Chen-Ruan Cohomology of Orbifolds

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Project Report

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Career Development Statement

The summer research scholarship provided me with an excellent platform to learn and engage with mathematics. I am especially grateful to my supervisor, Pedram Hekmati, for a hospitable and motivating environment, and in addition, for inviting me to the "Workshop on Poisson Geometry, Groupoids and Quantization", in which I encountered an incredible blend of geometry. The experience over the summer reinforced my will to pursue graduate studies in mathematics.

Summary of Research and its Significance - General Readership

Many objects in our universe are locally flat, but globally they may be curved or twisted; one might think of a soccer ball or even planets in our solar system. In mathematics, such objects are called 'manifolds.' Over the summer, I investigated a concept that generalises the notion of a manifold, that of an 'orbifold.' An orbifold is almost like a manifold, except we allow the presence of a few 'singularities.' A natural question to ask is to what extent the existence of these singularities affects the operations that we could usually perform. I investigated this question in this project.

Abstract

Orbifolds are spaces which locally look like the quotient $\mathbb{R}^n/G$, where G is a finite group. Consequently, they are examples of singular spaces; the singular points arising as fixed points of a finite group action. The concept arises quite naturally when one considers a quotient of the form M/G, where M is a smooth manifold and G a compact Lie group, for, if the G-action is not sufficient, the quotient space will fail to be a manifold, and it becomes of interest to know whether or not we can still peform any differential geometry on the resulting quotient. The formal notion was introduced by Ichirō Satake in 1956 under the title of a 'V-manifold' ([3]). Upon introducing the concept, Satake simultaneously provided natural generalizations of many standard tools and theorems from the differential geometry of manifolds, such as de Rham cohomology and the Gauss-Bonnet Theorem ([3], [4]). To explain the term that stuck, 'orbifold' was introduced via a democratic procedure in a course of William Thurston[1] in 1976-77, and evidently represents a contraction of the two words, 'orbit' and 'manifold'. In modern times, one might suggest that the best way to view an orbifold is as a special kind of differentiable stack $\mathfrak{X}$, which can be thought of as a collection of Lie groupoids, up to Morita equivalence, where a choice of Lie groupoid is akin to choosing an 'atlas'.

The purpose of this report is to summarise an encounter with the theory of orbifolds. A tentative outline is as follows. In the first section, we discuss orbifolds from the classical perspective, and then their incarnation as groupoids. In the second section, we motivate the Chen-Ruan cohomology by discussing Satake's de Rham Theorem for orbifolds. In the final section, we introduce the inertia orbifold and Chen-Ruan cohomology groups.

Contents

[1]Thurston used orbifolds in his geometrization program for 3-manifolds, and introduced the notion of the orbifold fundamental group.

1 Orbifolds - The Two Viewpoints

As discussed above, there are two ways to view an orbifold. The first is the classical approach, where we proceed locally and view an orbifold as a space equipped with an orbifold atlas; this construction is remniscent of the theory for smooth manifolds. The second is a modern take, using the language of groupoids to proceed globally, and view an orbifold structure on a space as a *Morita equivalence* class of sufficiently nice groupoids. It is the purpose of this section to discuss these two vantage points, and the bridge between them. We follow the first chapter of 'Orbifolds and Stringy Topology' ([1]) and the fourth chapter of 'Sasakian Geometry' ([9]) closely.

1.1 The Classical Perspective

Let us define an orbifold.

Definition 1.1. *Let X be a topological space. Fix $n \in \mathbb{N}$.*

1. *An orbifold chart of dimension n for an open subset $U \subseteq X$ consists of the following data; an open non-empty connected subset $\tilde{U} \subseteq \mathbb{R}^n$, a finite group G of smooth automorphisms of $\tilde{U}$, a G-invariant, continuous, and surjective map $\phi : \tilde{U} \to U$ which induces a homeomorphism from $\tilde{U}/G$ onto $U \subseteq X$. We shall frame this data as a tuple $(\tilde{U}, G, \phi)$, leaving U to be clear from notational convention. We say that U is uniformised by $(\tilde{U}, G, \phi)$.*

2. *By an embedding $\lambda : (\tilde{U}, G, \phi) \hookrightarrow (\tilde{V}, H, \psi)$ between two orbifold charts on X we mean a smooth embedding $\lambda : \tilde{U} \to \tilde{V}$ for which $\psi \circ \lambda = \phi$.*

3. *An n-dimensional orbifold atlas on X consists of a collection of n-dimensional orbifold charts $\mathcal{U} = \{(\tilde{U}, G, \phi)\}$ which cover X and are locally compatible in the following sense: for any two charts $(\tilde{U}, G, \phi)$, $(\tilde{V}, H, \psi)$ where $\phi(\tilde{U}) = U$ and $\psi(\tilde{V}) = V$, and a point $x \in U \cap V$, there exists an open neighbourhood $W \subseteq U \cap V$ of x and a chart $(\tilde{W}, K, \chi)$ for W such that we have two embeddings, $(\tilde{W}, K, \chi) \to (\tilde{U}, G, \phi)$ and $(\tilde{W}, K, \chi) \to (\tilde{V}, H, \psi)$.*

4. *We say that an atlas $\mathcal{U}$ refines another atlas $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{V}$. We call two atlases equivalent if they admit a common refinement.*

Using the notation as in the first point above, a group element $g \in G$ may be viewed as an embedding $g : (\tilde{U}, G, \phi) \to (\tilde{U}, G, \phi)$, for ϕ is G-invariant. We capture the information behind an orbifold chart $(\tilde{U}, G, \phi)$ on X with a diagram,

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\ \phi\ } & U \\ \downarrow & \nearrow_{\cong} & \\ \tilde{U}/G & & \end{array}$$

Definition 1.2 (Effective Orbifold)**.** *An (effective) orbifold $\mathcal{X}$ of dimension n is a paracompact Hausdorff topological space X equipped with an equivalence class of n-dimensional orbifold atlases.*

Remark 1.1. Let us explain the use of the word 'effective' in Definition (1.2) . The first point in Definition (1.1) implies that for each chart $(\tilde{U}, G, \phi)$ on X, the finite group G acts effectively on $\tilde{U}$ (i.e. if $g \in G$ fixes every element, then it is the identity). If instead we allow each group G to act ineffectively, then we arrive at the definition of an ineffective orbifold.

As we are (at least for now) only concerned with effective orbifolds, we shall simply say 'orbifold'. Before moving on, it is necessary to make a few technical remarks regarding the previous two definitions, as pointed out in page five of [2]. First, every orbifold atlas on X is contained in a unique maximal one, with two orbifold atlases equivalent if and only if they are contained in the same maximal one. We shall henceforth work implicitly with a maximal atlas. Finally, note that if $(\tilde{U}, G, \phi)$ and $(\tilde{V}, H, \psi)$ are two charts in a fixed atlas on X and $\tilde{U}$ is simply connected, then there exists an embedding $(\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ whenever $\phi(\tilde{U}) \subset \psi(\tilde{V})$. This allows us to recover Satake's original definition of an orbifold, except for the fact that Satake required in addition that for each chart $(\tilde{U}, G, \phi)$ in a fixed atlas $\mathcal{U}$ on X, the fixed point set of each $g \in G$ has codimension at least 2. Note that Satake's fixed point condition is automatically satisfied if one is working with an 'orientable' orbifold. By orientable, it is meant that we can assign an orientation to each $\tilde{U}$ such that all the embeddings (so this includes the group elements, $g \in G$) are orientation preserving.

