The Yamabe Invariant of 3-Manifolds

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Abstract

This dissertation lays out the theory of Riemannian geometry to understand smooth invariant values of 3-manifolds, namely, the Yamabe invariant. We will focus on cases of prime 3-manifolds, such as the 3-sphere $\mathbb{S}^3$ and the real projective 3-space $\mathbb{RP}^3$, in order to understand the Yamabe problem and its related object of interest, the Yamabe Invariant $\sigma(M)$, which is a smooth scalar invariant $\sigma : M \to \mathbb{R}$ that any compact 3-manifold may exhibit. The main goal of my thesis is to compute $\sigma(\mathbb{RP}^3)$ and understand its consequences for other prime 3-manifolds as explored by Bray and Neves in [1].

The Yamabe problem deals with being able to find metrics of constant scalar curvature for any compact, boundaryless Riemannian manifold (M, g). The purpose of this would intuitively be to simplify calculations relating to the general curvature of M, i.e., its scalar curvature tensor. But as we will see, this has ramifications when we consider conformal classes of metrics and analyse the Yamabe invariant.

The Yamabe Invariant $\sigma(M)$ corresponds to a real number, which is defined as the supremum over all conformal classes of metrics of the Einstein Hilbert functional once it is minimised over all metrics in one particular class via a *conformal factor*; hence the definition of a conformal class. One useful interpretation of this value is that it can tell us the types of metrics M can support by its size and sign.

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Chapter 1

Introduction

Riemannian geometry is the primary continuation of what we consider standard or extrinsic geometry. In the works of Gauss and the like, (differential) geometry is always dependent on the function/s that embed a topological space M within another space, which we call the ambient space. Eventually, one of Gauss' students, Bernhard Riemann, developed his own form of differential geometry which did not depend on an embedding map, but rather equips a smooth manifold M with a smooth function g_p which is also a positive-definite bilinear form at each $p \in M$, called the *metric tensor* or simply the *metric* on M. This construction clearly follows from Gauss' discoveries about the embedding-independent properties of surfaces.

Riemannian geometry will serve to pose the Yamabe problem and explain why we introduce conformal classes of metrics, hence the name of the *conformal Yamabe invariant* Y(g) by Bray-Neves. This conformal invariant is the stepping stone for the smooth Yamabe invariant or simply the Yamabe invariant $\sigma(M)$ once we consider

$$\sigma(M) = \sup_{[g]} Y(g).$$

A smooth invariant in differential geometry is a property of a manifold M that is preserved under diffeomorphisms. Conformal classes are defined by

$$[g] = \{\tilde{g} = e^{2f} g \mid f \in C^{\infty}(M)\}.$$

The construction of the Yamabe Invariant involves using the (normalised) Einstein-Hilbert functional

$$E(g) = \frac{\int_M R_g \, dV_g}{(\int_M dV_g)^{(n-2)/n}} \ ,$$

which is a scale-normalised version of the total scalar curvature on M, i.e. $E(cg) = E(g)$, $c > 0$. $E(g)$ is normalised this way so that it captures the aforementioned *stretching* differences between conformal equivalence classes $[g]$, rather than the more ambiguous case of looking at the value of $E(g)$ or $\int_M R_g dV_g$ over the space of all metrics. This demonstrates strong dependence for $\sigma(M)$ on the scalar curvature R_g, whose search of a conformal change to a constant $R_{\tilde{g}}$ is the main aim in the Yamabe problem.

Yamabe was the first one to attempt the theorem [2], and it was later proved by Aubin, Trudinger and Schoen [3] (cf. [12]), that every conformal class of metrics on a compact manifold admits a metric of constant scalar curvature. Subsequently, it was also shown that we can achieve a minimum value of $E(g)$ for each conformal class of metrics, and that this minimum is achieved by a metric of constant scalar curvature. $\sigma(M)$ is then defined as the smooth scalar invariant

$$\sigma(M) = \sup_{[g]} Y(M,[g]), \quad \text{where} \quad Y(M,[g]) = \inf_{\tilde{g}\in[g]} \frac{\int_M R_{\tilde{g}}\, dV_{\tilde{g}}}{\left(\int_M dV_{\tilde{g}}\right)^{(n-2)/n}} \ ,$$

where $n = dim(M)$. This value is very convenient, as it allows us to claim from the size and sign of $\sigma(M)$ whether the manifold supports at least one conformal structure [g] that admits a positive, flat, or negative total scalar curvature. $\sigma(M)$ can also tell us which types of topologies its value is related to, e.g., $\sigma(S^3) = \sigma(S^2 \times S^1) = \sigma(S^2 \tilde{\times} S^1) = \sigma_1$.

We will focus on the work of Bray and Neves, which computes the Yamabe invariant of the real projective 3-space $\mathbb{RP}^3$, as well as that of other prime 3-manifolds, such as the 3-sphere $\mathbb{S}^3$. Restricting to the study of 3-manifolds, we can appreciate the computational methods and study key properties of manifolds and metrics as we progress from Riemannian geometry through to the Yamabe problem and related research, such as Huisken-Ilmanen's [4], to give an in-depth understanding of the Yamabe invariant.

Chapter 2 deals with the groundwork by introducing smooth manifolds, tensors, Riemannian metrics, the Levi-Civita connection, and the three key curvature tensors: the Riemann curvature tensor, the Ricci tensor, and the scalar curvature tensor. Chapter 3 outlines some key definitions of topological 3-manifolds, such as the connected sum operation # and prime 3-manifolds. We then apply the tools we covered in Chapter 2 to find the curvature tensors of simple prime 3-manifolds like $S^3, T^3, \mathbb{R}^3, \mathbb{RP}^3$. Chapter 4 opens up with the Yamabe problem, tracing its development and important techniques used to find metrics of constant scalar curvature in conformal classes. Afterwards we work towards the construction of Yamabe invariant to 3-manifolds, and calculate it using the techniques of Bray and Neves. I conclude the paper by computing $\sigma(\mathbb{RP}^3)$ and alluding to Bray-Neves and a paper by Akutagawa-Neves [11] which comment on and extend the consequences of this result.

Chapter 2

Riemannian Geometry

Most of the following definitions can be found in [5].

2.1 Manifolds

Definition 2.1.1. A topological manifold M of dimension n is defined as a Hausdorff topological space with a countable basis. Additionally, M is equipped with a family of homeomorphisms:

$$\{u_\alpha : M_\alpha \to U_\alpha \subset \mathbb{R}^n\}_{\alpha \in A}$$

mapping open sets $M_\alpha \subset M$ to open sets $U_\alpha \subset \mathbb{R}^n$, where

$$\bigcup_{\alpha \in A} M_\alpha = M \ ,$$

and all overlap maps, also known as *transition* maps

$$u_\alpha u_\beta^{-1} : u_\alpha(M_\alpha \cap M_\beta) \to u_\beta(M_\alpha \cap M_\beta)$$

are also homeomorphisms.

The homeomorphisms and their domains, denoted in pairs as (u_α, M_α), serve as *local coordinate charts* for M, and the collection indexed by $\alpha \in A$ forms the *atlas* for M. Since M has a countable topological basis, it allows the use of a countable atlas A to chart the entire manifold.

These conditions maintain that M be structurally similar to $\mathbb{R}^n$, where every point in M has a neighborhood *topologically equivalent* to an open subset of $\mathbb{R}^n$. This provides manifolds with their *locally Euclidean* structure.

Definition 2.1.2. An atlas $(u_\alpha, M_\alpha)_{\alpha \in A}$ is said to be a differentiable atlas if, for every pair $(\alpha, \beta) \in A \times A$, the homeomorphism:

$$u_\beta \circ u_\alpha^{-1} : u_\alpha(M_\alpha \cap M_\beta) \to u_\beta(M_\alpha \cap M_\beta)$$

is a diffeomorphism.

Definition 2.1.3. Two atlases are said to be equivalent if their union is again a differentiable atlas.

Definition 2.1.4. M is a *differentiable manifold* if it is a topological manifold equipped with an equivalence class of differentiable atlases.

Remark 2.2.1. The notion of a differentiable manifold extends easily to the C^k case by requiring the above maps to be C^k. In practice, we assume that the manifold is C^∞, meaning it is infinitely differentiable, and refer to such manifolds as *smooth manifolds*.

A smooth manifold is a topological manifold M equipped with a *smooth structure*. A smooth structure on a manifold M is a collection of coordinate charts that cover M and are smoothly compatible with each other (i.e. when homeomorphisms between patch intersections are extended to diffeomorphisms); allowing us to define and perform calculus on the manifold via smooth functions, differentiation of arbitrary order, etc. Just as homeomorphisms preserve topological invariants, diffeomorphisms preserve smooth invariants that depend on the differential structure of the manifold, as we'll see in the following chapters.

Definition 2.1.5.

(i) Given a chart (u_α, M_α) for M and a point $p \in M_\alpha$, a vector $(u_\alpha(p), X_\alpha) \in T_{u_\alpha(p)}\mathbb{R}^n$ is a representative of a tangent vector to M at p.

(ii) If (u_α, M_α) and (u_β, M_β) are two charts and $p \in M_\alpha \cap M_\beta$, then the vectors $(u_\alpha(p), X_\alpha)$ and $(u_\beta(p), X_\beta)$ are said to be equivalent if:

$$X_\beta = d(u_\beta \circ u_\alpha^{-1})X_\alpha.$$

This defines an equivalence relation and the equivalence class of X_α is called a *tangent vector* at p.

(iii) The set T_pM of tangent vectors to M at p carries a vector space structure (induced from one, equivalently any, representative space $T_{u_\alpha(p)}\mathbb{R}^n$). This vector space T_pM is called the *tangent space* to M at p.

Definition 2.1.6.

(i) A smooth n-manifold M is said to be *orientable* if there exists an atlas $(u_\alpha, M_\alpha)_{\alpha\in A}$ such that:

$$\det(d(u_\beta \circ u_\alpha^{-1})) > 0 \quad \text{for all } (\alpha, \beta) \in A \times A \quad \text{such that} \quad M_\alpha \cap M_\beta \neq \emptyset.$$

An atlas that satisfies this condition is called an *oriented atlas*, and the manifold is said to be oriented.

(ii) Two orientable atlases have the same orientation if their union is oriented. The equivalence class of an oriented atlas is called an *orientation* of M.

(iii) An *oriented manifold* is a manifold equipped with an orientation.

Definition 2.1.7. A manifold is *connected* if it cannot be divided into two disjoint, non-empty open subsets. Intuitively, this means the manifold is in one piece.

Definition 2.1.8. A manifold is *compact* if it is topologically compact.

