Donaldson's Diagonalisibility Theorem for 4-Manifolds

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Abstract

In 1982 Freedman showed that compact, oriented, simply-connected topological 4-manifolds are classified up to homeomorphism by their intersection form on the middle dimensional cohomology, and a $\mathbb{Z}_2$-valued invariant called the Kirby-Siebenmann invariant. An open question is which topological 4-manifolds are smoothable? In 1983, Donaldson gave a partial answer to this question. His result, called Donaldson's theorem, states that definite, compact, oriented, simply-connected 4-manifolds are smoothable if, and only if, their intersection form is diagonalisable.

In this thesis, we review a proof of Donaldson's theorem using Seiberg-Witten theory, due to Kronheimer.

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Chapter 1

Introduction

It is known that compact, oriented 2-manifolds are classified up to homeomorphism and diffeomorphism by a single enumerative invariant called the genus. Similarly, compact 3-manifolds are classified up homeomorphism and diffeomorphism by Thurston's geometrisation conjecture. A natural question is whether higher dimensional manifolds admit such classifications? In dimensions five and above, techniques pioneered in the 1960's such as surgery theory provide the required classification. While these techniques do provide a classification for topological 4-manifolds, they fail for smooth 4-manifolds.

The problem of classifying topological 4-manifolds had a breakthrough in 1982 with Freedman's seminal paper [Fre82]. Freedman's main result stated that compact, oriented, simply-connected topological 4-manifolds are classified, up to homeomorphism, by their intersection form and a $\mathbb{Z}_2$-valued invariant called the Kirby-Siebenmann invariant. This was a strengthening of results by Milnor and Whitehead which proved that the intersection form provides a classification up to homotopy.

Freedman's result is powerful because it turns a problem in topology into a problem in arithmetic, with the resulting problem being to classify the intersection forms. But is this possible? Intersection forms are defined as being either definite or indefinite, with the indefinite forms having been classified previously by Serre. However, the definite forms are not classified. It is known that for a given rank, there are only finitely many equivalent forms; but this number grows rapidly with respect to the rank.

The situation for definite intersection forms of smooth 4-manifolds, however, is very different.

In Donaldson's 1983 paper [Don83], published only a year after Freedman's, Donaldson was able to use gauge theory to construct novel invariants for certain smooth 4-manifolds.

Explicitly, Donaldson studied the space of anti-self-dual connections, called instantons, on a principal SU(2)-bundle. These instantons have an action by gauge transformations, and Donaldson investigated the quotient space of this action. The resulting space is a moduli space of instantons.

Studying the moduli space of instantons proved fruitful; it encodes a large amount of information about the differential topology of the 4-manifold. In particular, Donaldson was able to use this to classify all possible definite intersection forms for compact, oriented, and simply-connected smooth 4-manifolds; they are diagonalisable. Therefore, the intractable cases of Freedman's theorem are removed, and we have a classification of compact, oriented, and simply-connected smooth 4-manifolds up to homeomorphism.

Donaldson's theory is not without limitations. One main drawback is the moduli space is not compact, and so an exorbitant amount of work is needed to find a suitable compactification.

However, Witten [Wit88] showed that Donaldson's theory can be realised as $N = 2$ supersymmetric Yang-Mills theory; a topological quantum field theory. In joint work with Seiberg, Witten extensively studied this Yang-Mills theory. This led to Witten's 1994 paper [Wit94], where he showed this Yang-Mills theory had an associated "Seiberg-Witten" theory which encapsulated many results proved earlier by Donaldson. However, unlike Donaldson's theory, the spaces associated to Seiberg-Witten theory are compact, and easier to work with.

Seiberg-Witten theory turned out be the key unlocking the door to many open problems concerning smooth 4-manifolds. In the years following [Wit94], papers from mathematicians such as Kronheimer, Mrowka, Taubes, etc, proved many conjectures in geometry not limited to just classifcation problems. One example of where Seiberg-Witten theory found use can be seen in the Thom conjecture [KM94].

In this thesis, we review the Seiberg-Witten theory proof of Donaldson's theorem due to Kronheimer.

1.1 Outline of Thesis

The following is an outline for the remainder of this thesis:

In Chapter 2 we state the main results concerning the classification of 4-manifolds.

In Chapter 3 we provide an introduction to spin geometry.

In Chapter 4 we define the Seiberg-Witten equations, and study the moduli space of solutions to said equations.

In Chapter 5 we prove Donaldson's theorem.

Finally, Appendix A provides a review of basic results and constructions from differential geometry.

Chapter 2

4-Manifolds

2.1 Manifolds

We first briefly recall the definition of a n-manifold. A more formal treatment can be found in [Lee12].

Definition 2.1.1. Let X be a Hausdorff, second countable topological space. We say X is a **topological n-manifold** if for every $x \in X$ there is an open neighbourhood U of x which is homeomorphic to an open subset of $\mathbb{R}^n$, with homeomorphism φ. We call the pair (U, φ) a **chart**, and a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of charts which covers X is an **atlas**.

