i \to \pi^{-1}(U_i)$. This shows that $\mathcal{F}$ is locally free of rank n.

Conversely, if $\mathcal{F}$ is locally free of finite rank n, we will proceed as follows. Let $\{U_i\}_{i\in I}$ be a cover of X such that $\mathcal{F}|_{U_i}$ is free. Now we take $E^\sharp := \bigsqcup_{i\in I} U_i \times \mathbb{R}^n$, with the product topology. Every element of $E^\sharp$ is of the form (i, P, x), where $i \in I$, $P \in U_i$ and $x \in \mathbb{R}^n$. Note that by the construction of sheafification in Theorem 1.1.14, we can interpret elements s of $\mathcal{F}(U_i)$ as functions $s : U_i \to \bigsqcup_{P\in U_i} \mathcal{F}_P$ where the mapping is given by $s : P \mapsto s_P$. Since $\mathcal{F}$ is locally free, $\mathcal{F}_P$ is naturally isomorphic to $(\mathcal{O}_{X,P})^n$, where $\mathcal{O}_{X,P}$ is the set of germs of smooth functions at P, and thus we have a natural map $(\mathcal{O}_{X,P})^n \cong \mathcal{F}_P \to \mathbb{R}^n$ defined by $s_P = (s_{1,P}, ..., s_{n,P}) \mapsto (s_{1,P}(P), ..., s_{n,P}(P))$. Composing this map with s itself, we may identify s with a natural map $s^\sharp : U_i \to E^\sharp$ given by $P \mapsto (i, P, s_{1,P}(P), ..., s_{n,P}(P)) \in E^\sharp$

We define the equivalence relation $\sim$ by $(i, P, x) \sim (j, Q, y)$ if $P = Q$ and if for every $s \in \mathcal{F}(U_i \cap U_j)$ the map $s^\sharp$ satisfies the property $s^\sharp(P) = x$ if and only if $s^\sharp(Q) = y$. Finally, we define E to be $E^\sharp / \sim$.

Now we will check that E is indeed a smooth vector bundle. To this end, take some $P \in M$, so that its fibre is $\pi^{-1}(P) = (\bigsqcup_i (i, P) \times \mathbb{R}^n)/ \sim$ where the disjoint union is taken over all indices i such that $P \in U_i$. It is easy to see that given i, j and any element (i, P, x) there exists a unique $x' \in \mathbb{R}^n$ such that $(i, P, x) \sim (j, P, x')$. This means that $\pi^{-1}(P)$ has a natural vector space structure. It remains to check local trivialisation. But this is obvious, indeed given $P \in M$ take any U_i such that $P \in U_i$ and by construction its preimage is isomorphic to $U_i \times \mathbb{R}^n$

Finally, we check that the sheaf associated to E is in fact isomorphic to $\mathcal{F}$. To this end, let $\mathcal{G}$ denote the sheaf of sections of E and fix some $U \subseteq M$ and some section $s \in \mathcal{F}(U)$. On each U_i such that $U_i \cap U \neq \emptyset$, the restriction $s|_{U_i \cap U}$ defines a smooth function $U_i \cap U \to \mathbb{R}^n$, and clearly this is compatible with restriction. Thus by the sheaf axioms, there exists some $s' \in \mathcal{G}(U)$ that defines the same smooth function, and thus we define the map $s \mapsto s'$. This gives us a morphism of sheaves. Since the induced map on stalks are clearly isomorphisms, this concludes the proof. $\square$

A natural generalisation then would be a sheaf that is not locally free, but locally presentable, by which we mean the following:

Definition 1.2.18. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *quasicoherent* if X can be covered by open subsets $\{U_i\}$ such that for every U_i there exists an exact sequence

$$\bigoplus_I \mathcal{O}_X|_{U_i} \to \bigoplus_J \mathcal{O}_X|_{U_i} \to \mathcal{F}|_{U_i} \to 0$$

and the sets I and J depend on i. If we require the I and J to be finite, then $\mathcal{F}$ is *coherent*.

Proposition 1.2.19. *Let $\mathcal{F}$ and $\mathcal{G}$ be quasicoherent (resp. coherent) sheaves. Then $\mathcal{F} \oplus \mathcal{G}$ is quasicoherent (resp. coherent).*

Proof. Choosing a fine enough cover, we may assume that $\mathcal{F}$ and $\mathcal{G}$ are globally presentable. Thus we have presentations

$$\bigoplus_{j \in J_1} \mathcal{O}_X \to \bigoplus_{i \in I_1} \mathcal{O}_X \to \mathcal{F} \to 0$$

and

$$\bigoplus_{j \in J_2} \mathcal{O}_X \to \bigoplus_{i \in I_2} \mathcal{O}_X \to \mathcal{G} \to 0$$

Then it is obvious that

$$\bigoplus_{j \in J_1 \sqcup J_2} \mathcal{O}_X \to \bigoplus_{i \in I_1 \sqcup I_2} \mathcal{O}_X \to \mathcal{F} \oplus \mathcal{G} \to 0$$

is exact. Running the same argument through and assuming the I_1, I_2 and J_1, J_2 are finite yields the result for coherent sheaves. $\square$

We conclude this section with a discussion about the dual sheaf of a locally free sheaf. A bit of notation: given sheaves $\mathcal{F}$ and $\mathcal{G}$, we denote $\mathfrak{Hom}(\mathcal{F}, \mathcal{G})$ to be the sheaf $U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ (it is easily checked to be a sheaf). If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, we similarly define $\mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

Definition 1.2.20. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ a locally free $\mathcal{O}_X$-module of finite rank n. We define the *dual sheaf* of $\mathcal{F}$, denoted $\mathcal{F}^\vee$ to be the sheaf $\mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

Proposition 1.2.21. *For any $\mathcal{O}_X$-module $\mathcal{F}$ we have $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$.*

Proof. We define the morphism $\varphi : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ as follows: given an open set U and some $s \in \mathcal{F}(U)$ we define $\varphi_U(s)$ to be the map $\mathrm{ev}_s : \mathcal{F}^\vee|_U \to \mathcal{O}_X|_U$; that is given $V \subseteq U$ and $t \in \mathcal{F}^\vee(V)$ we define $\mathrm{ev}_{s,V}(t^\vee)$ to be the section $t_V(s|_V)$. It is easily checked that this is a morphism of sheaves. To check that this is an isomorphism, we observe that at a given point the induced map of stalks is simply the canonical inclusion $\mathcal{F}_P \to (\mathcal{F}_P^\vee)^\vee$. Since $\mathcal{F}$ is locally free of rank n, this corresponds to the canonical inclusion $\mathcal{O}_{X,P}^n \to ((\mathcal{O}_{X,P}^n)^\vee)^\vee$, which is an isomorphism. $\square$

Proposition 1.2.22. *For any $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ we have $\mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \cong \mathcal{F}^\vee \otimes \mathcal{G}$. In particular, $\mathcal{F}^\vee \otimes \mathcal{F} \cong \mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$.*

Proof. We define the morphism $\varphi : \mathcal{F}^\vee \otimes \mathcal{G} \to \mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ as follows. Given an open set U and some $s^\vee \otimes t \in \mathrm{Hom}_{\mathcal{O}_{X,U}}(\mathcal{F}|_U, \mathcal{O}_X|_U) \otimes \mathcal{G}(U)$, we define $\varphi_U^\flat(s^\vee \otimes t) : \mathcal{F}|_U \to \mathcal{G}|_U$ to be the morphism given by $x \mapsto s_V^\vee(x)t|_V$ for any open $V \subseteq U$ and $x \in \mathcal{F}(V)$. As U varies, this defines a morphism of presheaves $\varphi^\flat$. By the universal property of sheafification (Theorem 1.1.14), this induces a morphism of sheaves $(\varphi^\flat)^\sharp : \mathcal{F}^\vee \otimes \mathcal{G} \to \mathfrak{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$. We define $\varphi := (\varphi^\flat)^\sharp$. To check that this is an isomorphism, we simply note that at a point P the induced map of stalks is simply the natural isomorphism $\mathcal{F}_P^\vee \otimes \mathcal{G}_P \to \mathrm{Hom}_{\mathcal{O}_{X,P}}(\mathcal{F}_P, \mathcal{G}_P)$. $\square$

1.3 Sheaf Cohomology

Recall (Proposition 1.1.20) that the sections functor $\Gamma_U(\cdot)$ is left-exact but not exact. In practice, this presents a difficulty when computing dimensions of global sections. It would therefore be convenient if there was a way to measure this obstruction. More precisely, given a short exact sequence of sheaves,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

we would like a long exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{G}) \longrightarrow H^0(X,\mathcal{H}) \\ \hookrightarrow H^1(X,\mathcal{F}) \longrightarrow H^1(X,\mathcal{G}) \longrightarrow H^1(X,\mathcal{H}) \ldots \end{array} \tag{1.1}$$

of abelian groups where $H^0(X,\mathcal{F}) = \Gamma_X(\mathcal{F})$, and there exists some $n \in \mathbb{N}$ such that $H^i(X,\mathcal{F}) = 0$ for all $i \geq n$. Fortunately, such a sequence does exist, and the $H^i(X,\mathcal{F})$ are called the *cohomology groups of* $\mathcal{F}$. We dedicate the remainder of this section to their construction.

The way we will construct these groups is Grothendieck's derived functor approach, which is the most general and can be generalised to construct other cohomology theories, for example étale cohomology. We will work over an arbitrary abelian category (see [5]), and by the Freyd-Mitchell embedding theorem [3, Thm 7.14] we may assume without loss of generality that the abelian category is a subcategory of the category of modules over some ring. In particular, this means that we may "diagram-chase" in our proofs. An **object** in this section will always mean an object in an abelian category.

Definition 1.3.1. Let A and B be abelian categories. A functor $F : \mathsf{A} \to \mathsf{B}$ is said to be *additive* if for every hom-set $\mathrm{Hom}_{\mathsf{A}}(A,B)$ in A the induced map $\mathrm{Hom}_{\mathsf{A}}(A,B) \to \mathrm{Hom}_{\mathsf{B}}(F(A),F(B))$ is a homomorphism of abelian groups.

Proposition 1.3.2. *Suppose $F : \mathsf{A} \to \mathsf{B}$ is an additive functor. Then for any objects A and B in A, we have*

$$F(A \oplus B) = F(A) \oplus F(B)$$

.

Proof. [9, p.197] □

Definition 1.3.3. A *(co-chain) complex* $A^\bullet$ is a collection of objects A^i for every integer i as well as maps $d_A^i : A^i \to A^{i+1}$ called *co-boundary* maps, with the property that $d_A^{i+1} \circ d_A^i = 0$ for all i. If there is no ambiguity, we will write d^i in place of d_A^i. For each i, we define the *i-th cohomology group* $H^i(A^\bullet)$ to be $\ker d^i / \operatorname{im} d^{i-1}$; this makes sense by the previous condition. If we only specify objects or co-boundary maps for some i, the rest are assumed to be 0. A *morphism* of complexes $f^\bullet : A^\bullet \to B^\bullet$ is a collection of morphisms $f^i : A^i \to B^i$ that commute with the co-boundary maps.

Note that a morphism of complexes $f^\bullet : A^\bullet \to B^\bullet$ naturally induces morphisms between the cohomology groups $H^i(f^\bullet) : H^i(A^\bullet) \to H^i(B^\bullet)$ given by $\overline{x} \mapsto \overline{f^i(x)}$, where $\overline{x}$ is the image of $x \in \ker d_A^i$ in $H^i(A^\bullet)$ and $\overline{f^i(x)}$ is the image of $f^i(x) \in \ker d_B^i$ in $H^i(B^\bullet)$. It is easily checked to be well-defined.

Theorem 1.3.4. *Given a short exact sequence of complexes*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

There exists a long exact sequence of cohomology groups

$$\begin{tikzcd} 0 \arrow[r] & H^0(A^\bullet) \arrow[r] & H^0(B^\bullet) \arrow[r] & H^0(C^\bullet) \arrow[dll, rounded corners] \\ & H^1(A^\bullet) \arrow[r] & H^1(B^\bullet) \arrow[r] & H^1(C^\bullet) \dots \end{tikzcd}$$

Proof. First consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & A^0 \arrow[r] \arrow[d, "a^0"] & B^0 \arrow[r] \arrow[d, "b^0"] & C^0 \arrow[r] \arrow[d, "c^0"] & 0 \\ 0 \arrow[r] & A^1 \arrow[r] \arrow[d, "a^1"] & B^1 \arrow[r] \arrow[d, "b^1"] & C^1 \arrow[r] \arrow[d, "c^1"] & 0 \\ 0 \arrow[r] & A^2 \arrow[r] & B^2 \arrow[r] & C^2 \arrow[r] & 0 \end{tikzcd}$$

Now the sequence

$$0 \to \ker a^0 \to \ker b^0 \to \ker c^0 \to \operatorname{coker} a^0 \to \operatorname{coker} b^0 \to \operatorname{coker} c^0$$

is exact by the Snake Lemma (Theorem $A.0.2$). Now since the 0th cohomology groups are simply the kernels of the 0th maps and since the 0th maps map into the kernel of the first map, we can restrict the cokernels of the 0th maps to the kernels of the first map while retaining exactness to get the exact sequence:

$$0 \to H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet) \to H^1(A^\bullet) \to H^1(B^\bullet) \to H^1(C^\bullet)$$

Now applying the Snake Lemma at some $i \geq 1$ we have an exact sequence:

$$\ker a^i \to \ker b^i \to \ker c^i \to \operatorname{coker} a^i \to \operatorname{coker} b^i \to \operatorname{coker} c^i$$

Then clearly the following sequence is also exact:

$$H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet)$$

Now it remains to check that $H^i(B^\bullet) \to H^i(C^\bullet) \to \operatorname{coker} a^i$ is still exact, but this follows from the surjectivity of $B^{i-1} \to C^{i-1}$ and the exactness of the original maps, and once again restricting the cokernel we get the exact sequence:

$$H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet) \to H^{i+1}(C^\bullet)$$

And thus by induction, we have the desired long exact sequence of cohomology groups. □

Definition 1.3.5. An object I is *injective* if if given any injective map $i : A \to B$ and morphism $\varphi : A \to I$ there exists a (not necessarily unique) $\varphi^\sharp : B \to I$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\varphi"] & B \arrow[dl, "\varphi^\sharp"] \\ I & \end{tikzcd}$$

Before we state the next result, recall that $\operatorname{Hom}(\cdot, A)$ is a contravariant left-exact functor.