An important technical result for the theory is,

Proposition 1.1. *For two embeddings* $\lambda, \mu : (\tilde{U}, G, \phi) \rightrightarrows (\tilde{V}, H, \psi)$, *there exists a unique* $h \in H$ *for which* $\mu = h \circ \lambda$. *In the special case for which we view an element* $g \in G$ *as an embedding of the chart* $(\tilde{U}, G, \phi)$ *into itself, the two embeddings* λ *and* $\lambda \circ g$ *yield a unique* $h \in H$ *for which* $\lambda \circ g = h \circ \lambda$. *We denote this* h *by* $\lambda(g)$, *and hence associate to our embedding* $\lambda : \tilde{U} \to \tilde{V}$ *an injective group homomorphism* $\lambda : G \to H$.

Proof. This result is proved in the appendix of a paper by Moerdijk and Pronk, [2]. □

The proposition above tells us that an embedding is equivariant with respect to its associated group monomorphism.

We now come to the notion of a smooth map. We shall follow Satake ([3]) and define a smooth map of orbifolds as follows.

Definition 1.3. *Let* $\mathcal{X} = (X, \mathcal{U})$ *and* $\mathcal{Y} = (Y, \mathcal{V})$ *be two orbifolds. A map* $f : X \to Y$ *is said to be a smooth map of orbifolds if for any point* $x \in X$, *there are charts* $(\tilde{U}, G, \phi)$ *around* x *and* $(\tilde{V}, H, \psi)$ *around* $f(x)$ *with the property that* f *maps* U *into* V *and can be lifted to a smooth map* $\tilde{f} : \tilde{U} \to \tilde{V}$ *with* $\psi \circ \tilde{f} = f \circ \phi$. *Smooth maps can be composed. We call* $\mathcal{X}$ *and* $\mathcal{Y}$ *diffeomorphic if there are smooth maps* $f : X \to Y$ *and* $g : Y \to X$ *which compose to the respective identity maps.*

Remark 1.2. Historically, there have been issues with the notion of a smooth map between orbifolds provided by Satake. For example, the desired property that the pullback of an orbifold vector bundle by a smooth map is an orbifold vector bundle, may not always hold (see section 4.4 of [8] and section 2.4 of [1]). Fortunately, the issues are resolved by introducing another notion of a map between orbifolds, that of the Chen-Ruan good map (see section 4.4 of [8]) or, equivalently, the Moerdijk-Pronk strong map (see, for example, section 5 of [2]). In this way, upon considering the correct notion of an orbifold morphism, the theory of orbifolds begins to distinguish itself from its manifold counterpart.

Suppose that the finite group actions on all the charts of $\mathcal{X}$ are free, clearly then X is a manifold, being in addition locally Euclidean. This implies that points with non-trivial isotropy are what distinguishes an orbifold from a manifold; for this reason, they are our so called singular points. Let us make this notion of a singular point precise.

Definition 1.4 (Local Group and Singular Set)**.** *Let $\mathcal{X} = (X, \mathcal{U})$ be an orbifold and $x \in X$. If $(\tilde{U}, G, \phi)$ is any local chart around $x = \phi(y)$, the local group at x is defined as*

$$G_x = \{g \in G \mid gy = y\}.$$

The local group is uniquely determined up to conjugacy, independent of our choice of chart and representative $y \in \tilde{U}$ (see Definition 1.5 in [1]). We define the singular set of $\mathcal{X}$ to be

$$\Sigma(\mathcal{X}) = \{x \in X \mid G_x \neq 1\}.$$

A point of the singular set $\Sigma(\mathcal{X})$ is called a singular point of $\mathcal{X}$, so that singular points are points with non-trivial local group.

As is well known, we are able to add some additional assumptions regarding the behaviour of the group actions on our charts. We summarise this in the following proposition,

Proposition 1.2. *Let $\mathcal{X} = (X, \mathcal{U})$ be an orbifold. For a given chart $(\tilde{U}, G, \phi) \in \mathcal{U}$ about $x \in X$, we may assume, $\tilde{U}$ contains the origin, and $\phi(0) = x$, G is isomorphic to the local group at x, and G acts orthogonally on $\tilde{U}$.*

Proof. We refer the reader to Proposition 1.1.14 in [10]. □

Let us now provide some examples of orbifolds.

Example 1.1 (Effective Quotient Orbifolds)**.** Let M be a smooth manifold of dimension n and G a compact Lie group which acts smoothly, effectively, and almost freely (i.e. finite stabilisers) on M. We shall equip the orbit space M/G with the structure of an effective orbifold. Of course, the underlying topological space is the orbit space M/G equipped with the quotient topology; under our assumptions on M and G, it is paracompact and Hausdorff ([12], page 38). Fix $x \in M$. By the differentiable slice theorem ([12], page 308), there exists a a G_x-invariant neighbourhood U of x along with a G-equivariant map $G \times_{G_x} U \to M$ which is a diffeomorphism onto an open neighbourhood N of the orbit of x (so N is a G-space). Identify U with an open subset of $\mathbb{R}^n$, denoted $\tilde{U}$, via a diffeomorphism $f : \tilde{U} \to U$. We let G_x act on $\tilde{U}$ so that f is G_x-equivariant. An orbifold chart about x is given by $(\tilde{U}, G_x, \varphi)$ where $\varphi : \tilde{U} \to M/G$ is defined as follows; observe that $(G \times_{G_x} U)/G$ is homeomorphic to N/G and the former is identified with U/G_x. Now by definition, φ must map onto an open subset of M/G, and so it is defined via $\tilde{U} \to U \to U/G_x \to N/G$. Collecting such charts as x runs over M, we obtain an orbifold atlas on the orbit space M/G, where local compatibility of our charts is taken care of by sufficiently shrinking our open sets U. The resulting orbifold, again denoted by M/G, is called an effective quotient orbifold. A special case is for which G is a finite group, in this case such an orbifold is called an effective global quotient.

Remark 1.3. It is in fact the case that all effective orbifolds are effective quotient orbifolds; this is the content of Corollary 1.24 in [1], precisely, an effective n-orbifold is diffeomorphic to a quotient orbifold for a smooth, effective, and almost free $O(n)$-action on a smooth manifold M. The smooth manifold M is the 'frame bundle' of $\mathcal{X}$. For more details we refer the reader to page 12 of [1].

Example 1.2 (Coordinate Reflection on the Torus)**.** We now specialise to examples from a class called Toroidal orbifolds, these are orbifolds where we consider a quotient of the n-torus by a finite subgroup $G \subset GL_n(\mathbb{Z})$ acting smoothly. Let $\mathbb{T}^n = (S^1)^n = (\mathbb{R}/\mathbb{Z})^n$ be the n-torus, and consider the action of $\mathbb{Z}_2$ on $\mathbb{T}^n$ generated by the involution τ which acts by complex conjugation on each coordinate. The resulting orbit space $\mathbb{T}^n/\mathbb{Z}_2$ is an orbifold with 2^n singular points; the singular points having coordinates chosen from $\{0, 1/2\}$. A special case is the so called 'Pillowcase'. View the torus $\mathbb{T}^2 = S^1 \times S^1$ as a submanifold of $\mathbb{R}^3$. Let $\mathbb{Z}_2$ act on $\mathbb{T}^2$ via $(z, w) \mapsto (\bar{z}, \bar{w})$. The orbifold

$\mathbb{T}^2/\mathbb{Z}_2$ has underlying topological space (homeomorphic to) $\mathbb{S}^2$, and four singular points, each with local group $\mathbb{Z}_2$. We may visually interpret this action as a rotation by π around an axis,

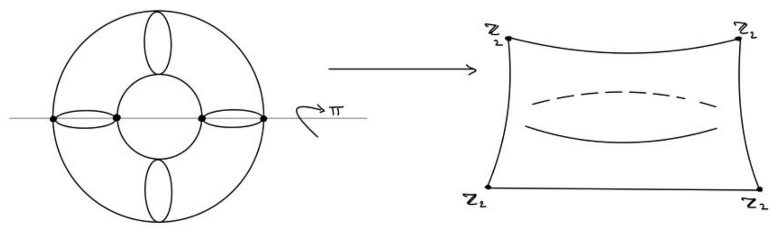

Figure 1.1: Our pillowcase.