From the definition it follows that for two charts (U_i, φ_i) and (U_j, φ_j) in an atlas for X, the map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j) \tag{2.1.1}$$

is a homeomorphism of open subsets of $\mathbb{R}^n$.

Notice that a n-manifold X need not be connected. However, if X is not connected then X is a disjoint union of the connected components. (If it was not disjoint, then neighbourhoods of the points of intersection would not be locally homeomorphic to $\mathbb{R}^n$.)

In this thesis we also wish to consider smooth 4-manifolds, which we now define. First, we say that the two charts (U_i, φ_i) and (U_j, φ_j) are **smoothly compatible** if either $U_i \cap U_j = \emptyset$, or (2.1.1) is a diffeomorphism of open subsets of $\mathbb{R}^n$. An atlas $\mathcal{A}$ for X where each pair of charts in $\mathcal{A}$ are smoothly compatible is called a **smooth atlas**. A smooth atlas $\mathcal{A}$ is **maximal** if it is not contained in a larger smooth atlas. Two atlases are equivalent if their union is an atlas. Hence, a maximal atlas is the same as an equivalence class of atlases.

Definition 2.1.2. A **smooth structure** $\mathcal{A}$ on a topological n-manifold X is a maximal smooth atlas. A **smooth n-manifold** X is a topological n-manifold with a smooth structure. If a topological manifold X admits a smooth structure, then we say that X is **smoothable**.

From the definition it is natural to ask whether every topological manifold admits a smooth structure? When $n = 1, 2, 3$ this turns out to be true. However, when $n \geq 4$ there are topological manifolds which are not smoothable; and we will construct such a manifold when $n = 4$ using theorems of Freedman and Rokhlin.

We now only consider manifolds which are compact, and oriented as these are the only manifolds we encounter in this thesis.

It is natural to ask whether we can classify such manifolds, either up to homeomorphism, or diffeomorphism in the smooth case. In low dimensions, $n = 1, 2, 3$, this is possible; and further the topological and smooth classifications coincide. If $n = 2$, then by the uniformisation theorem 2-manifolds are classified by their genus g. If $n = 3$, then the geometrisation conjecture of Thurston (proved by Perelman) also provides a classification, albeit it is much less explicit. When $n \geq 5$, advance techniques such as surgery theory provide classifications of smooth n-manifolds.

However, smooth 4-manifolds do not have such nice classifications. (Topologically 4-manifolds are classified by surgery, but it does not work smoothly.)

2.2 Intersection Forms

One of the remarkable results of the 20th century is Freedman's theorem, which states that compact, oriented, simply connected topological 4-manifolds are classified by their intersection form. This turns the problem of classifying manifolds into an arithmetic problem. In this section we define the intersection form, and state Freedman's theorem.

A space X is simply-connected if X is path-connected and $\pi_1(X) = 1$.

Let X be a compact, oriented, simply connected 4-manifold (topological or smooth). Under these assumptions it follows that X has the following integral cohomology groups:

$$H^0(X;\mathbb{Z}) \cong \mathbb{Z}, \qquad H^1(X;\mathbb{Z}) = 0, \qquad H^2(X;\mathbb{Z}) \cong \mathbb{Z}^{b^2(X)}, \tag{2.2.1}$$

where $b^2(X)$ is the second Betti number of X. By Poincaré duality

$$H^3(X;\mathbb{Z}) = 0, \qquad H^4(X;\mathbb{Z}) \cong \mathbb{Z}, \tag{2.2.2}$$

and as X is a 4-manifold, $H^n(X;\mathbb{Z}) = 0$ for all $n \geq 5$.

However, we know that cohomology comes equipped with a ring structure given by the **cup product**

$$\smile : H^i(X;\mathbb{Z}) \times H^j(X;\mathbb{Z}) \to H^{i+j}(X;\mathbb{Z}), \tag{2.2.3}$$

which makes $H^\bullet(X;\mathbb{Z})$ a graded-commutative ring; i.e., for $x \in H^i(X;\mathbb{Z})$ and $y \in H^j(X;\mathbb{Z})$ we have

$$x \smile y = (-1)^{ij} y \smile x. \tag{2.2.4}$$

Restricting the cup product to $H^2(X;\mathbb{Z})$, we have a symmetric map

$$\smile : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z}) \cong \mathbb{Z}. \tag{2.2.5}$$

Using this observation, we have the following definition.

Definition 2.2.1. The **intersection form** of X is the map

$$Q_X : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}, \tag{2.2.6}$$

given by the composition of the cup product, followed by the standard isomorphism of $H^4(X;\mathbb{Z}) \cong \mathbb{Z}$. If there is no risk of confusion, we drop the dependence on X and write Q for the intersection form.

Explicitly, we have

$$Q_X(x, y) = \langle x \smile y, [X]\rangle = \int_X (x \smile y), \tag{2.2.7}$$

where $[X]$ is the **fundamental class** of X, i.e. the generator of $H^4(X;\mathbb{Z})$.

The intersection form is a well-studied object in differential topology, and is the main object of consideration in the theorems of Freedman and Donaldson.