Proposition 1.3.6. *Let I be an object. Then the following are equivalent:*

(a) I is injective;

(b) The functor $\operatorname{Hom}(\cdot, I)$ is exact;

(c) Every short exact sequence of the form

$$0 \longrightarrow I \xrightarrow{f} A \longrightarrow B \longrightarrow 0 \tag{1.2}$$

splits.

Proof. We begin with $(c) \Rightarrow (a)$, which is the hardest direction. The following proof is based on the one found in [7, p.17]. Suppose we have an inclusion $i : A \to B$ and a morphism $\varphi : A \to I$. Now we have a map $A \to I \oplus B$ given by $a \mapsto (\varphi(a), -a)$. Let M denote the cokernel of this map, hence we have the following exact sequence:

$$A \longrightarrow I \oplus B \xrightarrow{\pi} M \longrightarrow 0$$

We claim $\pi|_I$ is injective. Indeed, if $\pi(x, 0) = 0$ then $(x, 0) = (\varphi(a), -a)$ for some $a \in A$, which means $a = 0$. Thus we have the following short exact sequence, which splits by assumption:

$$0 \longrightarrow I \overset{\pi|_I}{\longrightarrow} M \longrightarrow \operatorname{coker} \pi|_I \longrightarrow 0$$

Hence there exists a map $M \to I$ such that $I \overset{\pi|_I}{\to} M \to I = \mathrm{id}$. We claim that $B \overset{\pi|_B}{\to} M \to I$ is our desired map. Indeed, if $a \in A$, then

$$\pi(0, a) = \pi(0, a) + \pi(\varphi(a), -a) = \pi(\varphi(a), 0) \mapsto \varphi(a)$$

as required.

Next we establish $(b) \Rightarrow (c)$. We apply $\mathrm{Hom}(\cdot, I)$ to 1.2 to obtain the following sequence:

$$0 \longrightarrow \mathrm{Hom}(B, I) \longrightarrow \mathrm{Hom}(A, I) \overset{f^*}{\longrightarrow} \mathrm{Hom}(I, I) \longrightarrow 0$$

In particular, there exists a $g : A \to I$ such that $g \circ f = \mathrm{id}$. We construct a map $B \to A$ as follows: given $b \in B$, we define its image to be the unique x such that $f(x) = b$ and $x \in \ker g$. To see that such an element exists, we take any $b' \in A$ that maps to b and define $x := b' - f \circ g(b')$. To see that this element is unique, suppose $x' \in \ker g$ also satisfies $f(x') = b$. Then $x - x' \in \operatorname{im} f \cap \ker g$, which is easily seen to be 0. Thus we have have a map $I \oplus B \to A$, and this map is easily seen to be the unique map which commutes with g and $A \to B$, and thus A satisfies the required universal property.

The equivalence $(a) \Leftrightarrow (b)$ follows directly from the definitions. □

Example 1.3.7. An abelian group G (written additively) is said to be *divisible* if for every $y \in G$ and $n \in \mathbb{N}$ there exists $x \in G$ such that $nx = y$. We claim that G is injective if and only if it is divisible. Indeed, if G is injective, then given $n \in \mathbb{N}$ and $y \in G$, we define $\varphi : \mathbb{Z} \to G$ given by $1 \mapsto y$, and $i : \mathbb{Z} \to \mathbb{Q}$ to be the natural embedding. Since G is injective, the map φ lifts to $\varphi^\sharp : \mathbb{Q} \to G$, and the element $x := \varphi(1/n)$ has the desired property.

Conversely, if $i : A \to B$ is injective, and we have a map $\varphi : A \to G$, a Zorn's Lemma argument applied to pairs $(C, \varphi^\sharp_C)$ where $C \subseteq B$ is a subgroup containing A and $\varphi^\sharp_C : C \to I$ is a map such that $\varphi^\sharp_C \circ i = \varphi$, will yield the result.

Definition 1.3.8. Let A be an object. An *injective resolution* $A \to I^\bullet$ of A consists of a complex $I^\bullet$ such that each I^i is injective and an inclusion $A \to I^0$ such that the following sequence is exact:

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots$$

Example 1.3.9. An injective resolution of $\mathbb{Z}$ is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

An abelian category A has *enough injectives* if every object A can be embedded in an injective object. If A has enough injectives, then it is easy to see that every object has an injective resolution. Indeed, we can define I^0 to be some injective object into which A embeds, and I^1 to be some injective object into which I^0/A embeds and having defined I^i and I^{i-1} for $i \geq 1$, we define I^{i+1} to be an injective object into which I^i/I^{i-1} embeds.

Our further constructions will involve choosing and manipulating an injective resolution for each object, so naturally we have to check that these do not depend on the injective resolution we choose. To do this, we introduce the idea of a *homotopy of complexes*. Let $A^\bullet$ and $B^\bullet$ be two complexes with co-boundary maps d_A^i and d_B^i respectively for each i. We say two morphisms $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ are *homotopic* if there exists a collection $\Delta^i : A^i \to B^{i-1}$ of morphisms for each i (that do not necessarily commute with the co-boundary maps) such that $f^i - g^i = d_B^{i-1} \circ \Delta^i + \Delta^{i+1} \circ d_B^i$ for each i. Diagramatically:

$$\begin{tikzcd} & A^i \arrow[r, "d_A^i"] \arrow[dl, "\Delta^i"'] \arrow[d, "f^i - g^i"] & A^{i+1} \arrow[dl, "\Delta^{i+1}"] \\ B^{i-1} \arrow[r, "d_B^{i-1}"'] & B^i & \end{tikzcd}$$

It is readily seen that if f and g are homotopic, then they induce the same map on cohomology. Now we introduce a fundamental lemma, which will ensure that our further constructions are well-defined:

Lemma 1.3.10. *Let $A \to I^\bullet$ and $B \to J^\bullet$ be two injective resolutions and $f : A \to B$ a morphism. Then f induces a morphism of complexes $f^\bullet : I^\bullet \to J^\bullet$ which is unique up to homotopy.*

Proof. The proof for the dual result can be found at [13, p.50]. Applying the same argument with relevant arrows reversed yields this result. □

For the remainder of this section, F will be an additive left-exact covariant functor from an abelian category A with enough injectives into another such abelian category B.

Definition 1.3.11. We define the *right derived functors* of F as follows: for each object A we choose an injective resolution $A \to I^\bullet$. Then we apply F to our resolution and remove $F(A)$ to get a complex $F(I^\bullet)$ (which is not necessarily still exact), and then we define the right derived functors $R^iF(A) := H^i(F(I^\bullet))$. By the above lemma, any two injective resolutions will define canonically isomorphic right derived functors.

Theorem 1.3.12. *There is a natural isomorphism $F(A) \cong R^0F(A)$.*

Proof. By the left-exactness of F the following sequence is still exact:

$$0 \longrightarrow F(A) \longrightarrow F(I^0) \xrightarrow{F(d^0)} F(I^1)$$

Thus $\varepsilon : F(A) \to F(I^0)$ is injective and $\ker F(d^0) = \operatorname{im} \varepsilon \cong F(A)$. Hence $R^0F(A) = H^0(I^\bullet) = \ker F(d^0)/0 \cong F(A)$ as required. □

Theorem 1.3.13. *For any short exact sequence:*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

There exists a long exact sequence:

$$\begin{array}{l} 0 \longrightarrow R^0F(A) \longrightarrow R^0F(B) \longrightarrow R^0F(C) \longrightarrow \\ \hookrightarrow R^1F(A) \longrightarrow R^1F(B) \longrightarrow R^1F(C) \ldots \end{array}$$

Proof. We select injective resolutions $A \to I^\bullet$ and $C \to J^\bullet$ for A and C. Applying the Horseshoe Lemma Lemma $A.0.3$, we get an injective resolution $B \to (I \oplus J)^\bullet$ of B. In particular, we have a short exact sequence of complexes that splits.

$$0 \longrightarrow I^\bullet \longrightarrow I^\bullet \oplus J^\bullet \longrightarrow J^\bullet \longrightarrow 0$$

Applying F now, Proposition 1.3.2 implies that we have a short exact sequence of complexes:

$$0 \longrightarrow FI^\bullet \longrightarrow FI^\bullet \oplus FJ^\bullet \longrightarrow FJ^\bullet \longrightarrow 0$$

The Snake Lemma then implies the result. ☐

We will now show that this construction is unique in some way. We require a definition:

Definition 1.3.14. Let A and B be abelian categories. A *covariant δ-functor* from A to B consists of a collection of functors $T = (T^i)_{i \geq 0}$ for each nonnegative integer i such that given any short exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{1.3}$$

and $i \geq 0$ there exists a morphism $\delta^i : T^i(C) \to T^{i+1}(A)$ such that the resulting sequence is exact

$$\begin{array}{l} 0 \longrightarrow T^0(A) \longrightarrow T^0(B) \longrightarrow T^0(C) \longrightarrow \\ \xhookrightarrow{\delta^0} T^1(A) \longrightarrow T^1(B) \longrightarrow T^1(C) \ldots \end{array}$$

and given a morphism of the sequence 1.3 into another sequence,

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

the following square commutes:

$$\begin{array}{ccc} T^i(C) & \xrightarrow{\delta^i} & T^{i+1}(A) \\ \downarrow & & \downarrow \\ T^i(C') & \xrightarrow{\delta^i} & T^{i+1}(A') \end{array}$$

Proposition 1.3.15. *The collection of right derived functors is a covariant δ-functor.*

Proof Sketch. Note that we have already proved the existence of the long exact sequence, so it suffices to show that the square commutes. To this end, we simply chase through the following diagram:

$$\begin{tikzcd}
0 \arrow[rr] & & A \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[rr] \arrow[dr] \arrow[dd] & & C \arrow[rr] \arrow[dr] \arrow[dd] & & 0 & \\
& 0 \arrow[rr] & & A' \arrow[rr] \arrow[dd] & & B' \arrow[rr] \arrow[dd] & & C' \arrow[rr] \arrow[dd] & & 0 \\
0 \arrow[rr] & & I^0 \arrow[rr] \arrow[dr] & & I^0 \oplus J^0 \arrow[rr] \arrow[dr] & & J^0 \arrow[rr] \arrow[dr] & & 0 & \\
& 0 \arrow[rr] & & I^{0'} \arrow[rr] & & I^{0'} \oplus J^{0'} \arrow[rr] & & J^{0'} \arrow[rr] & & 0
\end{tikzcd}$$

□

Definition 1.3.16. A δ-functor T from A to B is *universal* if given another δ-functor T', and a natural transformation $f^0 : T^0 \to T^{0'}$, there exists a unique collection of natural transformations $f^i : T^i \to T^{i'}$ for each $i > 0$ which commute with the δ^{i-1}.

Note that if we fix T^0 (for example requiring $F = R^0(F)$), then if a universal δ-functor exists, then it is unique up to isomorphism.

We present a striking theorem of Grothendieck:

Theorem 1.3.17 (Grothendieck)**.** *Let $T = (T^i)_{i \geq 0}$ be a δ-functor from A into B. If for every object A in A and $i > 0$ there exists a monomorphism $u : A \to M$ such that $T^i(u) = 0$ then T is universal.*

Proof. [5, p.141] □

Corollary 1.3.18. *The right derived functors R^iF are universal.*

Proof. Fix an object A, and take a monomorphism into an injective object $u : A \to I$. Note that

$$0 \longrightarrow I \longrightarrow I \longrightarrow 0$$

is an injective resolution of I. In particular, $R^iF(I) = 0$ for all $i > 0$. Thus $F(u) = 0$. Then Theorem 1.3.17 implies the result. □

Finally, we will define the cohomology groups for sheaves. Let $(X, \mathcal{O}_X)$ be a ringed space. Note that the category of $\mathcal{O}_X$-modules is naturally abelian.

Theorem 1.3.19. *The category of $\mathcal{O}_X$-modules has enough injectives. In particular, if we take $\mathcal{O}_X$ to be the constant sheaf $\mathbb{Z}$, then the category of sheaves of abelian groups on X has enough injectives.*

Proof. [6, p, 207] □

Hence we see that if we define $H^i(X, \mathcal{F})$ to be the right derived functors of Γ_X, then we will have the desired long exact sequence of cohomology groups. We conclude this section with the statement of Grothendieck's Vanishing Theorem. Recall that the *Noetherian dimension* of a topological space X is the maximum natural number n such that there is a chain of irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$$

If no such natural number exists, then the dimension is taken to be ∞.