This example allows us to realise $\mathbb{S}^2$ as the underlying coarse space of a "flat" orbifold. (We refer the reader to `https://ncatlab.org/nlab/show/Riemannian+orbifold#kummer_surface`).

Example 1.3 (Teardrop). Let $p \in \mathbb{Z}_{>1}$. Thurston's famous teardrop is an orbifold $\mathcal{Q}$ such that; the underlying space is $\mathbb{S}^2$ and its singular set $\Sigma(\mathcal{Q})$ consists of just a single point, whose neighbourhood is modelled on $\mathbb{R}^2/\mathbb{Z}_p$, where $\mathbb{Z}_p$ acts by rotations. For a more explicit construction, we refer the reader to Example 4.1.4 in [8] or Example 1.3.1 at [11].

Example 1.4 (A Canonical Orbifold Structure on a Manifold With Boundary). Let M be an n-manifold with boundary. We may provide M with the structure of an orbifold as follows. We consider each point $x \in \partial M$ to be modelled on the quotient $\mathbb{R}^n/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is generated by reflection about the hyperplane in the half space model about x. The resulting orbifold has singular set ∂M, and the singular points are thought of as 'mirror' points, with local group $\mathbb{Z}_2$. (See Example 1.3.3 at [11]).

1.2 The Groupoid Viewpoint

We shall now introduce the incarnation of orbifolds as groupoids, and demonstrate, briefly, how one passes to the standing definition of an orbifold. We follow closely section 4.3 of [9], and section 1.4 of [1].

Definition 1.5. *A groupoid $\mathcal{G}$ is a (small) category in which every arrow is invertible.*

To be a little bit more precise, our groupoid $\mathcal{G}$ consists of a set of objects G_0 and set of arrows G_1, with five natural structure maps; the source and target maps $s, t : G_1 \rightrightarrows G_0$, a composition map $m : G_1 \times_{G_0} G_1 \to G_1$, a unit map $u : G_0 \to G_1$, and finally an inverse map $i : G_1 \to G_1$. For an arrow $g \in G_1$ with $s(g) = x$ and $t(g) = y$ we shall write $g : x \to y$. We write g^{-1} for $i(g)$ and $g \circ h$ for $m(g, h)$. The usual identities must be satisfied (see for example section 3 of [2]). A topological groupoid is a groupoid in which both the set of objects and arrows are topological spaces, and the structure maps are continuous. Going one step further,

Definition 1.6. *A Lie groupoid $\mathcal{G}$ is a groupoid whose objects G_0 and arrows G_1 both admit the structure of smooth manifolds, with the additional property that the structure maps of $\mathcal{G}$ are all smooth and further, our source and target maps $s, t : G_1 \to G_0$ are submersions.*

That the source and target maps are required to be submersions is so that the domain of the multiplication map $G_1 \times_{G_0} G_1$ is a manifold, and consequently it makes sense to say that the composition map m is smooth. We point out that sometimes it is useful to think of G_0 as a base space, and the groupoid $\mathcal{G}$ is written as $G_1 \implies G_0$. Let us provide some examples of Lie groupoids.

Example 1.5 (Action Groupoid)**.** Let a smooth manifold M be equipped with a smooth left action of a Lie group K. We define a Lie groupoid $K \ltimes M$ with objects $(K \ltimes M)_0 = M$ and arrows $(K \ltimes M)_1 = K \times M$. The source map $s : K \times M \to M$ is projection onto the second factor, the target map $t : K \times M \to M$ is the group action. Thus arrow $(k, x) \in (K \ltimes M)_1$ is of the form,

$$x \xrightarrow{(k,x)} k \cdot x$$

The composition map m is defined in the natural way, with respect to our action. We call such a Lie groupoid an action groupoid. Note that by specialising our Lie group or manifold in the obvious way, we may view a manifold as a Lie groupoid (the so called 'unit groupoid', whose arrows are all units), or alternatively, a Lie group as a Lie groupoid (the set of objects being a single point).

Example 1.6 (A Groupoid of Germs; see Example 5.32 in [7])**.** Let M be a smooth manifold. By a 'local transition' on M, we mean a diffeomorphism between two open subsets of M. For the set of all local transitions on M, we write C^∞_M. A pseudogroup of local transitions on M is a subset P of local transitions on M for which,

1. $\mathrm{Id}_V \in P$ for any open set $V \subseteq M$.
2. If $f, f' \in P$, then the composition $f' \circ f|_{f^{-1}(\mathrm{dom}(f'))} \in P$ and inverse $f^{-1} \in P$.
3. If f is a transition on M and (V_α) is an open cover of $\mathrm{dom}(f)$ for which each restriction $f|_{V_\alpha} \in P$, then $f \in P$.

For a pseudogroup of local transitions P, we can associate a groupoid $\Gamma(P)$ whose objects are points of M, and arrows between $x, y \in M$ are given by

$$\Gamma(P)_1(x, y) = \{\mathrm{germ}_x f \mid f \in P, x \in \mathrm{dom}(f),\ f(x) = y\}$$

Multiplication is defined naturally, by composing transitions. The set of arrows $\Gamma_1(P)$ may be equipped with the sheaf topology, upon doing so, the groupoid $\Gamma(P)$ becomes effective.

Example 1.7 (Fundamental Groupoid)**.** Suppose M is a connected manifold. The fundamental groupoid of M, denoted $\Pi(M)$, has as objects points of M, $\Pi(M)_0 = M$. An arrow $g \in \Pi(M)_1$ with $s(g) = x$ and $t(g) = y$ is given by a homotopy class of paths from x to y. Note then that composition is defined naturally, and inversion of an arrow is simply given by walking along in the opposite direction. If we consider all arrows with source and target $x \in M$ (i.e. self loops of x), then we capture the fundamental group of M, based at x, $\pi_1(M, x)$.

In the previous example we saw that self loops of an object had significance; we can make some general definitions and remarks regarding such loops.