Definition 2.1.9. A manifold is *closed* if it is compact and boundaryless, i.e.,

$$\partial M = \emptyset$$

Examples of Smooth Manifolds

1. $\mathbb{R}^n$: The simplest example of a smooth manifold is Euclidean space $\mathbb{R}^n$. It is globally homeomorphic to itself and thus requires only one chart, the identity map, to serve as an atlas. The entire space is covered by this single chart, making it a smooth manifold. It is non-compact, boundaryless and orientable.

2. S^n : The n-sphere S^n, defined as the set of points in $\mathbb{R}^{n+1}$ at a fixed distance from the origin. It is a smooth and compact manifold. It cannot be covered by a single chart but can be covered by two charts corresponding to stereographic projections from the north and south poles:

$$S^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + ... + x_{n+1}^2 = 1\}.$$

   It is closed and orientable.

3. T^n : The n-torus T^n is the Cartesian product of n circles,

$$T^n = \Pi_1^n S^1$$

   In n=2, it can be embedded as the surface of a doughnut in $\mathbb{R}^3$, although $\mathbb{R}^4$ may be thought as a more suitable ambient space for T^2 since either of the two circles S^1 can be embedded in $\mathbb{R}^2$ separately, thus avoiding self-intersections. The n-torus can be covered smoothly by at least 2^n charts. Like the sphere, it is a compact smooth manifold.

   It is closed, and orientable.

4. $\mathbb{RP}^n$: The real projective space $\mathbb{RP}^n$ is a smooth manifold that can be thought of as the set of all lines passing through the origin in $\mathbb{R}^{n+1}$. It is covered by $n+1$ charts, each corresponding to the coordinate hyperplanes in $\mathbb{R}^{n+1}$:

$$\mathbb{RP}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \sim,$$

   where $(x_1, \ldots, x_{n+1}) \sim \lambda(x_1, \ldots, x_{n+1})$ for $\lambda \neq 0$.

First Examples of Projective Spaces

1. $\mathbb{RP}^1$ is the set of all lines through the origin in $\mathbb{R}^2$ Each line corresponds to a point on a circle, meaning that $\mathbb{RP}^1$ is topologically equivalent to a circle S^1. $\mathbb{RP}^1$ is also called the real projective line. It is closed and orientable.
2. $\mathbb{RP}^2$ is the set of all lines through the origin in $\mathbb{R}^3$. Each line corresponds to a point on a hemisphere, but with the opposite points on the boundary identified. It is closed, but not orientable, as it can embed a Möbius strip.
3. $\mathbb{RP}^3$ is the set of all lines through the origin in $\mathbb{R}^4$. It is diffeomorphic to the special orthogonal group $SO(3)$. It can be written as the quotient (spherical) manifold $S^3/\mathbb{Z}_2$. It is closed and orientable.

Definition 2.1.10.

(i) The *Tangent Bundle* of an n-manifold is the set

$$TM := \bigsqcup_{p\in M} T_pM = \{(p,v) \mid p \in M,\ v \in T_pM\}$$

(ii) M induces a C^k structure on TM: For each atlas $(u_\alpha, M_\alpha)_{\alpha\in A}$ for M, there is a corresponding atlas on TM,

$$(Tu_\alpha, TM_\alpha)_{\alpha\in A}$$

defined by

$$TM_\alpha := \bigsqcup_{p\in M_\alpha} T_pM$$

and $Tu_\alpha : TM_\alpha \to Tu_\alpha$ is given by

$$T_pM \ni (p, X_p) \mapsto (u_\alpha(p), X_\alpha) \in T_{u_\alpha(p)}\mathbb{R}^n,$$

with X_α the coordinate representative of X.

(iii) With these definitions, $p : TM \to M$ is differentiable and TM is a $2n$-manifold.

We can interpret the Tangent Bundle as the space where we can define the smooth evolution of vectors and other linear-algebraic objects sitting *tangentially* on M, hence the following section.

2.2 Vectors and Tensors

Definition 2.2.1. A *vector field* X on M is a smooth section of the tangent bundle TM. That is, a smooth map:

$$X : M \to TM, \quad p \mapsto X(p) \in T_pM$$

with the canonical projection $\pi \circ X = id_M$.

For every $p \in M$, we have a unique tangent vector $X(p) \in T_pM$ which sits at the point $p \in M$. We also denote the collection of all smooth vector fields on M as $\Gamma(TM)$.

The vector field can be locally defined for some open $U \subseteq M$, with the local coordinates $x^i : U \to \mathbb{R}$, corresponding to the chart $\phi(p) = (x^1(p), ..., x^n(p)) \in \mathbb{R}^n$. We write:

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

where $X^i : U \to \mathbb{R}$ are the smooth component functions represented in our local coordinates.

Remark 2.2.1. It can be seen that the partial derivative operator works as a basis component in T_pM. X allows to determine a vector field action $X : C^\infty(U) \to C^\infty(U)$ by

$$X[f] = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i}, \text{ for } f : U \to \mathbb{R} \text{ smooth.}$$

At some $p \in U$, $X(p)$ is a priori expressed as:

$$X(p) = \sum_{i=1}^{n} X^i(p) e_i \in T_pM$$

with basis vectors $e_i \in T_pM$. Since $x^k : U \to R$ are smooth coordinate functions, the action $X[x^k]$ gives:

$$X(p)[x^k] = \sum_{i=1}^{n} X^i(p) \frac{\partial x^k}{\partial x^i} = \sum_{i=1}^{n} X^i(p) \delta_i^k = X^k(p) \ ,$$

that is, $e_i[x^k] = \dfrac{\partial x^k}{\partial x^i} = \delta_i^k = \begin{cases} 1 & , i = k, \\ 0 & , i \neq k. \end{cases}$

which uniquely determines the linear isomorphism that identifies $e_i \simeq \dfrac{\partial}{\partial x^i}$.

For convenience we may also write $\partial_i = \dfrac{\partial}{\partial x^i}$.

Einstein summation notation:

As can be seen above, it is common to use the Einstein summation convention for differential geometry to avoid carrying the summation signs everywhere. Our work deals exclusively with covariant and contravariant tensors. In this case, the position of the index (upper or lower) determines the type of tensor and how it may be contracted.

We write: $v^i w_i = \sum_{i=1}^{n} v^i w_i$ and $w^j = A_i^j v^i = \sum_{i=1}^{n} A_i^j v^i$ as valid objects. While $A_i^i v^i$ is not.

Definition 2.2.2. A *rank* (k, l) *tensor* T is a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ covectors}} \times \underbrace{V \times \cdots \times V}_{l \text{ vectors}} \to \mathbb{R},$$

where V^* is the dual space of V. Similarly to how matrices are representations of linear maps, tensors are their multilinear extensions.

Remark 2.2.2. The above definition can be and is often reinterpreted to form associated linear maps. For example, matrices can represent rank $(1, 1)$ tensors in a given basis.

Let $A = A_i^j e_j \otimes e^i \in V \otimes V^*$. Then $A : V \to V$ is possible, as well as $A : V^* \times V \to \mathbb{R}$. Fundamentally they are the same tensor but different representations for the choice of input. Another definition for T is

$$T \in \underbrace{V \times \cdots \times V}_{k \text{ vectors}} \times \underbrace{V^* \times \cdots \times V^*}_{l \text{ covectors}}.$$

Contraction:

Contraction of a vector field is analogous to taking the trace of a matrix, i.e.,
$C : A_j^i \mapsto A_i^i = \sum_{i=1}^n A_i^i$, which is a rank $(0, 0)$ tensor, i.e., a scalar.

Contraction over one pair of indices, one raised and the other lowered, that turns a rank (k, l) tensor to a rank $(k - 1, l - 1)$ tensor.

Definition 2.2.3. A *tensor field* of rank (k, l) on a manifold M is a smooth function

$$T : M \to \mathcal{T}^{k,l} M, T : p \mapsto T(p),$$

where

$$T(p) \in \bigotimes^k T_p M \otimes \bigotimes^l T_p^* M,$$
$$T(p) : T_p^* M \times \cdots \times T_p^* M \times T_p M \times \cdots \times T_p M \to \mathbb{R} \; .$$

And for local coordinates $(x^1, ..., x^n)$ on M,

$$T(p) = T_{i_1,\ldots,i_l}^{j_1,\ldots,j_k} \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_k}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_l}.$$

The *tensor bundle* $\mathcal{T}^{k,l} M$ is similar to the tangent bundle; it incorporates the tensor product with multiple tangent and cotangent spaces:

$$\mathcal{T}^{k,l} M = \bigsqcup_{p \in M} \bigotimes^k T_p M \otimes \bigotimes^l T_p^* M,$$

A tensor field may represent vector fields, e.g., rank $(1, 0)$ tensor fields with $X : M \to TM$, or may represent covector fields, e.g., rank $(0, 1)$ tensor fields with $w : M \to T^* M$.

Tensor fields extend the concept of vector fields onto a multilinear framework. We already saw the identification $e_i \simeq \partial_i$, so by similar analysis we define and identify the corresponding T_p^*M basis covector e^i to dx^i, i.e. dx^i is the *differential* of the coordinate function x^i.

Since $\frac{\partial}{\partial x^i} \in T_pM$ then $dx^i \in T_p^*M$, such that $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$.

Definition 2.2.4. We can use the language of tensors to define *p-forms*. The term "differential form" is also used, which is the general name given to any p-form. They are linear maps

$$\omega : T_xM \times^p \to \mathbb{R},$$

belonging to the space

$$\Lambda^p \, T_x^*M$$

of all *totally antisymmetric* rank $(0, p)$ tensors, which is clearly a subset of $\bigotimes^p T_x^*M$. Here, total antisymmetry means that for any swapping of arguments, ω changes its sign.

Remark 2.2.3. Similar to $\Gamma(TM)$, the space $\Lambda^p(M)$ is the space of all smooth $p-$forms on the manifold, i.e. $\Lambda^p(M) = \Gamma(\Lambda^p(T^*M))$, where T^*M is the cotangent bundle.

Let ω, α be 1-forms on T_xM, then we define a 2-form by:

$$\omega \wedge \alpha,$$

which defines the *wedge product* $\wedge$ by

$$(\omega \wedge \alpha)(X, Y) = \omega(X)\alpha(Y) - \omega(Y)\alpha(X),$$

where $X, Y \in T_xM$.

$p-$forms are the natural language for determining volume, curvature, orientation, integration, and calculus on manifolds.