Theorem 1.3.20. *[5, Théorème 3.6.5] Let X be a Noetherian topological space of Noetherian dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X we have $H^i(X, \mathcal{F}) = 0$.*

Chapter 2

Schemes

2.1 Definitions and Properties of Schemes

Definition 2.1.1. Let A be a ring. Then the *spectrum* of A, written $\operatorname{Spec} A$ is the set of prime ideals of A.

Let I be a subset of A. We write $\mathcal{V}(I)$ to be the set of prime ideals $\mathfrak{p}$ of $\operatorname{Spec} A$ such that $\mathfrak{p} \supseteq \langle I \rangle$.

Lemma 2.1.2. *Let A be a ring. Then the following hold:*

(i) $\mathcal{V}(0) = \operatorname{Spec} A, \mathcal{V}(1) = \emptyset$

(ii) If I and J are two ideals, then $\mathcal{V}(IJ) = \mathcal{V}(I) \cup V(J)$

(iii) If $\{I_\alpha\}$ is a family of ideals, then $\mathcal{V}(\sum I_\alpha) = \cap_\alpha(\mathcal{V}(I_\alpha))$

Proof. [6, p.70] □

It follows from these two observations that the sets $\mathcal{V}(I)$ taken across all ideals I form the closed sets of a topology on $\operatorname{Spec} A$. We call this topology the *Zariski topology*.

Definition 2.1.3. Let f be an element of A. The *distinguished open subset* generated by f, written $D(f)$ is the set $\operatorname{Spec} A \setminus \mathcal{V}(f)$

Proposition 2.1.4. *The distinguished open subsets form a base of the Zariski Topology on* $\operatorname{Spec} A$.

Proof. Let $U = X \setminus \mathcal{V}(I)$ be an open subset of $\operatorname{Spec} A$. Note that

$$I = \sum_{f \in I} \langle f \rangle$$

and by Lemma 2.1.2 we have

$$\mathcal{V}(I) = \bigcap_{f \in I} \mathcal{V}(f)$$

Taking complements and applying de Morgan's laws will yield the result. □

We now define a sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} A$ to make a ringed space. We proceed as follows: for each $D(f)$, we define $\mathcal{O}(D(f)) := A[f^{-1}]$, the localisation of A at the multiplicative set $\langle f \rangle$. Note that $D(f) \cap D(g) = D(fg)$ and thus we have a natural restriction map between base open subsets.

In particular, observe that the global section is isomorphic to A. We can check the sheaf axioms are satisfied (for example [2, pp.19-20]) and thus by Lemma 1.1.11 we have a sheaf. By abuse of notation, we will simply denote the pair $(\operatorname{Spec} A, \mathcal{O})$ as just $\operatorname{Spec} A$. It is a ringed space; in fact a locally ringed space:

Proposition 2.1.5. *Let* $\operatorname{Spec} A$ *be the spectrum of some ring* A*. Then for any* $\mathfrak{p} \in \operatorname{Spec} A$*, the stalk at* $\mathfrak{p}$ *is isomorphic to* $A_{\mathfrak{p}}$*.*

Proof. [6, p.71] □

We now come to the definition of a scheme:

Definition 2.1.6. An *affine scheme* is a locally ringed space that is isomorphic to the spectrum of some ring. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ that can be covered by open sets $\{U_\alpha\}$ such that for each U_α the pair $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is an affine scheme. Such a U_α is known as an *open affine subset*. A *morphism* of schemes is a morphism of locally ringed spaces between schemes.

Example 2.1.7. Let $X = \operatorname{Spec} \mathbb{Z}$. Then the elements of X are the ideals generated by prime numbers, and the zero ideal. Since $\mathbb{Z}$ is a PID, every open set is a distinguished open set, and on the open set $D(n)$, we have $\mathcal{O}_X(D(n)) = \mathbb{Z}[n^{-1}]$. Note that X does not have an analogy in the category of classical varieties.

Example 2.1.8. Let A denote the zero ring and let $X = \operatorname{Spec} A$. Since A has no prime ideals, X does not contain any points. The sheaf of rings is therefore empty too. Conversely, if X is a scheme and $\mathcal{O}_X(U)$ is the zero ring, then for any open affine subset V we have $\mathcal{O}_X(V) = 0$ and hence V is empty, and thus U is empty too.

Let k be an algebraically closed field. Then any affine variety V over k has a natural associated scheme, $\operatorname{Spec} k[V]$, where $k[V]$ is its co-ordinate ring. Note firstly that V is $\mathbf{T_1}$, whereas $\operatorname{Spec} k[V]$ is not always, for example, if V is the affine plane, then any prime ideal of $k[V] \cong k[x, y]$ that is not maximal will not be closed. (In fact, the point corresponding to the zero ideal is dense! Such a dense point is known as a *generic point*). However, the set of closed points of $\operatorname{Spec} k[V]$ is homeomorphic to V, and the sheaf of regular functions on V pushed forward via the inclusion is isomorphic to $\mathcal{O}_{\operatorname{Spec} k[V]}$. We will study another subtle difference below.

Example 2.1.9. As varieties, the equations $x = 0$ and $x^2 = 0$ define the same variety in $\mathbb{A}^1(k)$. Indeed, they both correspond to the point 0. However, as schemes, we observe that $X := \operatorname{Spec} k[x]/\langle x \rangle$ and $Y := \operatorname{Spec} k[x]/\langle x^2 \rangle$ are different. While topologically they are identical, consisting of only one point, we observe that the space of global sections (which in this occasion is also the local ring) of X is isomorphic to k whereas the space of global sections of Y is a two-dimensional k-algebra. This shows how schemes can retain information that is lost in the case of varieties.

Of course, not every scheme is affine, see for example Example 3.1.12.

Proposition 2.1.10. *Let* $\operatorname{Spec} A$ *be an affine scheme and* $g \in A$*. Then* $D(g)$ *with the restricted sheaf is isomorphic to* $\operatorname{Spec} A[g^{-1}]$*.*

Proof. The set $D(g)$ is the set of prime ideals that do not contain any power of g and are in one-to-one correspondence with the prime ideals of $A[g^{-1}]$. This defines a map $f : D(g) \to \operatorname{Spec} A[g^{-1}]$ which is clearly a homeomorphism. We now define a map $f^{\#} : \mathcal{O}_{\operatorname{Spec} A[g^{-1}]} \to f_* \mathcal{O}_{\operatorname{Spec} A[g^{-1}]}|_{D(g)}$. Indeed, given any base open set $D(h) \in \operatorname{Spec} A$, the corresponding base open set in $D(g)$ is $D(gh)$, and we have $\mathcal{O}_{\operatorname{Spec} A}(D(gh)) = A[(gh)^{-1}]$. On the other hand, the corresponding open set in $\mathcal{O}_{\operatorname{Spec} A[g^{-1}]}$ also has a section isomorphic to $A[(gh)^{-1}]$, and thus we have an isomorphism of sections. Clearly these commute with restriction, and thus we have a isomorphism of sheaves as required. □

Corollary 2.1.11. *The underlying topological space of every scheme has a base of open affine schemes.*

Proposition 2.1.12. *Every affine scheme is compact.*

Proof. Let $X = \operatorname{Spec} A$ be an affine scheme, and let $\{D(I_\alpha)\}$ be open sets which cover X. Then by definition, $\sum_\alpha I = A$ and in particular, there exist some finite collection $f_1, ..., f_n$ where $f_i \in I_{\alpha_i}$ such that $1 = \sum_{i=1}^{n} f_i$. Then the $\{D(I_{\alpha_1}), ..., D(I_{\alpha_n})\}$ cover X □

Theorem 2.1.13. *The category of affine schemes is equivalent to the opposite category of rings*

Proof. A morphism of rings $\varphi : A \to B$ induces a morphism of spectra $f : \operatorname{Spec} B \to \operatorname{Spec} A$ given by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. This in turn induces naturally a morphism of sheaves $f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}_{\operatorname{Spec} B}$. Details can be found in [6, p.73].

Conversely, given a morphism $(f, f^{\#}) : \operatorname{Spec} B \to \operatorname{Spec} A$, we have a morphism of sheaves $\mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$. Taking global sections we obtain a morphism $\varphi : A \to B$. We can then show that f is induced by φ. Details can be found in [6, p.73]. □

Example 2.1.14. The inclusion morphism $\operatorname{Spec} A[f^{-1}] \to \operatorname{Spec} A$ corresponds to the inclusion $A \to A[f^{-1}]$.

We will now introduce the Proj construction. This is a generalisation of projective varieties, in the same way affine schemes are generalisations of affine varieties. In order to do so, we will first need the concept of a graded ring.

Definition 2.1.15. Let S be a ring. A *grading* on S is a collection of subgroups S_i indexed over some commutative monoid G, often the nonnegative integers, such that $S = \bigoplus_{i \in G} S_i$ and $S_i S_j \subseteq S_{i+j}$ for any $i, j \in G$. A ring S equipped with such a grading is known as a *graded ring*. An element of such an S_i is known as a *homogeneous* element of degree i. A *homogeneous* ideal is an ideal that can be generated by homogeneous elements. If S is graded over the nonnegative integers, the ideal $S^+ = \bigoplus_{i>0} S_i$ is known as the *irrelevant ideal*. A *graded homomorphism* is a homomorphism of rings graded over the same monoid $\varphi : S \to T$ such that $\varphi(S_i) \subseteq T_i$.

Example 2.1.16. The canonical example is $S = k[x_1, ..., x_n]$ for some field k. The S_i are the group of homogeneous polynomials of degree i.

Example 2.1.17. Let S be a graded ring and T a multiplicative system of homogeneous elements. Then $T^{-1}S$ has a natural grading, where $\deg(s/t) = \deg(s) - \deg(t)$.

Definition 2.1.18. Let S be a ring graded over $\mathbb{Z}_{\geq 0}$. We define $\operatorname{Proj} S$ to be the set of homogeneous prime ideals of S that do not contain the irrelevant ideal.

Similar to Spec, we will introduce a topology on $\operatorname{Proj} S$. For a set of homogeneous elements I, we define $\mathcal{V}_+(I)$ to be the elements of $\operatorname{Proj} S$ that contain $\langle I \rangle$. The statements of Lemma 2.1.2 all hold here too, and thus the $\mathcal{V}_+(I)$ form a topology on $\operatorname{Proj} S$.

Definition 2.1.19. For a homogeneous $f \in S^+$, we define the *distinguished open homogeneous* subsets $D_+(f)$ to be $D_+(f) := \operatorname{Proj} S \setminus \mathcal{V}_+(f)$.

Proposition 2.1.20. *The $D_+(f)$ form a base of the topology on* $\operatorname{Proj} S$*. If S is generated by S_1 as an S_0-algebra, then the $D_+(f)$ for $f \in S_1$ form a base of the topology on* $\operatorname{Proj} S$*.*

Proof. The proof is identical to that of Proposition 2.1.4, except we sometimes add in the condition of homogeneity. □

And finally we define a sheaf of rings $\mathcal{O}$ on $\operatorname{Proj} S$. To do this, we define $\mathcal{O}(D_+(f)) := (S[f^{-1}])_0$, the elements of degree 0 in the ring $S[f^{-1}]$. Note that there is a natural homeomorphism between $D_+(f)$ and the underlying space of $\operatorname{Spec}(S[f^{-1}])_0$, and thus $(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec}(S[f^{-1}])_0$. This shows that $\operatorname{Proj} S$ is actually a scheme! We call a scheme of the form $\operatorname{Proj} S$ a *scheme of* Proj *type*.

Similar to Spec, we can show that Proj is a contravariant functor (only for graded homomorphisms); the proof is virtually identical.

Example 2.1.21. For any projective variety V with co-ordinate ring $S := k_+[V]$, the scheme $\operatorname{Proj} S$ is the scheme which is naturally associated to V.

Proposition 2.1.22. *Let V be a projective variety over k, let $S := k_+[V]$ and let $X := \operatorname{Proj} S$. Then $\Gamma_X(\mathcal{O}_X) \cong k$.*

Proof. By Lemma 1.1.11, we know that $\Gamma_X(\mathcal{O}_X) = \varprojlim \mathcal{F}(U)$ where the limit is taken across all the distinguished open subsets. It is easy to check that k satisfies the required univeral property. □

We will now study some general properties of schemes.

Let A be a ring and X a scheme. Given a morphism $f : X \to \operatorname{Spec} A$, we have an associated morphism of sheaves $f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_X$ and taking global sections we have a morphism $A \to \Gamma_X(\mathcal{O}_X)$. Thus we have a map $\alpha : \operatorname{Hom}(X, \operatorname{Spec} A) \to \operatorname{Hom}(A, \Gamma_X(\mathcal{O}_X))$.