Definition 1.7. *Let $\mathcal{G}$ be a Lie groupoid with objects G_0 and arrows G_1. For an object $x \in G_0$, the set of all arrows with source and target x is called (because of a canonical group structure) the isotropy group (or local group) at x, and is denoted by G_x. The set $ts^{-1}(x)$ of targets of arrows with source x is called the orbit of x. The orbit space $|\mathcal{G}|$ of $\mathcal{G}$ is by definition of the quotient space $G_0/\sim$ where $x \sim y$ if and only if x and y are in the same orbit (i.e. there is an arrow from x to y). We call $\mathcal{G}$ a (groupoid) represenation of $|\mathcal{G}|$.*

In order to make the connection to orbifolds, we must restrict our attention to classes of Lie groupoids. The classes of interest are as follows;

Definition 1.8. *Let $\mathcal{G}$ be a Lie groupoid, with set of objects G_0 and arrows G_1.*

1. *We call $\mathcal{G}$ proper if the map $(s, t) : G_1 \to G_0 \times G_0$ is a proper map (i.e. the preimage of any compact set is compact).*

2. *We call $\mathcal{G}$ a foliation groupoid if for each $x \in G_0$, the isotropy group G_x is discrete.*

3. *We call $\mathcal{G}$ étale if the source and target maps $s, t : G_1 \rightrightarrows G_0$ are local diffeomorphisms.*

Note that if $\mathcal{G}$ is étale, then $\dim \mathcal{G} = \dim G_0 = \dim G_1$ is well-defined. Next, for an arbitrary Lie groupoid $\mathcal{G}$ each isotropy group G_x is a Lie group. To see this, we note that $G_x = (s,t)^{-1}(x,x) = s^{-1}(x) \cap t^{-1}(x) \subset G_1$ and that, by hypothesis, s and t are submersions (i.e. their differential is everywhere onto), which implies that G_x is a smooth submanifold of G_1 (refer to Theorem 9.9 of [13]), the assumption that our structure maps are smooth implies that the natural group operations are smooth on G_x, so it is a Lie group. If we assume that $\mathcal{G}$ is proper, then each G_x is clearly a compact Lie group. A compact discrete Lie group is a finite group, so that if we assume $\mathcal{G}$ is a proper foliation Lie groupoid, then each G_x is a finite group. Clearly an étale Lie groupoid is a foliation groupoid, and so we have the following proposition,

Proposition 1.3. *If $\mathcal{G}$ is a proper étale Lie groupoid, then for each $x \in G_0$, the isotropy group G_x is finite.*

The reason for restricting to a special case in the above proposition will soon become apparent. A useful property of proper étale Lie groupoids is as follows.

Construction 1.1. Let $\mathcal{G}$ be a proper étale Lie groupoid. We shall describe a way in which the (finite) isotropy group G_x of $x \in G_0$ acts as a group of diffeomorphisms on a neighbourhood of x. Let $g \in G_x$ be fixed, then, because s and t are local diffeomorphisms, there exists an open neighbourhood V_g of $g \in G_1$ for which both s and t map V_g diffeomorphically onto an open neighbourhood U_x of x. Let $j : U_x \to V_g$ be the local inverse to the source map $s|_{V_g} : V_g \to U_x$. Define a diffeomorphism $\tilde{g} = t|_{V_g} \circ j : U_x \to U_x$. We obtain a group homomorphism $G_x \to \mathrm{Diff}(U_x)$ defined by $g \to \tilde{g}$. In this way, an arrow $g : x \to x$ yields a well defined germ of a diffeomorphism about x.

Definition 1.9. *An orbifold groupoid $\mathcal{G}$ is a proper étale Lie groupoid. We call an orbifold groupoid $\mathcal{G}$ effective if, for each $x \in G_0$, there exists an open neighbourhood U_x about x such that the associated group homomorphism $G_x \to \mathrm{Diff}(U_x)$ is injective.*

In what is to come, we shall justify the title 'orbifold groupoid'. In order to do so, we need the notion of Morita equivalence. First, a few definitions. Given that we view a Lie groupoid as a sort of 'smooth category', a homomorphism of Lie groupoids should be a smooth functor. Precisely,

Definition 1.10. *A homomorphism of Lie groupoids $\phi : \mathcal{K} \to \mathcal{G}$ consists of a pair of smooth maps $\phi_0 : K_0 \to G_0$, $\phi_1 : K_1 \to G_1$ which together commute with all the structure maps.*

If homomorphisms are functors, then we must have natural transformations. Let us quickly mention this, for completeness. If $\phi, \psi : \mathcal{K} \rightrightarrows \mathcal{G}$ are homomorphisms of Lie groupoids, a natural transformation α from ϕ to ψ, denoted $\alpha : \phi \implies \psi$, is given by a smooth map $\alpha : K_0 \to G_1$ for which $s \circ \alpha = \phi_0$ and $t \circ \alpha = \psi_0$. By natural, it is meant that if $k : x \to x'$ is an arrow in K_1, the following diagram commutes,

$$\begin{array}{ccc} \phi_0(x) & \xrightarrow{\phi_1(k)} & \psi_0(x) \\ {\scriptstyle \alpha(x)}\downarrow & & \downarrow{\scriptstyle \alpha(x')} \\ \psi_0(x) & \xrightarrow[\psi_1(k)]{} & \psi_0(x') \end{array}$$

Definition 1.11. *A homomorphism $\phi : \mathcal{K} \to \mathcal{G}$ of Lie groupoids is called an equivalence if,*

1. *(Essentially Surjective) The map*
$$t\pi_1 : G_{1s} \times_\phi K_0 \to G_0$$
defined on $G_{1s} \times_\phi K_0 = \{(g,k) \mid s(g) = \phi_0(k)\}$ *is a surjective submersion.*

2. *(Fully faithful) The diagram*

$$\begin{array}{ccc} K_1 & \xrightarrow{\phi_1} & G_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ K_0 \times K_0 & \xrightarrow[\phi_0 \times \phi_0]{} & G_0 \times G_0 \end{array}$$

is a fibered product of manifolds.

If we unwrap these conditions; the first means that any object in G_0 can be connected by an arrow in G_1 to the image of ϕ_0. The second condition means that ϕ produces a diffeomorphism

$$K_1(y,z) \to G_1(\phi_0(y), \phi_0(z))$$

from the space of arrows between y and z in K_0 and the space of arrows between $\phi_0(y)$ and $\phi_0(z)$ in G_0. Thus an equivalence is a smooth equivalence of categories. We call ϕ strong if $\phi_0 : K_0 \to G_0$ is a surjective submersion. We point out that an equivalence ϕ yields a homeomorphism of the underlying orbit spaces, $|\phi| : |\mathcal{K}| \to |\mathcal{G}|$. We now come to the notion of Morita equivalence.

Definition 1.12. *We say that two Lie groupoids $\mathcal{H}$ and $\mathcal{G}$ are Morita equivalent if there exists a third Lie groupoid $\mathcal{K}$ and two equivalences,*
$$\mathcal{H} \overset{\psi}{\leftarrow} \mathcal{K} \overset{\phi}{\rightarrow} \mathcal{G}$$

Let us make two remarks. If $\phi : \mathcal{K} \to \mathcal{G}$ is an equivalence, then $\mathcal{K}$ is Morita equivalent to $\mathcal{G}$ via strong equivalences (see Definition 1.43 in [1]). If ϕ is an equivalence of orbifold groupoids, then $\phi_0 : K_0 \to G_0$ is a local diffeomorphism (see Lemma 2.1 in [1]). We will now explain the connection between our standing definition of an orbifold (a space with charts) and the content of Definition 1.9, in which we called a proper étale Lie groupoid an 'orbifold groupoid'. We shall pass from an orbifold to an orbifold groupoid, and vice versa. Upon considering Morita equivalent Lie groupoids and isomorphic orbifolds, this passage is well defined.