It is known that, up to torsion elements, the intersection form is a non-degenerate, unimodular, symmetric bilinear form on $H^2(X;\mathbb{Z})$. As X is simply connected, $H^2(X;\mathbb{Z})$ has no torsion, and so the intersection form of X has these properties. (A proof of these statements can be found in [Sco05, Chapter 3].)

These properties imply that Q_X can be viewed as a $b^2(X) \times b^2(X)$ symmetric matrix with integer entries and determinant ± 1.

Remark 1. The intersection form has a geometric interpretation which provides an insight to its name. By Poincaré duality, a class $x \in H^2(X;\mathbb{Z})$ corresponds to a cycle in $\alpha \in H_2(X;\mathbb{Z})$; and as X has dimension less than 6, a theorem of Thom states α corresponds to an embedded surface S_α in X. So suppose we have $\alpha, \beta \in H^2(X;\mathbb{Z})$. These correspond to embedded surfaces S_α, S_β in X which, without a loss generality, intersect transversely. It turns out that $Q(\alpha, \beta)$ is precisely the intersection number of S_α and S_β. A more detailed explanation can be found in [Sco05, Moo01] ♦

We now recall properties of the intersection form which will be used extensively.

Theorem 2.2.2. *Suppose that X and Y are two compact, oriented, simply-connected 4-manifolds with intersection forms Q_X and Q_Y, respectively. Then*

(I) *If $\overline{X}$ is the manifold X with the reverse orientation, then the intersection form of $\overline{X}$ is $-Q_X$. i.e. changing orientation changes the sign of the intersection form.*

(II) *The connected sum $X \# Y$ has the intersection form $Q_X \oplus Q_Y$.*

(III) *If $f : X \to X$ is a orientation preserving homeomorphism, then $f^* Q_X = Q_X$. i.e. for all $x, y \in H^2(X; \mathbb{Z})$ we have $Q_X(f^*x, f^*y)_X = Q_X(x, y)$.*

We now define invariants of the intersection form itself.

Definition 2.2.3.

1. The **rank** of the intersection form Q is defined as

$$\operatorname{rk} Q \coloneqq \dim_{\mathbb{Z}} \left(H^2(X; \mathbb{Z})\right), \tag{2.2.8}$$

   i.e. $\operatorname{rk} Q = b^2(X)$, by non-degeneracy.

2. We say that Q is **even** if for all $x \in H^2(X; \mathbb{Z})$ we have $Q(x, x) \in 2\mathbb{Z}$. Otherwise, Q is **odd**.

3. Q is **positive-definite** if $Q(x, x) > 0$ for all non-zero $x \in H^2(X; \mathbb{Z})$. If $Q(y, y) < 0$ for all non-zero $y \in H^2(X; \mathbb{Z})$, then Q is **negative-definite**. If Q is neither positive-definite nor negative-definite, it is **indefinite**.

Consider the change of coefficients

$$H^2(X; \mathbb{R}) = H^2(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{b^2(X)}. \tag{2.2.9}$$

Then there exists a basis such that

$$Q = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}. \tag{2.2.10}$$

We set $b^2_+(X)$ to be the number of $+1$ entries, and $b^2_-(X)$ to be the number of -1 entries; and these numbers do not depend on the choice of basis. We define the **signature of** Q to be the integer $b^2_+(X) - b^2_-(X)$.

We define the **signature of** X to be the signature of its intersection form Q_X, i.e.

$$\tau(X) = b^2_+(X) - b^2_-(X). \tag{2.2.11}$$

From the definition we can see that the signature is the dimension of the maximal positive-definite subspace of Q minus the dimension of the maximal negative-definite subspace of Q.

Remark 2. There is another possible way to define the signature if X is a smooth manifold. Suppose X is smooth and let g be a Riemannian metric for X. The Riemannian metric allows us to define the Hodge star operator $\star$ on $\Omega^\bullet(X)$. It then follows that $b^2_+(X)$ is the dimension of the space of self-dual 2-forms, and $b^2_-(X)$ is the dimension of the space of anti-self dual 2-forms. We then define the signature of X to be

$$\tau(X) = b^2_+(X) - b^2_-(X). \tag{2.2.12}$$

We recall some basic facts about Hodge Theory in Appendix A ♦

2.3 Theorems of Freedman and Donaldson

One reason why the intersection form is important in differential topology is that it was shown by Milnor and Whitehead that the intersection form is a homotopy invariant [Mil58, Whi49]. So it is natural to ask whether the intersection form is actually a stronger invariant? A result of M. Freedman shows that this is indeed the case. However, before we state Freedman's theorem, we need one more definition.

Definition 2.3.1. The **Kirby-Siebenmann invariant** of X is the class $\kappa(X) \in H^4(X; \mathbb{Z}_2)$ such that

$$\kappa(X) = \begin{cases} 0 & \text{if } X \text{ is smoothable,} \\ 1 & \text{if } X \text{ is not smoothable.} \end{cases} \tag{2.3.1}$$

Remark 3. Strictly speaking the Kirby-Siebenmann invariant vanishes if, and only if, X admits a piece-wise linear structure. This distinction is not necessary for us, as in dimension 4 the category of smooth and piece-wise linear manifolds are equivalent, see [Mil11]. ♦

We are now ready to state Freedman's theorem.