Proposition 2.1.23. *The above map α is a bijection*

Proof. Given a homorphism $g : A \to \Gamma_X(\mathcal{O}_X)$, composing with the natural maps $\Gamma_X(\mathcal{O}_X) \to B_i$ for an affine open subset $\operatorname{Spec} B_i$ we get a morphism $g_i : A \to B_i$. Applying Spec, this in turn gives us morphisms $\operatorname{Spec} B_i \to A$ for every B_i. It suffices to show that this uniquely determines a morphism $f : X \to \operatorname{Spec} A$ such that $\alpha(f) = g$. To this end, we note that the $\operatorname{Spec} B_i$ cover X and they thus factoring through their inclusion maps, we get a map of topological spaces. We will show it is well defined; suppose $\mathfrak{p}$ is contained in both $\operatorname{Spec} B_i$ and $\operatorname{Spec} B_j$. Let $\operatorname{Spec} B$ be an open affine subset of $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j$ that contains $\mathfrak{p}$. Then the following diagram commutes:

$$\begin{tikzcd} & A \arrow[d] & \\ & \Gamma_X(\mathcal{O}_X) \arrow[dl] \arrow[dr] & \\ B_i \arrow[dr] & & B_j \arrow[dl] \\ & B & \end{tikzcd}$$

and thus tracing $\mathfrak{p}$ as a prime ideal of B back, we see that its image in $\operatorname{Spec} A$ is well-defined.

We will now define the map of sheaves $f^{\#} : \operatorname{Spec} A \to f_*(\mathcal{O}_X)$. But this follows since $\Gamma_{f^{-1}(U)}(\mathcal{O}_X) = \varprojlim_i B_i$ for each U, where the limit is taken over all affine open subsets $\operatorname{Spec} B_i$ of $f^{-1}(U)$. Clearly then $\alpha(f) = g$, since $\Gamma_X(\mathcal{O}_X) = \varprojlim_i B_i$ where the limit is taken over all i. Moreover, it is clear that this f is unique, since the induced map of global sections uniquely determine the maps $\operatorname{Spec} B_i \to \operatorname{Spec} A$, which uniquely determines the morphism, as we have seen. □

Corollary 2.1.24. *The scheme* $\operatorname{Spec} \mathbb{Z}$ *is final in the category of schemes. The scheme* $\operatorname{Spec} 0$ *is initial in the category of schemes.*

Proof. This follows from the above proposition and the fact that $\mathbb{Z}$ and 0 are respectively initial and final in the category of rings. □

2.2 $\mathcal{O}_X$-Modules on a Scheme

We have looked at $\mathcal{O}_X$-modules before in the setting of a ringed space, and we have shown, as an example, that a vector bundle on a manifold is a locally free sheaf. This motivated the definition of quasicoherent and coherent sheaves as a generalisation. In the setting of schemes , quasicoherent sheaves are much better behaved, and have many desirable properties. In fact, we will give a characterisation of all quasicoherent sheaves on an affine scheme, and all coherent sheaves for a class of affine schemes. We begin with a definition.

Definition 2.2.1. Let A be a ring and M an A-module. Then we define the *sheaf associated to* M on $\operatorname{Spec} A$, denoted $\tilde{M}$ to be the unique sheaf that takes on the values $M[f^{-1}]$ for all distinguished open subsets $D(f)$. It can be shown that this is a sheaf ([2, pp.19-20]).

Proposition 2.2.2. *If A is a ring and $X = \operatorname{Spec} A$ then $\tilde{A} = \mathcal{O}_X$*

Proof. Clear from the definitions □

Proposition 2.2.3. *The functor $M \mapsto \tilde{M}$ is fully faithful and exact.*

Proof. (Taken from [6, p. 111]) The first statement follows since $^\sim$ induces a map $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M},\tilde{N})$ for any pair of A-modules N and M. Then taking global sections gives us a map the other way, and these two maps are clearly inverses. The second statement follows because localisation is exact, and a sequence is exact if and only if the induced sequence of stalks is exact. □

We will now characterise quasicoherent sheaves on an affine scheme. We begin with a lemma

Lemma 2.2.4. *Let $X = \mathrm{Spec}\, A$ be an affine scheme. Then for any A-module M and $\mathcal{O}_X$-module $\mathcal{F}$ there is a natural isomorphism*

$$\mathrm{Hom}_A(M, \Gamma_X(\mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$$

Proof. We define $\alpha : \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M},\mathcal{F}) \to \mathrm{Hom}_A(M,\Gamma_X(\mathcal{F}))$ as follows: given a morphism $\varphi : \tilde{M} \to \mathcal{F}$ we define $\alpha(\varphi) := \varphi_X$, the morphism between global sections. This is clearly a group homomorphism.

To check that it is an isomorphism, we observe that any morphism of sheaves is uniquely determined by the morphisms on the distinguished open subsets. The morphism between global sections is also uniquely determined this way, by Lemma 1.1.11, and thus it follows that α is injective. To see that it is surjective, note that for any $f \in A$ and morphism $\varphi_X : M \to \Gamma_X(\mathcal{F})$, there exists a unique morphism $\varphi_{D(f)} : F[f^{-1}] \to \Gamma_{D(f)}(\mathcal{F})$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\varphi_X"] \arrow[d] & \Gamma_X(\mathcal{F}) \arrow[d] \\ M[f^{-1}] \arrow[r, "\varphi_{D(f)}"] & \Gamma_{D(f)}(\mathcal{F}) \end{tikzcd}$$

and hence any φ_X lifts to a morphism of sheaves $\varphi : \tilde{M} \to \mathcal{F}$. The naturality is easy to check. □

Theorem 2.2.5. *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if X can be covered by affine open subsets $U = \mathrm{Spec}\, A$ such that $\mathcal{F}|_U \cong \tilde{M}$ for some A-module M. Furthermore, if each A is Noetherian, then $\mathcal{F}$ is coherent if and only if $\mathcal{F}|_U \cong \tilde{M}$ for some finitely-generated A-module M.*

Proof. Since quasicoherence and coherence are local properties, we may assume X is affine and equal to $\mathrm{Spec}\, A$. Suppose $\mathcal{F} = \tilde{M}$ for some module M. Note that M is presentable, thus we have an exact sequence

$$\bigoplus_{i\in I} A \longrightarrow \bigoplus_{j \in J} A \longrightarrow M \longrightarrow 0$$

Applying $^\sim$ we deduce $\mathcal{F}$ is quasicoherent. If M is finitely generated, then the sets I and J are finite (we need the Noetherian hypothesis to deduce that the kernel of $\bigoplus_{j\in J} A \to M$ is finitely-generated), and thus $\mathcal{F}$ is coherent.

Conversely, suppose we have an exact sequence

$$\bigoplus_{i\in I} \mathcal{O}_X \longrightarrow \bigoplus_{j \in J} \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow 0 \tag{2.1}$$

Taking global sections, we have a morphism $A^I \to A^J$, and we define M to be the cokernel of this map. Applying $^\sim$, we have an exact sequence

$$\bigoplus_{i\in I} \mathcal{O}_X \longrightarrow \bigoplus_{j\in J} \mathcal{O}_X \longrightarrow \tilde{M} \longrightarrow 0 \tag{2.2}$$

The above lemma implies that the first morphisms in 2.1 and 2.2 are the same, and hence $\tilde{M}$ and $\mathcal{F}$ are cokernels of the same morphism, and are hence isomorphic.

If $\mathcal{F}$ is coherent, then I and J are finite, thus M is finitely generated. $\square$

Corollary 2.2.6. *There is an equivalence of categories between the category of A-modules and quasicoherent $\mathcal{O}_{\operatorname{Spec} A}$ modules. If A is Noetherian, then there is an equivalence of categories between the category of finitely-generated A-modules and the category of coherent $\mathcal{O}_{\operatorname{Spec} A}$ modules.*

Proposition 2.2.7. *Let $\varphi : A \to B$ be a homomorphism of rings, $f : \operatorname{Spec} B \to \operatorname{Spec} A$ the induced morphism of spectra, M an A-module and N a B-module. Write N_A for N as an A-module. Then $f_*(\tilde{N}) \cong \widetilde{N_A}$ and $f^*(\tilde{M}) \cong \widetilde{M \otimes_A B}$*

Proof. For the first statement, take a base open subset $D(g)$ of $\operatorname{Spec} A$. Pulling back, this is the base open subset $D(\varphi(g))$ of $\operatorname{Spec} B$, thus

$$f_*(\tilde{N})(D(g)) = \tilde{N}(D(\varphi(g)))_{A[g^{-1}]} = N_A[g^{-1}] = N_A(D(g))$$

It is clear that this isomorphism lifts to an isomorphism of sheaves.

For the second statement, we observe firstly that by Theorem $A.0.1$ there is a natural isomorphism

$$\operatorname{Hom}_B(M \otimes_A B, N) \cong \operatorname{Hom}_A(M, N_A)$$

Now by Lemma 2.2.4, we know that there are isomorphisms

$$\operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} B}}(\widetilde{M \otimes_A B}, \tilde{N}) \cong \operatorname{Hom}_B(M \otimes_A B, N)$$

and

$$\operatorname{Hom}_A(M, N_A) \cong \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} A}}(\tilde{M}, \widetilde{N_A})$$

and thus we have a natural isomorphism

$$\operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} B}}(\widetilde{M \otimes_A B}, \tilde{N}) \cong \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} A}}(\tilde{M}, \widetilde{N_A})$$

Since $\widetilde{N_A} \cong f_*(\tilde{N})$, that means $\widetilde{M \otimes_A B}$ satisfies the adjunction property of $f^*(\tilde{M})$, and thus they are isomorphic. $\square$

Next we introduce the following key theorem, which helps us characterise affine schemes.

Theorem 2.2.8. *Let X be an affine scheme and $\mathcal{F}$ a quasicoherent sheaf. Then for all $i > 0$ we have $H^i(X, \mathcal{F}) = 0$*

Proof. [4, p.432] □

Finally, we conclude this section with an analogous construction for Proj-type schemes.

Definition 2.2.9. Let $X = \operatorname{Proj} S$ be a projective scheme and M a graded S-module. We define the *sheaf associated to* M, denoted $\tilde{M}$ to be the sheaf such that for each $D_+(f)$, we define $\tilde{M}(D_+(f)) := M[f^{-1}]_0$, the degree zero elements of the localised module $M[f^{-1}]_0$.

Note that on $\operatorname{Spec} S[f^{-1}]_0$, the restriction of the sheaf $\tilde{M}$ is simply $\widetilde{M[f^{-1}]_0}$, and thus the sheaf $\tilde{M}$ is quasicoherent.

2.3 Morphisms

In this section, we will study some properties of morphisms. To begin, we will define two especially important classses of morphisms, known as *open immersions* and *closed immersions*. These provide us with a rigorous definition of subschemes.

Definition 2.3.1. Let X be a scheme and U an open subset of X. Then U equipped with the restricted sheaf $\mathcal{O}_X|_U$ is a scheme, by Proposition 2.1.4. An *open subscheme* is a scheme $(Y, \mathcal{O}_Y)$ that is isomorphic to $(U, \mathcal{O}_X|_U)$. An *open immersion* is an open subscheme Y equipped with a morphism $Y \to X$ that factors into an isomorphism $Y \to U$ followed by the inclusion $U \to X$.

Indeed, the inclusion $i : U \to X$ equipped with the natural morphism of sheaves $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_U$ is a morphism for any open set U.

Definition 2.3.2. Let X be a scheme. A *closed subscheme* of X is a scheme Y equipped with a morphism $Y \to X$, known as a *closed immersion* such that the map of spaces is a homeomorphism onto a closed subset of X and the morphism of sheaves is surjective.

For any open subset U of X, there is a unique open subscheme up to isomorphism. However, this is not true for closed subschemes. Consider the following example:

Example 2.3.3. Consider the affine line $\mathbb{A}^1_k := \operatorname{Spec} k[x]$. The quotient map $k[x] \to k[x]/\langle x \rangle$ corresponds to the closed immersion that maps the unique point of $\operatorname{Spec} k[x]/\langle x \rangle$ to the point $\langle x \rangle \in \mathbb{A}^1$, and the associated morphism of sheaves is defined in the obvious way. Similarly, $k[x] \to k[x]/\langle x^2 \rangle$ defines a closed immersion that maps the unique point of $\operatorname{Spec} k[x]/\langle x^2 \rangle$ to the point $\langle x \rangle \in \mathbb{A}^1$. However, observe that these two schemes are not isomorphic.

Proposition 2.3.4. *If X is a closed subscheme of $Y = \operatorname{Proj} k[x_0, ..., x_n]$ then $\Gamma_X(\mathcal{O}_X) = k$ if X is nonempty.*

Proof. Since $\Gamma_Y(\mathcal{O}_Y) = k$ and the map $\Gamma_Y(\mathcal{O}_Y) \to \Gamma_Y(\mathcal{O}_X)$ is surjective, it follows that $\Gamma_X(\mathcal{O}_X)$ is either zero or k. If $\Gamma_X(\mathcal{O}_X)$ is the zero ring, then $\Gamma_U(\mathcal{O}_X) = 0$ for any open set U, which would imply X is empty, contrary to the hypothesis. The result follows. □

Definition 2.3.5. Let $i : Y \to X$ be a closed immersion. We define the *sheaf of ideals* $\mathcal{I}_Y$ of Y to be the kernel of $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Y$.