First, we will show how one goes from an effective orbifold to an effective orbifold groupoid. Let $\mathcal{X} = (X, \mathcal{U})$ be an effective orbifold with a fixed atlas $\mathcal{U} = \{(\tilde{U}_i, G_i, \phi_i)\}$ on X. Define

$$\tilde{\mathcal{U}} = \coprod_i \tilde{U}_i.$$

Let $\mathcal{P}_\mathcal{X}$ denote the pseudogroup of local diffeomorphisms of $\tilde{\mathcal{U}}$ generated by the embeddings and their inverses. Let $\mathcal{G}(\mathcal{U})$ denote the groupoid of germs of diffeomorphisms of this pseudogroup $\mathcal{P}_\mathcal{X}$, as in Example 1.6, i.e. objects $\tilde{\mathcal{U}}$ and arrows germs of the embeddings. Consider the projection map $\phi : \tilde{U} \to X$ defined by taking the union of the ϕ_i. If $x_i \in \tilde{U}_i$ and $x_j \in \tilde{U}_j$ are such that $x_i \sim x_j$, then there is an embedding $\lambda_{ij} : \tilde{U}_i \to \tilde{U}_j$ for which $\lambda_{ij}(x_i) = x_j$, then, because $\phi_j \circ \lambda_{ij} = \phi_i$, we see that $\phi(x_i) = \phi(x_j)$. This implies that ϕ yields a well-defined map from the space of orbits $|\mathcal{G}(\mathcal{U})| \to X$. In this sense, we say that the groupoid $\mathcal{G}(\mathcal{U})$ represents the orbifold X. (A nice point to skip to now would be Definition 1.13). In fact, even more is true,

Proposition 1.4. *Let $\mathcal{X} = (X,\mathcal{U})$ be an effective orbifold with a fixed atlas $\mathcal{U}$, then $\mathcal{G}(\mathcal{U})$ is an effective orbifold groupoid. Moreover, if $\mathcal{X}' = (X',\mathcal{U}')$ is another effective orbifold with a fixed atlas $\mathcal{U}'$, then $\mathcal{G}(\mathcal{U})$ is Morita equivalent to $\mathcal{G}(\mathcal{U}')$ if and only if the orbifolds $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic.*

Proof. See Proposition 5.29 in [7]. □

Now we will show how to go from an effective orbifold groupoid to an effective orbifold. Let $\mathcal{G}$ be an effective orbifold groupoid. By Proposition 1.3, for each $x \in G_0$, the isotropy group G_x is finite. Futhermore, for any $x \in G_0$, there exists an open neighbourhood U_x of x in G_0 with an action of G_x such that there is an isomorphism of étale Lie groupoids,

$$\mathcal{G}|_{U_x} \cong G_x \ltimes U_x$$

(see for example, Corollary 5.31 in [7]). This allows us to construct an orbifold atlas on the orbit space $|\mathcal{G}|$, which is both Hausdorff and paracompact. Let $\pi : G_0 \to |\mathcal{G}|$ denote the quotient projection. For $x \in G_0$, we choose the neighbourhood U_x so that we have a diffeomorphism $\phi_x : U_x \to \tilde{U}_x \subseteq \mathbb{R}^n$, for $n = \dim \mathcal{G}$. Let G_x act on $\tilde{U}_x$ so that ϕ_x is G_x-equivariant. An orbifold atlas $\mathcal{U}$ on $|\mathcal{G}|$ consists of charts of the form,

$$\{(\tilde{U}_x, G_x, \pi \circ \phi_x^{-1})\}.$$

Embeddings of charts look as follows. If V_y and U_x are two such neighbourhoods and $V_y \overset{\iota_y}{\hookrightarrow} U_x$, then the embedding

$$\lambda_{xy} : (\tilde{V}_y, G_y, \pi \circ \psi_y^{-1}) \to (\tilde{U}_x, G_x, \pi \circ \phi_x^{-1})$$

is defined by $\lambda_{xy} = \phi_x \circ \iota_y \circ \psi_y^{-1}$. Note that the resulting orbifold represents the groupoid $\mathcal{G}$, for its underlying topoogical space is exactly $|\mathcal{G}|$. Our discussion may be summarised, along with a Theorem 1.45 from [1] (originally appearing in [2]),

Theorem 1.1. *If $\mathcal{G}$ is an effective orbifold groupoid, then its space of orbits $|\mathcal{G}|$ admits the structure of an effective orbifold. Two effective orbifold groupoids $\mathcal{G}$ and $\mathcal{H}$ represent the same effective orbifold up to isomorphism if and only if they are Morita equivalent.*

This roughly describes the bridge between the two vantage points. Equipped with our current theory, one may provide a new definition of an orbifold (one which makes it easy to drop the condition of an effective action). First, we specify the data akin to an atlas.

Definition 1.13. *An orbifold structure on a paracompact Hausdorff topological space X is given by an orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$. If $\phi : \mathcal{K} \to \mathcal{G}$ is an equivalence, then $|\phi| : |\mathcal{K}| \to |\mathcal{G}|$ is a homeomorphism, and $f \circ |\phi| : |\mathcal{K}| \to X$ is said to define an equivalent orbifold structure on X.*

The modern definition is now as follows.

Definition 1.14. *An orbifold $\mathcal{X}$ is a paracompact Hausdorff space X equipped with an equivalence class of orbifold structures. A specific choice of structure is given by the datum of an orbifold groupoid $\mathcal{G}$, and a homeomorphism $f : |\mathcal{G} \to X$, and is called a presentation of $\mathcal{X}$.*

2 A Recollection of Some General Theory

In this section, we will recall (in the atlas formalism) some general theory; and in particular attempt to motivate the Chen-Ruan cohomology. Our notion of bundle follows Satake ([4]).

Construction 2.1 (Tangent Bundle)**.** Given an n-dimensional orbifold $\mathcal{X} = (X, \mathcal{U})$, we may construct its tangent bundle $T\mathcal{X}$ as follows. Over each chart $(\tilde{U}, G, \phi) \in \mathcal{U}$ we have the corresponding tangent bundle $T\tilde{U} \to \tilde{U}$. Note $T\tilde{U} \cong \tilde{U} \times \mathbb{R}^n$. We let G act on the total space $\tilde{U} \times \mathbb{R}^n$ as follows; for $(\tilde{x}, v) \in \tilde{U} \times \mathbb{R}^n$ and $g \in G$,

$$g(\tilde{x}, v) = (g(\tilde{x}), J_g(\tilde{x})v) \in \tilde{U} \times \mathbb{R}^n$$

where $J_g(\tilde{x})$ is the Jacobian matrix of g at $\tilde{x} \in \tilde{U}$. With this prescribed G-action, the projection map $\tilde{U} \times \mathbb{R}^n \to \tilde{U}$ is then G-equivariant. To obtain the tangent bundle of $\mathcal{X}$, we glue together the tangent bundles $T\tilde{U} \cong \tilde{U} \times \mathbb{R}^n$ according to the underlying embedding data. Precisely, the underlying space is

$$TX = \left(\coprod_{\tilde{U} \in \mathcal{U}} \tilde{U} \times \mathbb{R}^n \right) \Big/ \sim$$

where $(\tilde{x}, v) \sim (\tilde{y}, w)$ whenever we have an appropriate embedding λ for which $\lambda(\tilde{x}) = \tilde{y}$ and $J_\lambda(\tilde{x})v = w$. Note that the chain rule implies that for composable embeddings $\mu \circ \lambda$, one has $J_{\mu\circ\lambda}(\tilde{x}) = J_\mu(\lambda(\tilde{x})) \circ J_\lambda(\tilde{x})$, i.e. a cocyle condition. An orbifold atlas $\mathcal{U}^*$ on TX is given by charts of the form $(\tilde{U} \times \mathbb{R}^n, G, \phi^*)$ where ϕ^* is defined as the canonical map $\tilde{U} \times \mathbb{R}^n \to TX$. We note $TU := \phi^*(\tilde{U} \times \mathbb{R}^n) \cong (\tilde{U} \times \mathbb{R}^n)/G$, and that embedding data is lifted up from the underlying orbifold. We thus have an orbifold $T\mathcal{X} = (TX, \mathcal{U}^*)$, called the tangent bundle of $\mathcal{X}$. It is an orbifold of dimension $2n$. Note that the projections over each chart yield a smooth map of orbifolds $p : T\mathcal{X} \to \mathcal{X}$ which is defined so that, for each chart $(\tilde{U}, G, \phi)$, the following diagram commutes,

$$\begin{array}{ccc} \tilde{U} \times \mathbb{R}^n & \longrightarrow & \tilde{U} \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi} \\ TX \supset TU & \underset{p}{\longrightarrow} & U \subset X \end{array}$$