Theorem 2.3.2 (Freedman, [Fre82])**.** *Compact, oriented, simply connected topological 4-manifolds are in bijective correspondence with pairs* (Q, κ)*, where* Q *is a non-degenerate, symmetric, unimodular, bilinear form. With the condition that if* Q *is even, then*

$$\frac{\tau(Q)}{8} = \kappa \pmod 2.$$

Thus if given a symmetric, non-degenerate, bilinear, and unimodular form Q there is a 4-manifold such that its intersection form is Q. Now if Q is even, there is a unique such manifold; and if Q is odd there are two possible manifolds, one smooth, and one topological.

We now present some examples of intersection forms and the manifolds they correspond to.

Example 2.3.3. As $H^2(S^4; \mathbb{Z}) = 0$, it follows that the zero form (0) is the intersection of the 4-sphere S^4. ◀

Example 2.3.4. The definite form (1) is odd, and is realised by the manifold $\mathbb{C}P^2$. Hence by Theorem 2.2.2 it follows that (-1) is realised by $\overline{\mathbb{C}P}^2$ ($\mathbb{C}P^2$ with the opposite orientation); and the form

$$\operatorname{diag}(\overbrace{1, \ldots, 1}^{\ell}, \overbrace{-1, \ldots, -1}^{m}) \tag{2.3.2}$$

with $\ell, m \geq 0$ is realised by the manifold $\#^\ell \mathbb{C}P^2 \#^m \overline{\mathbb{C}P}^2$. ◀

Example 2.3.5. The form

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.3.3}$$

is indefinite, even, and is realised by the manifold $S^2 \times S^2$. ◀

Example 2.3.6. The intersection form given by the Cartan matrix of E_8

$$E_8 = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & 0 & -1 \\ & & & & -1 & 2 & -1 & 0 \\ & & & & 0 & -1 & 2 & 0 \\ & & & & -1 & 0 & 0 & 2 \end{bmatrix} \tag{2.3.4}$$

is definite, even, and is realised by the **topological** E_8-manifold, M_{E_8}. ◀

Remark 4. The emphasis that E_8 gives rise to a topological manifold in Example 2.3.6 is important; as M_{E_8} is not smoothable by Rokhlin's theorem, Theorem 3.5.9, and Proposition 5.1.3. ⧫

Freedman's theorem is remarkable because if we have a classification of forms then we have a corresponding classification of topological 4-manifolds. Therefore, Theorem 2.3.2 turns the classification of topological 4-manifolds from a problem in geometry to a problem in arithmetic.

So does such a classification of forms exist? This is where the distinction between definite and indefinite forms is important. If the form is indefinite, then we have such a classification.

Theorem 2.3.7 (Serre, [Ser73])**.** *Indefinite, symmetric, unimodular, bilinear forms are classified by their rank, signature, and parity. Explicitly:*

(I) *If the form is odd, then it is diagonalisable. i.e.*

$$Q = \oplus^{\ell}(1) \oplus^{m} (-1) = \operatorname{diag}(1, \dots, 1, -1, \dots, -1). \tag{2.3.5}$$

(II) *If the form is even, then*

$$Q = \oplus^{\ell} H \oplus^{m} E_8. \tag{2.3.6}$$

There is no such classification of definite forms, but for a given rank, there are finitely many distinct isomorphism classes. However, the number grows rapidly with respect to the rank. For example, there are over 10^7 distinct even negative-definite forms of rank 32, and over 10^{51} distinct even negative-definite forms of rank 40.

The situation is very different for smooth 4-manifolds due to following theorem of Donaldson.

Theorem 2.3.8 (Donaldson, [Don83])**.** *Let X be a compact, oriented, simply-connected smooth 4-manifold with negative definite intersection form $\big(H^2(X,\mathbb{Z}), Q\big)$. Then Q is $\mathbb{Z}$-diagonalisable, i.e. $Q = -I = \operatorname{diag}(-1, -1, \dots, -1)$.*

Hence by combining Theorem 2.3.2, Theorem 2.3.7, and Theorem 2.3.8 we have a classification of compact, oriented, simply-connected smooth 4-manifolds up to homeomorphism.

Chapter 3

Spin Geometry

In this chapter we give an introduction to spin geometry. However, we only present enough material that is required for our needs, focusing only on the case when $n = 4$. We refer to [LM89, Fri00] for a more general approach.

We follow the exposition presented in [Moo01].

