Introduction to Topological Quantum Computation

Marc Lescano

Department of Mathematics
The University of Auckland

Supervisor: Pedram Hekmati Co-supervisor: Jurij Volčič

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Abstract

In 2008, Alexander Kitaev and Chris Laumann discovered a fault-tolerant quantum computing model —topological quantum computing. The model is predicated upon a quasiparticle called an anyon, which is a two dimensional extension of fermions and bosons - exhibiting exotic spin statistics. The mathematical habitat of these particles is called a unitary modular tensor category (UMTC), encapsulating the representation of the adiabatic exchange operation of such particles in what is called, the braid group. This dissertation introduces the physical and mathematical realisation of anyons, other quantum computing models using adiabaticity and holonomy to give intuition and motivation on how UMTCs model the braiding of anyons to yield fault-tolerant topological quantum computations. Here, we show the usage of Fibonacci anyons and Ising anyons as examples of how one may form quantum gates and make a universal topological quantum computer.

To Pedram, and Jurij,
To Baylee, Jake, and the rest of my friends in academia,
To Cambell, Sam, Max, Jeffers, and the rest of my close friends,
To my parents, my sister, and my family,

Thank you.

Contents

Introduction

Today, quantum computers (QCs) have become a well-funded field of research in mathematics, physics, and engineering due to their overwhelmingly powerful potential in computational ability and efficiency. In contrast to classical computers, which typically compute in polynomial time, QCs can execute computations and algorithms that would otherwise take until the end of the universe to complete. This is due to the philosophically nonsensical, yet naively simple, properties of the quantum world. Such properties include *superposition* and *entanglement*.

The concept of superposition relies entirely on the fact that the ones interacting with the system — a collection of objects at a particular location at a given time, being the main point of study — do not know in any way what state the system has chosen. Instead, we take the collection of such possible outcomes for the system and *assume* it encompasses all such states, with some states being more probable than others. Such a mathematical description of a quantum system is referred to as a wavefunction. A classic example would be Schrödinger's cat. [1]

Similarly, classical bits are either 0 or 1. In quantum mechanics, we can assume that it can be both at the same time. However, if one were to observe and/or make some perturbation in the immediate surroundings of the QC, then the superposition property collapses. This is known as *wavefunction collapse*, and it poses significant challenges for creating such a computer.

In this paper, we aim to introduce a fault-tolerant model of quantum computing discovered by Kitaev and Laumann called topological quantum computing (TQC) [Kit97]. This model uses a new type of particle called an *anyon*; a two-dimensional extension of the more commonly known particles, the fermion and boson. Anyons were mathematically discovered by Wilczek [Wil91]. They were then physically observed by Stormer, Tsui and Laughlin in 1998 from the fractional quantum Hall effect [STG99]. Unlike its predecessors, anyons have frac-

[1] A nuclear isotope with a $1/2$ chance of decaying in an hour is placed next to a Geiger counter. which is connected to a contraption containing a vial of toxic bromine gas. Upon activation, the vial is dropped and released inside a closed box containing a cat. There is no way for anyone to know if the cat is dead or alive with some special tool, and so without opening the box, the cat has a $1/2$ chance of being either dead or alive. Quantum mechanics dictates that the cat is both dead and alive, *at the same time*.

tional spin. Since these particles are only realisable on a plane, we can track how they exchange with each other over time using the braid group.

We start by introducing the overarching postulates of quantum mechanics in the context of discrete states. Realistically, when one engineers a quantum computer, one cannot ignore the fact that the computer will interact with the system. Despite little description in this respect, it would be unfair not to introduce how one accounts for this decoherence. Density states are the formalism of experimental physicists and the engineers of this field, building actual quantum computers for differing models [Pac12]. We then introduce an abstract structure called a category. If group theory is the study of symmetry, category theory is the study of functions. Here, one can study the similar relationships of abstract structures and the main map of concern. Like groups and group homomorphisms, or topological spaces and the continuous maps between them.

Secondly, we introduce physical ideas that regard the notions of anyons. In the theoretical side of things, we look mainly towards the Aharonov-Bohm effect. Its importance is derived from showing how, on a two-dimensional space, anyons are just a charge q and a flux ϕ. We introduce other physical concepts from here, like the non-Abelian Berry phase. Together, these physical ideas unveil how the spin exchange statistics of anyons behave - collecting phase factors of moduli one upon exchanging. These exchanges are captured in holonomy theory and in turn, geometric phases.

Then, after introducing what quantum computing is in general, defining notions of a qubit — the quantum bit — and the fundamental model of QC, namely the quantum circuit model (QCM). Examples which are similar to topological quantum computing are then briefly introduced, modelled from the properties of adiabatic evolution and holonomic parallel transport. Such models are rightfully called adiabatic quantum computing and holonomic quantum computing. Adiabatic evolution is essentially the evolution of a wavefunction formed entirely on a degenerate ground state space, and the idea that one can evolve to *stay* in said space.

A world-line for our anyons restricted to a two-dimensional surface is a time-like curve in $\mathbb{R}^{2+1}$. We then introduce the mathematical playground of anyons, known as unitary modular tensor categories (UMTCs). These categories algebraically capture the fusing and braiding operation of the anyons. Showcased to avoid and simplify a potential jumble of symbols, we illustrate these operations in a graphical calculus, justified that we are merely illustrating the anyon's world lines. Turaev showed that UMTCs have a one-to-one correspondence to topological quantum field theories in $(2+1)$ dimensions ($(2+1)$-TQFT) [Tur94]. Some examples of UMTCs are provided, namely Fibonacci and Ising UMTCs.

Briefly, we give a short description of what a quantum field theory is and, more importantly, state the axioms that form an anomaly-free $(2+1)$-TQFT. In condensed matter physics, anyons are *physically* realised in a phenomena called the fractional quantum hall effect. The Hall effect forms a linear relationship between a metal plate with a current in a magnetic field. The quantum Hall effect is a quantisation of this linear relationship, where the conductivity R_{xy} is related to what is called the Landau level ν by

$$R_{xy} = \nu^{-1} \frac{h}{q_e^2}. \tag{0.0.1}$$

Each ν gives a different anyon, each with its unique magnetic flux and charge. For $\nu = 5/2,\ 12/5$, it is hypothesised that one may find Ising and Fibonacci anyons, respectively. The state of matter that physically realises these anyons is called a fractional quantum hall liquid (FQH liquid). Mathematically, we describe how, for an FQH liquid restricted onto a two-dimensional surface, we can separate the properties of the surface according to its global and local properties. This idea of *separating the local and global* properties is justified in the categorical language of anyons, as it is in the global factor of our space and in UMTCs that we can conduct measurements of braiding and fusing.

Finally, we reach the pinnacle, describing what exactly topological quantum computing is. Provided with examples using Fibonacci and Ising anyons. How then is this model fault-tolerant? The problem of wavefunction collapse due to local interactions with the immediate surroundings is a local operation. But in separating the surface and rewriting the surface as a product of two spaces, containing the global and local properties, the local interactions are local operations on the surface. Since we have built the habitat where anyons reside, in the space concerning the global properties, such interactions are deemed irrelevant, i.e., fault-tolerant. A physical attempt of engineering an actual QC uses a kind of particle called a Majorana fermion. This particle is infact, modelled by Ising anyons.

Chapter 1

Preliminaries

1.1 The Postulates of Quantum Mechanics

We shall introduce the postulates of quantum mechanics as given in [BD06]. With these postulates, we can mathematically describe, and more importantly *play*, with our quantum systems. Dirac notation will be utilised. The following postulates describe a quantum system.

Note that we present these postulates for discrete systems; for this paper, it will suffice. Should one try to obtain a continuous definition, alongside some adjustments, one can swap the discrete sums for integrals.

Let $\mathcal{H}$ be a topologically separable complex Hilbert space with an inner product $\langle \phi \,|\, \psi \rangle$.

Postulate 1.1.1. *The state of an isolated physical system is mathematically represented by a state vector $\psi \colon \mathbb{R} \to \mathcal{H}$. This vector $|\psi(t)\rangle$ belongs to a Hilbert space $\mathcal{H}$, which one often refers to as the state space.*

Postulate 1.1.2. *The Hilbert space of a composite system is the tensor product of the state spaces, being Hilbert spaces, associated with the component systems. The component systems are the individual particles for a non-relativistic system consisting of a finite number of particles.*

In physics, it is common practice to notate the tensor product by disregarding it and write

$$|\psi\rangle \otimes |\phi\rangle = |\psi\rangle|\phi\rangle = |\psi\phi\rangle.$$

Definition 1.1.3. *A state from a composite system is **quantum entangled** if it cannot be factored as a tensor product of states from its constituents.*

Example 1.1.4. *Let $\mathcal{H} = span_{\mathbb{C}}(|0\rangle, |1\rangle) \otimes span_{\mathbb{C}}(|0\rangle, |1\rangle)$. Then the state $|\psi\rangle = (|0\rangle \otimes |0\rangle)/\sqrt{2} + (|1\rangle \otimes |1\rangle)/\sqrt{2}$ cannot be factored into the form $|\phi\rangle \otimes |\varphi\rangle$, ergo quantum entangled.*

The next lot of postulates introduces and develops the notion of measurement.

Postulate 1.1.5. *Every measurable physical quantity $\mathcal{A}$ is described by some Hermitian operator $A\colon \mathcal{H} \to \mathcal{H}$. The eigenvectors of A form an orthonormal basis for $\mathcal{H}$, and the values in the spectrum of A are the physical quantities one can measure.*

The expectation value of a quantum system $|\psi\rangle \in \mathcal{H}$, with respect to the observable A, can be written as $\langle\psi\rangle_A = \langle\psi|A|\psi\rangle$.

Definition 1.1.6. *Let $\mathcal{H} = span(\{\psi_i\}_{i\in\mathbb{N}})$. We call elements of this Hilbert space $|\psi\rangle \in \mathcal{H}$* ***wavefunctions****. They take the form*

$$|\psi\rangle = \sum_{i\in\mathbb{N}} c_i|\psi_i\rangle.$$

For this to be mathematically viable to allow for physical interpretation, we must normalise the wavefunction such that $|\langle\psi \,|\, \psi\rangle|^2 = 1$.

Postulate 1.1.7. *When the physical quantity $\mathcal{A}$ is measured on a system with the wavefunction $|\psi\rangle$, the probability of obtaining an eigenvalue a_n or α - discrete and continuous spectra respectively - of the corresponding observable A is given by the amplitude squared of the appropriate wavefunction (projection onto the corresponding eigenvector)*

$$\mathbb{P}(a_n) = |\langle a_n \,|\, \psi\rangle|^2 \qquad \text{——}\textit{Discrete, non-degenerate spectrum} \tag{1.1.1}$$

$$\mathbb{P}(a_n) = \sum_{i=1}^{g_n} |\langle a_{ni} \,|\, \psi\rangle|^2 \qquad \text{——}\textit{Discrete, degenerate spectrum} \tag{1.1.2}$$

Where for each eigenvalue a_n of A, there are up to g_n eigenstates.

Remark 1.1.8. *For some Hilbert state space and $|\psi\rangle$ a wavefunction, $|\psi\rangle \in \mathcal{H}^*$ is a linear functional mapping $\mathcal{H} \to \mathbb{C}$ with the property $\langle\psi| = |\psi\rangle^\dagger$. Then we define the* ***density state*** *as $p = |\psi\rangle\langle\psi|$. Physically, this formalism of states incorporates the environmental contribution - as will be shown below. Regarding the experimental side of the paper (which will be slightly talked about) the experimental formalism of quantum computing is spoken in the language of density states.*

Postulates 1.1.4 and 1.1.6 together form what is more commonly known as **Born's Rule**.

Postulate 1.1.9. *(**Wavefunction Collapse**) If the measurement of the physical quantity $\mathcal{A}$ on the system in the state $|\psi\rangle$ yields some eigenvalue a_n, then the system's state, immediately after the measurement, is the normalised projection of $|\psi\rangle$ onto the eigensubspace correlated with the eigenvalue a_n - where P_n is the projection operator for a_n.*

$$|\psi\rangle \xrightarrow{a_n} \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}} \tag{1.1.3}$$

Remark 1.1.10. *If the eigenvalue a_n has some degeneracy with orthonormal eigenstates, call them $\{|a_{ni}\rangle\}_{i=1}^{m}$, then the projection operation becomes*

$$P_n = \sum_{i=1}^{m} |a_{ni}\rangle\langle a_{ni}| \tag{1.1.4}$$

The following postulate regards the time evolution of the quantum system. There are two such postulates, but they are equivalent. Both are stated for their importance in quantum mechanics, but the latter will be used more.

Postulate 1.1.11. *Let $\psi\colon [0.\infty) \to \mathcal{H}$ and $H(t)\colon \mathcal{H} \to \mathcal{H}$ is Hermitian. The time evolution for some $t > 0$ of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger Equation,*

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{1.1.5}$$

Postulate 1.1.12. *The time evolution of a closed system is governed by a unitary matrix $U(t) = \exp(i\mathcal{H}t)$ on an initial state $|\psi(t_0)\rangle$.*

$$|\psi(t)\rangle = U(t)|\psi(t_0)\rangle \tag{1.1.6}$$

The postulate regards the type of particles in *everyday life*. However, the extension of this postulate allows this paper to delve into particles called *anyons* naively.

Postulate 1.1.13. *For a system consisting of N identical particles, in a 3D space, the wavefunction is either symmetric or anti-symmetric, with respect to particle exchange. In this way, we define the fundamental particles **bosons** and **fermions**.*

Let $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ be wavefunctions and let $P\colon \mathcal{H} \to \mathcal{H}$ by $|x\rangle \otimes |y\rangle \mapsto |y\rangle \otimes |x\rangle$. Then we have:

$$P(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle \text{—— } \textit{Bosons} \tag{1.1.7}$$
$$P(|\psi\rangle \otimes |\phi\rangle) = -|\phi\rangle \otimes |\psi\rangle \text{—— } \textit{Fermions} \tag{1.1.8}$$

In a many-body system, there are often too many parameters. Thus, making numerical analysis, let alone closed-form analysis, impractical for making time-local deterministic calculations. However, if one treats the many-body system as one system, one can form a notion of a ground state, and the first few excited states are easier to find. Therefore, we make a definition.

Definition 1.1.14. *A **quasiparticle** is a low-lying excitation of a many-body system which exhibits particle-like properties and can be mathematically treated as a particle.*

1.2 Category Theory

Anyons, the necessary quasiparticles for topological quantum computation (TQC), exist in an abstract structure called a ribbon fusion category (RFC). We follow the notes from [Awo10].

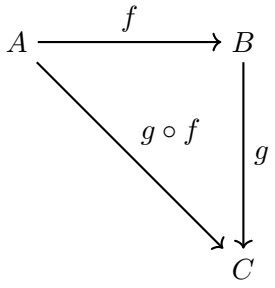
Definition 1.2.1. *A **category** $\mathcal{C}$ contains a collection of objects $Ob(\mathcal{C})$ (or just $\mathcal{C}_0$) and morphisms (arrows) $hom(\mathcal{C})$ which have the following properties*

- *For each morphism f, it has the following data:*

$$dom(f) \qquad cod(f)$$

  where for $f\colon A \to B$, $A, B \in Ob(\mathcal{C})$, $A = dom(f)$ and $B = cod(f)$.

- *Let f, g be morphisms in $\mathcal{C}$ such that $f\colon A \to B$ and $g\colon B \to C$. Define a composition operation $\circ\colon hom(\mathcal{C}) \times hom(\mathcal{C}) \to hom(\mathcal{C})$ such that $(g \circ f)\colon A \to C$*



- *There is a morphism called the identity morphism 1_A for each $A \in Ob(\mathcal{C})$ such that $1_A\colon A \to A$.*

- *The composition operator is associative. That is, for any f, g, h morphisms in $\mathcal{C}$, where $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$ and A, B, C, D are objects ,*

$$(h \circ g) \circ f = h \circ (g \circ f) \tag{1.2.1}$$

- *The identity morphism behaves as a unit, that is, for a morphism $f\colon A \to B$,*

$$f \circ 1_A = f = 1_B \circ f \tag{1.2.2}$$

As intuition, some examples would be the collection of groups as objects and homomorphisms as our morphisms, the collection of topological spaces and continuous maps, and the collection of rings and ring homomorphisms.

1.2.1 Functors and Morphisms

One can then consider a category where our objects are categories. The morphisms of this category are called functors.

Someone interested in confusing people may ask: "Is there a way to go from one category to another, preserving the morphisms?" There is!

Definition 1.2.2. *Let $\mathcal{C}, \mathcal{D}$ be categories. Then we define $F\colon \mathcal{C} \to \mathcal{D}$ so objects in $\mathcal{C}$ maps to objects in $\mathcal{D}$ and morphisms in $\mathcal{C}$ map to morphisms to $\mathcal{D}$ such that:*

- $F(f\colon A \to B) = F(f)\colon FA \to FB$
- $F(1_A) = 1_A$
- $F(g \circ f) = F(g) \circ F(f)$

*Such an F is called a **functor**.*

These levels of abstraction can continue. For instance, we define the following.

Definition 1.2.3. *Let $\mathcal{C}, \mathcal{D}$, be categories and $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. We define a **natural transformation** $\vartheta\colon F \to G$ as a family of arrows in $\mathcal{D}$*

$$(\vartheta_C\colon FC \to GC)_{C \in \mathcal{C}_0} \tag{1.2.3}$$

such that, for $f\colon C \to C'$ a morphism in $\mathcal{C}$, we have that $\vartheta_{C'} \circ F(f) = G(f) \circ \vartheta_C$ and the following diagram commutes:

$$\begin{array}{ccc} FC & \xrightarrow{Ff} & FC' \\ {\scriptstyle \vartheta_C}\downarrow & & \downarrow{\scriptstyle \vartheta_{C'}} \\ GC & \xrightarrow[Gf]{} & GC' \end{array}$$

Definition 1.2.4. *Let F, G be functors. A **natural isomorphism** is a natural transformation $\vartheta\colon F \to G$ that is an isomorphism in the functor category $F(\mathcal{C}, \mathcal{D})$.*

One can use the following lemma to characterise the notion of a natural isomorphism.

Lemma 1.2.5. *A natural transformation $\vartheta\colon F \to G$ is a natural isomorphism if and only if, for all $C \in \mathcal{C}_0$, $\vartheta_C\colon FC \to GC$ is an isomorphism.*

The following proof was left as an exercise.

Proof. Let F, G be functors and $\vartheta\colon F \to G$ be a natural transformation.

($\Rightarrow$): Let $\vartheta\colon F \to G$ be a natural isomorphism. Then there is a $\varphi\colon G \to F$ such that $\vartheta^{-1} = \varphi$. Then since φ is a family of natural transformations $(\varphi_C\colon FC \to GC)_{C\in\mathcal{C}_0}$, we have that for any $C \in \mathcal{C}_0$, $\varphi_C = \vartheta_C^{-1}\colon F \to G$. So we have that $\vartheta_C \circ \varphi_C = 1_{GC}$, $\varphi_C \circ \vartheta_C = 1_{FC}$ and since $C \in \mathcal{C}_0$ is arbitrary, this holds for all $C \in \mathcal{C}_0$.