For $x \in X$, we may consider the fiber $p^{-1}(x)$. Let $(\tilde{U}, G, \phi)$ be a chart about x, with $\phi(\tilde{x}) = x$. That $\tilde{U}/G \cong U$ means that we may associate $G\tilde{x} \leftrightarrow x$. The subset $\{G(\tilde{x}, v) \,|\, v \in T_{\tilde{x}}\tilde{U}\} \subset (\tilde{U} \times \mathbb{R}^n)/G$ projects to $G\tilde{x} \in \tilde{U}/G$, and by commutativity of the diagram above, is uniquely identified with $p^{-1}(x)$. We define a map $f : p^{-1}(x) \to T_{\tilde{x}}\tilde{U}/G_{\tilde{x}}$ by $f(G(\tilde{x}, v)) = G_{\tilde{x}}v$ where $G_{\tilde{x}}$ is the isotropy subgroup at $\tilde{x}$. We claim that f is a homeomorphism. Indeed, $f(G(\tilde{x}, v)) = G_{\tilde{x}}v = G_{\tilde{x}}w = f(G(\tilde{x}, w))$ if and only if there exists $g \in G_{\tilde{x}}$ for which $J_g(\tilde{x})(v) = w$, which is equivalent to $g \in G$ for which $g(\tilde{x}, v) = (\tilde{x}, w)$, and this happens if and only if $G(\tilde{x}, v) = G(\tilde{x}, w)$. Both directions together prove that f is well-defined and injective. Finally, f is clearly surjective and continuous, with a continuous inverse. The above discussion means that the fibers of $T\mathcal{X}$ look like $\mathbb{R}^n$ modulo a finite subgroup of $GL(n, \mathbb{R})$, upon considering linear charts.

A section s of the tangent bundle $T\mathcal{X}$ is given by a collection of sections $s_{\tilde{U}} : \tilde{U} \to T\tilde{U} \cong \tilde{U} \times \mathbb{R}^n$ over each chart $(\tilde{U}, G, \phi) \in \mathcal{U}$, such that, if $\lambda : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is an embedding, then the following diagram commutes,

$$\begin{array}{ccc} \tilde{U} \times \mathbb{R}^n & \overset{\lambda^*}{\longrightarrow} & \tilde{V} \times \mathbb{R}^n \\ \uparrow{\scriptstyle s_{\tilde{U}}} & & \uparrow{\scriptstyle s_{\tilde{V}}} \\ \tilde{U} & \underset{\lambda}{\longrightarrow} & \tilde{V} \end{array}$$

where $\lambda^* : \tilde{U} \times \mathbb{R}^n \to \tilde{V} \times \mathbb{R}^n$ is the lifted embedding, defined by $\lambda^*(\tilde{x}, v) = (\lambda(\tilde{x}), J_\lambda(\tilde{x})(v))$. Note that this means that each section $s_{\tilde{U}}$ is G-invariant. The collection of sections patch together to form a smooth map $s : \mathcal{X} \to T\mathcal{X}$ for which $p \circ s = \mathrm{Id}_{\mathcal{X}}$. As usual, we call a section of $T\mathcal{X}$ a vector field on $\mathcal{X}$.

Remark 2.1. The tangent bundle of an orbifold is an example of an orbifold vector bundle, the typical fibre being the quotient of a finite-dimensional vector space. In a natural way, we may provide a general definition of an orbifold fibre bundle (see [4]).

In similar fashion to the tangent bundle, for an orbifold $\mathcal{X}$ we may construct its cotangent bundle $T^*\mathcal{X}$ and its exterior powers $\bigwedge^k T^*\mathcal{X}$. In particular, a differential k-form on $\mathcal{X}$ consists of a collection of (invariant) k-forms $\omega_{\tilde{U}}$ over each chart $(\tilde{U}, G, \phi)$. As usual, we write the space of k-forms on $\mathcal{X}$ as $\Omega^k(\mathcal{X})$. The wedge product, $\wedge$, of forms on an orbifold is defined, furthermore, by naturality, we have a well-defined exterior derivative $d : \Omega^k(\mathcal{X}) \to \Omega^{k+1}(\mathcal{X})$, and in particular, taking the cohomology of the complex

$$\cdots \xrightarrow{d} \Omega^{k-1}(\mathcal{X}) \xrightarrow{d} \Omega^k(\mathcal{X}) \xrightarrow{d} \Omega^{k+1}(\mathcal{X}) \xrightarrow{d} \cdots$$

we obtain the de Rham cohomology $\mathcal{X}$ of an orbifold, $H^*_{dR}(\mathcal{X})$. Let us now discuss the integration of differential forms over an oriented n-orbifold $\mathcal{X}$. It is akin to integration on a manifold. Let $U \subset X$ be uniformised by $(\tilde{U}, G, \phi)$. A compactly supported n-form on U is, by definition, a compactly supported G-invariant n-form $\tilde{\omega}$ on $\tilde{U} \subseteq \mathbb{R}^n$. The integration of ω on U is defined by,

$$\int_U^{orb} \omega := \frac{1}{|G|} \int_{\tilde{U}} \tilde{\omega}.$$

Let us now consider the global case. We have $\mathcal{X}$ with a cover $\{U_\alpha\}$ of uniformised open sets; we may choose, via paracompactness, (see Lemma 3.4.1 in [11]) a smooth partition of unity $\{\rho_\alpha\}$ subordinate to this cover, then integrate a compactly supported n-form ω on $\mathcal{X}$ as,

$$\int_{\mathcal{X}}^{orb} \omega := \sum_\alpha \int_{U_\alpha}^{orb} \rho_\alpha \omega.$$

In exactly the same way as for manifolds, this definition is independent of the choice of partition of unity. Now, for a chart $(\tilde{U}, G, \phi)$ on X, if we allow $\tilde{U} \subseteq \mathbb{R}^n_+$ with $g \in G$ satisfying $g(\partial\tilde{U}) = \partial\tilde{U}$, we introduce the notion of an orbifold with boundary (see Remark 4.3.1 in [8]). We have (see Theorem 3.4.2 in [11]),

Theorem 2.1 (Stokes' Theorem)**.** *Let $\mathcal{X}$ be an oriented n-dimensional orbifold with boundary, and $\omega \in \Omega^{n-1}(\mathcal{X})$ a compactly supported $(n-1)$-form. Then,*

$$\int_{\mathcal{X}}^{orb} d\omega = \int_{\partial\mathcal{X}}^{orb} \omega.$$

Proof. This is a trivial consequence of Stokes' theorem in the setting of manifolds. □

We shall now state a collection of classical results generalised to the setting of orbifolds.