3.1 The Spin and Spin$^{\mathrm{c}}$ Groups

We can identity $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, which have a representation as complex 2×2 matrices. Hence we have the map

$$\kappa : \mathbb{R}^4 \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2), \tag{3.1.1}$$

defined by

$$\kappa(x) = \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \tag{3.1.2}$$

where $x = (a, b, c, d) \in \mathbb{R}^4$. As $\det \kappa(x) = 0$ if, and only if, $x = 0$; κ is injective. Hence, if $V = \operatorname{im} \kappa$ then $\kappa : \mathbb{R}^4 \to V$ is an isomorphism. The subspace V is generated by the matrices

$$\kappa(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \kappa(e_2) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \kappa(e_3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \kappa(e_4) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \tag{3.1.3}$$

where e_1, e_2, e_3, e_4 is the standard basis of $\mathbb{R}^4$. Note that for $x \in \mathbb{R}^4$,

$$\det \kappa(x) = a^2 + b^2 + c^2 + d^2 = |x|^2 = \langle x, x \rangle, \tag{3.1.4}$$

where $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{R}^4$. Thus, it follows that the unit sphere S^3 in $\mathbb{R}^4$ can be identified with the special unitary group SU(2).

Definition 3.1.1. The 4-dimensional **spin group** is the direct product of two copies of SU(2), i.e.

$$\mathrm{Spin}(4) = \mathrm{SU}_+(2) \times \mathrm{SU}_-(2) = \left\{ \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} \mid A_\pm \in \mathrm{SU}_\pm(2) \right\}. \tag{3.1.5}$$

An element in Spin(4) is denoted by (A_+, A_-), where $A_\pm \in \mathrm{SU}_\pm(2)$.

We have a (real) representation of the Spin(4) group

$$\begin{gathered} \rho : \mathrm{Spin}(4) \to \mathrm{GL}(V), \\ \rho(A_+, A_-)(\kappa(x)) = A_-\kappa(x)(A_+)^{-1}. \end{gathered} \tag{3.1.6}$$

As both A_+ and A_- have determinant 1,

$$\left\langle A_-\kappa(x)(A_+)^{-1}, A_-\kappa(x)(A_+)^{-1} \right\rangle = \det(A_-\kappa(x)(A_+)^{-1}) = \det \kappa(x) = \langle x, x \rangle\,. \tag{3.1.7}$$

Thus the representation preserves the inner product, and so ρ maps into $\mathrm{SO}(4) \subset \mathrm{GL}(V)$. We show that ρ actually surjects onto SO(4), and this is done by examining the action of ρ.

Consider the element

$$A = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \in \mathrm{SU}(2). \tag{3.1.8}$$

Then

$$\begin{aligned} \rho(A, I) \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} &= \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \\ &= \begin{bmatrix} e^{i\theta}(a+bi) & e^{-i\theta}(c+di) \\ e^{i\theta}(-c+di) & e^{-i\theta}(a-bi) \end{bmatrix} \end{aligned} \tag{3.1.9}$$

Therefore $\rho(A, I)$ rotates the (a, b) and (c, d) planes by the same angle in the opposite directions. Similarly

$$\begin{aligned} \rho(I, A) \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} &= \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} \\ &= \begin{bmatrix} e^{i\theta}(a+bi) & e^{i\theta}(c+di) \\ e^{-i\theta}(-c+di) & e^{-i\theta}(a-bi) \end{bmatrix}, \end{aligned} \tag{3.1.10}$$

and so $\rho(I, A)$ rotates the (a, b) and (c, d) planes by the same angle in the same direction. Note that every element A in SU(2) is conjugate to a matrix of the form (3.1.8). Hence for a given orthonormal basis (e_1, e_2, e_3, e_4) for V, $\rho(A, I)$ rotates the plane $e_1 \wedge e_2$ and $e_3 \wedge e_4$ in the opposite direction, while $\rho(I, A)$ rotates $e_1 \wedge e_2$ and $e_3 \wedge e_4$ in the same direction.

By the Cartan-Dieudonné theorem, SO(4) is generated by the rotations constructed above, so $\mathrm{SO}(4) \subset \operatorname{im}\rho$ and ρ is surjective. Moreover, as both Spin(4) and SO(4) are Lie groups of dimension 6, ρ induces an isomorphism their respective Lie algebras. Thus $\ker\rho$ is a finite subgroup, and ρ is a covering map. Further, as SU(2) is homeomorphic to S^3, it follows that Spin(4) is simply connected and is a universal cover of SO(4).

As ρ is a universal cover of SO(4), there is a long exact sequence of homotopy groups which includes the sequence

$$1 \longrightarrow \pi_1(\mathrm{SO}(4), I) \longrightarrow \ker\rho \longrightarrow 1, \tag{3.1.11}$$

and so $\ker\rho \cong \pi_1(\mathrm{SO}(4), I) \cong \mathbb{Z}_2$. Explicity, we have $\ker\rho = \{(I, I), (-I, -I)\}$. One of the more important aspects of Spin(4) is that it sits inside a Lie group of dimension 7 called the $\mathrm{Spin}^c(4)$ group.

Definition 3.1.2. The $\mathrm{Spin}^c(4)$ group is defined as

$$\mathrm{Spin}^c(4) = \left\{ \begin{bmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{bmatrix} \mid A_\pm \in \mathrm{SU}_\pm(2), \lambda \in \mathrm{U}(1) \right\}. \tag{3.1.12}$$

We denote an element of $\mathrm{Spin}^c(4)$ by $[A_+, A_-, \lambda]$.