($\Leftarrow$) : Fix $C \in \mathcal{C}_0$, let $\vartheta_C\colon FC \to GC$ be an isomorphism. Then there, is a two-sided morphism $\varphi_C\colon GC \to FC$ such that $\varphi_C = \vartheta_C^{-1}$. We want to show now that φ_C is a natural transformation. Since ϑ is natural, we have that $\vartheta_{C'} \circ F(f) = G(f) \circ \vartheta_C$ for any $C \in \mathcal{C}_0$. By pre-composing and post-composing accordingly, we have $F(f) \circ \vartheta_C^{-1} = \vartheta_{C'}^{-1} \circ G(f)$. Observe that $\vartheta_C^{-1} = \varphi_C$. So we are left with $F(f) \circ \varphi_C = \varphi_{C'} \circ G(f)$ and the collection of all such φ_C is a natural transformation φ. By construction, we have that $\varphi \circ \vartheta = 1_F$ and $\vartheta \circ \varphi = 1_G$; thus φ is a two-sided inverse of ϑ, i.e., ϑ is a natural isomorphism.

□

1.2.2 Types of categories

Definition 1.2.6. *A **fusion category** $\mathcal{C}$ is a rigid semisimple $\mathbb{C}$-linear monoidal category with finitely many isomorphism classes of simple objects such that the monoidal unit is simple.*

That is, a fusion category is a category where the following properties hold:

Definition 1.2.7. *A **monoidal/tensor category** is a category equipped with*

- *the functor, often called the tensor product,*

$$\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \tag{1.2.4}$$

- *the unit object $1 \in \mathcal{C}$,*
- *the associator a, which is a natural isomorphism such that, for all $x, y, z \in \mathcal{C}_0$,*

$$a_{x,y,z}\colon ((x \otimes y) \otimes z) \xrightarrow{\cong} (x \otimes (y \otimes z)) \tag{1.2.5}$$

- *the left unitor λ, which is a natural isomorphism such that, for all $x \in \mathcal{C}_0$,*

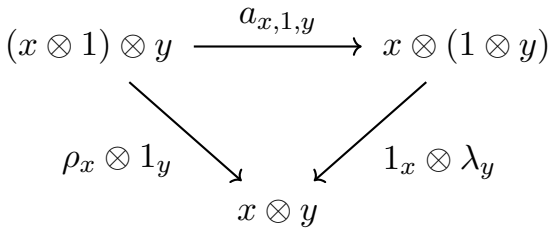
$$\lambda_x \colon 1 \otimes x \xrightarrow{\cong} x \tag{1.2.6}$$

- *the right unitor ρ, which is a natural isomorphism such that, for all $x \in \mathcal{C}_0$,*

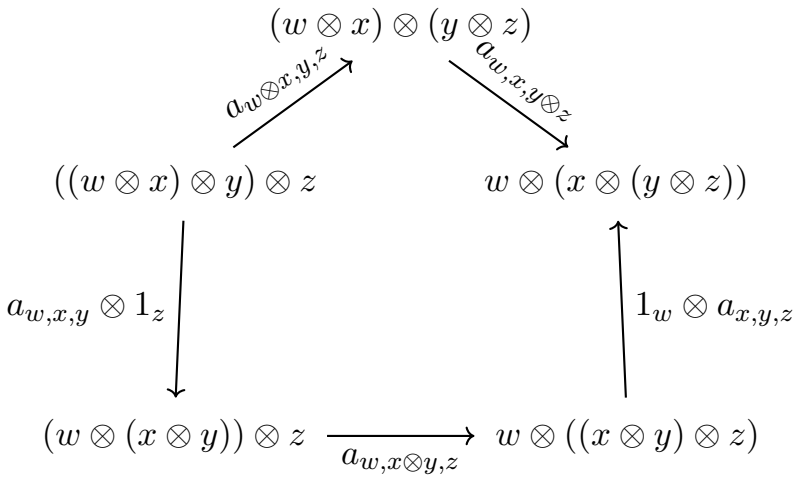
$$\rho_x \colon x \otimes 1 \xrightarrow{\cong} x \tag{1.2.7}$$

Which allow the following diagrams to commute

- *the triangle identity*



- *the pentagon identity*



Definition 1.2.8. *A **right rigid monoidal category** $\mathcal{C}$ is one where every object x in the category has a right dual $x*$ and is equipped with the natural isomorphisms $ev_x \colon x \otimes x^* \to 1$ and $coev_x \colon 1 \to x^* \otimes x$ such that*

$$(ev_x \otimes 1_x) \otimes (1_x \otimes coev_x) = 1_x \qquad (1_x \otimes coev_{x^*}) \otimes (ev_{x^*} \otimes 1_x) = 1_{x^*} \tag{1.2.8}$$

The definition of a left rigid monoidal category follows mutatis mutandis.

Definition 1.2.9. *A monoidal category that is left and right rigid is called* ***rigid****.*

Definition 1.2.10. *A* ***semisimple*** $\mathbb{C}$***-linear category*** *is a category in which all objects in the category can be expressed as a direct sum of simple objects, given that the direct sum exists - this is semisimple property. For all objects* $x, y \in \mathcal{C}_0$*, the hom-set between two objects,* $Hom(x, y) \cong \mathbb{C}^d$ *- where* d *is the number of morphisms between two objects - this is* $\mathbb{C}$*-linearity.*

Remark 1.2.11. *Let* $\mathcal{C}$ *be a category. Then an object* $x \in \mathcal{C}_0$ *is simple, if and only if,* $Hom(x, x) \cong \mathbb{C}$*.*

The developments to ribbon fusion categories, pivotal categories and the like, all the way to a unitary modular tensor category, will be done in Chapter 4 to marry the physics and the mathematics together to motivate the entire process.

Chapter 2

Geometrical and Topological Phases

2.1 Phase factors from Gauge Fields

The Aharonov-Bohm effect (ABE) describes a phase shift when the charged particle circulates a magnetic flux tube, inaccessible to the particle.

2.1.1 Charged particle in a magnetic field

Let $\mathbf{A}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the magnetic vector potential and define the magnetic field $\mathbf{B}$ as

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.1.1}$$

In fact, we can introduce an $\omega\colon \mathbb{R}^3 \to \mathbb{R}$ such that

$$\mathbf{B} = \nabla \times (\mathbf{A} + \nabla\omega) = \nabla \times \mathbf{A}. \tag{2.1.2}$$

This holds due to the identity $\nabla \times \nabla\omega = 0$. The fact that $\mathbf{B}$ is invariant by our choice of ω is called a **gauge invariance**, and $\mathbf{A}$ is called a **gauge field**. On that note, if there are two magnetic vector potentials $\mathbf{A}$ and $\mathbf{A}'$ that both yield the same magnetic field upon being curled upon, then they are **gauge equivalent**.

Consider a particle with charge q at position $\mathbf{r} = (x, y, z)$ moving along a looping trajectory in a magnetic field. The non-relativistic Hamiltonian operator is given by:

$$\mathcal{H}^A = -\frac{\hbar}{2m}\left(\nabla - i\frac{q}{c\hbar}\mathbf{A}\right)^2. \tag{2.1.3}$$

This is the minimal coupling prescription; a Hamiltonian used in the context of electromagnetism. Suppose we have an eigenbasis $|\psi(\mathbf{r})\rangle$ from our Hamiltonian operator when $\mathbf{A} = 0$.

By perturbing the system with a magnetic vector potential, one can show that

$$|\psi^A\rangle = \exp\left(i\frac{q}{c\hbar}\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'\right)|\psi\rangle, \tag{2.1.4}$$

where $\mathbf{r}_0$ is an arbitrary reference point and the integral is evaluated along a path from $\mathbf{r}_0$ to $\mathbf{r}$. We take $|\psi\rangle$ to be in the ground state. As the particle moves, it obtains a momentum, thus it acquires kinetic energy. If it moves fast enough, it acquires too much energy and goes into an excited state. Hence, we assume that the particle moves adiabatically, i.e., slow enough so as to stay in the ground state. If the particle moves in a loop C instead, then the wavefunction obtains a phase

$$\varphi = \frac{q}{c\hbar}\oint_C \mathbf{A}\cdot d\mathbf{r}. \tag{2.1.5}$$

Invoking Stokes' theorem, the phase can be written as

$$\varphi = \frac{q}{c\hbar}\iint_{S(C)}(\nabla\times\mathbf{A})\cdot d\mathbf{s} = \frac{q}{c\hbar}\underbrace{\iint_{S(C)}\mathbf{B}\cdot d\mathbf{s}}_{\phi,\text{ Magnetic flux}} = \frac{q}{c\hbar}\phi. \tag{2.1.6}$$

Where $d\mathbf{r}$ is an elementary segment of the loop C, $S(C)$ is a surface enclosed by C and $d\mathbf{s}$ is a surface element of $S(C)$ and ϕ is the magnetic flux of the magnetic field $\mathbf{B}$ going through $S(C)$. In natural units, $\hbar = c = 1$, one has

$$\varphi = q\phi. \tag{2.1.7}$$

Observe that φ is gauge-invariant under $\mathbf{A}$. Also, observe that ϕ depends on the geometry of our loop C, but is invariant under deformations of C that keep the ϕ fixed.

2.1.2 Aharonov-Bohm Effect

Consider the same system as in section 2.1.1. Restrict the particle with charge q on an infinitesimally thin plane $(x, y, 0) \in \mathbb{R}^3$. The magnetic field is confined to an infinitely long, infinitely thin and impenetrable tube that is orthogonal to the plane. Then a vector potential that matches is:

$$\mathbf{A}(\mathbf{r}) = \left(\frac{-y\Phi}{2\pi r^2}, \frac{x\Phi}{2\pi r^2}, 0\right) \tag{2.1.8}$$

Using Equation 2.1.2, one yields

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}}\Phi\delta(r) \tag{2.1.9}$$

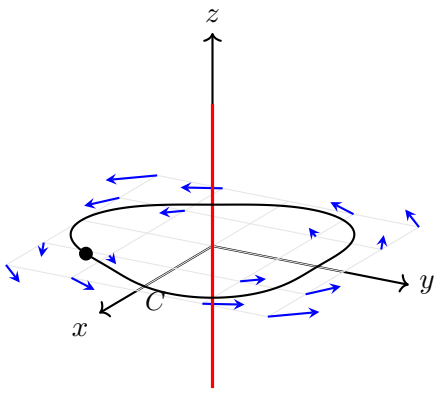


Figure 2.1: Visual aid for the setup of the ABE. The blue arrows indicate the magnetic field and the red line pointing upwards indicates the magnetic field. The black dot being the particle, is shown to traverse around the magnetic field. The magnetic field going through the loop forms a flux ϕ.

Where $|\mathbf{r}| = r$. Consequently, the magnetic field is null everywhere except $r = 0$ and the vector potential is parallel to the plane. Since this is a spatial restriction to the scenario before, all the equations prior still hold. Since the magnetic field is confined in a tube, we refer to the magnetic field by its flux ϕ. Every circulation the particle of charge q undertakes, a integer multiple of a phase factor $\varphi = q\phi$ is accumulated. That is, following from equations 2.1.4 and 2.1.7,

$$|\psi^A\rangle = e^{inq\phi}|\psi\rangle, \qquad n \in \mathbb{N}. \tag{2.1.10}$$

Observe that the overall phase factor is dependent only on the winding number, charge and flux. That is, the shape of the path is irrelevant.

The main idea with the ABE is that potentials, in quantum mechanics and quantum field theory, are more fundamental than the fields themselves. The charged particle, as it travels in its loop, interacts with the vector potential in a region with no magnetic field. As loops around the flux are equivalent up to homotopy, Peshkin and Tonomura observed the ABE by making a loop large enough to observe in a macroscopic world, thus proving this phenomenon in 1989 [Ton89].

2.1.3 Anyons and the Aharonov-Bohm Effect

Definition 2.1.1. *An **anyon** is a quasiparticle that is confined to a plane characterised by the duple (q, ϕ), where ϕ is the flux and q is the charge such that*

$$|\psi_1\psi_2\rangle = e^{2i\varphi}|\psi_2\psi_1\rangle. \tag{2.1.11}$$

Here, we assume $\varphi = q\phi$ is non-trivial, i.e., $\varphi \neq 0, \pi$.

Suppose we have two anyons of identical charge and flux. One needs only to consider the fundamental group of the punctured disk, i.e., $\mathbb{Z}$ and this can faithfully describe the statistics of the particles. We discuss this more in Chapter 4.

As a thought experiment, consider a configuration in which two anyons are next to each other. Then if one anyon, say anyon a, circles the other anyon b, then a obtains a phase factor. But since the flux of a circled the charge of b, then the system overall obtains 2 phase factors.

By using Maxwell's equations and Coulomb's law, one can use the charges of the anyons to show that the angular momentum from the electric field is $\mathbf{J}_E = \frac{q\Phi}{2\pi c}\hat{\phi}$. The total angular momentum is then $\mathbf{J} = n\hbar\hat{\phi} + \mathbf{J}_E$ — the left term is the canonical angular momentum. If there is no contribution from the canonical angular momentum, i.e., we set $n = 0$, then since spin is intrinsic then it must be that the spin of the particle is

$$s = \frac{q\Phi}{2\pi}. \tag{2.1.12}$$

Here, s is the effective, non-trivial spin — holds only when either q or Φ is fractionalized.

2.2 Geometric phases and Holonomies

In 1984, Michael Berry showed that there are physical cases where gauge transformations cannot remove non-dynamic phase factors - implying that a physical intuition is required for such phase factors.

2.2.1 Spin-1/2 Particles in a Magnetic Field

Consider a magnetic field $\mathbf{B}(\theta, \phi)$, with direction parameterised by the longitudinal and azimuthal angles ϕ and θ, and with a constant non-zero magnitude B. We place a particle of spin 1/2 at the origin, orienting the spin vector to be parallel to the magnetic field. Note, the magnetic field is here for the sole purpose of controlling the orientation of the spin. The Hamiltonian for such a particle interacting with the magnetic field is

$$\mathcal{H} = -\boldsymbol{\sigma} \cdot \boldsymbol{B}(\theta, \phi) = -\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}(\theta, \phi)B \tag{2.2.1}$$

where $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ gives the direction of the magnetic field, and $\boldsymbol{\sigma}$ are the usual Pauli spin matrices. Another way to write the Hamiltonian is

$$\mathcal{H} = -\mathcal{U}(\theta, \phi)\sigma^2\mathcal{U}^\dagger(\theta, \phi)B = \mathcal{U}(\theta, \phi)\mathcal{H}_0\mathcal{U}(\theta, \phi) \tag{2.2.2}$$

where $\mathcal{H}_0 = -\sigma^2 B$ and $\mathcal{U} \in SU(2)$ such that

$$\mathcal{U}(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{2.2.3}$$

The eigenstates and eigenvalues $E_\uparrow$ and $E_\downarrow$, for some orientation of $B(\theta,\phi)$, are

$$|\uparrow(\theta,\phi)\rangle = \mathcal{U}(\theta,\phi)|\uparrow\rangle, \qquad E_\uparrow = B \tag{2.2.4}$$
$$|\downarrow(\theta,\phi)\rangle = \mathcal{U}(\theta,\phi)|\downarrow\rangle, \qquad E_\downarrow = -B \tag{2.2.5}$$

Prepare a system with an initial direction of the magnetic field at some (θ,ϕ), and a particle in its up state $|\uparrow(\theta_0,\phi_0)\rangle$. When we change the orientation magnetic field slow enough, the system will adapt to this change as $|\uparrow(\theta,\phi)\rangle$. This is true provided adiabatic movement. This was given by Born and Fock in 1928 [Max28]. When the magnetic field returns to its original direction, the Hamiltonian is as what it was before. Using the Schrödinger equation and given that the Hamiltonian is identical when the particle returns to its initial state, up to a phase factor, one has

$$e^{i\phi} = e^{\oint_C \boldsymbol{A}\cdot d\boldsymbol{r}} e^{iE_\uparrow T} \tag{2.2.6}$$

With

$$A_\mu = \langle\uparrow|\mathcal{U}^\dagger(\theta,\phi)\frac{\partial}{\partial\lambda^\mu}\mathcal{U}(\theta,\phi)|\uparrow\rangle \tag{2.2.7}$$

where $\mathbf{A}$ is the Berry connection, playing a similar role to the magnetic vector potential. Indexed by $\mu = 1, 2$, $\lambda^\mu = \{\theta,\phi\}$, and C is a cyclic path with T as the total time evolution.

The first term, the geometrical phase, regards the path spanned in the parametric space. More importantly, it is independent on $\mathcal{H}_0$. The second term depends on $E_\uparrow$. If we add a constant to the Hamiltonian, we can omit this term.

We are now in a position to find the geometric phase. We have it that

$$A_\theta = \langle\uparrow|\mathcal{U}^\dagger(\theta,\phi)\frac{\partial}{\partial\theta}\mathcal{U}(\theta,\phi)|\uparrow\rangle = 0 \tag{2.2.8}$$

and

$$\langle\uparrow|\mathcal{U}^\dagger(\theta,\phi)\frac{\partial}{\partial\phi}\mathcal{U}(\theta,\phi)|\uparrow\rangle = \frac{i}{2}(1-\cos\theta) \tag{2.2.9}$$

For the down state, the derivation follows mutatis mutandis but with an additional factor of -1. The field strength is given by the curvature of the vector potential

$$F_{\theta\phi} = \partial_\theta A_\phi - \partial_\phi A_\theta = \frac{i}{2}\sin\theta \tag{2.2.10}$$

Using Stokes' Theorem, it yields

$$\phi_g = -i \oint \mathbf{A} \cdot d\mathbf{r} = -i \int\int_{S(C)} \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2} \int\int_{S(C)} d\theta d\phi \sin\theta = \frac{1}{2}\Omega(C) \tag{2.2.11}$$

where $\Omega(C)$ is the solid angle of the loop spanned on the unit sphere, by the normal vector.

Let $\mathbf{A}$ be the Berry connection and $|\Psi\rangle$ be gauge transformed by:

$$\mathbf{A}(\theta,\phi) \to \mathbf{A}(\theta,\phi) - \nabla\omega(\theta,\phi), \tag{2.2.12}$$

$$|\Psi(\theta,\phi)\rangle \to e^{i\omega(\theta,\phi)}|\Psi(\theta,\phi)\rangle \tag{2.2.13}$$

where $\omega(\theta,\phi)$ is a scalar function. The Berry phase $\phi = \phi_g$ is gauge invariant. If the vector potential has no force, i.e., $\mathbf{F} = 0$ for all $\phi \in [0, 2\pi)$ and $\theta \in [0, \pi)$, then we say that the potential is a *pure gauge*.

Consider the geometry of the space of states $|\Psi\rangle$ parameterised by λ^μ. Take B, the magnitude of the magnetic field, as the radius of the sphere on which we define Ω. If $B = 0$, then the up and down states are degenerate. We require such points to create non-trivial curvature in the Hilbert space.

Similarly, but not completely identical, the process to achieve the geometrical phase is equivalent to that mentioned with the ABE, yielding the phase factor. However, the same topological properties hold. The distinction between the two phases is the choice of coordinates. In the ABE, the coordinates are Cartesian. Here, they are spherical, with the property that one can create sufficiently complex Hamiltonians so that the coordinates of a quasiparticle form this parametric space.

2.2.2 Non-Abelian Geometric Phases

Previously, for our minimal coupling prescription Hamiltonian, our solutions were a product of a complex phase on the eigenbasis of the unperturbed system. In particular, if we keep making the particle loop around, we can just keep multiplying by the exponential term. Since it is a complex number, it commutes - the order in which the particle traverses is irrelevant. This means our phase is abelian. Now, we introduce the notion of a Berry phase to introduce the non-Abelian phase. We can also use operators which correspond to the system's evolution given that it is an adiabatic and cyclic process.