Theorem 2.2. *The following results were proved by Satake in [3];*

1. *For $\mathcal{X}$ a compact, orientable, n-orbifold the pairing,*

$$\int : H^k_{dR}(\mathcal{X}) \otimes H^{n-k}_{dR}(\mathcal{X}) \to \mathbb{R}$$

$$(\omega, \tau) \mapsto \int_{\mathcal{X}}^{orb} \omega \wedge \tau$$

is non-degenerate. In particular, compact orientable orbifolds satisfy Poincaré duality, $H^k_{dR}(\mathcal{X}) \cong (H^{n-k}_{dR}(\mathcal{X}))^*$.

2. *For an orbifold* $\mathcal{X}$*, with underlying topological space* X*, there is an isomorphism,*

$$H^*_{dR}(\mathcal{X}) \cong H^*(X; \mathbb{R})$$

where the right hand side denotes the singular cohomology of the underlying topological space, with real coefficients.

The second point above is a de Rham Theorem in the setting of orbifolds. It implies, in particular, that the orbifold de Rham cohomology does not detect singular points (for example, consider a point with the trivial action of a finite group; all the group data is lost upon passing to the orbifold de Rham cohomology). Taking the perspective that our orbifold $\mathcal{X}$ consists of two pieces of data;

1. Geometric data; the underlying topological space X.
2. Singular data; the set of all points in X with non-trivial local group, $\Sigma(\mathcal{X})$.

We see that the orbifold de Rham cohomology is insufficient; it simply loses too much information. This suggests we search for an alternative cohomology theory, one which at least detects the presence of singular points. This brings us to the following philosophy (inspired by 'Introduction to Differentiable Stacks', Section 4.3, [14])

- "The correct characteristic zero (co)homology invariants of an orbifold $\mathcal{X}$ are those of its inertia orbifold (possibly up to some regrading)"

We will now investigate this further.

3 The Chen-Ruan Cohomology

The Chen-Ruan cohomology of an orbifold is the 'honest' cohomology of the corresponding inertia orbifold. The inertia orbifold is decomposed into sectors, the cohomology of which identifies the singular data. We shall assume that our orbifolds admit an almost complex structure. In this section we follow closely the original paper by Chen and Ruan ([5]) and the PhD thesis of Fabio Perroni ([6]). Let us mention that it is possible to cast the theory (briefly) discussed here in the language of groupoids; this is the approach taken in chapter four of [1].

Remark 3.1. From now on, we shall freely assume that for a chart about a point, say p, it is of the form $(\tilde{U}_p, G_p, \phi_p)$, where $\tilde{U}_p$ is an open subset containing the origin, G_p fixes 0, and $\phi_p(0) = p$. In other words, G_p 'is' the local group at p. Recall that the local group is well-defined up to conjugacy.

3.1 The Inertia Orbifold

For an orbifold $\mathcal{X} = (X, \mathcal{U})$, we are going to define an orbifold $\tilde{\mathcal{X}}$, equipped with a smooth map $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$. It will be called the inertia orbifold of $\mathcal{X}$. For a point $p \in X$, let $(\tilde{U}_p, G_p, \phi_p)$ be a chart about p. We define the underlying set,

$$\tilde{X} = \{(p, (g)_p) \,:\, p \in X, (g)_p \subset G_p \text{ is a conjugacy class in the local group}\}.$$

Clearly, we have a surjective function $\tilde{X} \to X$ defined by $(p, (g)_p) \to p$. We now have the following result (see Lemma 3.1.1 in [5]).

Lemma 3.1. *The set $\tilde{X}$ admits the structure of an orbifold, with orbifold charts of the form*

$$\{(\tilde{U}_p^g, C(g), \pi_{p,g})\}_{(p,(g))\in\tilde{X}}$$

where $\tilde{U}_p^g$ is the fixed point set of g in $\tilde{U}_p$, $C(g)$ the centraliser of g in G_p, and $\pi_{p,g} : \tilde{U}_p^g \to \tilde{U}_p^g/C(g)$ projection. The set $\tilde{X}$ together with this orbifold structure is denoted $\tilde{\mathcal{X}}$, called the inertia orbifold of $\mathcal{X}$. There is a smooth map of orbifolds $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ whose underlying map is defined by $(p,(g)_p) \to p$.

Remark 3.2. Note that the inertia orbifold is not, in general, an effective orbifold (if g is not the unit element, then $g \in C(g)$ acts trivially on $\tilde{U}_p^g$). It is of course effective in the case that $\mathcal{X}$ is a manifold, for then so is $\tilde{\mathcal{X}}$.

We shall now introduce an equivalence relation on the set $\tilde{X}$, which allows us to analyse the connected components of the inertia orbifold of $\mathcal{X}$. For $p \in X$, let $(\tilde{U}_p, G_p, \phi_p)$ be a chart about p. For $q \in U_p$, we choose a chart $(\tilde{W}_q, K_q, \mu_q)$ about q for which we have $W_q \subset U_q$ and an embedding $\lambda : (\tilde{W}_q, K_q, \mu_q) \to (\tilde{U}_p, G_p, \phi_p)$ with an associated injective group homomorphism $\lambda : K_q \to G_p$. For a conjugacy class $(g)_q \subset K_q$, it is naturally identified with the conjugacy class $(\lambda(g))_p \subset G_p$, which is independent of the choice of embedding (see Proposition 1.1.7 in [6], originally appearing as Proposition A.1 in [2]). This induces an equivalence relation on $\tilde{X}$ where we set $(q,(g)_q) \sim (p,(\lambda(g))_p)$. We let T denote the set of equivalence classes, and, for example, write (g) for the equivalence class to which $(g)_q$ belongs. We now have a decomposition into connected components,

$$\tilde{X} = \coprod_{(g)\in T} X_{(g)}$$

where $X_{(g)} = \{(x,(g')_x) \mid g' \in G_x, (g')_x \in (g)\}$. We have the following definition,

Definition 3.1. *For $g \neq 1$, we call $X_{(g)}$ a twisted sector. For $g = 1$, we call $X_{(1)} = X$ the non-twisted sector.*

In the case of a global quotient, $X = Y/G$, one is able to identify the inertia orbifold $\tilde{X}$ with,

$$\coprod_{(g),g\in G} Y^g/C(g)$$

where Y^g is the fixed point set in G. (See Example 3.1.3 of [5]). A few special cases are of interest.

Example 3.1. If $\mathcal{Q}$ is a point p equipped with the trivial action of a finite group, then, for $g \in G$, the sector $\mathcal{Q}_{(g)}$ is a point with the trivial action of $C(g)$, namely $\{(p,(g)\}$. In particular, we may view the sectors of the inertia orbifold as conjugacy classes in G. Consider the following visualisation,

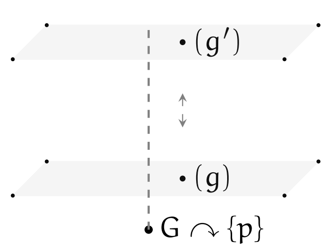

Figure 3.1: The inertia orbifold of $\mathcal{Q} = \{p\}$ with the trivial action of a finite group G. The 'fibres' are visualised as 'slices', which represent conjugacy classes in G.