The representation ρ extends to a representation

$$\rho : \mathrm{Spin}^c(4) \to \mathrm{GL}(V), \tag{3.1.13}$$

defined by

$$\rho^c \begin{bmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{bmatrix} \kappa(x) = (\lambda A_-)\kappa(x)(\lambda A_+)^{-1}. \tag{3.1.14}$$

There also exists a group homomorphism $\pi : \mathrm{Spin}^c(4) \to \mathrm{U}(1)$ given by

$$\pi \begin{bmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{bmatrix} = \det(\lambda A_+) = \det(\lambda A_-) = \lambda^2. \tag{3.1.15}$$

Thus the Spin and Spin^c groups fit into the exact sequences

$$\begin{gathered} 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(4) \xrightarrow{\quad\rho\quad} \mathrm{SO}(4) \longrightarrow 1, \\ 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(4) \xrightarrow{\rho^c \times \pi} \mathrm{SO}(4) \times \mathrm{U}(1) \longrightarrow 1, \end{gathered} \tag{3.1.16}$$

and as a corollary we have the Lie algebras of the Spin and Spin^c groups are given by $\mathfrak{spin}(n) = \mathfrak{so}(4)$, and $\mathfrak{spin}^c(4) = \mathfrak{so}(4) \oplus \mathfrak{u}(1)$. These facts will be important when defining Spin and Spin^c connections.

Let W_+ and W_- be two copies of $\mathbb{C}^2$, each with the standard Hermitian metric representing $\mathrm{SU}_\pm(2)$. Then Spin(4) descends to an action on $W_\pm$ by

$$\rho_\pm \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} w_\pm = A_\pm w_\pm. \tag{3.1.17}$$

Similarly, Spin^c(4) acts on $W_\pm$ by

$$\rho^c_\pm \begin{bmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{bmatrix} w_\pm = \lambda A_\pm w_\pm. \tag{3.1.18}$$

The representations ρ^c_+ and ρ^c_- are called the **positive** and **negative spinor representations** associated to the representation ρ^c.

It is easy to see that the actions $\rho^c_\pm$ and $\rho_\pm$ preserve the standard Hermitian metrics on W_+ and W_- (this follows from $A_\pm$ being unitary and $\lambda \in \mathrm{U}(1)$). Hence, there is an isomorphism

$$V \otimes \mathbb{C} \cong \mathrm{Hom}_{\mathbb{C}}(W_+, W_-). \tag{3.1.19}$$

Since unit length vectors in $\mathbb{R}^4$ are represented by unitary matrices in V, they acts as isometries from W_+ to W_-.

3.2 Spin and Spin^c Structures

Let X be a 4-dimensional orientable Riemannian manifold with Riemannian metric g. We wish to define a Spin^c-structure on X, which is a principal Spin^c-bundle on X.

Let $\mathrm{Fr}(X)$ denote the frame bundle of X. It follows that $\mathrm{Fr}(X)$ is a principal $\mathrm{GL}(4,\mathbb{R})$-bundle, and via the Riemannian metric we can reduce the structure group to O(4). Further, as X is orientable, the structure group can be reduced to SO(4). Thus, by the previous section it could be possible to lift the frame bundle to a principal Spin^c(4) (or Spin(4)) bundle.

Definition 3.2.1. A **Spin^c-structure** on an oriented Riemannian manifold X is a lift of the frame bundle $\mathrm{Fr}(X)$ to a principal Spin^c(4)-bundle $P \to X$. A **Spin-structure** on X is a lift of the frame bundle to a Spin(4)-bundle.

Thus P is Spin^c-structure on X if the following diagram commutes:

$$\begin{array}{ccc} P \times \mathrm{Spin}^c(4) & \longrightarrow & P \\ \downarrow & & \downarrow \\ \mathrm{Fr}(X) \times \mathrm{SO}(4) & \longrightarrow & \mathrm{Fr}(X) \\ & & \downarrow \\ & & X \end{array} \tag{3.2.1}$$

It is clear that every Spin-structure determines a Spin^c-structure under the inclusion $\mathrm{Spin}(4) \hookrightarrow \mathrm{Spin}^c(4)$.

Let $\mathcal{S}(X)$ denote the space of all Spin^c-structures on X. Using to the map $\pi : \mathrm{Spin}^c(4) \to \mathrm{U}(1)$ defined in (3.1.15), we can associate to a Spin^c-structure a principal U(1)-bundle L^2. This line bundle is called the **determinant line bundle**. (The notation of L^2 for the determinant line bundle will be explained later in the section.)

We now look at the construction of a Spin^c-structure locally. Given that X is an oriented Riemannian manifold, there is a reduction of the structure group from $\mathrm{GL}(4, \mathbb{R})$ to SO(4). This implies there exists a trivialisation of the tangent bundle TX with an open cover $\{U_\alpha\}_{\alpha \in A}$ of X and transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(n), \tag{3.2.2}$$

satisfying the cocycle condition:

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1. \tag{3.2.3}$$

A Spin^c-structure is therefore a collection of maps

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}^c(4) \tag{3.2.4}$$

which satisfy the cocycle condition, and $\rho^c \circ \tilde{g}_{\alpha\beta} = g_{\alpha\beta}$. Similarly, a Spin-structure is a collection of maps $\tilde{g}_{\alpha\beta}$ which map into the Spin(4) group and satisfy the cocycle condition.