Theorem 2.3.6. *The sheaf of ideals $\mathcal{I}_Y$ is quasicoherent. Conversely, any quasicoherent sheaf of ideals on X is the sheaf of ideals of a unique closed subscheme up to isomorphism.*

Proof. [6, p. 116] □

Corollary 2.3.7. *Let $X = \operatorname{Spec} A$ be an affine scheme. Then there is a one-to-one correspondence between ideals of A and closed immersions of A. In particular, every closed subscheme of A is affine.*

Proof. Given any ideal $\mathfrak{a}$, we observe that the projection $A \to A/\mathfrak{a}$ identifies a closed immersion.

Conversely, let $f : Y \to X$ be a closed immersion. Then ideal sheaf of Y is quasicoherent, so it is of the form $\widetilde{\mathfrak{a}}$ for some ideal $\mathfrak{a}$. Thus we have the following short exact sequence

$$0 \to \widetilde{\mathfrak{a}} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to 0$$

By Theorem 2.2.8, taking global sections, we obtain

$$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$$

Now taking Spec, we see $\widetilde{\mathfrak{a}}$ is the sheaf of ideals of $\operatorname{Spec} A/\mathfrak{a}$, and thus by the above theorem $Y \cong \operatorname{Spec} A/\mathfrak{a}$. By Proposition 2.1.23, the projection $A \to A/\mathfrak{a}$ identifies the morphism $Y \to \operatorname{Spec} A$ uniquely. It is clear that this process is the inverse of the one defined in the first paragraph. □

Now we come to the definition of a scheme over another scheme.

Definition 2.3.8. Let Y be a scheme. A *scheme over Y* is a scheme X equipped with a morphism $f : X \to Y$. If A is a ring, we will say X is a *scheme over A* if X is a scheme over $\operatorname{Spec} A$.

Example 2.3.9. An affine variety V over k comes with an inclusion $k \to k[V]$. Applying Spec to this map, we see that the scheme associated any affine variety is a scheme over k.

Example 2.3.10. If V is a projective variety with projective coordinate ring $k_+[V]$, then by Proposition 2.1.22 and Proposition 2.1.23 there is a natural map $\operatorname{Proj} k_+[V] \to \operatorname{Spec} k$ induced by the identity map on k, thus the scheme associated to projective varieties are schemes over k.

Definition 2.3.11. A morphism $f : X \to Y$ is *locally of finite type* if Y can be covered by open affine subsets $V_i = \operatorname{Spec} B_i$ such that $f^{-1}V_i$ can be covered by $U_{ij} = \operatorname{Spec} A_j$ such that A_j is a finitely generated B_i-algebra. The morphism f is *of finite type* if $f^{-1}V_i$ can be covered by a finite number of the U_{ij}.

Example 2.3.12. If V is an affine variety with coordinate ring $A := k[V]$, then the morphism $\operatorname{Spec} A \to \operatorname{Spec} k$ is of finite type.

Example 2.3.13. If V is a projective variety with projective coordinate ring $S := k_+[V]$, then the morphism $\operatorname{Proj} S \to \operatorname{Spec} k$ is of finite type. Indeed, $\operatorname{Proj} S$ can be covered by a finite number of the sets $\operatorname{Spec} S[f^{-1}]_0$, and since each $S[f^{-1}]_0$ is finitely generated as an algebra over k, the result follows.

We conclude this section with some important results about coherent sheaves on a projective scheme over a field. Before we begin, first observe that if $S = k[x_1, ..., x_n]$ is the polynomial ring over a field k then there is a natural map $X = \operatorname{Proj} S \to \operatorname{Spec} k$, induced by the identity map $k \to k = \Gamma_X(\mathcal{O}_X)$ (Proposition 2.1.23).

Definition 2.3.14. Let X be a scheme over a field k. We say X is *projective* over k if the morphism $\varphi : X \to \operatorname{Spec} k$ factors into a closed immersion $X \to \operatorname{Proj} k[x_1, ..., x_n]$ for some n followed by the map $\operatorname{Proj} k[x_1, ..., x_n] \to \operatorname{Spec} k$ as described above.

Example 2.3.15. Let $S = k[x, y, z]/\langle y^2 - xz\rangle$. Then the natural map $k[x, y, z] \to k[x, y, z]/\langle y^2 - xz\rangle$ induces a closed immersion $\operatorname{Proj} S \to \operatorname{Proj} k[x_1, ..., x_n]$. Composing with the projection we observe that $\operatorname{Proj} S$ is projective over k.

Theorem 2.3.16. *Let X be a projective scheme over k and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. For any $i \geq 0$ we have $\dim_k H^i(X, \mathcal{F}) < \infty$.*

Proof. [6, p.228]. □

Definition 2.3.17. Let X be a projective scheme over k and $\mathcal{F}$ a coherent sheaf. We define the *Euler characteristic*, $\chi(\mathcal{F})$ of $\mathcal{F}$ to be the following quantity:

$$\sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F})$$

Proposition 2.3.18. *Let X be a projective scheme over k and suppose $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are coherent sheaves such that the following sequence is exact:*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

Then $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$.

Proof. Since $\operatorname{Proj} k[x_0, ..., x_n]$ has dimension n (see, for example, [6, p.12]), and X is a closed subset of $\operatorname{Proj} k[x_0, ..., x_n]$ for some n, it follows that X is finite-dimensional (and thus Noetherian as a topological space). By Theorem 1.3.20, it follows that $H^i(X, \mathcal{F}) = H^i(X, \mathcal{G}) = H^i(X, \mathcal{H}) = 0$ for $i > n$. Thus the long exact sequence of cohomology 1.3.13 is zero for all but finitely many terms, and since the H^i are finite-dimensional vector spaces, the result follows from the Rank-Nullity Theorem. □

2.4 Varieties as Schemes

As seen in the previous sections, schemes enlarge the class of varieties by a lot. There are many schemes which do not have analogous varieties (for example, $\operatorname{Spec} \mathbb{Z}$). In this section, we will define a class of schemes known as abstract varieties which share familiar properties. However, being schemes there will naturally be differences, for example the existence of generic points. These will be studied.

We first consider a basic example. Recall that in Example 2.1.9 we noted that schemes can keep track of multiplicity. The reason is that $k[x]/\langle x^2\rangle$ is not the co-ordinate ring of any affine variety; since any affine variety is irreducible, and thus its defining polynomial must be irreducible. In particular, note that this means that any co-ordinate ring is an integral domain. This motivates the following definitions:

Definition 2.4.1. Let X be a scheme. We say that X is *irreducible* if its underlying space is irreducible.

Definition 2.4.2. Let X be a scheme. We say X is *reduced* if for every open subset U, the ring $\mathcal{O}_X(U)$ is a reduced ring (that is it has no nilpotent elements). We say X is *integral* if for every open subset U, the ring $\mathcal{O}_X(U)$ is an integral domain.

Note then that integrability implies reducedness. In fact:

Proposition 2.4.3. *Let X be a scheme. Then X is integral if and only if it is reduced and irreducible.*

Proof. [6, p.82] □

For affine schemes, these properties are much more simple.

Proposition 2.4.4. *Let $X = \operatorname{Spec} A$ be an affine scheme. Then*

(a) X is irreducible if and only if $\operatorname{nil} A$ is prime.

(b) X is reduced if and only if $\operatorname{nil} A = 0$.

(c) X is integral if and only if A is an integral domain.

Proof. If $\operatorname{nil} A$ is prime, then any closed set that contains $\operatorname{nil} A$ is necessarily X. Conversely, if $\operatorname{nil} A$ is not prime, then there exists elements a, b such that ab is nilpotent but a and b are both not. Then $X = \mathcal{V}(ab)$ but $\mathcal{V}(a)$ and $\mathcal{V}(b)$ are both proper. This proves (a).

The forward implication of (b) is trivial. Conversely, suppose $\operatorname{nil} A = 0$. Note that if $x \in \mathcal{O}_X(U)$ is nilpotent for some open set U then the image of x is nilpotent in $\mathcal{O}_X(D(f))$ for any f such that $D(f) \subseteq U$. In particular, that means $f^n x^m = 0$ for some m, n and supposing without loss of generality that $m \geq n$ it follows $(fx)^m = 0$ in A and thus $fx = 0$, hence $x = 0$ in $\mathcal{O}_X(D(f))$. Since the $D(f)$ cover $\mathcal{O}_X(U)$ by the sheaf axiom it follows that $x = 0$ in $\mathcal{O}_X(U)$. This proves (b).

Part (c) follows from (a) and (b) and Proposition 2.4.3. □

As a quick application, we address the issue of closed subschemes not being unique.

Theorem 2.4.5. *Let X be a scheme and V a closed subset. Then there exists a unique reduced closed subscheme Y such that the associated closed immersion $f : Y \to X$ is a homeomorphism onto V.*

Proof Sketch. First, suppose X is affine and equal to $\operatorname{Spec} A$. Then V is a set of prime ideals of A. Set

$$\mathfrak{a} := \bigcap_{\mathfrak{p} \in V} \mathfrak{p}$$

and let $Y := \operatorname{Spec} A/\mathfrak{a}$, and let f be the closed immersion $f : Y \to X$ induced by the projection $A \to A/\mathfrak{a}$. Since $\mathfrak{a}$ is clearly radical, the ring $A/\mathfrak{a}$ is reduced, and hence Y is reduced. Uniqueness is clear in this case.

If X not affine, we cover X with affines, apply the above process and then ”glue” them together to form Y. Details can be found at [6, p. 70]. □

Definition 2.4.6. The closed subscheme Y described above is known as the *reduced subscheme associated to* V.

Now we define an abstract variety:

Definition 2.4.7. An *abstract variety* is an integral scheme of finite type over a field k.

Recall that a *generic point* of a topological space X is a point which is dense in X. As remarked, a classical variety does not have generic points. However, on an abstract variety (and in general a “nice” scheme), generic points are very well-behaved. We give some properties.

Proposition 2.4.8. *Let X be a scheme. Then every irreducible closed subset has a unique generic point.*

Proof. Let V be an irreducible closed subset of X. We put the reduced subscheme structure on V. Then any open subset of V (in the induced topology) is also irreducible. Let $\operatorname{Spec} A$ be an open affine subset of V. Now $\operatorname{Spec} A$ is also irreducible, and thus A has a prime nilradical ζ, whose closure in X is the closure of $\operatorname{Spec} A$ in X. But since V is irreducible, $\operatorname{Spec} A$ is dense in V as well, which means the closure of $\operatorname{Spec} A$ in X is V. We have shown that ζ is the generic point of V.

To prove uniqueness, suppose ζ and η are two generic points of V. Since the boundary of any set is closed, they must be in the interior V°. Now let $\operatorname{Spec} A$ be an open affine neighbourhood of ζ contained in V°, $\operatorname{Spec} B$ be an open affine neighbourhood for η and let $\operatorname{Spec} R$ be an affine open subset of the nonempty intersection $\operatorname{Spec} A \cap \operatorname{Spec} B$. Then the inclusion morphism $\operatorname{Spec} R \to \operatorname{Spec} A$ induces a morphism of rings $\varphi : A \to R$, and clearly $\operatorname{nil} A = \varphi^{-1}(\operatorname{nil} R)$. In particular, this means that the inclusion maps the generic point of $\operatorname{Spec} R$ to the generic point of $\operatorname{Spec} A$. Since the same argument is true for $\operatorname{Spec} B$, we must conclude that ζ and η are the same point. □

Proposition 2.4.9. *Let X be an integral scheme with a unique generic point ζ. Then the local ring $\mathcal{O}_{\zeta,X}$ is a field.*

Proof. Since the local ring depends only on a neighbourhood of X, we may assume that X is affine. Let $X = \operatorname{Spec} A$. Then A is an integral domain by Proposition 2.4.4. Now ζ corresponds to the zero ideal of X and thus $\mathcal{O}_{\zeta,X}$ is the localisation of A at 0, which is in fact the fraction field of A. □

Corollary 2.4.10. *Let X be an integral scheme with unique generic point ζ, and let $\operatorname{Spec} A$ and $\operatorname{Spec} B$ be open affine subschemes. Then A and B have isomorphic fraction fields, which are isomorphic to the local ring at ζ.*

We conclude this section with a discussion about the dimension of a scheme. In general, dimensions are not always well-behaved, but for abstract varieties, their properties are as expected.

Definition 2.4.11. Let X be a scheme. The *dimension* of X is its Noetherian dimension. If Y is an irreducible closed subset, then the *codimension of* Y in X, denoted $\operatorname{codim}_X(Y)$ is defined to be the largest integer n such that there exists a chain

$$Y = Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n$$

where each Y_i is closed and irreducible, and ∞ if no such chain exists. If Y is not irreducible, then

$$\operatorname{codim}_X(Y) := \min_{Z \subseteq Y}(\operatorname{codim}_X(Z))$$

where the minimum is taken across all irreducible closed subsets of Y. Finally, if Y is not closed, we define $\operatorname{codim}_X(Y) := \operatorname{codim}_X(\overline{Y})$.