Example 2.2.1. If V is an $n-$dimensional vector space, then the space of $p-$forms Λ^pV^* has dimension $\binom{n}{p}$

Example 2.2.2. For local coordinates $(x^1, \ldots, x^n)$ on a manifold M, the differential or covector field dx^i is a $1-$form. $p-$Forms may be expressed as

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} \omega_{i_1 \cdots i_p} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

Where $\omega_{i_1 \cdots i_p} : M \to \mathbb{R}$.

Differential Forms in Riemannian Geometry and Exterior Calculus

Example 2.2.3. A canonical, $p-$degree *volume form* Vol_g can be naturally induced by the metric tensor g on a Riemannian manifold M. These are natural tools for establishing local orientation, and if M is globally orientable, then constructing Vol_g is possible over any region of M. The volume form allows to perform analysis on the geometry of M, and to define the integrals of functions on M.

Example 2.2.4. The *exterior derivative* is a very important operator for $p-$forms. In the spirit of advanced calculus, it generalises the concept of divergence, gradient, curl, flux, etc.

It is denoted by d, corresponding to the linear map $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$.

For a 1-form locally expressed as $\omega = \omega_i dx^i$, its exterior derivative is given by

$$d\omega = (d\omega_i) \wedge dx^i \in \Lambda^2(M).$$

d can be inductively defined so that $d \circ d = 0$ and

(i) $d(f\omega) = df \wedge \omega + f d\omega$

(ii) $d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^p \omega \wedge d\alpha$

Where $\omega \in \Lambda^p(M), \alpha \in \Lambda^q(M), f \in C^\infty(M)$.

2.3 Riemannian Geometry

All smooth manifolds can be equipped with various kinds of smooth inner products for the tangent space T_pM at each point of the manifold $p \in M$. One such inner product is the *Riemannian metric tensor*, or simply *Riemannian metric* g, which allows for measurements of lengths and angles of vectors in T_pM.

Every smooth manifold M equipped with a Riemannian metric g is called a *Riemannian Manifold* and it is usually written as the pair (M, g).

2.3.1 The Riemannian Metric and its Role

This particular structure is the main object of study in Riemannian Geometry. The metric g allows the measurement of the lengths of curves on M, volumes of areas or submanifolds, the sizes of tangent vectors, and the angles between them. With g one can also define objects such as geodesics, curvature tensors, and geometric invariants.

This lays the foundation for a theory that joins topology and geometry independently of any embedding into some ambient space for M.

Remark 2.3.1. There is a lot of freedom to the metrics that an arbitrary smooth manifold M may admit, but the topology of M does restrict cases which may produce contradictions.

As a non-example, the sphere S^2 equipped with the Euclidean metric $g = \delta_{ij}dx^i dx^j$ describes a flat geometry like that of Euclidean space or the torus, which gives a Gaussian curvature of $K = 0$, but by the Gauss-Bonnet theorem, which shows how topology determines the possible geometries of a manifold, the Euler characteristic of a sphere is non-zero.

That is, for a compact $2-$dimensional manifold without boundary we have

$$\int_M K \, dA = 2\pi\chi(M).$$

But in this case,

$$\int_M K\, dA = 0\ ,\ 2\pi\chi(M) \neq 0.$$

Hence the sphere cannot admit a Riemannian metric with $K = 0$. This is a more intuitive example, as in higher dimensions the possible geometries of a manifold are less determined by their topology than when $n = 2$.

Definition 2.3.1. The *Riemannian metric* g is a smooth map

$$g : M \to \mathbb{R},$$

which also defines a bilinear map

$$g_p : T_pM \times T_pM \to \mathbb{R}\ ,\ \forall p \in M.$$

It has special properties:

(i) Symmetry: $g_p(X, Y) = g_p(Y, X)$

(ii) Non-degeneracy: $g_p(X, Y) = 0,$ if $X = 0$ or $Y = 0$.

(iii) Positive definiteness: $g_p(X, X) > 0$.

For all $p \in M;\ X, Y \in T_pM$.

Smoothness allows the components of g_p vary smoothly as $p \in M$ varies, i.e.,

$$g_p(X, Y) = g(X(p), Y(p)) = f(p)$$

is a smooth function on M, for any choice of $X, Y \in \Gamma(TM)$.

We can use the definition of the Riemannian metric to represent it as a $(0, 2)-$tensor field in local coordinates $(x^1, \ldots, x^n)$:

$$g = g_{ij}(x)\, dx^i \otimes dx^j,$$

Where $g_{ij} \in C^\infty(M)$, and g can be written as a matrix (g_{ij}) which satisfies symmetry and positive-definiteness. The inverse matrix of the metric is written as (g^{ij}). g and g^{-1} allow the raising and lowering of indices of tensors; something that cannot be done without a non-degenerate metric on a manifold.

The purpose of raising or lowering indices is that when we have a special tensor on M, which describes important geometric properties, we can raise and lower indices to provide a different representation of T while it remains the same geometric object.

In other words, g provides an isomorphism between T_pM and T_p^*M.

Example 2.3.1.

1. $T^i = g^{ij} T_j$
2. $T_i = g_{ij} T^j$
3. $T^k_{jli} = g^{km} g_{ip} g_{lq} T^{pq}_{mj}$, (ignoring symmetries).

Symmetry of Tensors

The metric g provides useful properties that allow the isomorphism between tangent and cotangent spaces on a manifold, hence we can represent matrix symmetries, or lack thereof, and analogous properties in tensors in a basis-independent fashion. For a $(1,1)-$Tensor:

$$(T^i{}_j)^T := T_j{}^i$$

Where $T^i{}_j$ represents T^{ij} with the right index lowered, or T_{ij} with the left index raised, and inversely for $T_j{}^i$. In the case where T is symmetric:

$$T^i{}_j = T_j{}^i = T^i_j$$

The difference is needed so that we know what order the raising or lowering uses even if T is not symmetric:

$$T_j{}^i = g_{jk} T^{ki} = T_{jk} g^{ki}$$
$$T^i{}_j = T^{ik} g_{kj} = g^{ik} T_{kj}$$

Hence the top and bottom indices preserve the order they had before they where raised or lowered.

The actual placement of g_{ij} in the summation is arbitrary, since this is just a scalar component and commutes in multiplication left and right with the tensor component. However, the notation is a convenient method for memorisation.

Examples of Riemannian Manifolds

We consider introductory examples of Riemannian metrics which demonstrate their use:

1. $(\mathbb{R}, \delta_{ij})$: The *Euclidean metric* on $\mathbb{R}^n$ is the canonical metric derived from the dot product on $\mathbb{R}^n$. We express it in Cartesian coordinates $(x_1, \dots, x_n)$ as:

   $$g = dx_1^2 + dx_2^2 + \cdots + dx_n^2,$$

   i.e. $g_{ij} = \delta_{ij}$. And dx_i is the differential of the coordinate x_i. The flatness can be seen in how this is just Pythagoras' theorem for n dimensions.

2. (S^2, g_0) : The 2-sphere can be equipped with the *round metric* g_0 which can be studied using spherical coordinates (although only well-defined if we use two charts). Let $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$:

   $$g_0 = r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2.$$

   The geometry given by this metric is isotropic because the curvature tensors that we can form with g_0 describe the paths of minimal curvature as great circles of S^2.

3. $(\mathbb{H}^2, h)$: The hyperbolic plane $\mathbb{H}^2$ is based on the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric:
$$h = \frac{dx^2 + dy^2}{y^2}.$$
This gives a constant, negative Gaussian curvature $R = -2$.

These examples demonstrate how the geometric information of Riemannian manifolds is induced by the chosen metrics, with each metric serving as a prototype for a possible class of metrics.

2.3.2 Connections and Covariance

A connection properly defines the concept of transporting geometric objects on a manifold. This is a necessary tool, because unlike the case $\mathbb{R}^n \cong T_p\mathbb{R}^n, \forall p \in \mathbb{R}^n$, we do not have an implied isomorphism among the tangent spaces of an arbitrary manifold M, hence the need for introducing a structure which allows to make these comparisons.

Definition 2.3.2. A *connection* ∇ on a manifold M is a bilinear operator which satisfies

(i) *Linearity over Tensor Fields:* Let X be a vector field, T and S tensor fields of types (k, l) and (p, q) respectively, then:
$$\nabla_X(aT + bS) = a\nabla_X T + b\nabla_X S,$$
where $a, b \in \mathbb{R}$.

(ii) *Leibniz rule over scalar multiplication:* Let $f \in C^\infty(M)$, X a vector field, and S a tensor field, then:
$$\nabla_X(fS) = f\nabla_X S + (Xf)S \text{ , where } (Xf) = X[f].$$

(iii) *Linearity in Direction of Differentiation:* Let Y be a vector field, then:
$$\nabla_a X + bYT = a\nabla_X T + b\nabla_Y T.$$

(iv) *Leibniz rule for Tensor Products:*
$$\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S).$$

Connections are tools to measure and compare tensor fields at different tangent or cotangent spaces. This is a broad generalisation of the directional derivative on $\mathbb{R}^n$

The first input is some vector field X, which represents the direction of integration. The second input can be a tensor field Y, the object which is being differentiated along the direction specified by X.

The output is expressed as $\nabla_X Y$, a new tensor field of the same type as Y, called the *Covariant Derivative* of Y (along some vector field X).

Definition 2.3.3. An *affine connection* is a type of connection defined as the map:

$$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y, \end{aligned}$$

where $X, Y, \nabla_X Y \in \Gamma(TM)$ are rank $(1,0)$ tensor fields, i.e., vector fields. Hence the first three conditions above are well-defined for ∇ affine.

We can define *parallel transport* using affine connections in order to study vector fields that follow the same trajectory as other vector fields or curves on M.

Definition 2.3.4. Given an affine connection ∇, a vector field X is *parallel transported along a curve*

$$c : [0,1] \to M,$$

if

$$\nabla_{\dot{c}} X = 0.$$

Where $\dot{c}$ is the tangent vector along the curve.

Definition 2.3.5. A *geodesic* on a smooth manifold M with affine connection ∇ is defined as a curve $\gamma(t)$ such that the parallel transporting of its tangent vector along itself is null, i.e.,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

Definition 2.3.6. In local coordinates the connection may be expressed:

$$\nabla_i \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

With other alternative notations $\nabla_i = \nabla_{e_i} = \nabla_{\partial_i}$. Here, the terms Γ^k_{ij} are called the *connection coefficients*, which de facto describe the output of ∇ explicitly as a vector field. These coefficients are coordinate dependent, and thus are *not* tensorial objects.