Example 3.2. Let $\mathcal{Q}$ be the $\mathbb{Z}_n$-teardrop orbifold (i.e. underlying space homeomorphic to S^2, and a single singular point with local group $\mathbb{Z}_n$). The inertia orbifold $\tilde{\mathcal{Q}}$ has n connected components; the non-twisted sector yields $\mathcal{Q}$, and the other $n-1$ components are all given by a point with a trivial action of $\mathbb{Z}_n$.

We will now discuss the degree shifting. From here on out, it is necessary to assume that our orbifolds admit an almost complex structure. Recall,

Definition 3.2. *Let $\mathcal{X}$ be an orbifold and $T\mathcal{X}$ its tangent bundle. An almost complex structure on $\mathcal{X}$ is an endomorphism $J : T\mathcal{X} \to T\mathcal{X}$ such that $J^2 = -Id$.*

Carrying through our notation from above, we let $p \in X$ be a point, $(\tilde{U}_p, G_p, \phi_p)$ a chart about p. The action of G_p on $\tilde{U}_p$ fixes the origin, and hence employing the tangent functor yields an action of G_p on the tangent space $T_0\tilde{U}_p$. Through the almost complex structure, we obtain a group homomorphism,

$$\rho_p : G_p \to GL(n, \mathbb{C})$$

where $n = \dim_{\mathbb{C}}(\mathcal{X})$. Noting that each $g \in G_p$ has finite order, we write $\rho_p(g)$ as a diagonal matrix of the form

$$\begin{pmatrix} e^{2\pi i \frac{m_{1,g}}{m_g}} & & \\ & \ddots & \\ & & e^{2\pi i \frac{m_{n,g}}{m_g}} \end{pmatrix}$$

where m_g is the order of $\rho_p(g)$ and each $m_{k,g}$ an integer for which $0 \le m_{k,g} < m_g$. It can be shown that this matrix depends only on the conjugacy class of g in G_p. We thus define a function, $\iota : \tilde{\mathcal{X}} \to \mathbb{Q}$ by

$$\iota(\,(p,(g)_p)\,) = \sum_{k=1}^{n} \frac{m_{k,g}}{m_g}.$$

This function restricted to the component $X_{(g)}$ is constant, and we denote this constant by $\iota_{(g)}$ (see for example, Lemma 2.2.1 in [6]). We call $\iota_{(g)}$ a **degree shifting number**. A natural direction is now, in what situation are the degree shifting numbers integral? We have,

Lemma 3.2. *The degree shifting number(s) satisfy the following properties,*

1. $\iota_{(g)} \in \mathbb{Z}$ *if and only if* $\rho_p(g) \in SL(n, \mathbb{C})$.
2. $\iota_{(g)} + \iota_{(g^{-1})} = rank(\rho_p(g) - I_n)$

Proof. Write,

$$\det(\rho_p(g)) = \prod_{k=1}^{n} e^{2\pi i \frac{m_{k,g}}{m_g}} = e^{2\pi i \iota(g)}$$

from which the first point follows. For the second point, clearly $m_g = m_{g^{-1}}$, and further, $\rho_p(g)\rho_p(g^{-1}) = I_n$ yields, for each $1 \le k \le n$, $(m_{k,g} + m_{k,g^{-1}})/m_g \in \mathbb{Z}$ with $0 \le m_{k,g}, m_{k,g^{-1}} < m_g$. Note,

$$e^{2\pi i \frac{m_{k,g}}{m_g}} = 1$$

if and only if $m_{k,g} = 0$. The rank of $\rho_p(g) - I_n$ is precisely the number of entries of $\rho_p(g)$ distinct from 1, and hence is exactly counting the number of $m_{g,k}$ for which $m_{g,k} \neq 0$. Now, if $m_{k,g} = 0$, then $m_{k,g^{-1}} = 0$, so

$$\frac{m_{k,g} + m_{k,g^{-1}}}{m_g} = 0.$$

If $m_{k,g} \neq 0$, then

$$\frac{m_{k,g} + m_{k,g^{-1}}}{m_g} = 1.$$

Now,

$$\iota(g) + \iota(g^{-1}) = \sum_{k=1}^{n} \frac{m_{k,g} + m_{k,g^{-1}}}{m_g}.$$

The second point follows. □

We call an orbifold for which each $\rho_p(g) \in SL(n, \mathbb{C})$ an SL-orbifold. As pointed out after definition 3.2.2 in [5], the degree shifting numbers $\iota_{(g)}$ are independent of the choice of almost complex structure J. We are now ready to define the cohomology groups.

3.2 The Chen-Ruan Cohomology Groups

Definition 3.3. *Let $\mathcal{X}$ be an orbifold. We define the Chen-Ruan orbifold cohomology group of degree* d *by*

$$H^d_{CR}(\mathcal{X}) = \bigoplus_{(g)\in T} H^{d-2\iota_{(g)}}(X_{(g)})$$

where $H^*(X_{(g)})$ denotes the singular cohomology of the sector $X_{(g)}$ with real coefficients. The total orbifold cohomology group of $\mathcal{X}$ is,

$$H^*_{CR}(\mathcal{X}) = \bigoplus_{d} H^d_{orb}(\mathcal{X})$$

and the orbifold Betti numbers are $b^d_{orb} = \sum_{(g)} \dim H^{d-2\iota_{(g)}}(X_{(g)})$.

In general, the orbifold cohomology groups $H^d_{\mathrm{orb}}(\mathcal{X})$ are rationally graded. However, they are integrally graded if the degree shifting numbers are integral, which is the case if and only if $\mathcal{X}$ is an SL-orbifold. If $\mathcal{X}$ is an SL-orbifold, this means that the canonical bundle $K_{\mathcal{X}} = \bigwedge^n_{\mathbb{C}} T^*\mathcal{X}$ (a complex orbibundle over $\mathcal{X}$) is an honest line bundle (see page 15 of [1]). Moreover, any Calabi-Yau orbifold is an SL-orbifold. Recall that we say an orbifold is Calabi-Yau if its canonical bundle $K_{\mathcal{X}}$ is a trivial line bundle.

Example 3.3. Let $\mathcal{Q} = \{\mathrm{pt}\}$ with the trivial action of a finite group G. In Example 3.1 we saw that each twisted sector $\mathcal{Q}_{(g)}$ was a point with a trival action of $C(g)$. We observe that all the degree shifting numbers $\iota_{(g)}$ are zero, and in particular,

$$H^d_{\mathrm{CR}}(\mathcal{Q}) = \begin{cases} \mathbb{R}^n & d = 0 \\ 0 & d \neq 0 \end{cases}$$

where n is the number of conjugacy classes in G. (See Example 5.4 in [5]).

Example 3.4. Consider the $\mathbb{Z}_n$-teardrop orbifold $\mathcal{Q}$ as in Example 3.2. We saw that its inertia orbifold had n connected components, the non-twisted sector $\mathcal{Q}$ and $n-1$ copies of the singular point with a trivial action of $\mathbb{Z}_n$. The degree shifting numbers are i/n for $1 \leq i \leq n-1$. We compute,

$$\dim H^0_{\mathrm{CR}}(\mathcal{Q}) = \dim H^2_{\mathrm{CR}}(\mathcal{Q}) = \dim H^{2\frac{i}{n}}_{\mathrm{CR}}(\mathcal{Q}) = 1.$$

Note that the Chen-Ruan cohomology classes corresponding to the twisted sectors have rational degrees (see, for example, page 97 [1], or Example 5.3 in [5]).

These final examples provide (some) justification for the philosophy discussed at the end of section two, namely that the 'correct' cohomology of an orbifold is the honest cohomology of its inertia orbifold.

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