Theorem 2.4.12. *Let $X = \operatorname{Spec} A$ be an affine scheme. Then there is a one-to-one inclusion-reversing correspondence between prime ideals of A and irreducible closed subsets of X.*

Proof. Let

$$Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n$$

be a chain of irreducible closed subsets. Given the reduced subscheme structure, each Y_i is integral by Proposition 2.4.3, and hence isomorphic to $\operatorname{Spec} A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. If $i < j$, it is clear that $\mathfrak{p}_i \supsetneq \mathfrak{p}_j$, and thus we have a chain of prime ideals

$$\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \dots \supsetneq \mathfrak{p}_n$$

Conversely, given a chain of prime ideals

$$\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \dots \supsetneq \mathfrak{p}_n$$

we have a chain of integral (hence irreducible) subschemes

$$Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n$$

where $Y_i := \operatorname{Spec} A/\mathfrak{p}_i$. □

Corollary 2.4.13. *Let $X = \operatorname{Spec} A$ be an affine scheme. Then $\dim X = \dim A$.*

Corollary 2.4.14. *Let $X = \operatorname{Spec} A$ be an affine scheme and $Y = \operatorname{Spec} A/\mathfrak{p}$ be an integral closed subscheme. Then $\operatorname{codim}_X Y = \operatorname{ht} \mathfrak{p}$.*

Where $\dim A$ is the Krull Dimension of A and $\operatorname{ht} \mathfrak{p}$ is the height of $\mathfrak{p}$.

Proposition 2.4.15. *Let X be an abstract variety over a field k, and Y a closed subscheme. Then $\dim(X) = \dim(Y) + \operatorname{codim}_X(Y)$.*

Proof. [12, pp.311-312] □

Chapter 3

Geometric Constructions on Schemes

3.1 Divisors

In this section, we develop the theory of Weil and Cartier divisors, which are, roughly speaking, objects that encode the intrinsic geometry of an underlying scheme. In particular, they allow us to develop two invariants, known as the divisor classes. We will introduce Weil Divisors in the context of smooth abstract varieties, then we will introduce Cartier Divisors in the context of schemes, and state a partial equality. A *variety* in this section will always mean an abstract variety.

We begin with two key definitions:

Definition 3.1.1. Let X be an integral scheme. We define the *function field* of X, denoted $K(X)$ to be the local ring at the generic point of X.

Definition 3.1.2. Let X be a variety. We say X is *smooth* if all its local rings are regular rings.

Now we define the notion of a Weil divisor.

Definition 3.1.3. Let X be a smooth variety. A *prime divisor* on X is a closed subvariety of codimension one. The *group of Weil divisors* $\operatorname{Div} X$ is the free abelian group generated by the prime divisors. A *Weil divisor* is an element of $\operatorname{Div} X$.

We will next define a subgroup of $\operatorname{Div} X$ that captures the geometric information of the underlying variety. In order to do so, we require the following result:

Proposition 3.1.4. *Let X be a smooth variety, Y a prime divisor and η its generic point. Then the local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring, with residue field $K(X)$, the function field of X.*

Proof. We may assume X is an affine scheme $X = \operatorname{Spec} A$ where A is a finitely-generated k-algebra that is also an integral domain. Then Y is an integral closed subscheme and hence $Y = \operatorname{Spec} A/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$. Corollary $2.4.14$ implies that $\operatorname{ht} \mathfrak{p} = 1$. Since $\mathfrak{p}$ is dense in $\operatorname{Spec} A/\mathfrak{p}$ (being the zero ideal), we have $\eta = \mathfrak{p}$ and thus $\mathcal{O}_{X,\eta} = A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is regular by assumption, it follows that $\mathfrak{p}$ is principal. [11, Ch. 1, Proposition 2] then implies that $A_{\mathfrak{p}}$ is a discrete valuation ring, and it is clear that the residue field equal to $K(X)$. $\square$

Let X be a smooth variety with function field K, Y be a prime divisor with generic point η and $f \in K^*$ a function. We define the *valuation of f at Y*, denoted $v_Y(f)$ to be the valuation of f in the discrete valuation ring $\mathcal{O}_{X,\eta}$.

Proposition 3.1.5. *We have $v_Y(f) = 0$ for all but finitely many Y.*

Proof. [6, p.131] □

Definition 3.1.6. Let X be a smooth variety with function field K, Y be a prime divisor with generic point η and $f \in K^*$ a function. We define the *divisor associated to* f to be

$$\operatorname{div} f := \sum v_Y(f)Y$$

taken across all prime divisors Y. By the above proposition, this sum is finite, so this divisor is well-defined.

The map $f \mapsto \operatorname{div} f$ gives us a homomorphism $K^* \to \operatorname{Div} X$. We call the image of K^* the *group of principal divisors*. An element in this group is a *principal divisor*.

Definition 3.1.7. The group of Weil divisors quotiented by the group principal divisors is known as the *divisor class group of* X, denoted $\operatorname{Cl} X$.

We will now study some examples.

Example 3.1.8. Let $X = \mathbb{A}_k^n = \operatorname{Spec} k[x_1, ..., x_n]$. Then prime divisors are in one-to-one correspondence with principal prime ideals of $k[x_1, ..., x_n]$ (which, in turn, are in one-to-one correspondence with irreducible polynomials) and the function field of X is equal to $k(x_1, ..., x_n)$. If $D = \sum n_i P_i$ is a Weil Divisors, and f_i is the polynomial which generates the principal prime ideal associated to P_i, then $\operatorname{div} \prod f_i^{n_i} = D$. In particular, $\operatorname{Cl} X = 0$.

Example 3.1.9. Let $X = \operatorname{Spec} k[x, y]/\langle y - x^2\rangle$. Now prime divisors on X are simply points, and in particular they are in one-to-one correspondence with the polynomials $x - a$ for $a \in k$. Thus given a divisor $D = \sum n_i P_i$, we may associate $a_i \in k$ to each P_i, and $\operatorname{div} \prod f_i^{n_i} = D$. Once again $\operatorname{Cl} X = 0$. In fact, it can be shown that if A is a UFD and $\operatorname{Spec} A$ is normal, meaning that all its local rings are integrally closed integral domains, then $\operatorname{Cl}(\operatorname{Spec} A) = 0$ ([6, Ch. II, Proposition 6.2]).

Let X be a scheme, recall that it has a base of open affine subsets. We define a sheaf $\mathcal{K}_X$ on it as follows: let $U = \operatorname{Spec} A$ be one such base open affine subset. We define $K(A)$ to be the localisation of A at the set of non zero-divisors (if A is an integral domain, then $K(A)$ is just the field of fractions $\operatorname{Frac} A$), and define $\mathcal{K}_X(U)$ to simply be $K(A)$. If $V = \operatorname{Spec} A[f^{-1}]$ is a base open affine subset of $\operatorname{Spec} A$, then we have a natural map $\mathcal{K}(U) \to \mathcal{K}(V)$ and it is easy to see that it will agree for all such U and V as they vary. Thus we have a sheaf. Furthermore, we define the sheaf $\mathcal{K}_X^*$ as follows: for an open affine subset $U = \operatorname{Spec} A$, we take $\mathcal{K}_X^*(U)$ to be the multiplicative group of invertible elements of $\mathcal{K}_X(U)$. Now observe that $\mathcal{O}_X$ is a subsheaf of $\mathcal{K}_X$ (by which we mean $\mathcal{O}_X(U) \subseteq \mathcal{K}_X(U)$ for any open set U), and in similar fashion we can define $\mathcal{O}_X^*$. It is a subsheaf of $\mathcal{K}_X^*$.

Definition 3.1.10. Let X be a scheme. A *Cartier divisor* is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$. A *principal Cartier divisor* is an element in the image of $\mathcal{K}_X^*(X) \to (\mathcal{K}_X^*/\mathcal{O}_X^*)(X)$. We define the *Cartier divisor class group* of X denoted $\operatorname{CaCl} X$ to be the group $(\mathcal{K}_X^*/\mathcal{O}_X^*)(X)/\mathcal{K}^*(X)$.

Theorem 3.1.11. *Let X be a smooth variety. Then there is an isomorphism* $\operatorname{Cl} X \cong \operatorname{CaCl} X$.

Proof Sketch. Given a Cartier divisor D_{Cart}, we may write D_{Cart} as $\{(U_i, f_i)\}$ where the U_i cover X and the image of $f_i \in \mathcal{K}_X^*(U_i)$ in $(\mathcal{K}_X^*/\mathcal{O}_X^*)(U_i)$ is equal to $D_{\text{Cart}}|_{U_i}$. Then for each prime divisor Y, we define n_Y to be $v_Y(f_i)$ where i is any index such that $U_i \cap Y \neq \emptyset$. It can be shown that this does not depend on i and that the $n_Y = 0$ for all but finitely many Y; and thus we have a Cartier Divisor $\sum n_Y Y$. It is clear that this is a homomorphism. Details can be found at [6, p. 141].

Conversely, given a Weil divisor D, we can take a small enough open subset U such that the restriction of D is principal, and is associated to some f. Then as U varies, we recover a global section, which can be shown to be a Cartier divisor. It is easy to check that these processes are inverses. Details can be found at [6, p. 141]. □

Example 3.1.12. To see how this works, we compute an example. Let $X = \operatorname{Proj} k[x,y,z]/\langle yz - x^2\rangle$ and define the following points $P := (3,9,1) = \langle x - 3z\rangle, Q := (2,4,1) = \langle x - 2z\rangle$ and $R := (0,0,1) = \langle y\rangle$. Let

$$D := 3P - 4Q + R$$

We will compute the Cartier Divisor associated to D as follows: On the open set $U_P := D(yz(x - 2z))$, the restriction of D is simply $3P$. Now note that this is the same as the principal divisor generated by $f^P := ((x-3z)/z)^3 \in K(X)$. Similarly, we can take $f^Q := ((x-2z)/z)^{-4} \in K(X)$ on U_Q, $f^R := (y/z) \in K(X)$ on U_R and $f^z := 1$ on $U_z := D(y(x-2z)(x-3z))$. Now these open subsets cover X and the note that the image of f^i on $U_i \cap U_j$ is invertible in $\mathcal{O}_X(U_i \cap U_j)$ if $i \neq j$ and is thus the identity in $\mathcal{K}_X^*/\mathcal{O}_X^*$, and so we have a global section, which is a Cartier Divisor.

Remark 3.1.13. Note that we have an example of a surjective map of sheaves such that the induced map of sections is not surjective. This means the presheaf image is not equal to the image sheaf. In particular, this means that $H^1(X, \mathcal{O}_X) \neq 0$, hence X is not affine by Theorem 2.2.8.

3.2 Invertible Sheaves

We saw in a previous section that locally free sheaves of finite rank on a smooth manifold are ”the same” as smooth vector bundles. In this section, we will study locally free sheaves on a scheme, though we restrict ourselves to locally free sheaves of rank one. Such sheaves are also known as *invertible sheaves*. We will first explain this term, then we will introduce twisting sheaves on a Proj-type scheme as an example. This will be followed by a gneeral study of invertible sheaves and their relation to divisors.

Proposition 3.2.1. *Let X be a scheme and $\mathcal{F}$ be a locally free sheaf of rank 1 on X. Then $\mathcal{F} \otimes \mathcal{F}^\vee = \mathcal{O}_X$. If $\mathcal{G}$ is another locally free sheaf of rank 1, we then so $\mathcal{F} \otimes \mathcal{G}$ is also locally free of rank 1.*

Proof. (Based on the proof in [6, p.143]) We know from Proposition 1.2.22 that $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathfrak{Hom}(\mathcal{F}, \mathcal{F})$, so it suffices to show that $\mathfrak{Hom}(\mathcal{F}, \mathcal{F}) \cong \mathcal{O}_X$. Now observe that there is a natural map $\mathcal{O}_X \to \mathfrak{Hom}(\mathcal{F}, \mathcal{F})$, given by scalar multiplication. At the level of the stalks, the induced morphism is the natural homomorphism $\mathcal{O}_{X,P} \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_{X,P}, \mathcal{O}_{X,P})$, which is well known to be an isomorphism. This proves the first statement.

The second statement is clear, since $\mathcal{F}$ and $\mathcal{G}$ are locally free of rank 1. □

Definition 3.2.2. An *invertible sheaf* on a ringed space $(X, \mathcal{O}_X)$ is a locally free sheaf of rank 1.

The above proposition shows that the isomorphism classes of sheaves on a ringed space $(X, \mathcal{O}_X)$ forms a group, known as the *Picard Group*, denoted $\operatorname{Pic} X$.

Definition 3.2.3. Let S be a graded ring, graded over some monoid G, and M an S-module. A *grading* on M is a collection of submodules M_i, indexed over G such that $M = \bigoplus_{i \in G} M_i$, and $S_i M_j \subseteq M_{i+j}$ for any $i, j \in G$. A module M equipped with sich a grading is known as a *graded ring*. An element of M_i is known as a *homogeneous* element of degree i.

Definition 3.2.4. Let S be a graded ring, and M a graded S-module. We define $M(n)$ the *n-th twist* on M to be M with the following grading: $M(n)_m := M_{n+m}$.

Unless stated otherwise, a graded ring S will always be graded over $\mathbb{Z}_{\geq 0}$ in the rest of this section. However, we will sometimes twist by negative integers. In order for this to make sense, we re-interpret the grading on S as over $\mathbb{Z}$, but $S_n = 0$ if $n < 0$.