2.2.2.1 Holonomy

Definition 2.2.1. *A unitary evolution, manifested mathematically as a a time-dependent unitary operator $U(t)$, is **isospectral** if the spectrum of $U(t)$ is constant.*

Consider a parametric space $\mathcal{M} = \{\lambda^\mu : \mu = 1, \ldots, d\}$ with λ^μ being the parameters, and the D-dimensional Hamiltonian

$$\mathcal{H}(\lambda(t)) = \mathcal{U}(\lambda(t))\mathcal{H}_0\mathcal{U}^\dagger(\lambda(t)). \tag{2.2.14}$$

Here $U(\lambda(t)) \in \mathrm{U}(D)$ is a time-dependent unitary rotation. The parameters λ^μ can be controlled externally as they are classical parameters. We assume that the ground state of $\mathcal{H}_0$ consists of an n-dimensional degenerate subspace, $\mathbb{H}_0 = \{|\Psi^\alpha\rangle : \alpha = 1, \ldots, n\}$, with $E_0 = 0$, and $\Delta E = E_1$ separating the subspace from the excited states. We also ensure that the velocities of the parameters abide by the adiabatic requirement.

Observe that as the particle travels along C in $\mathcal{M}$ the parametric space, it is equivalent to a unitary matrix $\Gamma_{\boldsymbol{A}}(C)$ acting on the initial state. That is,

$$|\Psi(C)\rangle = \Gamma_{\boldsymbol{A}}(C)|\Psi(0)\rangle, \tag{2.2.15}$$

where $|\Psi(C)\rangle, |\Psi(0)\rangle \in \mathbb{H}_0$, $\Gamma_{\boldsymbol{A}}(C)$ is the non-abelian geometric phase. This is also known as a **holonomy**, defined as

$$\Gamma_{\boldsymbol{A}}(C) = \mathbf{P}\exp\left(\oint_C \mathbf{A}\cdot d\boldsymbol{\lambda}\right) \tag{2.2.16}$$

where $\mathbf{P}$ is a path ordering. The path ordering is essential in quantum field theory since our parameters are dependent on time, such that

$$\mathbf{P}\exp\left(\oint_C \mathbf{A}\cdot d\boldsymbol{\lambda}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\int\cdots\int}_{n\text{ times}} d\lambda_1\ldots d\lambda_n A(\lambda(t_1))\ldots A(\lambda(t_n)) \tag{2.2.17}$$

The time ordering symbol rearranges the multiplication order according to the time. Notice there is no i as it is absorbed into $\boldsymbol{A}$. So $\boldsymbol{A}$ is an anti-Hermitian operator defined as

$$(A_\mu)^{\alpha\beta} = \langle\Psi^\alpha|\,\mathcal{U}^\dagger(\lambda)\frac{\partial}{\partial\lambda^\mu}\mathcal{U}(\lambda)\,|\Psi^\beta\rangle \tag{2.2.18}$$

This is the non-abelian generalisation of the Berry phase. Note that if the matrix is 1×1, then we have the Berry phase given prior. This was discovered by Wilczek and Zee in 1984 [WZ84] and the derivation can be found in [Pac12]

2.2.3 Properties of geometric evolutions

Since the geometric evolutions resemble interactions with gauge fields, they **should** (and they do) have gauge-invariants. These evolutions are parametrised by loops C; C thus inherits holonomic properties. More so, the structure of the parametric space $\mathcal{M}$ determines the form of the connection $\mathbf{A}$.

2.2.3.1 Gauge Transformations

Consider a Hamiltonian with isospectral transformations as in Equation 2.2.2:

$$\mathcal{H}(\lambda) = \mathcal{U}(\lambda)\mathcal{H}_0\mathcal{U}^\dagger(\lambda) \tag{2.2.19}$$

and, for $g(\lambda) \in U(n)$ a gauge transformation, the unitary transformation is

$$U(\lambda) \to U^g(\lambda) = g^{-1}(\lambda)U(\lambda)g(\lambda) \tag{2.2.20}$$

leaves the Hamiltonian $\mathcal{H}(\lambda)$ invariant. For non-trivial transformations, this only happens if $g(\lambda)$ is acting exclusively on $\mathbb{H}_0$, the degenerate ground state-space of the Hamiltonian. On this subspace, the transformation should act trivially. I.e., for all $\lambda \in \mathcal{M}$ we have

$$g^{-1}(\lambda)\mathcal{H}_0 g(\lambda) = \mathcal{H}_0. \tag{2.2.21}$$

In the perspective of $U(\lambda)$, the action of $g(\lambda)$ is simply the reparametrisation of the λ's. Thus one yields the following gauge transformations due to their commutativity:

$$\mathbf{A} \to A^g = g^{-1}\mathbf{A}g + g^{-1}\mathbf{d}g. \tag{2.2.22}$$

For the holonomy,

$$\Gamma_{\mathbf{A}} \to \Gamma_{\mathbf{A}^g} = g^{-1}\Gamma_{\mathbf{A}}g. \tag{2.2.23}$$

Thus, in these new coordinates due to transformation,

$$|\Psi\rangle \to |\Psi^g\rangle = g^{-1}|\Psi\rangle \tag{2.2.24}$$

Implying that the action of holonomic evolution on a state has a coordinate-free formalisation.

2.2.3.2 Loop parametrisation

Let λ_0 be a point in $\mathcal{M}$. Then $\mathcal{L}_{\lambda_0} = \mathcal{L}$ is the space of all loops fixed at λ_0. That is, for any $C \in \mathcal{L}$, $C(0) = C(T) = \lambda_0$.

Proposition 2.2.2. *$\mathcal{L}$ fixed at $\lambda_0 \in \mathcal{M}$ is a group with the binary operation defined as:*

$$(C_2 \cdot C_1)(t) = \theta\left(\frac{1}{2} - \frac{t}{T}\right) C_1(2t) + \theta\left(\frac{t}{T} - \frac{1}{2}\right) C_2(2t - T) \tag{2.2.25}$$

for any $C_1, C_2 \in \mathcal{L}$.

Heuristically, we have that closure follows from the fact that the step functions make it so each loop is traversed entirely, and the other loop has no contribution when one loop is at focus. The identity loop is $C_0(t) = \lambda_0$ for all t. The inverse of a loop $C \in \mathcal{L}$ is the same loop, but traversed oppositely. That is, $C^{-1}(t) = C(T - t)$. Associativity in this setting asks effectively,"does it matter if we post-concatenate or pre-concatenate with the product $C_1 \cdot C_2 \cdot C_3$?" In which case, it does not.

Let $\Gamma_{\mathbf{A}}(C) = \mathbf{P} \exp\left(\oint_C \mathbf{A} \cdot d\boldsymbol{\lambda}\right)$. Then consider $C_1, C_2 \in \mathcal{L}$. Then

$$\begin{aligned}
\Gamma_{\mathbf{A}}(C_1 \cdot C_2) &= \mathbf{P} \exp\left(\oint_{C_1 \cdot C_2} \mathbf{A} \cdot d\boldsymbol{\lambda}\right) \\
&= \mathbf{P} \exp\left(\oint_{C_1} \mathbf{A} \cdot d\boldsymbol{\lambda} + \oint_{C_2} \mathbf{A} \cdot d\boldsymbol{\lambda}\right) \\
&= \mathbf{P} \exp\left(\oint_{C_1} \mathbf{A} \cdot d\boldsymbol{\lambda}\right) \cdot \mathbf{P} \exp\left(\oint_{C_2} \mathbf{A} \cdot d\boldsymbol{\lambda}\right) \\
&= \Gamma_{\mathbf{A}}(C_1)\Gamma_{\mathbf{A}}(C_2)
\end{aligned}$$

So $\Gamma_{\mathbf{A}}(C)$ is a group homomorphism. Moreover, $\Gamma_{\mathbf{A}}(C_0) = \mathbb{1}$, $\Gamma_{\mathbf{A}}(C^{-1}) = \Gamma_{\mathbf{A}}^{-1}(C)$ and for $f\colon [0,T] \to [0,T]$ monotone, we have $\Gamma_{\mathbf{A}}(C \circ f) = \Gamma_{\mathbf{A}}(C)$

2.2.3.3 Holonomies as Unitary Matrices

Definition 2.2.3. *From the path dependence of the holonomies, we have it that Hol($\boldsymbol{A}$) = $\Gamma_{\boldsymbol{A}}(\mathcal{L}) \leq U(n)$. This subgroup is called the holonomy group of the connection $\boldsymbol{A}$. When Hol($\boldsymbol{A}$) = $U(n)$, then $\boldsymbol{A}$ is an irreducible connection.*

To consolidate whether this is the case, consider the curvature, ie., the effective magnetic field **F** from the connection **A** with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{2.2.26}$$

If all $F_{\mu\nu}$ can span the whole $U(n)$ algebra, then we say **A** is irreducible [Nak03]. This question of irreducibility is correlated with the universality of quantum computers. If a quantum computer uses holonomies to perform computations, we need to know if we can build any algorithm by simply combining different holonomic "paths" (or sequences of them).

Definition 2.2.4. *For a given model of quantum computing. If, by some change in parameters unique to the model, it is possible to create any quantum gate, then we say that the model is **universal**.*

This is not to be confused with a universal set of gates.

We now provide a few properties. Given that the holonomies act on the degenerate space of ground states, regarding time evolution under adiabatic processes, if certain states are **not** involved in the system, then the holonomy $\Gamma_{\mathbf{A}}(\mathcal{L})$ will act trivially on it. I.e., if $\mathcal{U} \in U(n)$ acts trivially on an element of the degenerate subspace, then so will the correlated holonomy.

Suppose that we remove the adiabatic requirement. Then the connection $\mathbf{A}$ is not projected to any subspace of states, from the Hamiltonian. Therefore, it is a pure gauge that gives the trivial holonomy group. Only when $\mathbf{A}$ is non-trivially projected to a subspace of states, give rise to a non-trivial geometric phase.

2.2.4 Anyons and Geometric Phases

In this chapter, we introduced the ABE, giving a mathematical foundation to understand the abelian phase. We can generalise our understanding of the phase with the Berry phase to introduce the non-Abelian phase. These notions of phase are well-founded in TQC as braiding. In this setting, however, the obtained holonomy depends on the path the particle traverses. One can, in fact, form a model of quantum computing using holonomies. By changing the loop, we obtain different holonomies, i.e., we obtain different unitary matrices. This model we call universal. That is, we need only change or alter the loops that the anyon spans. These loops are unique parameters to the model, yielding desired gates and algorithms.

Regarding adiabaticity, the idea is that for a fixed Hamiltonian, we can make the anyon travel along the ground state at a slow enough rate, keeping our anyon in the degenerate ground state. Instead of making the anyon travel in a fixed Hamiltonian, we slowly change the Hamiltonian by slowly changing a certain parameter (say $\lambda \in [0, 1]$). In this new parameterised Hamiltonian $\mathcal{H}(\lambda)$, we define a degenerate ground state space for all λ — this is our computational space. In fact, by constructing a parameterised Hamiltonian with such a ground space, we can *set* the initial state as an eigenstate of the initial Hamiltonian and our solution to be an eigenstate of the final Hamiltonian. This model of quantum computing is in fact, universal, as we shall see in Chapter 3.

What is captured here that is carried over in TQC is the notion of our anyons braiding each other to obtain a phase. Instead of the phase being dependent on (say) the path, as in the Berry phase, the phase is entirely intrinsic to what type of anyon we are dealing with. As with the abelian phase, it is path independent, but with non-abelian phases. Furthermore, a method of fault-tolerant quantum computing is the energy gap. This gap must be non-zero and we assure that we do not cross this gap by moving our anyons along the path adiabatically. We also want an efficient mechanism for anyonic statistics that can be physically interpreted in topological systems.

Chapter 3

Quantum Computation

Modern computational models are based on the universal (classical) Turing machine; conceptualised by Alan Turing in 1937. Such a machine can simulate any other machine capable of carrying out an algorithmic process. The idea of a quantum Turing machine was proposed by David Deutsch (1985). Around the same time, Feynman — wanting to simulate quantum systems using classical Turing machines — proposed that such a machine would be more effective than the classical kind.

Instead of the classical bit, we have a qubit; a linear superposition of $|1\rangle$ and $|0\rangle$. Quantum mechanics also allows superpositions of multiple qubits and entangled states. Such properties dramatically increases the dimension of the encoding space.

Evolution/manipulation of qubits is done by quantum gates. These gates are unitary operations, performing any desired state transformation.

The problem is physically making them. One needs to be very accurate when initializing states, acting quantum gates and obtaining a final state. Errors can come in from inaccuracies in performing such actions, or environmental perturbations. Shor, 1995, and Steane, 1996, theoretically proposed a quantum error correction model - correcting errors to perform meaningful quantum computations. But this model is still technologically unrealizable as of this date. Hence, the motivation to propose a model for quantum error correction.

Some models include those that are founded on adiabatic transitions [Far+01], geometric phases [ZR99], and main model of concern for this paper, topological evolutions [Kit97].

In this chapter, we wil follow from [Pac12].

3.1 Qubits and their manipulations

3.1.1 Quantum Bits

Definition 3.1.1. *A **qubit** is the Hilbert space $\mathcal{H} = span_{\mathbb{C}}(|0\rangle, |1\rangle)$ and a **qubit state** is a wavefunction $|\phi\rangle \in \mathcal{H}$. A **qudit** is the Hilbert space $\mathcal{H}_d = span_{\mathbb{C}}(|0\rangle, |1\rangle, \ldots, |d\rangle)$ and a **qudit state** is a normalized wavefunction in said space.*

Qubits are written as a linear superposition of 0 and 1 by

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \tag{3.1.1}$$

where $|a_0|^2 + |a_1|^2 = 1$. One can compose qubits together to yield the following wavefunction:

$$|\psi\rangle = \sum_{i_1,\ldots,i_n \in \mathbb{Z}_2^n} a_{i_1,\ldots,i_n} |i_1, \ldots, i_n\rangle. \tag{3.1.2}$$

Here, $|i_1, \ldots, i_n\rangle = |i_1\rangle \otimes \cdots \otimes |i_n\rangle$ with complex coefficients.

Corollary 3.1.2. *A tensor product of qubits has 2^n basis vectors. So for each new qubit, the dimension grows exponentially. Classical computers have dimension $2n$. Making precise the fact that quantum computers operate at exponential time as opposed to classical computers in polynomial time.*

An immediate consequence of this is that tensoring is the emergence of entangled qubit states. An example is the following maximally entangled qubit state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{3.1.3}$$

3.1.2 Decoherence and mixed states

For motivation: when the quantum state of a system is not exactly known, then classical probabilities mix with the amplitudes of the quantum states. Quantum decoherence is the effect of the quantum system interacting with the surroundings, resulting in the system *leaking* its quantum properties.

If we omit the environmental contributions in the quantum system, we have a system subject to unitary evolutions. If we do take such contributions into account, then the evolutions of the system are non-unitary. Such systems are described by so-called density matrices.

Consider an orthonormal basis of states $\{|\psi_i\rangle\}$ of some system. Then we have the density matrix given by

$$\rho = \sum_{i,j} p_{ij} |\psi_i\rangle\langle\psi_j| \tag{3.1.4}$$

Where probability that the state $|\psi_i\rangle$ occurs is p_{ii}. Implying that $\sum p_{ii} = 1$, giving $\text{tr}(\rho) = 1$. The expectation value of an operator $\mathcal{O}$ with respect to ρ is

$$\langle\mathcal{O}\rangle = \text{tr}(\rho\mathcal{O}) = \sum_i p_i \langle\psi_i|\mathcal{O}|\psi_i\rangle \tag{3.1.5}$$

These matrices give the most general description of a quantum system. For a system in some pure state $|\psi\rangle$,

$$\rho = |\psi\rangle\langle\psi| \tag{3.1.6}$$

On the experimental aspect of quantum computing, the most common formalism uses density matrices. This is due to the natural interaction of the qubits with the immediate surroundings and environment. This is a major contributor to error.

Analogous to 1.1.3, if the density matrix cannot be written like (3.1.6), then we say the system is in a mixed state.

Example 3.1.3. *Recall that a qubit is a Hilbert space over $\mathbb{C}$ spanned by $|0\rangle$ and $|1\rangle$. Then take the pure state*

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$$

So for $\theta \in [0, 2\pi)$, the density matrix is

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix}$$

As the system evolves, the state ρ will evolve and it will interact with the immediate surroundings. Then the phase θ is perturbed uniformly over the interval $[0, 2\pi)$. The measurement that one observes is the expected value. But the average of a collection of points that lie on the circle is 0. So we yield

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

. Observe that this is not a pure state. Suppose $|\psi\rangle = A|0\rangle + B|1\rangle$, $A, B \in \mathbb{C}$. Then the off diagonals are $A^\dagger B$ and $B^\dagger A$. If the expected value was 0, then it implies that A or B is zero. For our chosen pure state, this is clearly not he case. In fact, this means that the density matrix no longer corresponds to a wavefunction. Rather, it reflects the classical scenario in which there is a $1/2$ chance of either measuring $|0\rangle$ or $|1\rangle$.

Remark 3.1.4. *If the computational state becomes mixed due to the interactions with the environment (or from lack of knowledge about the control procedure thereof) then we will yield erroneous results in our quantum computations.*

3.1.3 Quantum Gates and Projectors

Quantum gates in general are 2×2 unitary matrices.

Example 3.1.5. *Examples of quantum gates are the three Pauli spin matrices. Observe that the matrix* $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *is the classical NOT gate, so* $|0\rangle \xrightarrow{\sigma^x} |1\rangle$ *and* $|1\rangle \xrightarrow{\sigma^x} |0\rangle$

Example 3.1.6. *(Hadamard Gate) A more quantum mechanically relevant gate is the Hadamard gate*

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.1.7}$$

, where

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

If we want to create entanglement between two qubits, we can introduce the following collection of gates denoted with CU. Such gates treat one qubit as a controller, and the other the target. Where if we act a gate from CU on $|0\rangle$, then nothing happens and if it acts on $|1\rangle$, a unitary transformation is acted upon the state. There are known as control gates.

Let $\mathcal{H}$ be spanned by $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ over $\mathbb{C}$. Control gates take the form $\text{diag}(\mathbb{1}_2, U)$, where $U \in U(2)$.

Example 3.1.7. *(controlled-NOT gate, (CNOT)) Take* $U = \sigma^x$*, such that*

$$\textit{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.1.8}$$

Example 3.1.8. *(controlled-phase gate (CP)) Take* $U = diag(1, -1)$*, such that*

$$\textit{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.1.9}$$

To remark, CU gates are unitary. Also, due to the "controlled" nature of the gates, one can generate entangled states.

Example 3.1.9. *Let* $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$. *Then*

$$\textit{CNOT}|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{3.1.10}$$

Observe that this is a maximally entangled state of two qubits.

Another useful gate from classical machines is the SWAP gate such that $|ij\rangle \xrightarrow{\text{SWAP}} |ji\rangle$.

We can also have multi-qubit gates with more than one target and/or control qubit. When we say we can realise a set of gates, it means that we can apply them to any qubits we want - in the system.

Besides unitary operators, we can also have projectors. This is a set of operators $\{P_i\}$ such that

$$P_i^2 = P_i \text{ and } P_iP_j = 0,\ i \neq j. \tag{3.1.11}$$

These operators give back the component attached to each state. They are used as mathematical tools for measuring qubits.