Remark 2.3.2. By applying Leibniz rule we can see the corresponding coefficients of $\nabla_i X$:

$$\nabla_i X = \nabla_i (X^k \frac{\partial}{\partial x^k}) = \frac{\partial X^k}{\partial x^i} \frac{\partial}{\partial x^k} + X^k \Gamma^j_{ik} \frac{\partial}{\partial x^j} = \Big(\frac{\partial X^k}{\partial x^i} + \Gamma^k_{ij} X^j \Big) \frac{\partial}{\partial x^k}$$

The connection coefficients Γ^k_{ij} allow us to explicitly write the covariant derivative. This object is an invariant of the manifold, as it does not depend on choice of coordinates. We can interpret ∇X by analogy as the projection onto $T_p M$ of the ambient space derivative of X in the direction e_i, highlighting the importance of ∇ in intrinsic structures.

Definition 2.3.7. On a Riemannian manifold (M, g) we can obtain a unique, canonical connection called the *Levi-Civita connection.* This is an affine connection uniquely determined by the Riemannian metric providing 2 conditions,

(i) *Metric compatibility:* $\forall X, Y, Z \in \Gamma(TM)$:

$$X[g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

This makes ∇ compatible with g to preserve the inner product structure on every tangent space.

(ii) *Zero Torsion:* $\forall X, Y \in \Gamma(TM)$, and the torsion tensor T(X,Y):

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0 \ ,$$

$$\nabla_X Y - \nabla_Y X = [X, Y] \ . \text{ Where}$$

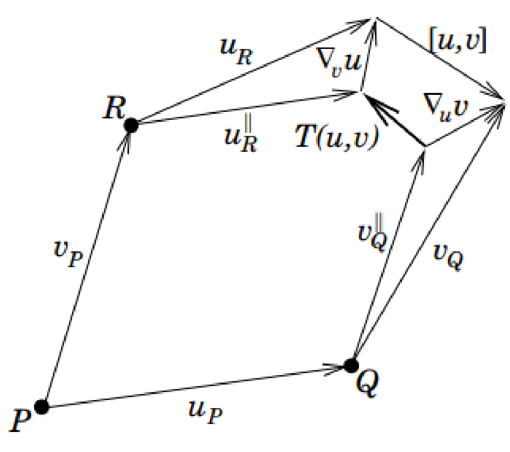
$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \ ,$$

where $[X, Y]$ is the the Lie Bracket, and the torsion tensor measures any twisting that the connection contributes to. By looking at the components of T, we can visualise how T measures the failure of parallel transport to be symmetric:

$$T^k(X, Y) = X^i Y^j (\Gamma^k_{ij} - \Gamma^k_{ji}) = 0, \text{ hence } \Gamma^k_{ij} = \Gamma^k_{ji}.$$

Which is a symmetry condition that ensures the connection does not admit any skew-symmetric components in the covariant derivative. This is due to the symmetry of $g_{ij} = g_{ji}$.

Remark 2.3.3. $T = 0$ implies that the connection adds no extra twisting of the vectors other than the one induced by the topology and measured by the Lie-bracket.



See [6].

We have the clear case that for the coordinate basis, zero torsion implies $\nabla_i \partial_j = \nabla_j \partial_i$, since $[\partial_i, \partial_j] = 0$. This can be described as the basis vectors of T_pM forming a closed parallelogram when comparing pairs of parallel transports.

Additionally, even though $[X, Y] \neq 0$ for arbitrary vectors, the torsion-free property still ensures the closure of the parallelogram formed by the parallel transport of both vectors.

Exclusively in structures with the Levi-Civita connection, we refer to the connection coefficients as the *Christoffel Symbols*, and the covariant derivative $\nabla_X Y$ in the Riemannian context implies the one derived from the Levi-Civita connection, expressed using the Christoffel Symbols.

Definition 2.3.8. The metric compatibility and zero torsion properties allow us to determine a closed expression for Γ^k_{ij} in local coordinates:

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{k\ell} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right).$$

The Levi-Civita connection ∇ is a central tool for the study of Riemannian geometry. Metric compatibility means that the vector lengths $\sqrt{g(V, V)}$, metric inner products $g(V, W)$, and angles between vectors are constant along parallel transport.

2.3.3 Curvature Tensors

Once a Riemannian metric g and its Levi-Civita connection ∇ have been introduced, one can analyze how ∇ behaves when moving around the manifold M. Nontrivial curvature arises when parallel transport around a loop fails to return a vector to its original direction. For example, parallel transporting vectors starting at $(0, 0, 1)$ in S^2 and traversing a loop along a top octant of S^2 (a 270° triangle), leading to the resulting and initial vectors facing in different directions even though parallel transport was preserved. One can quantify this failure of flatness via the curvature tensors derived from ∇.

Definition 2.3.9. The most important curvature tensor of Riemannian geometry is the *Riemann curvature tensor*,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$\forall X, Y, Z \in \Gamma(TM)$.

It explicitly measures the failure for vectors to be preserved under parallel transport. In other words, R a $(1, 3)$-tensor field that measures the deviation from flatness of M.

We can explicitly write R using the Christoffel symbols with local coordinates $(x^1, \ldots, x^n)$:

$$R^l{}_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{jk}}{\partial x^i} + \Gamma^l_{jm}\Gamma^m_{ik} - \Gamma^l_{im}\Gamma^m_{jk}.$$

The Riemann curvature tensor is an intrinsic Riemannian-geometric object in the sense that it represents the same object encoding the curvature of (M, g) despite its explicit components depending on the choice of coordinates.

Definition 2.3.10. Contracting the Riemann curvature tensor gives a $(0,2)-$tensor field known as the *Ricci tensor*. Formally defined as:

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} R(E_i, X, E_i, Y),$$

where $\{E_i\}_{i=1}^{n}$ is any orthonormal basis for T_xM.

More succinctly, local coordinates give the expression:

$$\mathrm{Ric}_{ij} = R^{m}_{imj} = g^{ml}R_{milj}.$$

The Ricci tensor only measures a fraction of the total information of the Riemann curvature tensor. In particular, it encodes the change of the volume of M under geodesic flow.

Definition 2.3.11. A further contraction of the Ricci tensor gives the *scalar curvature*,

$$R_g = g^{ij}\mathrm{Ric}_{ij}.$$

At each point on M, this curvature value is a single scalar measure of the curvature of M. The scalar curvature is very useful for defining geometric invariants because it summarizes as determines very succinctly the types of geometries possible on M. It is thus no surprise that in $n = 2$, the Gaussian curvature K is

$$K = \frac{R_g}{2}.$$

Chapter 3

Analysis of 3-Manifolds

3.1 Key Definitions

I remind the reader of a few important definitions and theorems, and present some new ones.

Definition 3.1.1. A compact manifold is a 3-manifold M such that M is closed and bounded as a topological space.

Definition 3.1.2. Compact, and closed vs. non-closed manifolds in the boundary sense:

(i) *Without boundary*: A compact 3-manifold where each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^3$. This is called a **closed** manifold.

(ii) *With boundary*: A compact 3-manifold that allows points whose neighborhoods are homeomorphic to the upper half-space,

$$\mathbb{R}^3_{\geq 0} = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\} = \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 \mid z < 0\}.$$

Whence the boundary ∂M is a compact 2-manifold. This is a **non-closed** manifold.

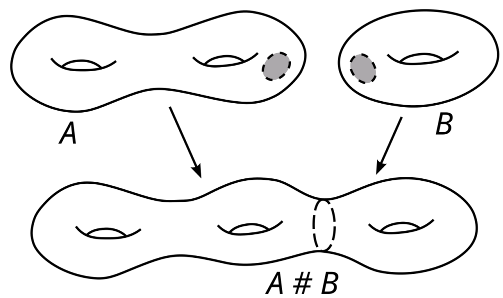
Definition 3.1.3. A 3-manifold M is **orientable** if it admits a globally consistent orientation on its tangent spaces. That is, there exists a nowhere-vanishing volume form ω that is globally defined. If no such global orientation exists, M is **non-orientable**.

Definition 3.1.4. The connected sum # of two 3-manifolds M_1, M_2 is the formation of a new 3-manifold by removing a ball from each manifold and gluing together the boundaries, which in this case are S^2.
The gluing is a diffeomorphism $\phi : S^2 \to S^2$ with the convention that it preserve orientation. More conditions may be required depending on the complexity of the manifolds, e.g, cases of non-orientability.

$$M_1 \# M_2 := (M_1 \setminus D^3) \cup_{\partial D^3} (M_2 \setminus D^3)$$

Such that $M_1 \# M_2$ is unique up to diffeomorphism.

Note that S^3 works as an identity under # .



See [7].

Definition 3.1.5. A prime 3-manifold is a manifold that cannot be expressed as a non-trivial connected sum:

$$M \neq M_1 \# M_2 \text{ unless } M_1 \text{ or } M_2 = S^3$$

Definition 3.1.6. A 3-manifold is said to be irreducible if any embedded 2-sphere bounds some embedded 3-ball.

Theorem 3.1.1. Every compact orientable 3-manifold M can be uniquely written (up to order and homeomorphism) as a connected sum of prime 3-manifolds:

$$M = M_1 \# ... \# M_k$$

Proof. See [8]. □

Theorem 3.1.2. If P is a prime 3-manifold, then either:

(i) P is $S^2 \times S^1$ (the orientable S^2-bundle over S^1),

(ii) P is $S^2 \tilde{\times} S^1$ (the non-orientable S^2-bundle over S^1), or

(iii) P is irreducible.

Proof. See [9]. □

3.2 Canonical Examples of Prime 3-manifolds

Euclidean Space $\mathbb{R}^3$

$\mathbb{R}^3$ canonically defines flat geometry $g_{ij} = \delta_{ij}$, is non-compact, and does not have a boundary. Conversely, the upper-half space $\mathbb{R}^3_{\geq 0}$ does have a boundary but it is again non-compact. Both are clearly orientable, and their scalar curvatures are $R_g = 0$ since all derivatives of δ_{ij} are 0 and hence all the Christoffel symbols are 0.

3-Torus T^3

$T^3 = \mathbb{R}^3/\mathbb{Z}^3$ is compact and without boundary. It inherits a flat geometry from $\mathbb{R}^3$, but T^3 is not homeomorphic to $\mathbb{R}^3$ due to compactness. T^3 is orientable and its scalar curvature is $R_g = 0$.

3-Sphere S^3

S^3 is compact and without boundary, hence closed, and it is also orientable. The most common metric is the round metric g_0. On spherical coordinates (χ, θ, ϕ) :

$$g_{11} = 1\ ,\ g_{22} = sin^2\chi\ ,\ g_{33} = sin^2\chi sin^2\theta$$

This metric mirrors the strong symmetry seen in the embedding of the lower dimensional case (S^2) into $\mathbb{R}^3$. Note that the parametrisation involved using hyperspherical coordinates allows us to express the metric and metric tensors but it does not define a proper chart of S^3 in the definitional sense due to coordinate singularities for g^{ab}.