If M and N are graded modules, there is a natural grading on $M \otimes N$ where if $u \in M, v \in N$ are homogeneous of degree m and n respectively, then $u \otimes v$ is of degree $m + n$. Since any element of M (resp N) can be written uniquely as a sum of homogeneous elements, this is well-defined. Note that $M_0 \otimes N_0 \cong (M \otimes N)_0$

Lemma 3.2.5. *For any graded module M, we have $M(n) \otimes M(m) = M(m + n)$.*

Proof. Clear from the definitions. □

Lemma 3.2.6. *Let S be a graded ring, $X = \operatorname{Proj} S$ and suppose S is generated by S_1 as an S_0-algebra. Then $(M \otimes_S N)^\sim \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.*

Proof. Since $(M[f^{-1}] \otimes N[f^{-1}])_0 \cong M[f^{-1}]_0 \otimes N[f^{-1}]_0$ for any $f \in S_1$, we have a natural isomorphism on the $D_+(f)$, which clearly commute. Since the $D_+(f)$ form a base of the topology, the result follows □

Definition 3.2.7. Let S be a ring graded over the nonnegative integers and $X = \operatorname{Proj} S$. We define the *twisted structure sheaf of degree n*, denoted $\mathcal{O}_X(n)$ to be the sheaf $S(n)^\sim$. If $\mathcal{F}$ is any $\mathcal{O}_X$-module, we define the *twisting of $\mathcal{F}$ of degree n*, denoted $\mathcal{F}(n)$ to be $\mathcal{F} \otimes \mathcal{O}_X(n)$.

Proposition 3.2.8. *Let S be a graded ring and $X = \operatorname{Proj} S$ and suppose S is generated by S_1 as a S_0-algebra. Then $\mathcal{O}_X(n)$ is locally free of rank 1 for any $n \in \mathbb{Z}$.*

Proof. Taken from [6, p.117]. Suppose $f \in S_1$. We will show that $\mathcal{O}_X(n)|_{D_+(f)} \cong \mathcal{O}_X|_{D_+(f)}$. By definition, we know $\mathcal{O}_X(n)|_{D_+(f)} = S(n)^\sim$ and $\mathcal{O}_X|_{D_+(f)} = S[f^{-1}]_0$ on $D_+(f) \cong \operatorname{Spec} S(n)[f^{-1}]_0$. Since f is invertible, multiplication by f^n induces an isomorphism of *modules* $S[f^{-1}]_0 \to S(n)[f^{-1}]_0$. Applying $\sim$ we obtain an isomorphism $\mathcal{O}_X|_{D_+(f)} \to \mathcal{O}_X(n)|_{D_+(f)}$. Since S is generated by S_1 as an S_0-algebra, the $D_+(f)$ cover X and the result follows. □

Remark 3.2.9. Note that while $\mathcal{O}_X(n)$ and $\mathcal{O}_X$ have isomorphic stalks, they are not isomorphic (indeed, they behave differently with respect to tensor products). This is because the isomorphisms of modules induced by multiplication by f in the above proof is not compatible between different distinguished open sets; in other words it does not commute with the restriction maps.

Remark 3.2.10. Note that multiplication by f only induces an isomorphism of *modules*, not rings.

We will now associate an invertible sheaf to a divisor on a scheme, in such a way that is compatible with the group structure. We will work with Cartier Divisors, but this works with Weil Divisors too, as we will see.

Definition 3.2.11. Let X be a scheme and let D_{Cart} be a Cartier Divisor. We define $\mathcal{L}(D_{\text{Cart}})$ to be the $\mathcal{O}_X$-module as follows: on an open set U, let $f \in K^*(U)$ represent $D_{\text{Cart}}|_U$ (recall that D_{Cart} is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$). We define $\mathcal{L}(D_{\text{Cart}})(U) := \langle f^{-1} \rangle$. This is a submodule of $\mathcal{K}_X(U)$. Note that this is well-defined, since if f' also represents D_{Cart} on U', then by definition f/f' is invertible in $\mathcal{O}_X(U \cap U')$ and thus they generate the same module. The obvious restriction maps turn this into a sheaf.

If X is a smooth variety and $D = \sum n_i Y_i$ is a Weil Divisor (where the sum is taken across all prime divisors Y_i, but all but finitely many n_i are 0), then we can alternatively define $\mathcal{L}(D)$ as follows:

$$\mathcal{L}(D)(U) := \{f \in \mathcal{K}_X(U) \mid v_{Y_i}(f) + n_i \geq 0 \text{ for each } i \text{ such that } Y_i \cap U \neq \emptyset\}$$

If D_{Cart} is the Cartier Divisor associated to D, then it is easy to see that $\mathcal{L}(D_{\text{Cart}}) = \mathcal{L}(D)$, since the divisor D is obtained from D_{Cart} by taking valuations of D_{Cart} at all prime divisors of X, thus an element of $\mathcal{L}(D)(U)$ would be a multiple of some f satisfying $v_{Y_i}(f) = -n_i$. But such an f is a generator of $\mathcal{L}(D_{\text{Cart}})(U)$. Conversely any multiple of such an f is an element of $\mathcal{L}(D_{\text{Cart}})$.

Theorem 3.2.12. *Let X be a scheme and D a Cartier Divisor. Then $\mathcal{L}(D)$ is invertible and the mapping $D \mapsto \mathcal{L}(D)$ is an injective group homomorphism* $\operatorname{CaCl} X \to \operatorname{Pic} X$. *If X is integral, then it is an isomorphism.*

Proof. [6, p. 144] □

Example 3.2.13. Let X be a smooth variety of dimension 1 and P a prime divisor. Then P is a point, by Proposition 2.4.15. The structure sheaf of P, considered as a closed subscheme, is the skyscraper sheaf $k(P)$ (Example 1.1.23). Now observe that there is a canonical injection $\mathcal{L}(-P) \to \mathcal{O}_X$. We will define a map $\mathcal{O}_X \to k(P)$ as follows: given an open set U, if $U \ni P$ we take the mapping $\mathcal{O}_X(U) \to k(P)(U)$ to be $f \mapsto \bar{f}_P$, where f_P is the image of $f \in \mathcal{O}_X(U)$ in $\mathcal{O}_{X,P}$ and $\bar{f}_P$ is the image of f_P in the residue field of $\mathcal{O}_{X,P}$, which is isomorphic to $k(P)(U) = k$. If U does not contain P then $f \mapsto 0$. We claim that the following sequence is exact

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{O}_X \longrightarrow k(P) \longrightarrow 0 \tag{3.1}$$

To check this, observe that $\mathcal{L}(-P)_P = \{f_P \in \mathcal{O}_{X,P} \mid v_P(f) \geq 1\}$, thus $\bar{f}_P = 0$ for all $f_P \in \mathcal{L}(-P)$. Conversely, if $\bar{f}_P = 0$ for some $f_P \in \mathcal{O}_{X,P}$ then $v_P(f_P) \geq 1$, thus $f_P \in \mathcal{L}(-P)_P$. Exactness at $\mathcal{L}(-P)_P$ and $k(P)_P$ are easily checked, hence the induced map of stalks is exact at P. If $Q \neq P$ is a point, then $\mathcal{L}(-P)_Q = \mathcal{O}_{X,Q}$ and $k(P)_Q = 0$, and thus the induced sequence of stalks is exact at Q as well, and thus the sequence is exact.

Example 3.2.14. More generally, if D is any Weil Divisor and P is a point, then the following sequence is exact.

$$0 \longrightarrow \mathcal{L}(D) \longrightarrow \mathcal{L}(D+P) \longrightarrow k(P) \longrightarrow 0 \tag{3.2}$$

Indeed, tensoring 3.1 with $\mathcal{L}(D+P)$, we obtain the following sequence.

$$0 \longrightarrow \mathcal{L}(-P) \otimes \mathcal{L}(D+P) \longrightarrow \mathcal{O}_X \otimes \mathcal{L}(D+P) \longrightarrow k(P) \otimes \mathcal{L}(D+P) \longrightarrow 0 \tag{3.3}$$

We will show this is exact. Indeed, since tensor products commute with direct limits, at the level of the stalks the induced sequence is

$$0 \longrightarrow \mathcal{L}(-P)_Q \otimes \mathcal{L}(D+P)_Q \longrightarrow \mathcal{O}_{X,Q} \otimes \mathcal{L}(D+P)_Q \longrightarrow k(P)_Q \otimes \mathcal{L}(D+P)_Q \longrightarrow 0$$

Because tensoring is right exact, and since exactness is measured at the stalks, this means the sequence 3.3 is exact everywhere except possibly at $\mathcal{L}(-P) \otimes \mathcal{L}(D+P)$. Thus we need only check that $\mathcal{L}(-P) \otimes \mathcal{L}(D+P) \to \mathcal{O}_X \otimes \mathcal{L}(D+P)$ is injective. To see this, simply observe that the following diagram commutes

$$\begin{tikzcd}
\mathcal{L}(-P) \otimes \mathcal{L}(D+P) \arrow[r] \arrow[d] & \mathcal{O}_X \otimes \mathcal{L}(D+P) \arrow[d] \\
\mathcal{L}(D) \arrow[r] & \mathcal{L}(D+P)
\end{tikzcd}$$

where the downward pointing arrow on the left is an isomorphism by Theorem 3.2.12, the downward pointing arrow on the right is the canonical isomorphism, and the right-pointing arrow on the bottom is the canonical injection.

Finally, we identify the objects of 3.3 with those of 3.2. The isomorphisms $\mathcal{L}(-P) \otimes \mathcal{L}(D+P) \to \mathcal{L}(D)$ and $\mathcal{L}(D+P) \to \mathcal{O}_X \otimes \mathcal{L}(D+P)$ are discussed already. Now note that there is a canonical injection $k(P) \to k(P) \otimes \mathcal{L}(D+P)$. Since $\mathcal{L}(D+P)$ is locally free of rank 1, the induced map of stalks is the canonical isomorphism $k(P)_P \to k(P) \otimes_{\mathcal{O}_{X,P}} \mathcal{O}_{X,P}$ at P, and the identity on zero elsewhere, and thus the map $k(P) \to k(P) \otimes \mathcal{L}(D+P)$ is an isomorphism and the result follows.

3.3 Sheaves of Differentials

In this section, we will develop the theory of differentials, which is allows us access to the tools of calculus, similar to their use in differential geometry. In particular, we will define the sheaf of relative differentials, which is analogous to the sheaf of differential 1-forms (or equivalently the cotangent bundle, through the association in Theorem 1.2.17), on a smooth manifold. We will use this to define the tangent sheaf and the canonical sheaf.

A *variety* in this section will always refer to an abstract variety. We begin with a review of the theory of differentials:

Definition 3.3.1. Let A be a ring, B an A-algebra and N a B-module. An A-*derivation* of B is a map $d : B \to N$ such that

(a) $d(b_1 + b_2) = d(b_1) + d(b_2)$

(b) $d(b_1 b_2) = b_2 d(b_1) + b_1 d(b_2)$

(c) $d(a) = 0$ for all $a \in A$.

Example 3.3.2. Let $A := \mathbb{R}$, $B := D[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is differentiable}\}$ and $N := \mathrm{Hom}_{\mathsf{Sets}}([0,1], \mathbb{R})$. Then differentiation is an example of an A-derivation of B.

Example 3.3.3. Let M be a smooth manifold. A vector field X on M can be interpreted as a map $d_X : C^\infty(M) \to C^\infty M$ given by $f \mapsto Xf$, where Xf may be interpreted as the directional derivative of f along X (here we adopt the notation of [8]). This is an $\mathbb{R}$-derivation of $C^\infty(M)$; indeed,

$$X(fg) = fX(g) + gX(f)$$

See [8, pp. 180-182] for details.

Example 3.3.4. Continuing with the previous example, given an element $f \in C^\infty(M)$, define $\hat{f} \in \mathcal{T}^* M$, given by $\hat{f}(X) = X(f)$ where $\mathcal{T}^* M$ is the space of smooth 1-forms. By the previous example, the map $f \mapsto \hat{f}$ is a derivation.

Example 3.3.4 will be our motivating example. Given a morphism of schemes $f : X \to Y$, we will attempt to construct a sheaf on X, in a way that if we take $Y = \operatorname{Spec} k$ for some field k then the sections of the sheaf behave like differential forms. We require a module contruction first:

Theorem 3.3.5. *Let B be an A-algebra. Then there exists a B-module $\Omega_{B/A}$ and an A-derivation $d : B \to \Omega_{B/A}$ such that if N is another B-module with a derivation $d' : B \to N$ there exists a unique morphism of B-modules $f : \Omega_{B/A} \to N$ such that the following diagram commutes:*

$$\begin{tikzcd} B \arrow[d, "d"] \arrow[dr, "d'"] & \\ \Omega_{B/A} \arrow[r, "f"] & N \end{tikzcd}$$

Proof Sketch. We define $\Omega_{B/A}$ to be the free module generated by the set of symbols $\{db \mid b \in B\}$, subject to the relations:

- $d(b_1 + b_2) = d(b_1) + d(b_2)$
- $d(b_1 b_2) = b_2 d(b_1) + b_1 d(b_2)$
- $da = 0$ for any $a \in A$.

Details can be found at [6, pp. 172-173]. □

Definition 3.3.6. Let B be an A-algebra. We define the *module of relative differentials* of B over A to be the module $\Omega_{B/A}$ in the previous theorem, equipped with the derivation $d : B \to \Omega_{B/A}$.