Example 3.1.10. *Consider the following operator*

$$P_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.1.12}$$

For wavefunctions of the form $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$, *one yields* $|a_0|^2 = tr(|\psi\rangle\langle\psi|P_0)$. *For* $P_1 = |1\rangle\langle 1|$ *then, we have* $P_1 = \mathbb{1}_2 - P_0$.

Example 3.1.11. *For a projection across a more general direction, we have* $P = |\psi\rangle\langle\psi|$ *with* $|\psi\rangle = \cos\theta|0\rangle + e^{i\theta}\sin\theta|1\rangle$. *The identity operator is as follows*

$$\mathbb{1} = \sum_{n=1}^{N} |\psi_n\rangle\langle\psi_n|, \tag{3.1.13}$$

where $\{|\psi_n\rangle, n \in \mathbb{N}_{\leq N}\}$ *is a complete orthonormal set of basis states.*

Example 3.1.12. *Let* $P = |0\rangle\langle 0| + |1\rangle\langle 1|$ *and* $|\psi\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle$, $|a_1|^2 + |a_2|^2 + |a_3|^2 = 1$, *then rank*$(P) = 2$ *such that*

$$P|\psi\rangle = \frac{a_0|0\rangle + a_1|1\rangle}{\sqrt{|a_1|^2 + |a_2|^2}}$$

Example 3.1.13. *Let $P = |\psi\rangle\langle\psi|$ with wavefunction $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$. Then any two-qubit state is projected onto this maximally entangled state.*

Another application for using projector operators is to mathematically define Hamiltonians with a very specific property(s).

Example 3.1.14. *Let $|\psi_0\rangle$ be a ground state for some Hamiltonian $\mathcal{H}$. Suppose we know a set of Hermitian projectors $\{P_i\}$ for $i \in \mathbb{N}_{\leq k}$. These projectors project onto subspaces which are not orthogonal to each other. If they all share a common state, namely the ground-state $|\psi_0\rangle$ such that $P_i|\psi_0\rangle = |\psi_0\rangle$ for all i, then the Hamiltonian takes the form*

$$\mathcal{H} = -\sum_{i=1}^{k}(\mathbb{1} - P_i), \tag{3.1.14}$$

where the $|\psi_0\rangle$ is the ground state with null-energy as its eigenvalue.

3.2 Quantum Circuit Model

We present the overarching model of quantum computing. It is the main goal of one who formulates a model of quantum computing to show its equivalence to this model.

In classical computation, information is represented with either a 0 or a 1 called *bits*. A *gate* is just a function $f\colon \mathbb{Z}_2 \to \mathbb{Z}_2$. One can take multiple bits into consideration and generalize to functions of the form $f\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^m$. A composition of such gates is referred to as a *circuit* and the act of giving an input and yielding an output is called a *computation*.

We can alter this idea to that of quantum computers. First we have qubits being superpositions of $|0\rangle$ and $|1\rangle$ i.e., qubit states. Then we have unitary matrices acting on our qubits, these are the quantum gates. Then a product of quantum gates leads to a quantum circuit, and the actual act of giving and receiving a qubit state from our quantum circuit is a quantum computation. The question however, is how does one physically realize this.

3.2.1 Algorithms and Quantum Universality

Definition 3.2.1. *An **algorithm** in quantum computing is a methodical sequence of quantum gates on a qubit state with the goal of solving a specific problem.*

What one wants would be a finite set of unitary matrices such that any algorithm can be built. For that matter, we first want that any quantum gate can be built given our finite set of matrices. So we want a notion of universality. For n qubits, we define the following.

Definition 3.2.2. *A set of quantum gates $\{U_i\}_{i\in I} \subset U(2^n)$ is called **universal** if it is able to approximate any $\tilde{U} \in U(2^n)$ by matrix multiplication within some error $\delta > 0$, that is,*

$$|U_{i_{k_1}} U_{i_{k_2}} \dots U_{i_{k_m}} - \tilde{U}| < \delta \tag{3.2.1}$$

for some elements $U_{i_{k_1}} U_{i_{k_2}} \dots U_{i_{k_m}}$ in the set.

In practice, it suffices to do analysis on $\mathrm{SU}(2^n)$ as measured states in quantum computing are identical up to a phase factor. The following theorem, corollary, and their respective proofs are keys to proving universality for models of quantum computing.

Theorem 3.2.3. ***(Solovay-Kitaev)*** *Let G be a finite set of elements in $SU(2)$ such that $\overline{\langle G \rangle}$ is dense in $SU(2)$. For any $\varepsilon > 0$, there is a $c > 0$ so that for any $U \in SU(2)$, there is a product of elements $V \in H$ of length $O(\log^c(1/\varepsilon))$ such that $\|V - U\| < \varepsilon$.*

Proof. Given in [Kit97]. □

Corollary 3.2.4. *The Solovay-Kitaev theorem can be extended to $G \subset SU(d)$.*

Proof. Given in [BG21]. □

We provide some well-known algorithms in quantum computing, one of which is Shor's factoring algorithm. It allows one to determine the prime factors of any integer in $O((\log(N))^3)$. This was given in 1997 by Shor. This algorithm is especially important because most encryption methods rely heavily on prime factorisation such as RSA-encryption. As of now, there exists no efficient classical algorithm capable of finding prime factors. [Sho97]. Another algorithm is Grover's searching algorithm. As opposed to the linear time dependence on classical computers, Grover's algorithm allows one to find the correct answer in $O(\sqrt{N})$ [Gro96].

3.2.2 Computational Complexity

To recap, there are two classes of problems with classical computers. Suppose we have a problem with some computational input size. If the problem can be solved in polynomial time, it is considered a **P** problem. The other class is consists of problems that can be solved only in exponential time. A larger class of problems are those whose solutions can be verified in polynomial time, but do not necessarily have an algorithm that can solve them in polynomial time. Such problems are called **NP**.

With the mathematical formulation of quantum computers, we have a new class of problems — those that are polynomially easy to solve with a quantum computer, **BQP**. However, the non-emptiness of this class has been proven difficult to answer. To do so requires a quantum computer, which is still under development. The relationship of BQP type problems with other

classes has also proven to be difficult to establish. For instance, are NP problems also BQP problems?

3.3 Other Computational Models

As we have seen, one can form an algorithm by implementing fixed unitary matrices representing time evolutions via fixed quantum gates in sequence, as given in the quantum circuit model. Surprisingly, this is not the only way to form a quantum algorithm. This is due to the exotic properties of quantum mechanics.

Some models are of importance to us due to their similarities and capabilities of intertwining with topological properties to form new computational methods. Some include adiabatic quantum computation from Farhi et al. in 2001 [Far+01] and holonomic quantum computation from Zanardi and Rasetti in 1999 [ZR99].

3.3.1 Adiabatic Quantum Computation

In this setting, we process algorithms using the adiabatic process. We take a Hamiltonian with a non-degenerate ground state and some energy gap to the first excited state. We encode information in the initial ground state and as time progresses for the Hamiltonian, the final ground state yielded is the desired result from the algorithm.

The way to make this process adiabatic is through the rate at which we change the parameter to reach the final ground state. The rate of change of the state of the qubit with respect to time corresponds to a kinetic energy, and this must be very small relative to the energy gap to the first excited state. In this way, the initial states traverse to the final state entirely in the ground state.

To be more precise, consider the initial and finial Hamiltonians $\mathcal{H}_i$ and $\mathcal{H}_f$ and define a parametrised Hamiltonian $\mathcal{H}(\lambda)$ as follows:

$$\mathcal{H}(\lambda) = (1-\lambda)\mathcal{H}_i + \lambda\mathcal{H}_f, \tag{3.3.1}$$

where $\lambda \in [0,1]$ is monotonic in time, like $\lambda = t/T$. $\mathcal{H}_i$ is a simple and known so $|\psi_i\rangle$ is the initial state and $\mathcal{H}_f$ is formed such that $|\psi_f\rangle$ is the final state that is the desired result of the algorithm. The requirement of adiabaticity is that the particle moves slowly enough to not become excited. But if the Hamiltonian changes, then the energy gap changes as well. So we require instead the Hamiltonian to transition slow enough such that the energy gap is kept

relatively constant. One way of keeping this is to make T big.

An important remark is that the energy gap between the ground state of $\mathcal{H}$ and the first excited state must always be non-zero for any $\lambda \in [0,1]$. With our time-dependent state, we require that $|\psi(0)\rangle = |\psi_i\rangle$. Due to adiabaticity, for some large enough T, we have that $|\psi(T)\rangle$ is close to $|\psi_f\rangle$ — our target state.

Example 3.3.1. *Consider the following initial and final Hamiltonians, respectively*

$$\mathcal{H}_i = -\sigma^z \otimes \mathbb{1} - \mathbb{1} \otimes \sigma^z \text{ with our ground state } |\psi_i\rangle = |00\rangle \tag{3.3.2}$$

$$\mathcal{H}_f = -\sigma^z \otimes \sigma^z - \sigma^x \otimes \sigma^x \text{ with our ground state } |\psi_f\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right). \tag{3.3.3}$$

Define then the parametrised Hamiltonian to be

$$\mathcal{H}(\lambda(t)) \coloneqq (1-\lambda)(-\sigma_1^z - \sigma_2^z) + \lambda(-\sigma_1^z\sigma_2^z - \sigma_1^x\sigma_2^x). \tag{3.3.4}$$

By increasing λ slowly from 0 to 1 relative to the smallest energy gap ΔE from $\mathcal{H}(\lambda)$, we get $|\psi_i\rangle$ into a maximally entangled $|\psi_f\rangle$. One can show that for all values of $\lambda \in [0,1]$, we have that $\Delta E \neq 0$ as required.

Proof. We proceed like a usual eigenvalue and eigenvector problem. Hence

$$\mathcal{H} = (1-\lambda)\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -2+\lambda & 0 & 0 & -\lambda \\ 0 & \lambda & -\lambda & 0 \\ 0 & -\lambda & \lambda & 0 \\ -\lambda & 0 & 0 & 2-3\lambda \end{pmatrix}$$

for $\lambda \in [0,1]$. We have our eigenstates of $\mathcal{H}(\lambda)$ as

$$\left(\frac{2-2\lambda+\sqrt{5\lambda^2-8\lambda+4}}{\lambda}, 0, 0, 1\right),$$
$$\left(\frac{2-2\lambda-\sqrt{5\lambda^2-8\lambda+4}}{\lambda}, 0, 0, 1\right),$$
$$(0,-1,1,0),$$
$$(0,1,1,0)$$

with respective eigenvalues $-\sqrt{5\lambda^2-8\lambda+4}-\lambda, \sqrt{5\lambda^2-8\lambda+4}-\lambda, 2\lambda, 0$. Observe that the first eigenstate is the ground state (unnormalized). Normalizing, we have that $|\psi(\lambda)\rangle = \frac{f(\lambda)}{\sqrt{(f(\lambda))^2+1}}|00\rangle + \frac{1}{\sqrt{(f(\lambda))^2+1}}|11\rangle$ - where $f(\lambda) = \frac{2-2\lambda+\sqrt{5\lambda^2-8\lambda+4}}{\lambda}$. When $\lambda \to 0^+$, $f(\lambda) \to$

∞. Hence $|\psi(\lambda \to 0)\rangle = |00\rangle = |\psi_i\rangle$ as expected. We also have $f(1) = 1$ which gives $|\psi(1)\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ also as required.

Notice that the corresponding eigenvalue is bounded above by -1 and the other eigenvalues are bounded below by 0. Hence there is always a non-zero energy gap for all $\lambda \in [0, 1]$.

□

Instead of being measured in terms of the number of quantum gates used, the computational complexity of an adiabatic quantum computation is measured in terms of the overall time of the evolution T. This T is naturally dependent on how many qubits are used n.

Using algorithms such as Grover's algorithm, Roland and Cerf showed that the energy gap can decrease at some point in the transition [RC02]. For such moments, it is imperative that the transition be slowed down to be sure that adiabaticity is preserved throughout.

This form of quantum computation is indeed equivalent to the circuit model [Aha+05]. That is, algorithms that can be expressed in the circuit model can be realised in adiabatic quantum computation. Known algorithms for quantum computers are transcribed into a Hamiltonian like in 3.3.1.

When we come to discuss topological quantum computation (TQC), one may realise the similarity whereby we construct our computational space with a constant energy gap, and the quasiparticle's coordinates give the control parameters λ of the Hamiltonian. In TQC, we require that the anyons are evolved adiabatically so that they do not leave the degenerate ground state.

3.3.2 Holonomic Quantum Computation

From chapter two, recall that we obtained unitary transformation from holonomies along loops in the parametric space of some quantum system. The fundamental idea of holonomic quantum computation (HQC) is then that we use these holonomies as our quantum gates.

Consider a quantum system governed by a parametrised Hamiltonian $\mathcal{H}(\mathbf{z})$, where $\mathbf{z}$ is from our parametric space which we take to be $\mathbb{C}^2$. Quantum logic gates are given as non-Abelian geometric phases which act on the degenerate ground state of $\mathcal{H}(\mathbf{z})$. For one-qubit holonomic gates, we use a three-level system such that $|\alpha\rangle$, where $\alpha = 0, 1, 2$, is subject to the Hamiltonian

$$\mathcal{H}(\mathbf{z}) = \mathcal{U}(\mathbf{z})\mathcal{H}_0\mathcal{U}(\mathbf{z})^\dagger \text{ where we define } \mathcal{H}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.3.5}$$

where $\mathcal{U}(\mathbf{z}) \in U(2)$ is an arbitrary unitary matrix defined to be

$$\mathcal{U}(\mathbf{z}) = U_1(z_1)U_2(z_2), \tag{3.3.6}$$

and for each α, we have $U_\alpha(z_\alpha) = \exp(z_\alpha|\alpha\rangle\langle 2| - \overline{z}_\alpha|2\rangle\langle\alpha|)$. The non-trivial transformations governed by $\mathcal{H}_0$ which are isospectral, are parametrized by the rotations of states between $|0\rangle$, $|2\rangle$ and $|0\rangle$, $|1\rangle$. Here, $\alpha = 0, 1$ and $z_\alpha = \theta_\alpha e^{i\phi_\alpha}$. For some $\alpha = 0, 1$, we look at the parameters of z_α which can be written as a duple $(\theta_\alpha, \phi_\alpha)$. Let $\mathcal{C}$ be a loop in this parametric space. The following holonomies are yielded:

$$\Gamma_{\boldsymbol{A}}(C) = \mathbf{P}\exp\left(\oint_C \mathbf{A}\cdot d\boldsymbol{\lambda}\right) \text{ where } (A^\mu)_{\alpha\beta} = \langle\alpha|\,\mathcal{U}^\dagger(\lambda)\frac{\partial}{\partial\lambda^\mu}\mathcal{U}(\lambda)\,|\beta\rangle \tag{3.3.7}$$

Where $\mathbf{P}$ is the path-ordering symbol, $\alpha, \beta \in \{0, 1\}$ and $\lambda^\mu \in \{\theta_1, \theta_2, \phi_1, \phi_2\}$. The connection $\mathbf{A}$ is a vector, where the components are matrices. Since it is irreducible, the holonomy $\Gamma_{\mathbf{A}}(C)$ generates the whole group [Pac12].

The loop integral inside the exponent is of most importance here,

$$\oint_C \mathbf{A}\cdot d\lambda = \oint A_{\lambda^\mu} d\lambda^\mu. \tag{3.3.8}$$

In general, connection components do not commute, therefore one cannot simply evaluate the exponential because of the path-ordering symbol $\mathbf{P}$. However, one can consider specific loops which bypass this issue.

First, we choose a loop C which lies on the plane (λ^1, λ^2). Then we choose the position of the plane such that $\mathbf{A}\cdot d\lambda = A^{\lambda^2}d\lambda^2$. That is, $A^{\lambda^1} = 0$. Therefore, the two components of $\mathbf{A}$ commute, and hence we can drop the path-ordering symbol. This, however, does not mean that the curvature $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is null, and it is this that gives the non-trivial holonomy.

Using the hamiltonian in 3.3.5, we choose the following loops and their induced holonomies. Let $C_1 \in (\theta_a, \phi_a)$ with $a = 1, 2$ so

$$\Gamma_{\mathbf{A}}(C_1) = \exp(-i\Sigma_1\sigma^3_\alpha) \tag{3.3.9}$$

where $\sigma^3_\alpha = |\alpha\rangle\langle\alpha|$, $\alpha = 0, 1$ and Σ_1 is the area enclosed by C_1 when projected onto a sphere with coordinates $2\theta_a$ and ϕ_a. Then for $C_2 \in (\theta_1, \theta_2)$ with $\phi_1 = \phi_2 = 0$, we have

$$\Gamma_{\mathbf{A}}(C_2) = \exp(-i\Sigma_2\sigma^2) \tag{3.3.10}$$

where $\sigma^2 = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and Σ_2 is the area enclosed by C_2 when projected onto a sphere with coordinates θ_1, θ_2. With this, we have enough freedom to create any arbitrary one-qubit gate [ZR99].

If we have m subsystems, then this corresponds to having m qubits. Therefore, one finds it sensible to think that having a rotation between two qubits would, in turn, correspond to having a two-qubit gate.

Example 3.3.2. *Suppose we want to make a rotation between the states $|11\rangle$ and $|22\rangle$. Then we employ $U(z) = \exp(z|11\rangle\langle 22| - \overline{z}|22\rangle\langle 11|)$ where $z = \theta e^{i\phi}$. With this, we have the following connection components*

$$A_\theta = 0, \;\; A_\phi = \mathrm{diag}(0, 0, 0, -i\sin^2\theta) \tag{3.3.11}$$

with the following vectors $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$ as the basis. [1]

Proof. Observe that $U(z)$ is unitary since by expanding the exponential, we have

$$U^\dagger(\lambda) = e^{-A} = U^{-1}(\lambda)$$

where $A = z|11\rangle\langle 22| - \overline{z}|22\rangle\langle 11|$. Then observe that $U(\lambda)$ acts trivially on $\{|00\rangle, |01\rangle, |10\rangle\}$ and non-trivially on $|11\rangle$. So the connection components for θ and ϕ is zero everywhere, except for $A_\phi^{4,4}$. We compute this by hand giving, $\langle 11|U^\dagger(\lambda) = \cos\theta\langle 11| - e^{-i\phi}\sin\theta\langle 22|$ and $\partial_\phi U(\lambda) = ie^{-i\phi}\sin\theta|22\rangle$. Hence $A_\phi^{4,4} = \langle 11|U^\dagger(\lambda)\partial_\phi U(\lambda)|11\rangle = -i\sin^2\theta$.

□

Now, let C be a loop on the θ, ϕ-plane such that we have the following holonomy

$$\Gamma_{\mathbf{A}}(C) = \mathrm{diag}(1, 1, 1, e^{i\Sigma}), \; \Sigma = \int_{D(C)} \sin(2\theta)\, d\theta d\phi \tag{3.3.12}$$

Then if $\Sigma = \pi$ one has that the holonomy becomes the controlled-phase gate (CP-gate). By using CP-gates between any two qubits, and with the ability to form any one-qubit gate, we can have universality. This model was first presented by Pachos [Pac00]. A physical realization of this system using trapped ions was proposed by Duan et al. [DCZ01].