One may exploit the strong symmetry and isotropic properties of S^3 to find its scalar curvature faster, however for the sake of studying the tools of Riemannian geometry we will work towards the curvature tensor without any helping assumptions.

For ease of notation, we write $\frac{\partial g_{ab}}{\partial x^j} = g_{ab,\,j}$. Then to find Γ^k_{ij} we calculate:

$g_{11,1} = g_{11,2} = g_{11,3} = 0$

$g_{22,1} = 2cos\chi\ sin\chi$

$g_{33,1} = 2cos\chi\ sin\chi\ sin^2\theta$

$g_{33,2} = 2cos\theta\ sin\theta\ sin^2\chi.$

Giving the following:

$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{13} = \Gamma^1_{23} = 0$

$\Gamma^1_{22} = -cos\chi\ sin\chi$

$\Gamma^1_{33} = -cos\chi\ sin\chi\ sin^2\theta$

$\Gamma^2_{12} = \Gamma^3_{13} = cot\chi$

$\Gamma^2_{33} = -cos\theta\ sin\theta$

$\Gamma^3_{23} = cot\theta,$

where the other terms are derived from the symmetry condition $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Now that we have the Christoffel symbols we can calculate the Riemann curvature tensor. Such a calculation should follow in the same way the Christoffel symbols were calculated, however also requiring many more calculations which are a waste of space considering most terms go to 0.

We end up with the following non-zero Riemann curvature tensor components, written in fully covariant form $R_{abcd} = g_{am}R^m{}_{bcd}$:

$R_{1212} = sin^2\chi$, $R_{2323} = sin^2\theta\ sin^4\chi$, $R_{1313} = sin^2\chi\ sin^2\chi$

Which gives the non-zero Ricci tensor components:

$R_{11} = 2$, $R_{22} = 2sin^2\chi$, $R_{33} = 2sin^2\theta\ sin^2\chi$

And so

$$\begin{aligned} R_{g_0} &= g^{ab}R_{ab} \\ &= g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \\ &= 2 + 2 + 2 = 6. \end{aligned}$$

Real Projective Space $\mathbb{RP}^3$

Defined as the space of lines passing through the origin in $\mathbb{R}^4$, or equivalently, $\mathbb{RP}^3 = S^3/\sim$ where $x \sim -x$. The metric descends from S^3:

$$g_{\mathbb{RP}^3} = (d\pi^{-1})^* g_{S^3}\ \text{, where } \pi : p \mapsto [p] = \{p, -p\}.$$

Hence, $\mathbb{RP}^3$ is locally isometric to S^3, and we can naturally descend any metric from S^3 locally unto $\mathbb{RP}^3$. However, due to their distinct global nature, we conclude the set of all possible metrics on the real projective space is a strict subset to that for the 3-sphere. For example, if we descend g_0 from S^3 to $\mathbb{RP}^3$, then consequently $R_{g_{\mathbb{RP}^3}} = R_{g_0} = 6$, since the Riemann tensor itself is a local object.

It might take some convincing to see that $\mathbb{RP}^3$ is orientable; this boils down to the fact that for the volume forms,

$$\pi^*(\omega_{\mathbb{RP}^3}) = \omega_{S^3},$$

and that the antipodal map $A : p \mapsto -p$ associated with the quotient map $\pi(p)$ also preserves ω_{S^3}. For convenient use of notation, ignoring the following is a vanishing form,

$$A^*(\omega_{S^3}) = sin^2(\chi + \pi)\ sin\,\phi\ d(\chi + \pi) \wedge d\theta \wedge d\phi = sin^2\chi\ sin\,\phi\ d\chi \wedge d\theta \wedge d\phi = \omega_{S^3}\ .$$

By picking a suitable atlas for S^3 we get that the volume form $\omega_{\mathbb{RP}^3}$ is the corresponding non-vanishing 3-form on $\mathbb{RP}^3$, i.e., $\mathbb{RP}^3$ is orientable.

3.3 Some Topology

Theorem 3.2.1.

$$\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2.$$

Proof. From algebraic topology we can swiftly argue that since S^3 is the universal cover of $\mathbb{RP}^3$, then the group action $\mathbb{Z}_2$ on S^3 given by the quotient map $\pi : S^3 \to \mathbb{RP}^3$ corresponds to the fundamental group of the covered space $\mathbb{RP}^3$. □

See also [10, Example 1.43, Proposition 1.39].

Remark 2.3.1. The result above helps clarify that $\mathbb{RP}^3$ is the simplest non-trivial spherical 3-manifold. As we'll see in the next chapter, this result is related to a conjecture about the smooth invariants of spherical manifolds S^3/Γ_n, where Γ is a discrete group of order n acting freely on S^3.

Chapter 4

The Yamabe Invariant

The following problem was posed in 1960 by Hidehiko Yamabe [2] as a geometric problem o finding a conformal metric with scalar curvature.

The Yamabe Problem

Given a compact (closed) C^∞-Riemannian Manifold (M^n, g) with $n \geq 3$, can it be conformally deformed to a C^∞-Riemannian Manifold (M^n, h) which has constant scalar curvature? i.e., $R_h = const.$

Subtle Remarks: Why focus on $n \geq 3$?

Clearly, Riemannian Geometry is already trivial for $n = 1$ since the curvature tensors have no meaningful interpretation; for $n = 2$, the problem reduces essentially to the *Gauss-Bonnet theorem* and the *minimum* of Yamabe's associated variational function $F^{(q)}(u)$ is determined by M's topology, that is,

$$F^{(q)}(u) = \frac{\int \left(\frac{4(n-1)}{n-2}|\nabla u|^2 + Ru^2\right)dV}{\left(\int |u|^q dV\right)^{2/q}} \text{ , where } q \to 2n/(n-2).$$

If we consider the variational function in its original metric class (cf. [11, p. 1–2]) we get the normalized Einstein-Hilbert functional

$$E(g) = \frac{\int_M R_g \, dV_g}{(\int_M dV_g)^{(n-2)/n}} \text{ ,}$$

where $E(cg) = E(g), \forall c > 0$. For $n = 2$, the functional is entirely determined by the Gauss-Bonnet theorem as

$$E(g) = \int 2K_g dV_g = 4\pi\chi(M),$$

for Gaussian curvature K_g.

In dimensions $n \geq 3$, the problem becomes highly nontrivial due to the Riemannian structure and consequently R_g and $F^{(q)}(u)$ being less determined by the topology of M (observed in how higher dimensional analogues of Gauss-Bonnet cannot be identified with values exclusively from the topology like the Euler characteristic). However, thanks to the work of Aubin [3], along with Trudinger and Schoen, we know that the Yamabe conjecture is correct, and we can define conformal and even smooth invariants for n-manifolds in general.

Definition 4.0.1. On a closed n-manifold, two metrics g and $\tilde{g}$ are conformal if

$$\tilde{g} = u^{\frac{4}{n-2}} g \ , \ u > 0 \text{ in } C^{\infty}(M).$$

Yamabe derives this convenient equation from $\tilde{g} = e^{2f} g$, where $\inf f > -\infty$, to simplify the equation of the conformal change of R_g,

$$R_{\tilde{g}} = e^{-2f}\Big(R_g - \frac{4(n-1)}{n-2} e^{(2-n)/2} \Delta(e^{(n-2)/2})\Big).$$

In order to calculate smooth invariants, Bray and Neves use the Sobolev space H^1 instead of $C^{\infty}(M)$. This allows to find suitable conformal factors that significantly facilitate the computation of integrals; it is also the space that Aubin used to rephrase and then prove the Yamabe conjecture, i.e., $\varphi \in H^1, \varphi \neq 0$.

Remark 4.0.1. The effect on the scalar curvature of (M^n, g) is:

$$R_{\tilde{g}} = R_g u^2 + \frac{4(n-1)}{n-2} |\nabla u|^2.$$

4.1 Conformal Invariants and σ-invariants in $n = 3$

In [1], Bray and Neves remind the reader of the Yamabe problem, as well as the definition of the σ-invariant of a closed 3-manifold. Alluding to the conjecture by Yamabe, which later Trudinger, Aubin and Schoen proved, the minimizing function $\bar{u}$ for $F^{(q)}(u)$ is always realized in each conformal class of metrics, giving $R_{\tilde{g}} = \text{const.}$ [Ibid., p. 408].

In this paper, $F^{(q)}(u)$ takes the more important role of the conformal alteration of the normalised Einstein-Hilbert functional $E(g)$, otherwise called the *Yamabe functional*,

$$E(g) = \frac{\int_M R_g dV_g}{(\int_M dV_g)^{(n-2)/n}},$$

which measures the total integral of the scalar curvature after the metric is rescaled to a total volume of 1, avoiding the scaling of g by a constant quantity.

It is clear that the invariants in this paper are *smooth* invariants, which means that they are preserved under diffeomorphisms, i.e., any two manifolds equivalent up to a diffeomorphism have the same invariants.

Definition 4.1.1. (The Conformal Yamabe Invariant).

The variational function which Yamabe derived can be used to define a constant $Y(M, [g])$ for each conformal class of M, known as the *Conformal Yamabe Invariant*. It measures the infimum of the total scalar curvature with a volume-normalisation constraint, over a conformal class:

$$Y(M, [g]) := \inf\{E(\tilde{g}) \mid \tilde{g} = u^4 g \in [g], u > 0 \text{ in } C^\infty(M)\},$$

where

$$E(\tilde{g}) = \frac{\int_M \left(8|\nabla u|_g^2 + R_g u^2\right) dV_g}{\left(\int_M u^6 dV_g\right)^{1/3}}.$$

This is Yamabe's $F^{(q)}(u)$ function which corresponds to finding $\tilde{g} \in [g]$ s.t. $R_{\tilde{g}} = \text{const.}$, for $n = 3$.

In other words, we can write $Y(M, [g])$ as

$$Y(M, [g]) = \inf_u \frac{\int_M (8|\nabla u|^2 + R_g u^2)\, dV_g}{\left(\int_M u^6 dV_g\right)^{1/3}}.$$

Definition 4.1.2. (The Smooth Yamabe Invariant, σ-constant, or σ-invariant).

$$\sigma(M) = \sup_{[g]} Y(M, [g]),$$

as defined in a paper by Akutagawa and Neves [11, p. 2].

Remark 4.1.1.

Now that we know how $\sigma(M)$ is defined, we can begin to study it by calculating its value for S^3. This is a very convenient manifold due its simple topology and connection to $\mathbb{RP}^3$, whose σ value is the main object of interest in this paper.