It can be shown ([6, Ch. II Corollary 8.5]) that if B is a finitely-generated A-algebra, then $\Omega_{B/A}$ is a finitely-generated B-module

Definition 3.3.7. Let $f : X \to Y$ be a scheme. Let $U = \operatorname{Spec} A$ be an affine open subset of Y and let $V = \operatorname{Spec} B$ be an affine open subset of $f^{-1}(U)$. Then we define the *sheaf of relative differentials* $\Omega_{X/Y}$ of X over Y to be the following sheaf: on V we define $\Omega_{X/Y}(V) := (\Omega_{B/A})^{\sim}$. If $V' = \operatorname{Spec} B'$ is an open affine subset of V then we have a map $B \to B'$, and composing with the map $d' : B' \to \Omega_{B'/A}$ we have an A-derivation $B \to \Omega_{B'/A}$. By the above theorem, this induces a morphism $\Omega_{B/A} \to \Omega_{B'/A}$. We can check that this commutes with restriction, and that the sheaf axioms are satisfied. As U and V vary, the V form a base of X and thus we have a sheaf.

By construction, the sheaf $\Omega_{X/Y}$ is quasicoherent by Theorem 2.2.5. If X is a variety and $Y = \operatorname{Spec} k$, we will simply write $\Omega_{X/k}$ instead. Since, by assumption, X can be covered by affine open subsets $U = \operatorname{Spec} A$ where A is a finitely generated k-algebra, it follows that $\Omega_{A/k}$ is a finitely-generated A-module. A is also noetherian by Hilbert's Basis Theorem, thus it follows that $\Omega_{X/k}$ is coherent.

The sheaf $\Omega_{X/k}$ plays the role of the sheaf of differential 1-forms, which is the sheaf associated to the cotangent bundle on a manifold M (or equivalently it maps an open set U to the set of 1-forms on U). Indeed, note the following analogy with Example 3.3.4: we have seen that the map $f \mapsto \hat{f}$ is a derivation, and similarly we have a derivation $A \to \Omega_{A/k}$ for any affine open subset $\operatorname{Spec} A$ of X.

As another more subtle similarity: smoothness is, very loosely speaking, associated with having a well-defined, well-behaved tangent space at every point. For example, a smooth manifold M has a tangent bundle as a vector bundle, as we saw in Example 1.2.10. This means smoothness as defined in Definition 3.1.2 should in some way be encoded in the sheaf $\Omega_{X/k}$. Fortunately, this is the case. In fact:

Theorem 3.3.8. *Let X be a variety over k. Then X is smooth if and only if $\Omega_{X/k}$ is locally free.*

Proof. [6, pp. 177-178] □

By the association between locally free sheaves and vector bundles we saw in Theorem 1.2.17, this result agrees with our analogy.

To conclude this section, we will define two sheaves on a smooth variety.

Definition 3.3.9. Let X be a smooth variety over k. We define the *tangent sheaf* to be the sheaf $\mathcal{T}_X := \mathfrak{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$.

Since we saw that $\Omega_{X/k}$ is similar to the cotangent sheaf on a manifold, this definition makes sense.

Definition 3.3.10. Let X be a smooth variety of dimension n. We define the *canonical sheaf* ω_X on X to be $\bigwedge^n \Omega_{B/A}$, where $\bigwedge$ is the exterior power defined in Example 1.2.16. Note that this is invertible.

3.4 The Riemann Roch Theorem

In this section, we will provide a partial proof of the famous Riemann-Roch Theorem for curves, which relates the cohomology of invertible sheaves to an invariant known as the *genus*. We begin with some definitions.

Definition 3.4.1. A *curve* over a field k is a smooth abstract variety of dimension one, projective over k.

Note that since X is a curve, the canonical sheaf is simply equal to $\Omega_{X/k}$. However, we will still denote it ω_X.

Definition 3.4.2. Let X be a curve. We define the *genus* of X, denoted $p(X)$ to be $\dim_k \Gamma_X(\omega_X)$. Since ω_X is coherent, this is finite by Theorem 2.3.16.

Remark 3.4.3. The genus as defined above is often known as the *geometric genus*. There is another quantity known as the *arithmetic genus*, and for curves they are the same quantity ([6, Ch. IV, Proposition 1.1]). In higher dimensions, this may not hold.

The main ingredient in this proof is the Duality Theorem of Serre:

Theorem 3.4.4 (Serre Duality (Théorème 4 of [10])). *Let X be a curve over k, $\mathcal{F}$ a locally free sheaf and ω_X the canonical sheaf. Then for each i there is a natural isomorphism of vector spaces:*

$$H^i(X,\mathcal{F}) \cong H^{2-i}(X,\mathcal{F}^\vee \otimes \omega_X)^\vee$$

Now we state and prove the main theorem. The proof closely follows that of [6, p.295-296], but we will present it nonetheless, as it demonstrates how the techniques we have developed may be applied.

Theorem 3.4.5. *Let D be a divisor and K a canonical divisor on a curve X of genus g. Then:*

$$\dim \Gamma_X(\mathcal{L}(D)) - \dim \Gamma_X(\mathcal{L}(K-D)) = \deg D + 1 - g$$

Proof. Note that $\Gamma_X(\mathcal{L}(D)) = H^0(X,\mathcal{L}(D))$ by definition. Similarly, we have $\Gamma_X(K-D) = H^0(X,\mathcal{L}(K-D))$. By Theorem 3.2.12, it follows that $\mathcal{L}(K-D) = \omega_X \otimes \mathcal{L}(D)^\vee$. By the Serre Duality Theorem, we have $\dim H^0(X,\omega_X \otimes \mathcal{L}(D)^\vee) = \dim H^1(X,\mathcal{L}(D))$. Thus the formula reduces to $\chi(\mathcal{L}(D)) = \deg +1 - g$. We will prove this by induction on $\deg D$.

If $\deg D = 0$, then $D = 0$ and $\mathcal{L}(D) = \mathcal{O}_X$ by Theorem 3.2.12. Since ω_X is invertible, it follows from the Serre Duality Theorem that $g = \dim H^1(X,\mathcal{O}_X)$, and Proposition 2.3.4 implies $\dim H^0(X,\mathcal{O}_X) = 1$. Thus the formula evaluates to $g + 1 = g + 1 + 0$, which is obviously true.

Now suppose the formula holds for some D, and let P be a prime divisor, by Example 3.2.13 and Example 3.2.14 we have the exact sequences:

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{O}_X \longrightarrow k(P) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{L}(D) \longrightarrow \mathcal{L}(D+P) \longrightarrow k(P) \longrightarrow 0$$

By Proposition 2.3.18, we have

$$\chi(\mathcal{L}(D+P)) = \chi(\mathcal{L}(D)) + 1 \tag{3.4}$$

But by the inductive hypothesis, we have

$$\chi(\mathcal{L}(D)) = \deg D + 1 - g \tag{3.5}$$

and thus plugging 3.4 into 3.5 we obtain

$$\chi(\mathcal{L}(D+P)) = (\deg D + 1) + 1 - g = \deg(D+P) + 1 - g$$

and thus the formula is true for $D + P$.

Finally, we need to show that the formula holds for $D - P$. To do this, simply run the above argument through with D in place of $D + P$ and $D - P$ in place of D. The result follows. $\square$

Appendix A

Some Results from Algebra

Theorem A.0.1 (Adjoint Property of $\otimes_A B$)**.** *Let $A \to B$ be a ring homomorphism, M and A-module and N a B-module. Write N_A as N considered as an A-module via the homomorphism. Then $M \otimes_A B$ has a natural structure as a B-module and there is a natural isomorphism of groups*

$$\mathrm{Hom}_B(M \otimes_A B, N) \cong \mathrm{Hom}_A(M, N_A)$$

Proof. We can give $M \otimes_A B$ a natural B-module structure by defining, for $b \in B$ and $m \otimes b' \in M \otimes_A B$ multiplication as $b(m \otimes b') := m \otimes bb'$.

Now given some $\varphi : M \to N_A$, we define $\tau\varphi : M \otimes_A B \to N$ as $m \otimes b \mapsto b\varphi(m)$. Conversely, given $\psi : M \otimes_A B \to N$, we define $\sigma\psi : M \to N_A$ as $m \mapsto \psi(m \otimes 1)$.

We now verify that σ and τ are inverses of each other. Suppose $\varphi : M \to N_A$ is a module homomorphism, and define $\psi := \tau\varphi$. Then $\psi(m \otimes b) = b\varphi(m)$. But now $\sigma\psi(m) = \psi(m \otimes 1) = \varphi(m)$ so that $\sigma\psi = \sigma\tau\varphi = \varphi$.

Conversely, suppose $\psi : M \otimes_A B \to N$ is a module homomorphism, and now define $\varphi := \sigma\psi$ so that $\varphi(m) = \psi(m \otimes 1)$. Now observe that $\tau\varphi(m \otimes b) = b\varphi(m) = b\psi(m \otimes 1) = \psi(m \otimes b)$ so that $\tau\sigma\psi = \psi$ as required. □

Theorem A.0.2 (Snake Lemma)**.** *Let A, B, C, A', B', C' be objects and suppose we have the following commutative diagram,*

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

such that the rows are exact. Then there exists an exact sequence:

$$\begin{tikzcd}
\ker\alpha \arrow[r] & \ker\beta \arrow[r, "f"] & \ker\gamma \arrow[dll, rounded corners, "\delta"] \\
\operatorname{coker}\alpha \arrow[r, "g"] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma
\end{tikzcd}$$

Proof. The maps between the kernels and between the cokernels are straightforward. We construct δ as follows: we have $c \in \ker\gamma$. Then there exists $c_B \in B$ that maps to c in C by the surjectivity of $B \to C$. Now $\beta(c_B)$ is in the kernel of $B' \to C'$ since the image of c_B is 0 in C', and since the second row is exact, there exists some unique $c_A \in A'$ whose image in B' is $\beta(c_B)$. Moreover this c_B is defined uniquely up to addition by an element in the image of $A \to B$, and thus when pulled back to A', we see that c_A is defined uniquely as an element of $\operatorname{coker} \alpha$.

Now it remains to check that the resulting sequence is exact. Exactness everywhere except at $\ker \gamma$ and $\operatorname{coker} \alpha$ follow from the exactness of the first diagram. We now check exactness at the remaining two objects. Suppose firstly $c \in \operatorname{im} f$. Then there exists $x \in \ker \beta$ such that $f(x) = c$. In particular, that means $\beta(x) = 0$ and thus $\delta(c) = 0$ by the construction of δ. Conversely, suppose $c \in \ker \delta$ so that $\delta(c) = 0$ in the cokernel of α. We show that $c \in \operatorname{im} g$ as follows. By the surjectivity, we know that there exists some element $c_B \in B$ whose image in C is c. Then $\beta(c_B)$ as an element of A' is in the image of α, and thus there exists some $a \in A$ such that the image of a in B' is equal to $\beta(c_B)$, and thus we observe that $a - c_B$, as an element of B is in the kernel of β and clearly $f(a - c_B) = c$ as desired.

We now check exactness at $\operatorname{coker} \alpha$. Clearly $\operatorname{im} \delta \subseteq \ker g$. Now suppose $a \in \ker g$. Then $g(a) \in \ker \beta$ (here we are abusing notation and using $g(a)$ to denote both the element in $\operatorname{coker} \beta$ and a preimage in B'). We take a preimage b of $g(a)$ in $\ker \beta$, and clearly $f(b)$ is mapped to a by δ. This concludes the proof. $\square$

Lemma A.0.3 (Horseshoe Lemma). *Consider the diagram:*

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & I^0 \arrow[r] & I^1 \ldots \\
 & B \arrow[d] & & \\
0 \arrow[r] & C \arrow[r] \arrow[d] & J^0 \arrow[r] & J^1 \ldots \\
 & 0 & &
\end{tikzcd}$$

if the following sequence is exact:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and $A \to I^\bullet$ and $C \to J^\bullet$ are injective resolutions, there is an injective resolution $B \to (I \oplus J)^\bullet$ of B, where $(I \oplus J)^i = I^i \oplus J^i$.

Proof. We begin by showing that if I and J are injective, then so is $I \oplus J$. To see this, suppose we have an inclusion $i : A \to B$ and a map $\varphi : A \to I \oplus J$. Then composing φ with the projections $\pi_i : I \oplus J \to I$ and $\pi_j : I \oplus J \to J$ we have maps $\pi_i \circ \varphi : A \to I$ and $\pi_j \circ \varphi : A \to J$. By assumption, I and J are injective, so we have maps $B \to I$ and $B \to J$. By the universal property of products, this induces a map $B \to I \oplus J$. A diagram-chase will show that this map does indeed commute with i and φ.

Now we will check that $B \to (I \oplus J)^\bullet$ is an injective resolution. Firstly defining the map $B \to I^0 \oplus J^0$, we observe that since $A \to B$ is injective, there exists a map $B \to I^0$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & I^0 \\ B \arrow[ur] & \end{tikzcd} \tag{A.1}$$

And by the diagram, we have a natural map $B \to C \to J^0$. Thus by the universal property of products, this induces a map $\psi : B \to I^0 \oplus J^0$. To see that this map is injective, observe that if $b \in \ker \psi$, then the image of b is 0 in J^0, which means b is in the kernel of $B \to C$, since the map $C \to J^0$ is injective. This means $b \in A$. But since the kernel of $A \to I^0$ is trivial, hence $b = 0$ as desired. Exactness at $I^i \oplus J^i$ for $i > 0$ follows from the injectivity of the rows of A.1. $\square$

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