Realizing that both holonomic and adiabatic quantum computation require adiabaticity of the system, holonomic quantum computation is used in a degenerate space of states — to which this degeneracy allows for an interpretation of quantum gates. That is, it resembles the circuit model. To relate this to TQC, one can recognize that TQC is *effectively* HQC whose adiabatic

[1] This matrix acts also on tensor products including $|2\rangle$. Realistically, this should form a 9×9 matrix with basis vectors as a tensor product of $|0\rangle$, $|1\rangle$, and $|2\rangle$. But we fix only said basis vectors as it is the computational space, so we disregard the basis vectors $\{|02\rangle, |20\rangle, |21\rangle, |22\rangle, |12\rangle\}$.

evolutions have topological properties. However, holonomic quantum computation depends entirely on the path whereas TQC should not.

Chapter 4

Anyons and Categories

In chapters two and three, we introduced physical concepts and applied them in a way that allows one to model a quantum computer. In particular, these models are very similar to TQC in the sense that TQC effectively involves anyons confined to a two-dimensional surface, so when the anyons adiabatically evolve to exchange each other, they are forming braids in their spacetime. For this half of the dissertation, we build the categorical theory to understand and correlate it with topological quantum field theory (TQFT) and apply it to introduce what exactly a TQC is and provide a few different models according to the type of chosen anyon.

This chapter, in particular, aims to categorically build up to a *unitary modular tensor category* (UTMC). They are the natural algebraic description of anyons. Here, one can analyse the braiding character of these anyons with more mathematical rigour. The proof to show a one-to-one correspondence for (2+1)-TQFTs and UMTCs can be found in Turaev's book [Tur94].

This chapter is heavily based on chapter 4 in [Wan10]. However, this chapter is written in a way so one may see the deep connection between the two structures.

4.1 Fusion Categories and Label Sets

Previously, we have shown Let $\mathcal{C}$ be a category and L be a subset of simple objects - we call this the *label set*. From L, we equip it with the properties of a fusion category as per definition 1.2.6. Each $a \in L$ is called an anyon species that is uniquely determined by its charge and flux. More plainly, these are our anyons - up to isomorphism. From here, we can formally define the following.

Definition 4.1.1. *A **label set** L is a finite set with*

1. *A fusion rule* $\otimes\colon L \times L \to \mathbb{N}_0^L$ *where* $\mathbb{N}_0^L$ *is the set of all maps from* L *to* $\mathbb{N}_0$*. We write the fusion rule as follows:*
$$a \otimes b = \bigoplus_{c \in L} N_{ab}^c c, \tag{4.1.1}$$
where a natural number $N_{ab}^c = (a \otimes b)(c)$ *is assigned as a coefficient for each* $c \in L$ *such that the fusion operator is associative.*

2. *There is a unique unit* $1 \in L$ *and a notion of an antiparticle i.e., for all* $a \in L$ *there is an* $a^* \in L$ *satisfying the following conditions:*
$$N_{ab}^1 = N_{ba}^1 = \delta_{ba^*}, \tag{4.1.2}$$
$$N_{1a}^c = N_{a1}^c = \delta_{ac}, \tag{4.1.3}$$
that the fusion rule must abide by, where the δ *symbol denotes the Kronecker delta.*

For the physical intuition, suppose we have two anyons of some charge x and y. Then the physical idea of fusing such anyons is simply placing these two particles in a potential box and *treating* the box as a particle whose constituents are the two particles.

Following the definitions of a fusion category, recall that we have the associator, a natural isomorphism. Analogously, we can define the following. First, let $a, b \in L$. Then if there is a $c \in L$ such that $N_{ab}^c \neq 0$, the triple $(a, b, c) \in L^3$ is *admissible*. Next, the sextuple $(a, b, c, d, n, m) \in L^6$ is admissible if $(a, b, m), (m, c, d), (b, c, d)$ and (a, n, d) are all admissible. So we make the following definition.

Definition 4.1.2. *For a label set* L *and a fusion rule, we define the* $6j$*-symbol* $F\colon L^6 \to \mathbb{C}$*. The sextuple input must be admissible. Denote this* $6j$*-symbol as* $F_{d;nm}^{abc}$*, where* F_d^{abc} *is a matrix, with entries* n *and* m *ranging over all labels. If the input is not admissible, then one simply yields* 0*. We also require that* F_d^{abc} *satisfies the pentagon axiom: for all* $a, b, c, d, e, f, p, q, m \in L$*,*
$$\sum_n F_{q;pn}^{bcd} F_{f;qe}^{and} F_{e;nm}^{abc} = F_{f;qm}^{abp} F_{f;pe}^{mcd}. \tag{4.1.4}$$

In practice, we simply refer to F_d^{abc} as the F-matrix and if it has inputs $n, m \in L$, then $F_{d;nm}^{abc}$ is a $6j$-symbol. Should a label set exist with a $6j$-symbol, we call L a $6j$*-symbol system*.

Definition 4.1.3. *A* **6j-*fusion*** *system is a* $6j$*-symbol system satisfying the following:*

1. *The triangle axiom: If* $1 \in \{a, b, c\}$*, then* $F_d^{abc} = 1$.

2. *Rigidity: For any* $a \in L$*, let* $G_{a^*}^{a^* a a^*}$ *be the inverse of* $F_{a^*}^{a^* a a^*}$*. With an* (m, n)*-entry,* $G_{a^*;mn}^{a^* a a^*}$*, then* $G_{a^*;mn}^{a^* a a^*} = F_{a;mn}^{a a^* a}$.

Definition 4.1.4. *Let F and F' be fusion systems from the label set L. They are considered **gauge equivalent** should there be a gauge transformation $f\colon L^3 \to \mathbb{C}$ by $(a,b,c) \mapsto f_c^{ab}$ satisfying the following:*

1. *(a,b,c) is admissible iff $f_c^{ab} \neq 0$,*
2. *$f_a^{1a} = f_a^{a1} = 1$ for all $a \in L$,*
3. *and the rectangle axiom: for all $a,b,c,d,n,m \in L$*

$$f_n^{bc} f_d^{an} F_{d;nm}^{abc} = F'^{abc}_{d;nm} f_m^{ab} f_d^{mc} \tag{4.1.5}$$

Definition 4.1.5. 1. *An automorphism of a fusion rule is a label permutation α satisfying*

$$N_{\alpha(y)\alpha(z)}^{\alpha(x)} = N_{yz}^{x} \tag{4.1.6}$$

2. *For two $6j$ systems that have a common label set, they are equivalent if they are gauge equivalent up to label permutation.*

Theorem 4.1.6. *We have the following identification:*

1. *$6j$-fusion systems, up to equivalence, have a $1-1$ correspondence with fusion categories - up to $\mathbb{C}$-linear monoidal equivalence.*
2. *(Ocneanu Rigidity) There are only finitely many equivalence classes of fusion categories, given a fusion rule on a label set.*

Proof. For (1) refer to [YAM02] and for (2) the proof is in [FLW00; FLW02].

With this theorem, one can identify a $6j$-fusion system with a fusion category. Ultimately, this is progress to convince one that the mathematical habitat of anyons is in category theory.

Definition 4.1.7. *Let L be a label set. Then we define $|L|$ to be the **rank**.*

Definition 4.1.8. *A **braiding** on a $6j$-symbol system with label set L, is a function $L^3 \to \mathbb{C}$ taking $(a,b,c) \mapsto R_c^{ab}$, such that*

1. *$R_c^{ab} \neq 0$ if (a,b,c) is admissible*
2. *The hexagon axiom is satisfied, that is: For all $a,b,c,d,e,m \in L$,*

$$(R_e^{ac})^{\pm 1} F_{d;em}^{bac} (R_m^{ab})^{\pm 1} = \sum_n F_{d;en}^{bca} (R_d^{an})^{\pm 1} F_{d;nm}^{abc}. \tag{4.1.7}$$

This is analogous to the categorical definition of a braided fusion category, in that we have the following.

Definition 4.1.9. *A fusion category is* ***braided*** *if it is equipped with a natural isomorphism* $c_{xy}\colon x \otimes y \to y \otimes x$ *and* c *satisfies the Hexagon axiom.*

Despite this realisation that anyons live in fusion categories such that they are the simple objects in our label set, one can exploit this notion of *fusion* and *braiding* by instead drawing the paths of the anyons. Physically motivated by observing that this is equivalent to considering the world line of the anyons in $(2+1)$. This is similar to Feynman's rules and his diagrams and Dyson's mathematical formalism for said rules in quantum field theory.

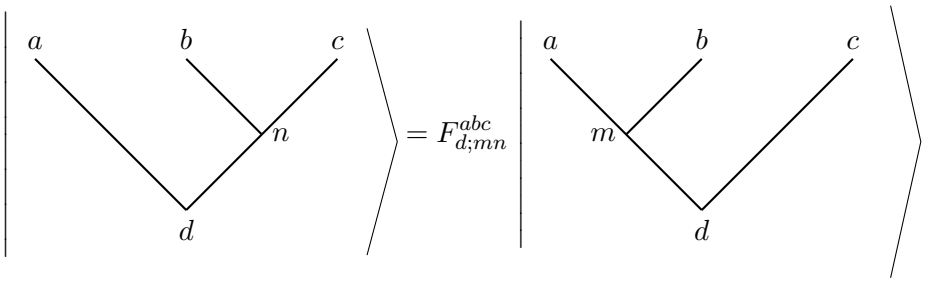
4.2 Graphical Calculus of Fusion Categories

Let $\mathcal{C}$ be a fusion category and let L be the label set from $\mathcal{C}$. Then since $\mathcal{C}$ is $\mathbb{C}$-linear, the Hom-sets for any fusion are isomorphic to an finite dimensional vector space over $\mathbb{C}$. Let $a, b, c, d \in L$ and assume appropriate admissibility. $\mathrm{Hom}(d, a \otimes b \otimes c)$ is spanned by the intermediate anyons formed depending on which order one decides to associate the product. In other words,

$$|b, c \to n; \nu\rangle = F^{abc}_{d;mn} |a, b \to m; \mu\rangle. \tag{4.2.1}$$

Here, $a, b, c, d, n, m \in L$ and $\mu \in \{1, \ldots, N^{ab}_m\}$ and $\nu \in \{1, \ldots, N^{bc}_n\}$.

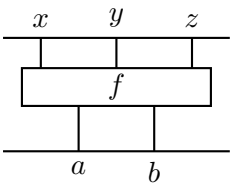
If we interpret time passing upwards, we can *unfuse* an anyon into a fusion product (as L is semi-simple) and graphically illustrate this as



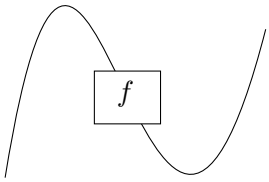
But $\mathrm{Hom}(d, (a \otimes b) \otimes c) = \mathrm{Hom}(d, a \otimes b \otimes c) = \mathrm{Hom}(d, a \otimes (b \otimes c))$. The first equality is the left state and the middle equality is the right state that is being acted upon by the F-matrix. Here, it is much easier to see exactly what the F-matrix is. It is a change of basis matrix, according to the association.

Regarding *right* rigidity, for $x \in L$ we choose $b_x \in \mathrm{Hom}(1, x \otimes x^*)$ and $d_x \in \mathrm{Hom}(x^* \otimes x, 1)$ which can be categorically identified as the coev_x and ev_x natural isomorphisms respectively.

Suppose we have a morphism $f\colon a \otimes b \to x \otimes y \otimes z$, where $a \otimes b, x \otimes y \otimes z \in L$. Then we can illistrate this as



Suppose we have the morphism $f\colon x \to y$. We can define the dual of this morphism as $f^*\colon y^* \to x^*$ or illustratively:



So one observes that * is a contravariant functor, that is for $F\colon \mathcal{C} \to \mathcal{D}$ a functor, a contravariant functor is simply $G : \mathcal{C} \to \mathcal{D}^{\text{op}}$. Where the opposite category $\mathcal{D}^{\text{op}}$ is defined so $\mathcal{D}_0 = \mathcal{D}_0^{\text{op}}$ and more importantly, for any morphism $f \in \mathcal{D}_1$ with (say) $f\colon X \to Y$, then one has $f' \in \mathcal{D}_0^{\text{op}}$ such that $f'\colon Y \to X$.

Notice that this is in the context of right rigidity. In order to define left rigidity, we cannot simply change all the words in the right rigidity definition to obtain a left rigidity definition. We must first have a notion of a *quantum trace* so that we can define left rigidity.

Definition 4.2.1. ***(Pivotal and Spherical)*** *Let $\mathcal{C}$ be a fusion category and $x, y \in L$.*

1. *If we have the isomorphisms $\phi_x\colon x \to x^{**}$ with the property that*

   (a) $\phi_{x\otimes y} = \phi_x \otimes \phi_y$,

   (b) $f^{**} = f$ *for any* $f\colon x \to y$,

   then we say that $\mathcal{C}$ is pivotal.

2. *We define a left and right trace in a pivotal category, but they are not necessarily equal. Let $f\colon x \to x$.*

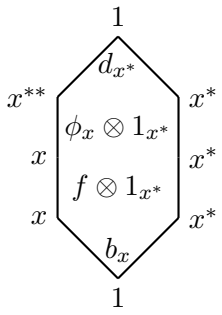
$$Tr^r(f) = d_{x^*} \circ (\phi_x \otimes 1_{x^*}) \circ (f \otimes 1_{x^*}) \circ b_x \tag{4.2.2}$$

$$1 \xrightarrow{b_x} x \otimes x^* \xrightarrow{f\otimes 1_{x^*}} x \otimes x^* \xrightarrow{\phi_x \otimes 1_{x^*}} x^{**} \otimes x^* \xrightarrow{d_{x^*}} 1$$

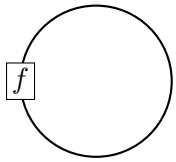
$$Tr^l(f) = d_x \circ (d_{x^*} \otimes f) \circ (1_{x^*} \otimes \phi_x^{-1}) \circ b_{x^*} \tag{4.2.3}$$

$$1 \xrightarrow{b_{x^*}} x^* \otimes x^{**} \xrightarrow{1_{x^*}\otimes\phi_x^{-1}} x^* \otimes x \xrightarrow{d_{x^*}\otimes f} x^* \otimes x \xrightarrow{d_x} 1$$

The graphical calculus for the right trace follows:



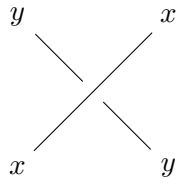
or simply:



and the graphical calculus for the left trace follows mutatis mutandis.

3. *If, for all f morphisms $Tr^r(f) = Tr^l(f)$, then we say that the category is spherical.*

If there is a braiding in the fusion category, then in the graphical calculus for $c_{x,y} \colon x \otimes y \to y \otimes x$, we have



One can also look at the notions of being braided and spherical from a $6j$-fusion system perspective.

Proposition 4.2.2. *Let $\mathcal{C}$ be a fusion category.*

1. *The associated $6j$-fusion system is **pivotal** if there is t_a that we can choose to be some root of unity such that for any $a \in L$, t_a satisfies the following pivotal axioms:*

$$t_1 = 1, \tag{4.2.4}$$
$$t_{a^*} = t_a^{-1}, \tag{4.2.5}$$
$$t_a^{-1} t_b^{-1} t_c = F_{1;a^*c}^{a,b,c^*} F_{1;a^*a}^{b,c^*,a} F_{1;a^*c}^{c^*,a,b}, \tag{4.2.6}$$

   *for each $(a, b, c) \in L^3$ admissible. These t_a will be called the **pivotal coefficients**.*

2. *The pivotal structure is spherical if all $t_a = \pm 1$.*

Definition 4.2.3. *A **ribbon** fusion category (RFC) is one that is braided and spherical.*

Among all the immense load of definitions, one can characterise a ribbon fusion category more nicely.

Proposition 4.2.4. *A RFC with multiplicity-free fusion rule is a collection of numbers $\{F_{d;mn}^{abc}\}$, $\{R_c^{ab}\}, \{t_i = \pm 1\}$ satisfying the pentagon, triangle, rigidity, hexagon and pivotal axioms.*

4.3 Unitary Fusion Categories

We denote right rigid isomorphisms with b_x and d_x and left rigid isomorphisms b'_x and d'_x for birth and death respectively.

Definition 4.3.1. *Let $\mathcal{C}$ be a fusion category.*

1. *A **conjugation** on $\mathcal{C}$ is an assignment to each $f \in Hom(x, y)$ a morphism, namely $\bar{f} \in Hom(y, x)$ which is conjugate, linear, and satisfies the following:*

$$\overline{\overline{f}} = f, \qquad \overline{f \otimes g} = \overline{f} \otimes \overline{g}, \qquad \overline{f \circ g} = \overline{g} \circ \overline{f}. \tag{4.3.1}$$

2. *A RFC $\mathcal{C}$ is **Hermitian** if there is a conjugation assignment for which*

   (a) $\overline{b_x} = d'_x$ and $\overline{d_x} = b'_x$,

   (b) $\overline{c_{x,y}} = c_{x,y}^{-1}$,

   (c) $\overline{\theta_x} = \theta'_x$.

3. *A RFC $\mathcal{C}$ is **unitary** if $\mathcal{C}$ is Hermitian and $Tr(f \circ \bar{f}) \geq 0$ for every $f \in Hom(x, y)$.*

Where if we define $\psi_x \colon x^{**} \to x$, then $\theta_x = \psi_x \phi_x$ where ϕ_x being a twist in the graphical calculus.

Recall that an object $x \in L$ is simple if and only if $\mathrm{Hom}(x, x) \cong \mathbb{C}$. The idea here is that we can make a morphism (say) $g \in \mathrm{Hom}(1, 1) \cong \mathbb{C}$ and since the identity is simple, we can establish a corresponding complex number so $g = \langle g \rangle 1_g$, where $\langle g \rangle \in \mathbb{C}$. This scalar is invariant under Reidermeister moves of the second and third kind.

Definition 4.3.2. *Let $\mathcal{C}$ be an RFC and L a strict label set of $\mathcal{C}_0$.*

1. *Let $x \in L$. Then we can define the **quantum dimension** of an anyon species as*

$$d(x) = d_x = Tr^L(1_x). \tag{4.3.2}$$

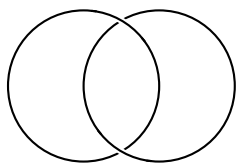
2. *Define the **global quantum dimension** of the category $\mathcal{C}$ as*

$$D = \sqrt{\sum_{i \in L} d_i^2}, \tag{4.3.3}$$

   we take $D > 0$ if $D \in \mathbb{R}$, without loss of generality.

3. *For all $i, j \in L$, we define $\tilde{S} = (\tilde{s}_{ij})$*

$\tilde{s}_{ij} =$



.