Bray and Neves stress the importance of the σ-invariant of $M = S^3$ equipped with the round metric g_0 on S^3:

$$E(g_0) = 6(2\pi^2)^{2/3} \equiv \sigma_1.$$

Akutagawa and Neves reference the result by Aubin [3] which clarifies this importance:

$$Y(M, [g]) \leq Y(S^3, [g_0]) = 6\text{Vol}_{g_0}(S^3)^{2/3} = \sigma_1 \ , \forall g, \ \forall M.$$

The inequality $Y(M, [g]) \leq Y(S^3, [g_0])$ is a consequence of being able to choose $u(x)$ close to zero except near a single point p, for any smooth metric g, such that the resulting metric is very close to (S^3, g_0) minus a neighbourhood of p.

Respectively, we can also consider how $E(g_0)$ describes a value of a manifold with maximally positive, yet constant, scalar curvature under normalisation of constant scalings.

Schoen [12] (cf. [1, p. 408]) thus defined the following expression

$$\sigma(M) = \sup\{Y(g) \mid g \text{ any smooth metric on } M\} \leq \sigma_1.$$

In the following section I will introduce some intermediate theorems and definitions which will help to calculate the above results. Theorems like the Obata theorem will be particularly helpful in calculating the σ-invariant of $\mathbb{RP}^3$.

Definition 4.1.3. A metric $\tilde{g} \in [g]$ is called a *Yamabe metric* if it realizes the infimum

$$Y(M, [g]) = E(\tilde{g})$$

The existence of such metrics on every conformally would satisfy Yamabe's initial goal of finding a metric of constant scalar curvature for any conformal class of M, given that $\tilde{g}$ is the metric corresponding to the infimum $Y(M, [g])$.

On a succession of groundbreaking papers (cf. [3]), Trudinger, Aubin, and Schoen managed to show that each conformal class $[g]$ does in fact contain such metrics.

Definition 4.1.4. A metric g is called *Einstein* if

$$\mathrm{Ric} = kg, \ \ k \in \mathbb{R}$$

Theorem 4.1.1. (Obata). Let h be an Einstein metric on a closed n-manifold M with $(n \geq 2)$, $\tilde{h} \in [h]$ a constant scalar curvature metric, and g_0 the round metric on S^n. Then, the following hold:

(i) If $(M, [h])$ is conformally equivalent to $(S^n, [g_0])$, then there exist a homothety (a diffeomorphism whose pulback is a constant scaling) $\Phi : (S^n, g_0) \to (M, h)$ and a conformal transformation (a diffemorphism whose pullback is conormally equivalent to the metric of its domain class) $\varphi \in \mathrm{Conf}(S^n, [g_0])$ such that $\Phi^*\tilde{h} = \varphi^*(\Phi^* h)$.

(ii) If $(M, [h])$ is not conformally equivalent to $(S^n, [g_0])$, then, up to constant rescaling, $\tilde{h} = h$.

Proof. See [13, 14] (cf. [15, p. 1]). □

Corollary 4.1.1. Let h be an Einstein metric on a closed n-manifold $(n \geq 3)$, then

$$Y(M, [h]) = E(h)$$

Proof. Case 1. Let $M = S^3$, $h \in [g_0] = [h]$. Then we are trying to calculate

$$Y(S^3, [g_0]) = E(h)$$

This follows from the fact that (S^3, h) is conformally flat given that $h \in [g_0]$, since h is also Einstein, it implies that it has a constant sectional curvature of 1 and hence is isometric to g_0. For further context this is alluded to in [13, Proposition 6.1.].

Case 2. Let $h \notin [g_0]$.

By Obata's theorem (ii) this means that $u =$ const. for $\tilde{h} = u^4 h$ where $\tilde{h} \in [h]$ is the Yamabe metric. Therefore,

$$\begin{aligned} Y(M,[h]) = E(\tilde{h}) &= \frac{\int_M \left(8|\nabla u|_h^2 + R_h u^2\right) dV_h}{\left(\int_M u^6 dV_h\right)^{1/3}} \\ &= \frac{\int_M \left(R_h u^2\right) dV_h}{\left(\int_M u^6 dV_h\right)^{1/3}} \\ &= \frac{u^2 \int_M R_h \, dV_h}{u^2 \left(\int_M dV_h\right)^{1/3}} \\ &= E(h) \end{aligned}$$

□

Corollary 4.1.2. $Y(S^3, [g_0]) = 6\mathrm{Vol}_{g_0}(S^3)^{2/3} = \sigma_1$

Proof. Follows directly from Obata's theorem and the fact that $R_{g_0} = 6$. □

Remark 4.1.2.

We have observed the satisfactory result $\sigma(S^3) = 6(2\pi^2)^{2/3} \equiv \sigma_1$. However, the σ-invariant is only known for very few 3-manifolds.

The following conjectures were posed or referenced by Bray-Neves and Akutagawa-Neves, which if proven to be true would significantly simplify the calculation of the σ-invariant for 3-manifolds or even more general n-manifolds. The conjectures provide insights into the tools one would need for a general theory of σ-invariants, primarily due to the efficiency of calculating σ more directly (e.g. as the integral value $E(g)$) instead of a minimisation/maximisation problem over conformal classes or over metrics inside one conformal class; as a consequence of this obstacle, Akutagawa and Neves [11, p. 6] commented on the elevated importance of asymptotically flat manifolds for studying Yamabe constants, since one can deform any (M_{AF}, g_0) with $E(g_0) = \inf E([g])$ into some $(\tilde{M}_{AF}, g_{AF})$ with $R = 0$ and $[g_{AF}] = [g_0]$, thus $Y(g)$ becomes entirely determined by the conformal factor $u(p)$ and the topology of M_{AF}. This allows for the introduction of functional analysis (Sobolev functions) as shown in [1] and in Section 4.2. of this paper.

Theorem 4.1.2. is an essential corollary from Bray-Neves' use of inverse mean curvature flow **(IMC flow)** to prove that asymptotically flat 3-manifolds, with certain topological constraints, exhibit a different σ value than S^3, and that it can be calculated for $\mathbb{RP}^3$.

Conjecture 4.1.1. (Akutagawa and Neves [11]).

If M is a closed 3-manifold which admits a constant scalar curvature metric h, then

$$Y(M) = E(h).$$

Conjecture 4.1.2. The following was proposed by Schoen [12].

If M admits a constant scalar curvature metric h, then $\sigma(M) = E(h)$.

Conjecture 4.1.3. (Bray-Neves).

If $M = S^3/G_n$ is a smooth quotient manifold and G_n a finite group of order n, then

$$\sigma(M) = \frac{\sigma_1}{n^{2/3}} \equiv \sigma_n.$$

Theorem 4.1.2. (Bray-Neves, Theorem 2.1.). A closed 3-manifold with $\sigma(M) > \sigma_2$ is either:

1. Diffeomorphic to S^3,
2. A connected sum with an S^2-bundle over S^1, or
3. Contains more than one non-orientable prime component.

Remark 4.1.3. Bray and Neves approached the result of this theorem as a corollary of more important theorems relating to values $\sigma(M) \leq \sigma_2$ [1, Theorem 2.12., Theorem 3.2.]. The following expands on their remarks of Theorem 2.1.

We note that if M is prime and simply connected, then it is orientable. Hence M is neither (2), nor (3); by the smooth Poincaré theorem we get a diffeomorphism to S^3.

If M is prime but not simply connected, then we have:

Case 1. M is $S^2 \times S^1$ and $S^2 \tilde{\times} S^1$ with Schoen's result: $\sigma(S^2 \times S^1) = \sigma(S^2 \tilde{\times} S^1) = \sigma_1$.

Case 2. M is not $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$, hence it is irreducible (and equivalent to being prime since $n = 3$).

This is where the use of analysis and stronger geometric tools becomes more helpful than topology or known theorems, hence the need for Bray and Neves to introduce Sobolev spaces and later on IMC flow techniques which require theorems of their own. We can nonetheless say that since M is not simply connected, it will be a manifold with $|\pi_1(M)| > 1$, which by the Thurston-Perelman geometrisation theorem we get that M must be a *spherical manifold*. This can help convince us that for such manifolds, $\sigma_1 > \sigma(M)$ because we have a strict subset of the possible metrics on S^3, as well as a strictly smaller volume than S^3.

Otherwise, if $\pi_1(M)$ is infinite, it falls outside of spherical geometries; we would thus expect $\sigma(M) \leq 0$.

If M is prime and non-orientable, then it is not simply connected and M falls into either of the two cases above.

4.2 Formulation of Results

In this chapter I will demonstrate the proof of Theorem 4.2.1. (Bray-Neves, Theorem 3.2.), which claims that the Sobolev constant of some (M^3, g) with constraints on its topology (asymptotically flat, satisfying properties A or B), as well as $R_g \geq 0$, is bounded above by $\sigma_2/8$. The importance of this result is clarified by comments made in Remark 4.1.2., which state that all asymptotically flat manifolds can be made to have $R_{\tilde{g}} = 0$. In turn, the problem of computing the Yamabe functional introduces functional analysis as a powerful tool, i.e.,

$$Y(g_{AF}) = \inf_u \frac{\int_M 8|\nabla u|^2 dV}{\left(\int_M u^6 dV\right)^{1/3}} .$$

This representation of the problem is Lemma 4.2.2., which links Theorem 4.2.1. with Theorem 4.2.2. The latter theorem generalizes the case to closed 3-manifolds satisfying properties A or B; by taking the supremum of the functional, we observe that the σ value of M is bounded above by σ_2. This is expected since $S(g)$ serves as an auxiliary value for $Y(g)$ and $S(g_{AF}) = Y(g_{AF})/8$ for an asymptotically flat g_{AF} in the same class as a g_0 that minimizes the $E([g])$ functional.

This construction is then applied to $\mathbb{RP}^3$, noting that the asymptotically flat metric of this manifold is the Riemannian Schwarzschild metric. This metric is important because it achieves the infimum $S = \sigma_2/8$, and by Obata's theorem,

$$S(g_{AF}) = Y(g_{AF})/8 = \sigma_2/8,$$

given that we're inside a class with an Einstein metric. To prove Theorem 4.2.1. we need to show we can always find a $u(x)$ that has Sobolev ratio $\sigma_2/8$ for *any* asymptotically flat M satisfying property A or B.

I will show how Bray and Neves utilize IMC flow and the work of Huisken-Ilmanen [4] to construct the desired conformal factor for (M, g) based on the factor u_0 of the asymptotically flat metric (See p. 37–38).