A natural question is whether every oriented Riemannian manifold X admits either a Spin or Spin^c-structure? Looking at the local description, by the work done in section 3.1 it is possible to lift the transition functions for the frame bundle $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{SO}(4)$ to maps $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}^c(4)$. However, a priori the lifts $\tilde{g}_{\alpha\beta}$ do not satisfy the cocycle condition; and hence may not define a Spin^c-structure on X. The obstruction to the lifts to satisfying the cocycle condition is entirely topological, and is described by the vanishing of the second **Stiefel-Whitney class** $w_2(TX) \in H^2(X; \mathbb{Z}_2)$.

Definition 3.2.2. An open covering $\{U_\alpha\}_{\alpha\in A}$ of X is a **good cover** if for every choice $(\alpha_1,\ldots,\alpha_n)$ the intersection

$$U_{\alpha_1}\cap\cdots\cap U_{\alpha_n} \tag{3.2.5}$$

is empty or diffeomorphic to $\mathbb{R}^4$.

Every manifold admits a good cover, for a proof see [BT82, p. 42]. Further, the short exact sequence (3.1.16) of coefficient groups induces a long exact sequence in Čech cohomology which contains

$$\cdots \longrightarrow \check{H}^1(X,\mathrm{Spin}(4)) \longrightarrow \check{H}^1(X,\mathrm{SO}(4)) \longrightarrow \check{H}^2(X,\mathbb{Z}_2) \longrightarrow \cdots . \tag{3.2.6}$$

Theorem 3.2.3. *Let X be an oriented Riemannian manifold. Then*

(I) *X admits a* Spin*-structure if, and only if, $w_2(TX)=0$.*

(II) *X admits a* Spin^c*-structure if, and only if, the second Stiefel-Whitney class $w_2(TX)$ is the reduction mod 2 of an integral class, i.e. $w_2(TX)=c \pmod 2$ for some $c\in H^2(X;\mathbb{Z})$.*

Proof. (I): Let $\{U_\alpha\}_{\alpha\in A}$ be a good cover of X, which trivialises TX. As $U_\alpha\cap U_\beta$ is contractible, each transition function

$$\check{H}^1(X,\mathrm{SO}(n)) \ni g_{\alpha\beta}: U_\alpha\cap U_\beta \to \mathrm{SO}(4), \tag{3.2.7}$$

can be lifted to functions

$$\check{H}^1(X,\mathrm{Spin}(4)) \ni \tilde{g}_{\alpha\beta}: U_\alpha\cap U_\beta \to \mathrm{Spin}(4). \tag{3.2.8}$$

However, these lifts may not satisfy the cocycle condition. Define

$$\eta_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}: U_\alpha\cap U_\beta\cap U_\gamma \to \mathrm{Spin}(4). \tag{3.2.9}$$

Then $\rho(\eta_{\alpha\beta\gamma})=1$, so $\eta_{\alpha\beta\gamma}\in\ker\rho=\mathbb{Z}_2$, which implies $\eta_{\alpha\beta\gamma}\in\check{H}^2(X,\mathbb{Z}_2)$. By exactness of (3.2.6), for the $g_{\alpha\beta}$ to lift to transition functions $\tilde{g}_{\alpha\beta}$, we require $\eta_{\alpha\beta\gamma}=1$. Let δ

denote the Čech differential. Then, as $\mathbb{Z}_2$ is abelian,

$$\begin{aligned}(\delta\eta)_{\alpha\beta\gamma\varepsilon} &= \eta_{\beta\gamma\varepsilon}\eta_{\alpha\gamma\varepsilon}^{-1}\eta_{\alpha\beta\varepsilon}\eta_{\alpha\beta\gamma}^{-1} \\ &= \eta_{\beta\gamma\varepsilon}\eta_{\alpha\beta\varepsilon}\eta_{\alpha\gamma\varepsilon}^{-1}\eta_{\alpha\beta\gamma}^{-1} \\ &= \tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\varepsilon}\tilde{g}_{\varepsilon\beta}\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\varepsilon}\tilde{g}_{\varepsilon\alpha}(\tilde{g}_{\alpha\gamma}\tilde{g}_{\gamma\varepsilon}\tilde{g}_{\varepsilon\alpha})^{-1}(\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha})^{-1} \\ &= \tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\varepsilon}\tilde{g}_{\varepsilon\beta}\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\varepsilon}\tilde{g}_{\varepsilon\alpha}(\tilde{g}_{\alpha\varepsilon}\tilde{g}_{\varepsilon\gamma}\tilde{g}_{\gamma\alpha})(\tilde{g}_{\alpha\gamma}\tilde{g}_{\gamma\beta}\tilde{g}_{\beta\alpha}) \\ &= \tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\varepsilon}\tilde{g}_{\varepsilon\beta}\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\varepsilon}\tilde{g}_{\varepsilon\gamma}\tilde{g}_{\gamma\beta}\tilde{g}_{\beta\alpha} \\ &= \tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\varepsilon}\tilde{g}_{\varepsilon\beta}\tilde{g}_{\alpha\beta}\eta_{\beta\varepsilon\gamma}\tilde{g}_{\alpha\beta} \\ &= \eta_{\beta\varepsilon\gamma}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\varepsilon}\tilde{g}_{\varepsilon\beta} \\ &= \eta_{\beta\varepsilon\gamma}\eta_{\beta\varepsilon\gamma}^{-1} \\ &= 1,\end{aligned} \tag{3.2.10}$$