4. *$\tilde{S}$ is the **modular $\tilde{S}$-matrix**, and $S = \frac{1}{D}\tilde{S}$ is the **modular S-matrix**.*

5. *If $\det S \neq 0$, then the RFC is **modular**.*

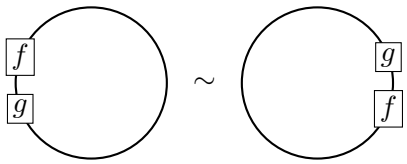
Lemma 4.3.3. *Let $\mathcal{C}$ be a ribbon fusion category.*

1. *Let $f \colon V \to W$ and $g \colon W \to V$. Then $Tr(f \circ g) = Tr(g \circ f)$.*

2. *Let $f, g \in End(\mathcal{C}_0)$. Then we have that $Tr(f \otimes g) = Tr(f)Tr(g)$.*

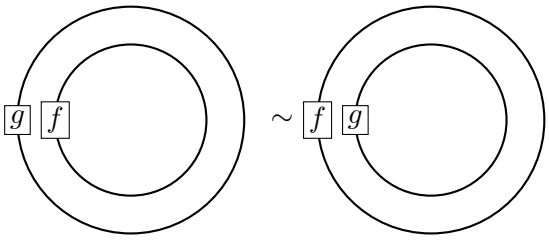
Proof. From the isotopic invariance of the graphical calculus of the trace function and a corollary - both in Turaev - we are able to make the following equalities

1. Consider Tr$(f \circ g)$ and its graphical calculus. Then by isotopic invariance of the graphical calculus of the trace function [Tur94], i.e., we can rotate the loop such that we have the following:



   The latter graphical calculus shows Tr$^r(g \circ f)$. But since $\mathcal{C}$ is spherical, the left and right traces are equal. So $Tr(f \circ g) =$ Tr$^r(g \circ f) =$ Tr$(g \circ f)$. □

2. By corollary in [Tur94], we have that



□

Lemma 4.3.4. *For any $x, y, z \in L$ such that (x, y, z) is admissible, we have that*

$$d_x d_y = \sum_z N_{xy}^z d_z. \tag{4.3.4}$$

Proof. Knowing that $1_x \otimes 1_y = 1_{x \otimes y}$, and from Lemma 4.3.3, this lemma immediately follows. □

A result that immediately follows is that $d_1 = 1$. As it turns out, from [Wan10] that for UMTCs with rank 4 or less, so there are at most 4 simple objects including the identity or there is four anyon species in the label set, that one has a TQFT for each UMTC of rank four or less.

4.4 Examples

A modular category is a specific type of RFC which satisfies a couple more axioms. However, in this setting they are effectively equivalent and for all practical purposes, we will treat them as such - modulo said axioms. Recall that another name for monoidal categories are tensor categories.

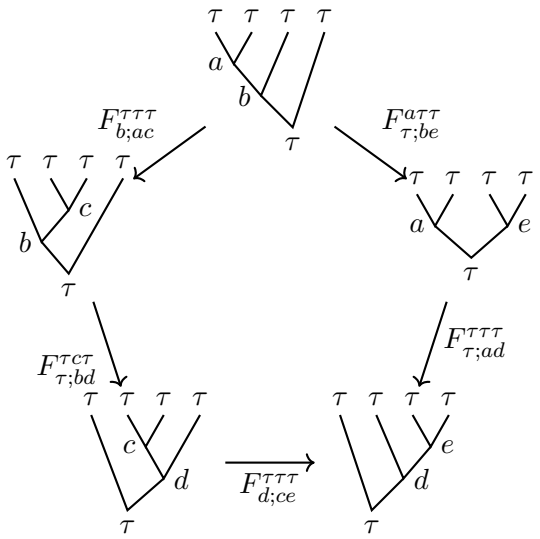


Figure 4.1: Visualisation of the pentagon axiom used to calculate the non-trivial F-matrix. The equation that this graphical calculus represents is in Equation 4.4.3

4.4.1 Fibonacci Anyons

Upon classification, the smallest non-trivial UMTC is the Fibonacci UMTC. Let $L = \{1, \tau\}$ be our label set and define our fusion rules as $1 \otimes 1 = 1$, $1 \otimes \tau = \tau \otimes 1 = \tau$ and $\tau \otimes \tau = \tau^2 = \tau \oplus 1$. By inductively fusing τ with itself, one finds the formulae

$$\tau^n = F_{n-2} 1 \oplus F_{n-1} \tau, \tag{4.4.1}$$

where F_n is the nth entry in the Fibonacci sequence $\{1, 1, 2, 3, 5, \dots\}$, hence the name. The quantum dimension of τ immediately follows from $\tau \otimes \tau = 1 \oplus \tau$ and Lemma 4.3.4 whereby one finds $d_\tau = \phi = \frac{1+\sqrt{5}}{2}$.

In Pachos, one finds the working for F and R-matrices with Ising anyons, not Fibonacci [Pac12]. As a completed exercise, we provide the construction of the F and R-matrices for Fibonacci anyons.

First, given the fusion rules for Fibonacci anyons and their respective N^c_{ab} values, we assume the that for our F-matrix F^{abc}_d, if a or b or c is 1, then the matrix is the scalar 1 for admissible sextuples in L^6. Similarly, if we have three τ anyons fusing to give the vacuum, then $F^{\tau\tau\tau}_1 = 1 \in \mathbb{R}$. On the other hand, entries like $F^{\tau 1 \tau}_{\tau;1\tau}$ is 0 since it is not admissible.

We take the pentagon expression where the bottom-most and top-most points are all τ. Let $\alpha, \beta, \gamma, \delta \in L$ be the end split points, $\varepsilon \in L$ be the starting point, and $a, b, c, d \in L$ be the intermediate splits. Then the pentagon axiom reads as

$$F^{\alpha\beta e}_{\varepsilon;ad} F^{a\gamma\delta}_{\varepsilon;be} = \sum_{c\in L} F^{\beta\gamma\delta}_{d;ce} F^{\alpha c\delta}_{\varepsilon;bd} F^{\alpha\beta\delta}_{b;ac} \tag{4.4.2}$$

$$F^{\tau\tau e}_{\tau;ad} F^{a\tau\tau}_{\tau;be} = \overbrace{F^{\tau\tau\tau}_{d;1e} F^{\tau 1\tau}_{\tau;bd} F^{\tau\tau\tau}_{b;a1}}^{c=1} + \overbrace{F^{\tau\tau\tau}_{d;\tau e} F^{\tau\tau\tau}_{\tau;bd} F^{\tau\tau\tau}_{b;a\tau}}^{c=\tau}. \tag{4.4.3}$$

See that $F^{\tau 1\tau}_{\tau;bd} = 0$ as it is not admissible if $b = 1$ and $d = \tau$. So with that in mind, a non-trivial combination of $a, b, d, e \in L$ is $d = a = \tau$ and $b = e = 1$. Then 4.4.3 reduces to

$$F_{11} = F_{\tau 1} F_{1\tau}. \tag{4.4.4}$$

To make it easier to read, set notate $F^{\tau\tau\tau}_{\tau;xy} = F_{xy}$. Together with 4.4.4 and acknowledging that F-matrices are unitary (so $FF^\dagger = 1$, we have the following system of equations:

$$\begin{cases} |F_{11}|^2 + |F_{1\tau}|^2 = 1, \\ F_{11}F_{\tau 1} = -F_{1\tau}F_{\tau\tau}, \\ |F_{11}F_{\tau\tau} - F_{1\tau}F_{\tau 1}| = 1, \\ F_{11} = F_{\tau 1}F_{1\tau}. \end{cases} \tag{4.4.5}$$

Which gives us, up to an arbitrary phase,

$$F^{\tau\tau\tau}_{\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}. \tag{4.4.6}$$

where ϕ is the golden ratio. Next we want to find the R-matrix. We do so mutatis mutandis first by finding the trivial braiding $R^{\tau 1}_{\tau} = R^{1\tau}_{\tau} = 1$, then with the hexagon axiom and set the outer points $\alpha = \beta = \gamma = \varepsilon = \tau$.

$$R^{\alpha\beta}_{c} F^{\beta\alpha\gamma}_{\varepsilon;ac} R^{\alpha\beta}_{a} = \sum_{b\in L} F^{\gamma\alpha\beta}_{\varepsilon;ab} R^{\gamma b}_{\varepsilon} F^{\alpha\beta\gamma}_{\varepsilon;bc}, \tag{4.4.7}$$

$$R^{\tau\tau}_{\tau} F_{ac} R^{\tau\tau}_{a} = F_{a1} R^{\tau 1}_{\tau} F_{1c} + F_{a\tau} R^{\tau\tau}_{\tau} F_{\tau c}. \tag{4.4.8}$$

The R-matrix is non-trivial only when we braid the fusion $\tau \otimes \tau$. So we find that R should be 2×2 and diagonal since the fusion can only have 1 or τ as outcomes. Respectively, we set $a = c = 1$, then set $a = 1,\ c = \tau$ and since $R^{\tau\tau}$ is unitary, we get the following equations

$$\begin{cases} (R^{\tau\tau}_1)^2 \phi^{-1} = \phi^{-2} + R^{\tau\tau}_{\tau}\phi^{-1}, \\ R^{\tau\tau}_1 R^{\tau\tau}_{\tau} \phi^{-1/2} = (1 - R^{\tau\tau}_{\tau})\phi^{-3/2}, \\ |R^{\tau\tau}_1 R^{\tau\tau}_{\tau}| = 1. \end{cases} \tag{4.4.9}$$

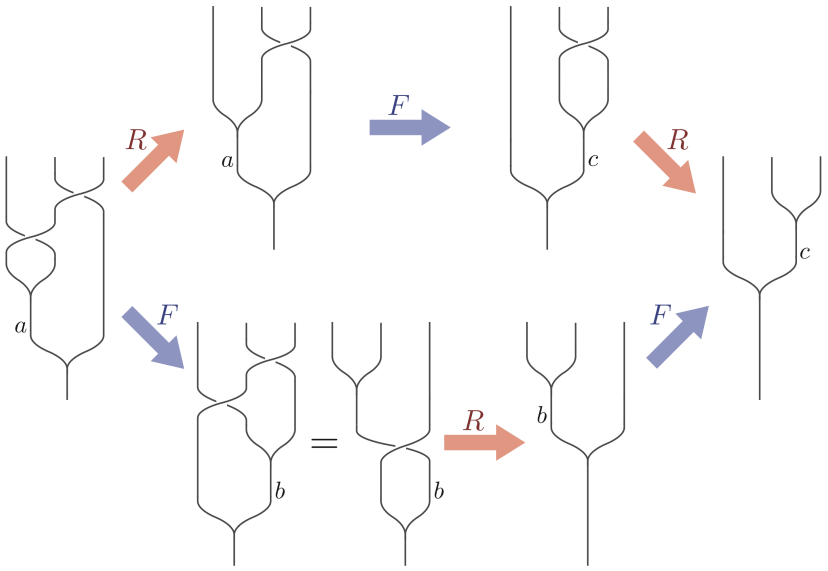


Figure 4.2: [Tre+08]Visual aid for the hexagon diagram. Note that the top points from left to right are $\alpha\ \beta\ \gamma$ and on the bottom, it starts at ε. From the left-most point going down via an F-matrix, the equality is justified by observing that there is topological invariance when γ crosses first β then α; this is the same as just crossing the fused anyon b.

This yields the matrix

$$R^{\tau\tau} = \begin{pmatrix} R_1^{\tau\tau} & 0 \\ 0 & R_\tau^{\tau\tau} \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & e^{-3\pi i/5} \end{pmatrix}. \tag{4.4.10}$$

4.4.2 Ising Anyons

A better well known UMTC is the Ising UMTC. Let $L = \{1, \sigma, \psi\}$ be the label set and define the non-trivial fusion rules as $\psi \otimes \psi = 1$, $\sigma \otimes \sigma = 1 \oplus \psi$ and $\sigma \otimes \psi = \sigma$. From Lemma 4.3.3, we can construct a system of equations to solve for the particles dimensions.

$$d_\sigma = \sqrt{2} \qquad d_\psi = 1 \tag{4.4.11}$$

In the similar way as in the Fibonacci UMTC, we use the pentagon axiom, and then the hexagon axiom to solve for the F and R matrices:

$$F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad R_{\sigma\sigma} = e^{i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{4.4.12}$$

Chapter 5

Anyons in Topological Quantum Field Theory

In quantum field theory (QFT), our particles exist in 3 spacial and 1 temporal coordinates, or a $(3+1)$-dimensional Lorentzian manifold which we interpret as spacetime. Anyons, particles which exhibit exotic spin statistics, only exist in $(2+1)$-dimensional manifolds [1]. In this chapter, we wish to give a small overview of QFT, introduce topological quantum field theory (TQFT) and relate this theory to familiar concepts to understand how anyons are represented on surfaces.

We will heavily follow the notes from chapter 5 and 6 from [Wan10].

5.1 Quantum Field Theory (an overview)

Nearly a century after Newton discovered mechanics by standard calculus and forces, Lagrange and Hamilton reformulated classical mechanics via a variational approach. Such formalisms allowed for physicists such as Feynman and Schwinger to quantise the fields in our standard model - thus the birth of QFT and quantum electrodynamics (QED). One approach is to use the Schwinger Action principle but here, we shall depend more heavily on Feynman's formalism using his path integral.

The idea is to take a classical system, then, via some quantisation procedure, one can obtain a corresponding quantum system; however, this procedure is often *not* obvious.

[1] Because 3 spacial coordinates makes loops around a point puncture equivalent, by homotopy, to the trivial loop and 1 spacial coordinate lacking degrees of freedom for anyons to be useful in any form.

In the language of physics, suppose we have a system consisting of n particles in some space X. The positions of the n particles are points in the configuration space $C_n(X) \subset X^n$. Then the joint trajectory of our particles is represented as a curve in $C_n(X)$. The Hamiltonian and Lagrangian formalisms of classical field theory, and QFT after quantisation, are defined on the cotangent and tangent bundles of the configuration space with coordinates (q, p) and $(q, \dot{q})$ — respectively. The proportionality constant between the momentum and velocity in their respective formalisms is referred to as the particle's rest mass. Both formalisms are equivalent in the classical setting for particles with non-zero rest mass via a Legendre transformation. However, in QFT and/or for particles with no rest mass (like photons), such a transformation might not be as straightforward or even impossible. An example would be TQFTs under the Lagrangian formalism.

For the Lagrangian formalism, we define a Lagrangian density to be a linear functional on the tangent bundle of our configuration space $\mathcal{L}\colon TC_n(X) \to \mathbb{R}$. The dynamics of the system is governed by a functional called the action — denoted by S which assigns a real number to paths $\gamma\colon I \to \mathbb{R}$ in our configuration space.

$$S[\gamma] \coloneqq \int_I \mathcal{L}(\dot{\gamma})dt \tag{5.1.1}$$

In physics, we want paths which minimise this action - the least action principle. So in this sense, we set $\delta S = 0$, the variation of S, and one yields the Euler-Lagrange equation.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{5.1.2}$$

To quantize this formalism, we use the Feynman path integral. Suppose we have a particle travelling from point x_a to x_b in some time t subject to a Hamiltonian $\mathcal{H}$. Then the amplitude, a number correlated to how likely a specific path will be taken, is

$$U(x_a, x_b; t) = \langle x_b | e^{\frac{-i}{\hbar}\mathcal{H}t} | x_a \rangle \tag{5.1.3}$$

Set $\hbar = 1$. Then we can define the path integral to be the sum over all possible paths the particle could take to get from the initial to the final point.

$$\sum_{\text{all paths}} e^{iS(\gamma)} \longrightarrow \int_{\text{all paths}} e^{iS(\gamma)}\, \mathcal{D}\gamma \tag{5.1.4}$$

Since there is an uncountable and infinite number of paths, we can change the discrete sum into an integral, with $\mathcal{D}\gamma$ as the measure for the paths. Note that this measure is often not rigorously defined. For all practical purposes, we set $A(\gamma)$ to be the action $S(\gamma)$. Quantisation is done this way as it preserves the classical symmetries and highlights the close relationship between

QFT and statistical mechanics. [2]

For field theory, we consider the following. Let X be a $(d+1)$-dimensional spacetime manifold and let $\Phi(X)$ be a space of data defined locally on X. We refer to elements of $\Phi(X)$ as *field configurations*. Some examples include $\Phi(X)$ being the space of all smooth maps $\phi\colon X \to \mathbb{R}^m$ and $\Phi(X)$ the space of all smooth functions $\phi\colon X \to M$, M a fixed Riemannian n-manifold. Next, for every $x \in X$ we choose some $F_x\colon \Phi(X) \to \mathbb{C}$ so that $F_x(\Phi)$ depends only on the behaviour of ϕ in some neighbourhood which contains our point x. Following from our previous examples, we can have $F_x(\phi) = \phi(x)$ and $F_x(\phi) = (f \circ \phi)(x)$, where $f\colon M \to \mathbb{C}$ is fixed.

If we wish to quantize a theory, we require a probability measure on our space $\Phi(X)$ so that we can evaluate the expectation value $\langle F_x \rangle$ and correlation functions $\langle F_{x_1} \dots F_{x_k} \rangle$ of our observables (locally) at the points $x_1, \dots, x_k$. These correlation functions, also known as Green's functions, contain all the physics in our theory. Should one find a probability measure, then when we evaluate the path integral with this measure over $\Phi(X)$, we are able to extract physically meaningful results - which is why the path integral is important. However, constructing such a measure turns out to be a very difficult task. [3]

5.2 Axioms of TQFT

In normal quantum field theory, one usually forms their theories in Fock spaces and Grassman algebras, to allow for mathematical manipulations of bosons and fermions with commutating and anti-commutating relations, all respectively. In TQFT, we want to find surfaces and somehow encode information on these surfaces such that one can exploit topological properties of these surfaces.

We introduce, heuristically, the modular functor V which maps compact and oriented surfaces to finite-dimentional vector spaces. For disjoint unions of such sufaces $X_1,\ X_2$, one has

$$V(X_1 \sqcup X_2) \cong V(X_1) \otimes V(X_2) \tag{5.2.1}$$

Orientation reversal is mapped such that for X a compact and oriented surface, denote $-X$ as the same X with opposite orientation, and

$$V(-X) = V(X)^*. \tag{5.2.2}$$

[2] To highlight the importance of this path integral, the path integral in QFT is analogous to the partition function in statistical mechanics and the equations of motion from classical mechanics.

[3] Its importance is ranked analogous to using the Schrödinger Equation to find the wavefunction of a system and Newton's second law or the Euler-Lagrange equation to find the equations of motion for a classical system.

For the trivial case, namely the empty set, $\emptyset \overset{V}{\mapsto} \mathbb{C}$. We have another functor, the partition functor Z such that for a compact and oriented surface X with a boundary, we have $Z(X) \in V(\partial X)$.

Definition 5.2.1. *Let X be an oriented 3-manifold with boundary $Y = \partial X$. A **Lagrangian subspace** λ is a maximal isotropic subspace of $H_1(Y;\mathbb{R})$ with respect to intersection pairing of $H_1(Y,\mathbb{R})$.*

We extend the category of surfaces (bordisms) to labelled extended surfaces.

Definition 5.2.2. *Let $\mathcal{C}$ be a strict fusion category, Y an oriented surface, and λ a Lagrangian subspace of $H_1(Y;\mathbb{R})$. Define a **labelled extended surface** as a triplet $(Y;\lambda,l)$, where l is an assignment of $U \in \mathcal{C}_0$ to each boundary circle.*

We want to introduce a notion of gluing our surfaces. Let γ_1, γ_2 be connected components of ∂Y and let $\phi_i \colon \mathbb{R} \to \gamma_i$ be diffeomorphisms, for $i = 1, 2$, and label each γ_1, γ_2 with $U, U' \in \mathcal{C}_0$ respectively. Define $\mathrm{gl} \colon \gamma_1 \to \gamma_2$ by $x \mapsto (\phi_2 \circ r \circ \phi_1^{-1})(x)$ where r is an involution. We define a gluing Y_{gl} as the factor space of Y by the identification of $\mathrm{gl} \colon \gamma_1 \to \gamma_2$. gl is a diffeomorphism which preserves orientation, boundary parametrisation, and labelling. Such diffeomorphisms are called labelled diffeomorphisms and they are the morphisms of the category of oriented labelled extended surfaces denoted by $\mathcal{X}^{2,e,l}$.