By proving Theorem 4.2.1. with the above steps, Theorem 4.2.2. will follow by Lemma 4.2.2. And finally, Theorem 4.1.2. will follow as a corollary of Theorem 4.2.2. by negating property A, thus concluding the set of tools required for calculating $\sigma(\mathbb{RP}^3)$.

Definition 4.2.1. A 3-manifold M^3 has Property A if M^3 is not S^3, nor $S^2 \times S^1$, nor $S^2 \tilde{\times} S^1$, and can be expressed as $P^3 \# Q^3$ where P^3 is prime and Q^3 is orientable.

Definition 4.2.2. $\alpha(Q^3)$ is defined as the supremum of the Euler characteristic $\chi(\partial M^3)$ of the boundary (not necessarily connected, e.g., a union of disjoint 2-manifolds) of all smooth connected regions with two-sided boundaries, whose complements are also connected.

Definition 4.2.3. A 3-manifold M^3 has Property B if M^3 is not S^3, nor $S^2 \times S^1$, nor $S^2 \tilde{\times} S^1$, and can be expressed as $P^3 \# Q^3$ where P^3 is prime and $\alpha(Q^3) = 2$.

Lemma 4.2.1. Property A implies property B.

Proof. See [1, Lemma 2.10.].

Definition 4.2.4. A Riemannian 3-manifold (M, g) is *asymptotically flat* if there is a compact set $C \subseteq M$ such that $M - C \overset{\phi}{=} \mathbb{R}^3 - \{|x| \leq 1\}$, where ϕ is a diffeomorphism. And that in the coordinate chart by ϕ we get

$$g = \sum_{i,j} g_{ij}(x) dx^i dx^j \ ,$$

where

$$g_{ij} = \delta_{ij} + O(|x|^{-1}), \quad g_{ij,k} = O(|x|^{-2}), \quad g_{ij,kl} = O(|x|^{-3}) \ .$$

Theorem 4.2.1. (Bray-Neves, Theorem 3.2.). Let (M, g) be an asymptotically flat 3-manifold with non-negative scalar curvature satisfying property A or B. Then

$$S(g) \leq \sigma_2/8 \ .$$

Where $S(g)$ is the Sobolev constant on (M, g). Note that the calculation of $S(g)$ is with respect to the conformal factor and not the metric itself. This is because we can construct an asymptotically flat metric g_{AF}, giving $R_{AF} = 0$, so the conformal factor becomes the main object of importance when taking the infimum of the Yamabe functional.

In the course of the following theorems we will see how this generalises over to all classes of metrics. Keep in mind that the condition $u > 0$ is primarily for integrability concerns, hence ignoring $u = 0$ at one-point sets, and that due to $R_{AF} = 0$ the asymptotically flat model gives the lowest possible Yamabe functional infimum out of any metric class $[g]$. This explains why the following theorem is entirely topological (not metric-dependent).

Theorem 4.2.2. (Bray-Neves, Theorem 2.12.) A closed 3-manifold M^3 with Property A or B has

$$\sigma(M^3) \leq \sigma_2.$$

In their work, they mention that Theorem 4.2.2. can be thought as the main theorem of their paper, whence Theorem 4.2.2. implies Theorem 4.1.2. by the negation of Property A.

Lemma 4.2.2. Theorem 4.2.1. implies Theorem 4.2.2.

Proof. (cf. [1, Chapter 3. The basic approach and some definitions]) Let M be a closed 3-manifold, suppose it has Property A or B, then to prove that $\sigma(M) \leq \sigma_2$ would follow if we proved that

$$Y(g) \leq \sigma_2,$$

for all conformal metric classes $[g]$ on M.

Case 1. $Y(g) \leq 0$, then we are done.

Case 2. $Y(g) > 0$, implying that the metric g_0 which minimizes $E(g)$ has constant, positive scalar curvature R_0.

We then define for (M, g_0) a conformal Laplacian for g_0,

$$L_0 := \Delta_0 - \frac{1}{8} R_0 \ .$$

Which allows us to define a Green's function $G_p(x)$ to L_0 at some point p, so that on $M-\{p\}$ we get

$$L_0 G_p = 0,$$
$$\lim_{q\to p} d(p,q)G_p(q) = 1 \ .$$

Since $R_0 > 0$, and by the maximum principle, we ensure that such a Green function exists.

We then define

$$g_{AF} := G_p(x)^4 g_0$$

on $M-\{p\}$, where $R_{AF} = -8G_p^{-5}L_0(G_p) = 0$ and $(M-\{p\}, g_{AF})$ is an asymptotically flat Riemannian manifold and $\lim_{q\to p}|\phi(q)| = \infty$, ϕ being the diffeomorphism between the manifold and $\mathbb{R}^3 - \{|x| \leq 1\}$.

We note that the following metrics are in the same conformal class, implying

$$Y(g) = Y(g_0) = Y(g_{AF}) \ .$$

Which allow to calculate the Sobolev constant

$$S(g_{AF}) = \frac{1}{8}\inf_u \frac{\int_M 8|\nabla u|^2 dV}{\left(\int_M u^6 dV\right)^{1/3}} \ .$$

Where $u \in H^1(M-\{p\}, g_{AF})$ has compact support.

The values of $Y(g)$ and $Y(g_0)$ are not affected by the compact support of conformal factors on $(M-\{p\}, g_{AF})$ when we consider that, as a consequence of compact support, the conformal factors u on (M,g) and (M,g_0) can be zero in an open, arbitrarily small neighbourhood around p. Hence as we approach the infimum, integrating over $M-\{p\}$ with $\{p\}$ as a zero-measure set hands over the same result as integrating over M, resulting in the desired $Y(g)$ value. □

Remark 4.2.1. (cf. [1, Chapter 4. Some intuition]) Bray and Neves prepare the reader regarding the approach for proving Theorem 4.2.1.

The Riemannian Schwarzschild metric on $\mathbb{RP}^3 - \{p\}$ is the only case which gives

$$S(g) = \sigma_2/8 \ .$$

This is because the Schwarzschild metric suitably describes an asymptotically flat manifold via

$$g_{ij} = \delta_{ij}(1 + 1/|x|)^4$$

on $\mathbb{R}^3 - \{0\}$.

This is thanks to the construction, as previously stated, of $g_{AF} = G(x)^4 g_0$ on $(S^3 - n_1 - n_{-1})$ where n_1, n_{-1} are antipodes. Removing the antipodes serves to describe the topological and symmetrical equivalent of removing a point on $\mathbb{RP}^3$ for asymptotic flatness.

Hence,

$$(\mathbb{R}^3 - B_1(0), \delta_{ij}(1 + 1/|x|)^4) := (L, h)$$

is an adequate model for the Schwarzschild metric on $\mathbb{RP}^3$. We can think of this as the stereographic projection of $\mathbb{RP}^3$ in $\mathbb{R}^3$. As this construction is unique, then $(\mathbb{RP}^3 - \{p\}, g_{AF})$ should be isometric to (L, h) up to constant scaling.

Using Obata's theorem, we can directly calculate $Y(g_0) = \sigma_2$, hence

$$S(g_{AF}) = Y(g_{AF})/8 = Y(g_0)/8 = \sigma_2/8$$

We then define a conformal factor u_0 by

$$\max(u_0) = 1 \text{ , and } (L, u_0(x)^4 h) \overset{isom.}{=} (\mathbb{RP}^3 - \{p\}, g_0)$$

By Lemma 4.2.2., it follows that this conformal factor achieves the infimum $S = \sigma_2/8$. Bray and Neves also especially note that u_0 has $O(3)$ symmetry.

Ultimately, the key to proving Theorem 4.2.1. is to show that on an asymptotically flat 3-manifold M with $R \geq 0$ satisfying either Property A or B, we can always construct some $u(x)$ that gives the *Sobolev ratio* $\sigma_2/8$, i.e., we can find such $u(x)$ for any $[g]$ on M.

This is sufficient to then prove Theorem 4.2.2., since when we take the infimum over u to give $S \leq \sigma_2/8$, we then get $\sigma(M) \leq \sigma_2$ via Lemma 4.2.2.

In the following statements we lay out the tools of IMC flow which Bray and Neves use to prove Theorem 4.2.1.

Definition 4.2.5. A 2-manifold is said to be a minimal surface when

$$H = g^{ij} \nabla_{e_i} e_j = 0 \ .$$

E.g., a minimal surface *in* a Riemannian 3-manifold (M, g) is a 2-*submanifold* $\Sigma \subset M$ with 0-mean curvature.

Remark 4.2.2. We define $\Sigma(0)$ to be the minimal sphere in (L, h). Flowing this sphere in the outward normal direction defines a family of surfaces $\Sigma(t)$ where at each t the speed of the flow is $1/H$, and H is the mean curvature of $\Sigma(t)$. This flow is defined for all $t > 0$, and requires that $\lim\limits_{t \to 0} \Sigma(t) = \Sigma(0)$.

Given that u_0 is a constant function on $\Sigma(t)$ due to its $u_0(x)^4 h \overset{isom.}{=} g_0$ and $O(3)$ properties, we define

$$f(t) = u_0(\Sigma(t)) \ , \ t \geq 0 \ .$$

For simplification Σ_t is also used. Bray and Neves refer to Huisken-Ilmanen [4] to use the weak formulation of inverse curvature flow. They introduce terminology which becomes important in proving Theorem 4.2.1.

The full context of the definitions and theorems they provide hereafter are outside of the scope of this paper, please see [1,4].

Some of the key terms from their work that we will be using are: Weak Mean Curvature, Hawking Mass, Minimizing Hull, and Theorem 5.2. of [1] which is about asymptotically flat manifolds providing Lipschitz functions given an open precompact minimizing hull in M.

Definition 4.2.6. The *weak mean curvature* H, of a C^1 surface Σ of a Riemannian 3-manifold (M, g), is $H \in L^1_{loc}(\Sigma)$ such that

$$\int_\Sigma \text{div}_M(\vec{X}) dA_g = \int_\Sigma H \langle \vec{X}, \vec{v} \rangle dA_g, \forall \vec{X} \in \Gamma(TM) \text{ with compact support.}$$

Where $\vec{v}$ is the exterior unit normal.

Definition 4.2.7. The *Hawking mass* of a compact C^1 hypersurface Σ with weak mean curvature $H \in L^2(\Sigma)$ is defined as

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{(16\pi)^3}} \Big(16\pi - \int_\Sigma H^2 dA_g\Big)$$

Definition 4.2.8. A compact set $E \subset M$ is said to be a minimising hull if

$$|\partial^* E| \leq |\partial^* F|$$

for all F s.t. $E \subseteq F$ and $F - E \subset\subset M$, where $\partial^* E$ is the reduced boundary of E (e.g. having removed sets of measure zero, etc.), and $\subset\subset$ denotes the ”well-contained” relation.