Given a particular partition function for some model of a (2+1)-TQFT, one would require invariance under gauge transformations. An example of such a transformation is the Lorentz transformation. However, this invariance is not intrinsically guaranteed. Rather, it is invariant up to some complex scalar which we refer to as an anomaly. We define an anomaly for a TQFT as a root of unity $\kappa = e^{i\pi c/4}$ where $c \in \mathbb{Q}$ mod 8 is referred to as the central charge. So if our TQFT is anomaly-free, then $c \equiv_8 0$. For our purposes, assume we have done the necessary steps of extending our bordisms to obtain an anomaly-free TQFT.

Definition 5.2.3. *(**$(2+1)$-TQFTs**) Let $\mathcal{C}$ be a strict fusion category with a strict label set L^{str}. An anomaly-free $(2+1)$-TQFT is a pair of functors (V,Z) where $V \colon \mathcal{X}^{2,e,l} \to Vec_{fd}$, the category of finite dimensional vector spaces. V is commonly referred to as the modular functor. Let X be an oriented 3-manifold and λ a Lagrangian subspace. Then we equip X with an extended boundary $(\partial X, \lambda)$ such that $Z(X;\lambda) \in V(\partial X;\lambda)$. If $Y = \partial X$, then we can extend the surface with the canonical Lagrangian subspace λ_X and we can simply write $Z(X) \in V(\partial X)$. We refer to Z as the partition functor.*

We have that V must satisfy the following axioms:

1. *Disk axiom:*

$$V(D;(l,1')) \cong \begin{cases} \mathbb{C} & l = 1 \\ 0 & \textit{else} \end{cases} \tag{5.2.3}$$

2. *Annulus axiom*

$$V(A; a, b') \simeq \begin{cases} \mathbb{C} & a = b \\ 0 & else \end{cases} \tag{5.2.4}$$

For $a, b \in L^{str}$.

3. *Disjoint union axiom:*

$$V(Y_1 \sqcup Y_2; \lambda_1 \oplus \lambda_2, l_1 \cup l_2) \cong V(Y_1; \lambda_1, l_1) \otimes V(Y_2; \lambda_2, l_2), \tag{5.2.5}$$

where the isomorphism is associative and compatible with actions from the mapping class group.

4. *Duality axiom:*

$$V(-Y; l) \cong V(Y, \hat{l})^* \tag{5.2.6}$$

Where the isomorphisms are compatible with actions from the mapping class group, orientation reversal. From here, we can state more properties of the disjoint union axiom:

   (a) *For* $f\colon (Y_1; l_1) \to (Y_2; l_2)$*, let* $\bar{f}\colon (-Y_1; \hat{l}_1) \to (-Y_2; \hat{l}_2)$*. One has that for* $x \in V(Y_1; l_1)$*,* $y \in V(Y_2; l_2)$*, that* $\langle x, y\rangle = \big\langle V(f)x, V(\bar{f})y\big\rangle$*.*

   (b) *If* $\alpha_1 \otimes \alpha_2 \in V(Y_1 \sqcup Y_2)$ *and* $\beta_1 \otimes \beta_2 \in V(-Y_1 \sqcup -Y_2)$*, then* $\langle \alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2\rangle = \langle \alpha_1, \beta_1\rangle \langle \alpha_2, \beta_2\rangle$

5. *Gluing axiom. Let* Y_{gl} *be an extended surface yielded from gluing two boundaries together from an extended surface* Y*. Then we have*

$$V(Y_{gl}) \cong \bigoplus_{l \in L} V(Y; (l, \hat{l})) \tag{5.2.7}$$

Where $l \in L$ *and the isomorphism is associative and compatible with actions from the mapping class group and duality as given: For*

$$\bigoplus_{j \in L} \alpha_j \in V(Y_{gl}; l) \qquad \bigoplus_{j \in L} \beta_j \in V(-Y_{gl}; \hat{l})$$

Then for each $l \in L$*, there is* $s_j \in \mathbb{R}^\times$ *such that*

$$\left\langle \bigoplus_{j \in L} \alpha_j, \bigoplus_{j \in L} \beta_j \right\rangle = \sum_{j \in L} s_j \langle \alpha_j, \beta_j\rangle$$

Additionally, Z *must obey the following axioms:*

1. *Disjoint union axiom: Let X_1, X_2 be disjoint, and oriented 3-manifolds. Then*

$$Z(X_1 \sqcup X_2) = Z(X_1) \otimes Z(X_2) \tag{5.2.8}$$

2. *Naturality axiom: Let X_1, X_2 be oriented 3-manifolds with extended boundaries $(\partial X_1, \lambda_1)$, $(\partial X_2, \lambda_2)$, and $f\colon (X_1, (\partial X_1, \lambda_1)) \to (X_2, (\partial X_2, \lambda_2))$ a diffeomorphism. Then we have $V(f)\colon V(\partial X_1) \to V(\partial X_2)$ by $Z(X_1, \lambda_1) \mapsto Z(X_2, \lambda_2)$.*

3. *Gluing axiom: Let X is an oriented 3-manifold and $Y_1, Y_2 \subset \partial X$ be disjoint and extended by $\lambda_1, \lambda_2 \subset \lambda_X$ respectively. Let $f\colon Y_1 \to Y_2$ be an orientation-reversing diffeomorphism mapping λ_1 to λ_2. Then for the isomorphism*

$$V(\partial X) \simeq \sum_{l_1,\, l_2} V(Y_1; l_1) \otimes V(Y_2; l_2) \otimes V(\partial X \backslash (Y_1 \cup Y_2); (\hat{l}_1, \hat{l}_2)) \tag{5.2.9}$$

   yielding

$$Z(X) = \bigoplus_{l_1,\, l_2} \sum_j \alpha^j_{l_1} \otimes \beta^j_{l_2} \otimes \gamma^j_{\hat{l}_1,\, \hat{l}_2} \tag{5.2.10}$$

   Moreso, if the diffeomorphism $f\colon Y_1 \to Y_2$ glues said surfaces along some boundary component to form a manifold X_f, then

$$Z(X_f) = \sum_{j,\, l} \left\langle V(f)\alpha^j_{l_1}, \beta^j_{l_2} \right\rangle \gamma^j_{\hat{l}_1,\, \hat{l}_2} \tag{5.2.11}$$

4. *Mapping Cylinder Axiom: Let Y be closed, extended by λ and $Y \times I$ extended by $\lambda \oplus (-\lambda)$. Then*

$$Z(\boldsymbol{I}_{id}, \lambda \oplus (-\lambda)) = \bigoplus_{l \in L(Y)} id_l \tag{5.2.12}$$

   where $\boldsymbol{I}_{id}$ is the mapping cylinder of $id\colon Y \to Y$ and id_l is the identity in $V(V; l) \otimes V(Y; l)^*$.

Example 5.2.4. *The fusion process is encoded in a pair of pants, with legs labelled as a and b and the hip labelled as c. Given the Lagrangian subspace λ_P, we have then the following space $V(P; (a, b, \hat{c}), \lambda_P)$. From a construction using the disk, annulus and gluing axioms above, one has that*

$$V(P^c_{ab}; (a, b, c')) \cong \begin{cases} \mathbb{C} & N^c_{ab} > 0 \\ 0 & N^c_{ab} = 0 \end{cases} \tag{5.2.13}$$

Proposition 5.2.5. *Some fundamental results that fall out of these axioms.*

1. $V(S^2) \cong \mathbb{C}$

2. $\dim(V(T^2))$ *is the number of labels*

Proof. As an exercise we give the proofs:

1. Recall that $S^2 = D \cup_D D$. So from the gluing and disk axiom, we have

$$V(S^2) \cong V(D \cup_D D) \cong V(D) \cong \mathbb{C}. \tag{5.2.14}$$

□

2. Take a pair of pants with label (a, b, c') and suppose we glue the boundaries a and c together. Then we have made T^2 with punctures with labelling (n, b'), for $n \in L^{\text{str}}$. Then suppose we take the hip of another pair of pants with label (x, y, z') and glue now the boundaries of x and c together. Then we have T^2 with punctures with labelling (n, b') and (m, x') and observe that there are two such labels. Inductively then, one can do this to construct any T^2 and for k gluing of pairs of pants, one has k labels. □

Let $a, b, c \in L^{\text{str}}$. Then we define/denote the following vector spaces. $V_a = V(B^2; a)$, $V_{ab} = V(A; (a, \hat{b}))$ and $V_{ab}^c = V(P_{ab}^c)$ where P_{ab}^c is a pair of pants with labels a, b at the legs and c at the hip.

5.3 Anyons and Topology

Let X be a space consisting of n particles. If n particles are pairwise distinct and cannot coincide in space, i.e., the particles are *hardcore*, then our configuration space $C_n(X)$ is written as the n-fold Cartesian product of X

$$C_n(X) = X^n \backslash \Delta \tag{5.3.1}$$

Where $\Delta = \{(x_1, \ldots, x_n) : x_i \neq x_j, \text{ for some } i \neq j\}$

If the particles are now identical, then we can construct the configuration space up to permutation. That is:

$$C_n(X) = {}^{(X^n \backslash \Delta)}\!/_{S_n}. \tag{5.3.2}$$

Suppose now that $X = \mathbb{R}^m$. Let $\mathbb{L}$ be the Hilbert space in which we describe the particle's quantum states. Let $\mathcal{H}$ be our Hamiltonian with λ_i as our eigenvalues. Set $0 = \lambda_0 < \lambda_1 < \ldots$ and deconstruct our Hilbert space as $\mathbb{L} = \bigoplus_i \mathbb{L}_i$ where $\mathbb{L}_i$ is an eigenspace associated with λ_i. $\mathbb{L}_0$ is the ground state and $\mathbb{L}_i$, for any $i > 0$, are called excited states.

Suppose we can write the ground state as $\mathbb{L}_0 \cong V_n \otimes W$ where V_n carries all the global properties of our configuration space and W contains all the local properties about each particle. One way to reinterpret this is V_n contains all the *global* degrees of freedom and *local*

degrees of freedom are contained in W, about the particle's positions. An example would be the number of particles there are, which is global degree of freedom, as any local operation does not change this fact. In some sense, it is global because one must alter the whole system for this degree of freedom to change. Suppose also that there exists a physical mechanism, like an energy difference $\Delta\lambda = \lambda_1$ that protects encoded global properties onto V_n.

Let $p_1, \ldots, p_n$ be the positions of n particles in V_n with global properties. These properties will be encoded into a state vector $|\psi_0(p_i)\rangle \in V_n$. Then suppose we let the system evolve such that the particle returns to its initial position after some time t. Call this state $|\psi_1(p_i)\rangle \in V_n$. If V_n is given an orthonormal basis $\{e_i = |\psi_0(p_i)\rangle\}_1^n$ then we can rewrite our final state vector as a linear combination of that basis:

$$|\psi_1(p_i)\rangle = \sum_{j=1}^{n} a_{ji}|\psi_0(p_i)\rangle. \tag{5.3.3}$$

Suppose that our global properties are topological and that the particles traverse in a loop b in the configuration space $C_n(\mathbb{R}^m)$ as given in 5.3.2; then the unitary matrix $U(b) = (a_{ij})$ associated with this loop depends solely on the homotopy class of b. Therefore, one yields a unitary projective representation of the fundamental group of our configuration space, $\pi_1(C_n(\mathbb{R}^m)) \to U(V_n)$. We refer to this as the *statistics* of the particles.

Definition 5.3.1. *Let $X = \mathbb{R}^m$ have n hardcore particles. Then we can represent the statistics of these particles by $\rho\colon \pi_1(C_n(\mathbb{R}^m)) \to U(V_n)$, where V_n is a Hilbert space. If $\dim(V_n) = 1$, then the anyon is **abelian**. Otherwise, the anyon is **non-abelian**.*

For n particles, it is well-known that

$$\pi_1(C_n(\mathbb{R}^m)) = \begin{cases} 1 & m = 1, \\ B_n & m = 2, \\ S_n & m = 3, \end{cases} \tag{5.3.4}$$

Where B_n is the braid group of order n, and S_n is the symmetry group of order n. Since our anyons reside only in TQFT's with 2 spatial coordinates, then by setting $m = 2$, it is clear that the braid group represents anyonic statistics.

5.4 Fractional Quantum Hall Effect

In a physical sense, imagine the page as a metal plate. Then, suppose one connects the top and bottom of the page with wires that are connected to a battery such that the flow of electrons goes up the page. Then suppose there is a magnetic field **B** pointing out of the page. Then by the Lorentz force $\mathbf{F} = q(\mathbf{v} \times \mathbf{B} + \mathbf{E})$, the flow of electrons get perturbed by the magnetic field

and organize themselves such that the left side of the page accumulates in electrons; becomes negatively charged and the right side thus becomes positively charged. This is the classical Hall effect.

A property that is greatly affected by this is the resistivity of the metal. Physicists such as Hall found that the resistance is linearly dependent on the strength of the magnetic field. However, quantisation has a significant effect when the magnetic field becomes too large, ≥ 30 Tesla. The relationship they found was the following:

$$R_{xy} = \nu^{-1} \frac{h}{q_e^2} \tag{5.4.1}$$

Where h is Planck's constant, q_e is the charge of the electron, R_{xy} is the resistance of the metal plate where we take the plate to be the $xy-$plane, and we to refer to ν as the *Landau level*. Stormer, Tsui and Laughlin found particles that resided with $\nu \in \{1/3, 2/5, 5/2, \text{and more}\}$; were then awarded Nobel prizes in 1998 for this [STG99]. Wilczek independently mathematically discovered these particles and coined the name **anyons** [Wil91]. All in all, different large magnetic field strengths gave different resistances, in turn gave different Landau levels ν. Formally, we state:

Definition 5.4.1. *An **anyon** is a quasi-particle found from electrons, confined on a plane, interacting according to the fractional quantum hall effect, yielding some charge q with a magnetic flux ϕ. Anyons are characterised by their unique Landau level ν.*

Currently, there are up to 50 different values of ν experimentally found. Ising anyons and Fibonacci anyons are conjectured to be at Landau levels of $5/2$ and $12/5$ respectively. These electrons, exhibiting exotic physical and topological properties are a new state of matter called **fractional quantum hall liquids** (FQH liquid) - as ν is expressed as a fraction. As far as it is understood, this is the only way to make such a state of matter which exhibits the fractional quantum hall effect.

5.4.1 Topological properties of FQH liquids

Let Σ be a closed and oriented surface and suppose we confine a FQH liquid on Σ. For the system governed by the Hamiltonian $\mathcal{H}$, let $L(\Sigma)$ form a Hilbert space consisting of the lowest energy states of the Hamiltonian. Suppose that $L(\Sigma) = V(\Sigma) \otimes V^{\text{local}}(\Sigma)$, where $V^{\text{local}}(\Sigma)$ contains the local information similar to how we deconstructed $\mathbb{L}_0$ with W. Here, $V(\Sigma)$ is degenerate and the dimension of $V(\Sigma)$ is a quantum number measuring the degeneracy of this Hilbert space. In the thermodynamic limit, i.e., $T \to 0$, FQH liquids have an energy gap, so there is a requirement for stability in the sense of adiabaticity. A key feature for FQH liquids is that the Hamiltonian is a constant. By appropriate normalising, we can just take $\mathcal{H} = 0$ - which is required for TQFT [Wan10]. The excitations of FQH liquids are the quasi-particles

which we call anyons. Evolution is induced through the system by making topological changes.

Let Σ be a surface, $L(\Sigma)$ a Hilbert space and suppose that at points $p_1, p_2, \ldots$ we have well-separated, localised elementary excitations. Small neighbourhoods around said points in $L(\Sigma)$ form a Hilbert space. Again, we suppose we can decompose the Hilbert space as $L(\Sigma; p_i) = V(\Sigma; p_i) \otimes V^{\text{local}}(\Sigma; p_i)$. If we then look at the boundary formed by removing the associated neighbourhood for each p_i, we label with the associated anyon and we yield the Hilbert space $V(\Sigma; p_1, \ldots, p_n)$. An example of a topological change would then be actions from mapping class groups of Σ, preserving labelling and boundaries. Funnily enough, if we take a disk with n punctures, we get B_n - the braid group of order n.

If we braid these anyons adiabatically, to ensure that we stay within the ground states, we yield unitary transformations from t_0 to t_1. It then becomes clear that $V(\Sigma; p_1, \ldots, p_n)$ is a projective representation of the mapping class group of the surface Σ. [Wan10; Kit97] So an anyonic system gives a way to assign anyons at $p_1, \ldots, p_n$ to a Hilbert space $V(\Sigma; p_1, \ldots, p_n)$ of topological ground states of a constant Hamiltonian, and braidings with mapping classes of $V(\Sigma; p_1, \ldots, p_n)$. To encapsulate the relationship better, we provide a table as found in [Wan10]

$UMTC$	**anyonic system**
simple object	anyon
label	anyon type or topological charge
tensor product	fusion
fusion rules	fusion rules
triangular space V_{ab}^{c} or V_{c}^{ab}	fusion/splitting space
dual	antiparticle
birth/death	creation/annihilation
mapping class group representations	anyon statistics
non-zero vector in $V(Y)$	ground state vector
unitary F-matrices	recoupling rules
twist $\theta_x = e^{2\pi i s_x}$	topological spin
tangles	anyon trajectories

Chapter 6

Topological Quantum Computing

Every anyonic model consisting of non-abelian anyons can form a model of a quantum computer. What makes this topological is the construction of a quantum gate as being unitary representation of the braid group.

In the quantum circuit model (QCM), it is often convenient to regard our Hilbert spaces V_n as having some sort of tensor product decomposition. Surprisingly, this is an inconvenience in TQC to the point where, if one forces such a decomposition, error arises. Suppose V_n has a tensor product decomposition. Then $\dim(V_n)$ is of the form k^n, $k \in \mathbb{N}$. In TQC, this is not guaranteed. Moreso, finding a model that satisfies this is rare.

We will heavily follow from the notes in Chapter 7 in [Wan10]. For a given UMTC, let L^{str} be the strict label set. Then for simpler UMTCs, one can perform inductive calculations on the given fusion rules to yield the dimensions of $V^a_{n;x}$, $x, a \in L^{\text{str}}$.Let $U_L \colon (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ a quantum circuit, $\iota \colon (\mathbb{C}^2)^{\otimes n} \to V^a_{n;x}$ be an embedding, and $\rho \colon B_n \to U(2^n)$ be a representation, where B_n is the braid group of order n. If we want quantum universality, we want to find

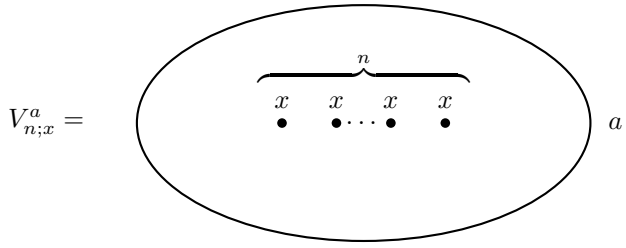
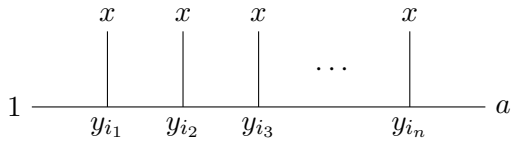


Figure 6.1: Diagramatic aid for how we arrange our n anyons, being the x species. Fusing them together, the anyons fuse to form a species of an anyon a

$b \in B_n$ such that the following diagram commutes.

$$\begin{array}{ccc} (\mathbb{C}^2)^{\otimes n} & \xrightarrow{\iota} & V^a_{n;x} \\ \Big\downarrow{\scriptstyle U_L} & & \Big\downarrow{\scriptstyle \rho(b)} \\ (\mathbb{C}^2)^{\otimes n} & \xrightarrow[\iota]{} & V^a_{n;x} \end{array}$$

This task is very difficult. In practice, it is easier to find a finite set of $b \in B_n$ such that the diagram commutes with some precision. For non-abelian anyons, one can guarantee universality by finding a braid group representation, for a finite set of braids in B_n, whose image is dense in $\mathrm{SU}(V^a_{n;x})$. But first we require some basis to make such representations make sense. The basis is derived from the fusion trees in $V^a_{n;x}$, - provided admissibility.



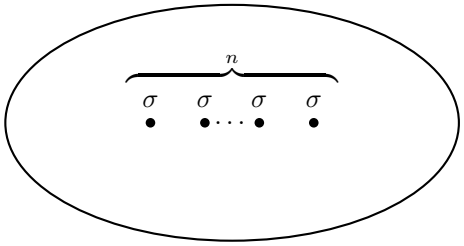
It is a matter of convention to have the vacuum state on the left, excluded from the total number of anyons. In Chapter 4, this diagram would have been drawn with a 1 on the far left vertex and the x anyons being drawn diagonally, meeting the line from 1 to a. Here, we draw it horizontally as it is easier to see the bit structure. Note that time would thus flow right to left as opposed to down to up in Chapter 4. Some examples from prior known UMTCs can follow

6.1 Ising Quantum Computation

The content here is heavily based from [FG10]. Recall that $L_{\text{Ising}} = \{1, \sigma, \psi\}$ and the main fusion rule of concern is $\sigma \otimes \sigma = \sigma^2 = 1 \oplus \psi$. Due to the dimension of the Ising anyon σ to be $\sqrt{2}$; in taking repetitive products of σ, we will have computational spaces with dimensions of powers of 2. So one *might* have hope in finding this model to be universal. Given the other fusion rule $\psi \otimes \sigma = \sigma \otimes \psi = \sigma$, the general formula for σ^n is

$$\sigma^n = \begin{cases} 2^{\frac{n-1}{2}}\sigma & n \text{ odd} \\ 2^{\frac{n-2}{2}}(1 \oplus \psi) & n \text{ even} \end{cases} \tag{6.1.1}$$

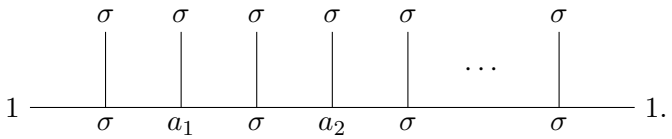
For the Hilbert space

$V^a_{n;\sigma} = V^a_n$ 

From 6.1.1, the dimensionality of V^a_n follows to be

$$\dim(V^a_n) = \begin{cases} 2^{\frac{n-2}{2}} & n \text{ even} \\ 2^{\frac{n-1}{2}} & n \text{ odd} \end{cases} \tag{6.1.2}$$

To form our computational space, we choose n to be even with $a = 1$ and thus define the basis we want to do computations on to consist of vectors of the form:

$e_{1\sigma a_1 \sigma a_2 \sigma \ldots \sigma 1} =$ 

Here, a_i is either 1 or ψ. For a single qubit, we take 4 anyons (the last being 1) and choose $|0\rangle = e_{1\sigma 1\sigma 1}$ and $|1\rangle = e_{1\sigma\psi\sigma 1}$. Let $\rho\colon B_3 \to U(2)$ be the representation by

$$\rho(\sigma_1) = \rho(\sigma_3) = e^{i\pi/8}\begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}, \qquad \rho(\sigma_2) = \frac{e^{-\pi i}}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{6.1.3}$$

With these representations, we have the following (Clifford) gates

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \rho(\sigma_1\sigma_2\sigma_1) \qquad \text{—— Hadamard Gate} \tag{6.1.4}$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \rho(\sigma_1^{-1}) \qquad \text{—— Phase Gate} \tag{6.1.5}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \rho(\sigma_2\sigma_2) \qquad \text{—— Pauli } x \tag{6.1.6}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \rho(\sigma_1\sigma_1\sigma_2^{-1}\sigma_2^{-1}) \qquad \text{—— Pauli } y \tag{6.1.7}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho(\sigma_1\sigma_1) \qquad \text{—— Pauli } z \tag{6.1.8}$$

If one were to have 2 qubits, then we require 6 anyons (the last being 1) and we choose the basis $|00\rangle = e_{1\sigma 1\sigma 1\sigma 1}, |01\rangle = e_{1\sigma\psi\sigma 1\sigma 1}, |10\rangle = e_{1\sigma 1\sigma\psi\sigma 1}$, and $|11\rangle = e_{1\sigma\psi\sigma\psi\sigma 1}$. Then for $\varrho\colon B_5 \to U(4)$,

$$\varrho(\sigma_1) = e^{\frac{i\pi}{8}}\,\mathrm{diag}(-1,-1,i,i) \tag{6.1.9}$$

$$\varrho(\sigma_2) = \frac{e^{\frac{i\pi}{8}}}{\sqrt{2}}\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix} \tag{6.1.10}$$

$$\varrho(\sigma_3) = e^{\frac{i\pi}{8}}\,\mathrm{diag}(-1,i,i,-1) \tag{6.1.11}$$

$$\varrho(\sigma_4) = -\frac{e^{\frac{i\pi}{8}}}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix} \tag{6.1.12}$$

$$\varrho(\sigma_5) = e^{\frac{i\pi}{8}}\,\mathrm{diag}(-1,i-1,i) \tag{6.1.13}$$

And, up to some phase factor, one can write that the $CNOT$ gate and CZ gate, where Z is the z-Pauli matrix, is:

$$\varrho(\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_3\sigma_4\sigma_3\sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{—— CNOT} \tag{6.1.14}$$

$$\varrho(\sigma_1\sigma_3^{-1}\sigma_5) = \mathrm{diag}(1,1,1,-1) \qquad \text{—— CZ} \tag{6.1.15}$$

Now, despite all these representations for our gates, the Ising anyonic model for quantum computing is not universal. In our basis, the image of the representation of braids into the unitary matrices is $\mathbb{Z}_2^{\frac{n}{2}-1} \rtimes S_n$ [Wan10; FG10].

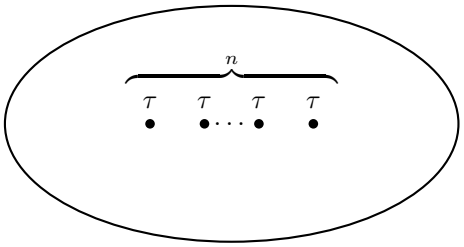
6.2 Fibonacci Quantum Computation

Recall the following setup for the Fibonacci model. $L_{\text{Fib}} = \{1, \tau\}$ with $\tau \otimes \tau = \tau^2 = 1 \oplus \tau$. We enumerate our well-loved Fibonacci sequence as $(F_n)_{n\in\mathbb{N}_0} := (1, 1, 2, 3, 5, \dots)$. Given the general formula from 4.4.1, the dimensionality for our space consisting of n τ type anyons is

$$\dim(V_n^a) = \begin{cases} F_{n-2} & a = 1 \\ F_{n-1} & a = \tau \end{cases} \tag{6.2.1}$$

where $F_{-1} = 0$. Recall that the dimensionality should be the power of some integer. But the Fibonacci sequence does not contain many such numbers, if any - let alone small ones. We

define the notion of *leakage* as when gates act on the space, but somehow manage to escape the space. If we require the quantum circuit model to be modelled in the model, we must choose a computational subspace. We choose the entirety of V_n^1 to be such space - that way there is no leakage into a superspace.

$V_{n;\tau}^a = V_n^a$  $a = 1$

For one qubit, we choose the basis for V_4^1 to be $|0\rangle = e_{1\tau 1\tau 1}$ and $|1\rangle = e_{1\tau\tau\tau 1}$ with the following braid representations

$$\rho(\sigma_1) = \begin{pmatrix} e^{\frac{-4\pi i}{5}} & 0 \\ 0 & e^{\frac{3\pi i}{5}} \end{pmatrix}, \qquad \rho(\sigma_2) = \begin{pmatrix} \phi^{-1}e^{\frac{-4\pi i}{5}} & \phi^{-1/2}e^{\frac{-3\pi i}{5}} \\ \phi^{-1/2}e^{\frac{-3\pi i}{5}} & \phi^{-1} \end{pmatrix} \tag{6.2.2}$$

For n qubits, we choose our computational subspace to be V_{2n+2}^1 and denote a qubit state $i_1 i_2 \dots i_n$ to be $e_{1\tau a_{i_1}\tau a_{i_2}\tau \dots a_{i_n}1}$ where $a_0 = 1$ and $a_1 = \tau$. The following theorem follows from the Solovay-Kitaev Theorem and from [FLW00; FLW02]

Theorem 6.2.1. *Let $U_L \colon (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ be a quantum circuit in $SU(2^n)$ and $\delta > 0$. Then there is a $b \in B_{2n+2}$ such that $|\rho(\sigma) - U_L| < \delta$, and σ can be computes by a Turing machine in time $O(n^k \frac{1}{\delta^m})$, for some $k, m \in \mathbb{N}$.*

For 2 qubits, we require 6 anyons on a 4-dimensional subspace of $V_6^1 \cong \mathbb{C}^5$. If a gate acts on a subspace of the whole computational space, then we indeed risk leakage. Take 3 qubits, i.e., $V_8^1 \cong \mathbb{C}^8$. As the model is universal, take the CNOT gate to act on the first 6 anyons. That is, on the first two qubits. Then there is a leak between V_6^τ and V_6^1. To get around this, one may introduce the Jones Representation. This yields a stronger density result, which allows us to approximate pairs $(A, B) \subset SU(5) \times SU(8)$ in the Jones representation of $V_6^1 \oplus V_6^1$. In this setting, if we use the gate CNOT $\oplus$ $\mathbb{1}$, then there is no leakage.

6.3 Fault Tolerance

Error arises in QC due to the decoherence of our qubits. That is a given model of QC (except TQC), the evolution of our computational space is encoded onto local degrees of freedom of the system. Take HQC for example. Suppose the loop C in the parametric space $\mathcal{M}$ encloses an area, and hence forms a flux ϕ. Perturbations in the loop are allowed in HQC provided that

approximately the same flux is yielded, so *approximately* the same phase is also yielded. But for a big enough perturbation, this fails. So if, for a given model of QC, one were to engineer the model physically, then the system couples to other unwanted things, such as the environment. These types of couplings form errors in our quantum computations.

In our topological model, the degrees are freedom are non-locally encoded onto our surfaces, so they are insensitive to local perturbations. The information is then encoded in the non-local degrees of freedom, and so is automatically protected against errors caused by local interactions with the environment. As Kitaev himself writes, this is "fault tolerance guaranteed at the hardware level with no further need of quantum error correction", [Kit97].

6.4 Recent Advancements

On February 2025, a team from Microsoft released a paper discussing a new quantum central processing unit (CPU) that they have engineered called Majorana 1 [Aas+25]. They used properties from superconductors to create a type of particle called a Majorana fermions. These fermions are laid out on wires and it turns out that they exhibit topological properties, such as fault-tolerance, to carry out computations - modelled using Ising anyons . We present a very brief overview of how superconductors are used to model Ising anyons to carry out quantum computations.

6.4.1 Majorana Fermions in Superconductors

6.4.1.1 Majorana Fermions

To introduce Majorana fermions, we will be using the language of spinors. Physically, Majorana fermions are defined by the property that they are their own anti-particle. For a more comprehensive description of these fermions, refer to Chapter 36 of [Sre07]. Let $\psi\colon \mathbb{R}^{1+1} \to \mathbb{C}^2$ be a left-handed spinor field. The Lagrangian density for ψ is

$$\mathcal{L}[\psi] = i\psi^\dagger(\bar{\sigma}^\mu\partial_\mu)\psi - \frac{m}{2}(\psi\psi + \psi^\dagger\psi^\dagger), \tag{6.4.1}$$

where $m \in \mathbb{R}$ is the mass of the particle. The equations of motion that correspond to Equation 6.4.1 according to the least action principle are:

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot{c}}\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{a}c}\partial_\mu & m\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \psi_c \\ \psi^{\dagger\dot{c}} \end{pmatrix} = 0. \tag{6.4.2}$$

Note that each entry in the matrix is a 2×2 complex matrix, indexed by a and c, and the *Majorana field* $\Psi \coloneqq \begin{pmatrix} \psi_c & \psi^{\dagger\dot{c}} \end{pmatrix}^T$ has four entries. We can condense this by defining the γ

matrices:

$$\gamma^\mu \coloneqq \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\dot{a}c\mu} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{6.4.3}$$

So we can rewrite Equation 6.4.2 in the following form, commonly known as the Dirac equation:

$$(-i\gamma^\mu \partial_\mu + m)\Psi = 0. \tag{6.4.4}$$

Now, consider two left handed spinor fields $\psi_1, \psi_2 \colon \mathbb{R} \to \mathbb{R}$ with the Lagrange density as:

$$\mathcal{L}[\psi] = \sum_{i=1,2} \psi_i^\dagger (\bar{\sigma}^\mu \partial_\mu)\psi_i - \frac{m}{2}(\psi_i \psi_i + \psi_i^\dagger \psi_i^\dagger). \tag{6.4.5}$$

This Lagrangian is invariant under the following $\mathrm{SO}(2)$ transformation,

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{6.4.6}$$

where $\alpha \in \mathbb{R}$. We get an equivalent description by defining $\chi, \xi \colon \mathbb{R} \to \mathbb{C}$ by

$$\chi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \tag{6.4.7}$$

$$\xi = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2). \tag{6.4.8}$$

The Lagrangian density, in terms of these new fields, is now

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m(\chi\xi + \xi^\dagger \chi^\dagger), \tag{6.4.9}$$

and similarly, we can obtain equations of motion like in equation 6.4.2:

$$\begin{pmatrix} m\delta_c{}^a & -i\sigma^\mu_{a\dot{c}}\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{a}c}\partial_\mu & m\delta^{\dot{a}}{}_c \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} = 0. \tag{6.4.10}$$

Here, we yield the *Dirac field* $\Phi \coloneqq \begin{pmatrix} \chi_c & \xi^{\dagger\dot{c}} \end{pmatrix}^T$. Observe that the $\mathrm{SO}(2)$ transformations act on χ and ξ as follows:

$$\chi \to e^{-i\alpha}\chi, \tag{6.4.11}$$

$$\xi \to e^{+i\alpha}\xi. \tag{6.4.12}$$

Define an operator called the *charge conjugation operator* C which satisfies the property:

$$\chi^C(x) = C^{-1}\chi(x)C = \xi(x), \tag{6.4.13}$$

$$\xi^C(x) = C^{-1}\xi(x)C = \chi(x). \tag{6.4.14}$$

Additionally, it follows from Equations 6.4.7 and 6.4.8 that we can write Majorana fields in terms of the Dirac fields by

$$\psi_1 = \frac{1}{\sqrt{2}}(\chi + \xi), \tag{6.4.15}$$

$$\psi_2 = \frac{-i}{\sqrt{2}}(\chi - \xi). \tag{6.4.16}$$

In this way, it is clear that these Majorana fields satisfy the property that they are their own anti-particle. That is, $\psi_i^C = \psi_i$, for $i = 1, 2$. In a physical interpretation, Dirac fields represent Dirac fermions (spin $1/2$ particles such as electrons), and Majorana fields represent Majorana fermions.

6.4.1.2 Superconductors

Physically, a *superconductor* is a material that conducts electricity with no electrical resistance when below a certain "critical temperature". In this discussion, we are concerned with how one prepares the superconductor to physically realise Majorana fermions (MFs), which are computationally modelled by Ising quantum computation. A more detailed description of the physical mechanisms behind this discussion is provided here [Lut10].

By some physical mechanism consisting of a quasiparticle called a phonon, one can couple electrons in superconductors to form a *Cooper pair*. A *hole* is another type of quasiparticle, representing the absence of an electron. They are usually depicted as positrons — the anti-particle of an electron. By coupling a hole with a Cooper pair in a superconductor, the hole can be occupied by either electron in the Cooper pair. So overall, we have an "electron" from fusing two electrons with a hole. But we do not know which electron occupied the hole. For either electron then, it is in an equal superposition of fusing with the hole, or not. We can "correspond" the state of not fusing with the hole with ξ and the state of fusing with the hole with χ. That way, it is clear to see that we have made two MFs.

Moreover, MFs in this setting are modelled by Ising anyons, where we take the MFs to be the Ising anyon σ. Recall that MFs are their own anti-particle. So they can fuse to the vacuum. But we have also demonstrated that they can fuse to form a Dirac fermion. This is well-aligned with the non-trivial fusion rule for Ising anyons, $\sigma^2 = 1 \oplus \psi$. We assign the fusion of two MFs to the vacuum anyon 1 with $|0\rangle$, and $|1\rangle$ for a Dirac fermion ψ. In Chapter 6 of [Pac12], they demonstrate that one can derive the F and R matrices given in Equation 4.4.12 using the Kitaev honeycomb model.

A more primitive model than the honeycomb model is the Kitaev toy model [Kit01]. Consider five MFs, each coupled to another MF due to their prior coupling as Cooper pairs. In this

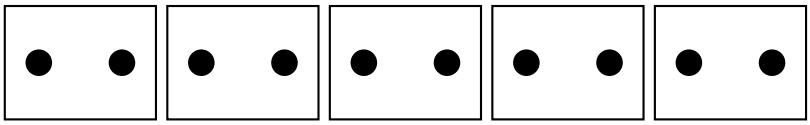


Figure 6.2: Visual description of Kitaev's toy model. The black dots represent MFs and the boxes around them show which MFs are paired together to form an ordinary Dirac fermion, an electron.

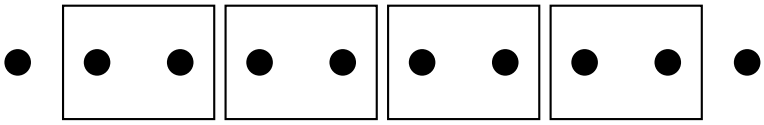


Figure 6.3: A rearangement of the boxes from Figure 6.2

model, we have the following arrangements of both fermions, illustrated in Figure 6.2. Then by "pushing the arrangements of boxes across" [1], we are able to isolate two MFs on either side and fuse them - as illustrated in Figure 6.3. In having multiple of these arrangements, we can take the MFs on either side and exchange them, adiabatically, to yield quantum gates. For example, consider two lots of the arrangement given in 6.3, placed on a line. Then if we adiabatically exchange the two adjacent MFs, as in 6.4, then in fact this braiding corresponds to the Hadamard gate H as given in Equation 3.1.6 [Lut10].

[1] This model is subject to the BCS Hamiltonian. Cooper pairs correspond to ground states in the BCS Hamiltonian, but the electron that MFs fuse to are an excited state. This inaccuracy allows us to shift the boxes [Zag12].

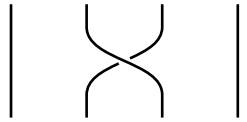


Figure 6.4: Time passes from the bottom going up. Enumerate the points left to right, $(1-2, 3-4)$. Label the fusion of the left pair as A, and the right pair as B. At the top we yield $(1-3, 2-4)$, due to the braiding in the middle.

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