Proof of Theorem 4.2.1.

Let (M, g) be an asymptotically flat 3-manifold with $R_g \geq 0$ having either property A or B. We will follow from assuming property B, since A implies B also.

We define the concept of an ”outermost minimal surface” for M as a minimal surface whose compact region it bounds contains the compact regions bounded by all other minimal surfaces. Meeks, et al. [17] show that as long as $M \neq \mathbb{R}^3$, and equivalently by homotopy $M \neq S^3 - \{p\}$, then we can always find such outermost minimal surface, and it is the disjoint, finite union of weakly embedded 2-spheres. These weak embeddings can be thought as the limits of smooth-embedded 2-spheres. Their result further states that the exterior region to this outermost surface is topologically equivalent to $\mathbb{R}^3$, minus a finite number of balls.

From property B follows the existence of an outermost minimal surface, since $M \neq S^3$ and hence no construction of $S^3 - \{p\} \cong \mathbb{R}^3$ can be made from the definition of an asymptotically flat M. Also, M is not part of a connect sum with S^2 over S^1, hence M bounds a connected and compact region. We finally observe that the prime manifold P of $M = P \# Q$ with $\alpha(Q) \leq 2$ is completely inside one of the outermost minimal surface spheres. Define this sphere as Σ.

Bray and Neves note this is the main object of study for their IMC flow and that the flow is always inside a subset of Q since P is already entirely in the region bounded by Σ.

Σ is the boundary of a region that is a *minimizing hull* which Bray-Neves label as E; thus allowing for the application of Huisken-Ilmanen’s theorem [1, Theorem 5.2.] which proves the existence of a "precompact" locally Lipschitz function ϕ that generously specifies the properties of Σ_t and claims that $\Sigma_0 = \partial E_0$ where E_0 is an open precompact minimising hull in M.

We crucially note that in the IMC flow, $\chi(\Sigma_t) \leq 2$, which is proven by Huisken-Ilmanen and explained by Bray-Neves. This condition is satisfied via the Gauss-Bonnet theorem requiring

$$\int_{\Sigma_t} K \leq 4\pi$$

in order to prove that the *Hawking mass* is nondecreasing.

For all t, the interior and exterior regions of Σ_t remain connected. This is ensured for the interior by Huisken-Ilmanen's energy minimisation argument; for the exterior region Bray-Neves argue that if more than one component was developed, filling in the components disconnected from infinity would contradict Σ being enclosed by surfaces of less area since the area of Σ would decrease.

Σ_t thus splits M, and consequently Q, into connected interior and exterior regions. The flow also occurs within a subset of Q entirely, using Property B we guarantee that $\chi(\Sigma_t) \leq 2$ since clearly $\alpha(Q) \leq 2$.

We recall the precompact locally Lipschitz ϕ mentioned above, taking part in an IMC flow that starts at Σ, and locally define a Lipschitz function u as

$$u(x) := \begin{cases} f(0) & \text{if } \phi(x) \leq 0 \\ f(\phi(x)) & \text{if } \phi(x) > 0 \end{cases}$$

where $f(t) = u(\Sigma(t))$ as in Remark 4.2.2. Hence the goal is to show this general u exhibits Sobolev ratio

$$\frac{\int_M |\nabla u|^2 dV_g}{\left(\int_M u^6 dV_g\right)^{1/3}} \leq \frac{\sigma_2^2}{8}.$$

and also that $u(x)$ approaches 0 at infinity fast enough. Given that the Sobolev constant is the infimum of above integral ratio, Theorem 4.2.1. follows.

For the integral calculations the reader is referred to Bray and Neves' paper [1], as these require intricate applications of Theorem 5.2. and collateral lemmas which lie outside the scope of this paper. We skip to the final calculation, where Bray-Neves mention that the integrals need not be directly computed, but rather that the inequality is a given by the model case of a Schwarzschild metric on $\mathbb{RP}^3 - \{p\}$:

$$\frac{\int_M |\nabla u|^2 dV_g}{\left(\int_M u^6 dV_g\right)^{1/3}} \leq \frac{(16\pi)^{2/3} \int_0^\infty f'(t)^2 \sqrt{e^t - e^{t/2}}\, dt}{\left(\int_0^\infty f(t)^6 e^{2t} (e^t - e^{t/2})^{-1/2}\, dt\right)^{1/3}}$$
$$= \frac{\sigma_2}{8},$$

since $f(t)$ gives the optimal test function, and so the Sobolev constant for the Schwarzschild metric on $\mathbb{RP}^3$ is $\sigma_2/8$. Hence no need to actually calculate $f(t)$ explicitly. Finally, Bray-Neves verify that $u(x)$ decays at infinity sufficiently fast. We consider the clear observation that in the asymptotically flat coordinate chart, and by the maximum principle, Σ_t lies in $\{x \mid c_1 < |x|e^{-t/2} < c_2\}$, for sufficiently large t and $c_1, c_2 \in \mathbb{R}$. Ergo, $\exists k_1, k_2 > 0$ such that

$$k_1 \leq u(x)|x| \leq k_2 \ ,$$

for a large enough t. This satisfies $\lim_{x\to\infty} u(x)|x|^{1/2} = 0$, where $u(x)$ doesn't require compact support but more generally $u \in H^1_{loc} \cap L^6$, which also aligns with the $u(x)$ constructed in Lemma 4.2.2. □

Remark 4.2.3.

For readers interested in direct calculation, Bray and Neves provide

$$f(t) = \frac{1}{\sqrt{2e^t - e^{t/2}}},$$

which clarifies and confirms the first non-trivial case in Conjecture 4.1.3., that is,

$$\sigma_2 = \frac{\sigma_1}{2^{2/3}}.$$

This value can now be used with Obata's theorem to show that $\sigma(\mathbb{RP}^3) \geq \sigma_2$.

4.3 The σ Invariant of $\mathbb{RP}^3$

Corollary 4.3.1.

$$\sigma(\mathbb{RP}^3) = \sigma_2.$$

Proof. Firstly, $\sigma(\mathbb{RP}^3) \leq \sigma_2$ since $\mathbb{RP}^3$ is none of the manifolds posited in Theorem 4.1.2. and $\sigma(\mathbb{RP}^3) \geq \sigma_2$ is given by Obata's theorem when we descend the round g_0 from S^3 to $\mathbb{RP}^3$. We only need to prove explicitly this second part.

Let h be the descended g_0 metric. Then all tensorial objects relating to h and g will be equivalent, since they are *local objects* and $\pi : S^3 \to \mathbb{RP}^3$ is a local isometry. We also note that u cancels out and $\nabla u = 0$ since u is constant by Case 2. of Obata's theorem. Then,

$$E(h) = Y(\mathbb{RP}^3, [h])$$

$$E(h) = \frac{\int_{\mathbb{RP}^3} R_h \, dV_h}{\left(\int_{\mathbb{RP}^3} \, dV_h\right)^{1/3}} = \frac{\frac{1}{2}\int_{S^3} R_{g_0} \, dV_{g_0}}{\left(\frac{1}{2}\int_{S^3} \, dV_{g_0}\right)^{1/3}}$$

$$E(g_0) = 2^{-1} \cdot 2^{1/3} \cdot \sigma_1 = \frac{\sigma_1}{2^{2/3}} = \sigma_2.$$

Therefore, $Y(h) = \sigma_2$ and by definition,

$$\sigma(\mathbb{RP}^3) \geq Y(h) = \sigma_2.$$

□

Chapter 5

Conclusion and Further Work

This dissertation investigated the Yamabe problem and its associated Yamabe invariants (conformal $Y([g])$ and smooth $\sigma(M)$). We focused on the computation of the σ value for the real projective space $\mathbb{RP}^3$ and established that

$$\sigma(\mathbb{RP}^3) = \sigma_2 = \frac{\sigma_1}{2^{2/3}} ,$$

where

$$\sigma(S^3) = \sigma_1 = 6(2\pi^2)^{2/3}.$$

We derived this using Bray and Neves' techniques of IMC flow on the Riemannian Schwarzschild metric of an asymptotically flat $\mathbb{RP}^3$. This metric is in fact the only case giving the equality in Theorem 4.2.1. (See [1, p. 414]), justifying its importance for constructing a general conformal factor that has Sobolev ratio $\sigma_2/8$.

As we saw that $\sigma_2 = \sigma_1/2^{2/3}$ in Remark 4.2.3., this hints to the possibility of Conjecture 4.1.3. being true, which is a generalization of a conjecture by Schoen on Lens spaces:

$$\sigma(L(p,q)) = \frac{\sigma_1}{p^{2/3}} = \sigma_p.$$

We can observe that if one wanted to investigate these conjectures further while utilising the same tools found in this paper, they would need to apply narrower constraints on Properties A or B. This is because the asymptotically flat property determines Yamabe functional exclusively by the topology of its manifold and the conformal factors acting on it. In this manner, it is conjectured that one could lower the supremum of

$$\frac{\int_M |\nabla u|^2 dV_g}{\left(\int_M u^6 dV_g\right)^{1/3}}$$

by narrowing the type of topology of M but following the general theme of Theorem 4.2.1., noting that we would also need a generalisation of the (asymptotically flat) Riemannian Schwarzschild metric beyond that of $\mathbb{RP}^3$, in order to find an u which achieves the desired new Sobolev ratio $\sigma_p/8$.

Moreover, in Remark 4.1.2. I noted that σ is not known for many manifolds, but the work of Akutagawa and Neves [11] welcomes some optimism by extending the findings of Bray and Neves to find the σ invariant of (somewhat) arbitrary connected sums of manifolds whose σ invariants are already known.

In [16], Kobayashi shows that for two non-negative σ-invariants $\sigma(M_1)$, $\sigma(M_2)$, we have $\sigma(M_1 \# M_2) \geq min(\sigma(M_1), \sigma(M_2))$. Furthermore, given $\sigma(M) > 0$ one can deduce that this is equivalent to M admitting a Yamabe metric of positive scalar curvature. As a counter-example, since for any 3-manifold M, $T^3 \# M$ doesn't admit a metric with positive scalar curvature, and since we know that the Euclidean metric gives $Y(T^3, g_E) = 0$, then it follows that $\sigma(T^3) = 0$ (See [1, p. 409]). This is a useful method of deduction to find σ for more complex manifolds.

Further research could build from the results discussed in this paper by calculating σ of other prime 3-manifolds, or even using Akutagawa and Neves' methods for σ on connected sums of manifolds.

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