which shows that $\eta_{\alpha\beta\gamma}$ is a Čech cocycle. Since the cover $\{U_\alpha\}_{\alpha\in A}$ is good, $\eta_{\alpha\beta\gamma}$ defines a class in $H^2(X;\mathbb{Z}_2)$. The cohomology class is precisely the second Stiefel-Whitney class $w_2(TX)$.

If X has a Spin-structure, then we can choose $g_{\alpha\beta}$'s that satisfy the cocycle condition. Hence all the $\eta_{\alpha\beta\gamma}$ are trivial and so $w_2(TX) = 0$. Conversely, suppose $w_2(TX) = 0$. Then $\eta_{\alpha\beta\gamma}$ is a coboundary, i.e. there exists maps

$$\eta_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}_2 \tag{3.2.11}$$

such that

$$(\delta\eta)_{\alpha\beta\gamma} = \eta_{\alpha\beta}\eta_{\beta\gamma}\eta_{\gamma\alpha} = \eta_{\alpha\beta\gamma}. \tag{3.2.12}$$

We show that $\{\eta_{\alpha\beta}\tilde{g}_{\alpha\beta}\}$ satisfy the cocycle condition, and hence defines a Spin-structure on X. Indeed, a quick calculation gives

$$(\eta_{\alpha\beta}\tilde{g}_{\alpha\beta})(\eta_{\beta\gamma}\tilde{g}_{\beta\gamma})(\eta_{\gamma\alpha}\tilde{g}_{\gamma\alpha}) = \eta_{\alpha\beta\gamma}\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = \eta_{\alpha\beta\gamma}^2 = 1, \tag{3.2.13}$$

as $\eta_{\alpha\beta\gamma} \in \mathbb{Z}_2$.

(II): In the case of Spin^c-structures, we are interested in lifts of the functions $g_{\alpha\beta}$ to $\tilde{g}_{\alpha\beta}$ which map into Spin$^c(4)$ and satisfy the cocycle condition. By the working in (I), for a good cover we can always find a lift, but it may not satisfy the cocycle condition. Assume that $w_2(TX) = c \pmod 2$ for some $c \in H^2(X;\mathbb{Z})$, and let $\eta_{\alpha\beta\gamma}$ be the Čech cocycle representing $w_2(TX)$. There is a lift of $\eta_{\alpha\beta\gamma}$ to

$$\tilde{\eta}_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z} \tag{3.2.14}$$

such that

$$\exp(i\pi\tilde{\eta}_{\alpha\beta\gamma}) = \eta_{\alpha\beta\gamma}, \tag{3.2.15}$$

and the cocycle condition is now

$$\tilde{\eta}_{\beta\gamma\varepsilon} - \tilde{\eta}_{\alpha\gamma\varepsilon} + \tilde{\eta}_{\alpha\beta\varepsilon} - \tilde{\eta}_{\alpha\beta\gamma} = 0, \tag{3.2.16}$$

which can be seen by exponentiating (3.2.12). Suppose that $\{\psi_\alpha\}_{\alpha\in A}$ is a partition of unity subordinate to the good cover $\{U_\alpha\}_{\alpha\in A}$. Define

$$f_{\beta\gamma} : U_\beta \cap U_\gamma \to \mathbb{R}, \tag{3.2.17}$$

by

$$f_{\beta\gamma} = \sum_{\alpha\in A} \psi_\alpha \tilde{\eta}_{\alpha\beta\gamma}. \tag{3.2.18}$$

Then it follows that

$$\begin{aligned} f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} &= \sum_{\varepsilon\in A} \psi_\varepsilon \tilde{\eta}_{\varepsilon\alpha\beta} + \sum_{\varepsilon\in A} \psi_\varepsilon \tilde{\eta}_{\varepsilon\beta\gamma} + \sum_{\varepsilon\in A} \psi_\varepsilon \tilde{\eta}_{\varepsilon\gamma\alpha} \\ &= \sum_{\varepsilon\in A} \psi_\varepsilon (\tilde{\eta}_{\varepsilon\alpha\beta} + \tilde{\eta}_{\varepsilon\beta\gamma} + \tilde{\eta}_{\varepsilon\gamma\alpha}) \\ &= \sum_{\varepsilon\in A} \psi_\varepsilon \tilde{\eta}_{\alpha\beta\gamma} = \tilde{\eta}_{\alpha\beta\gamma}. \end{aligned} \tag{3.2.19}$$

Set

$$h_{\alpha\beta} = \exp(\pi i f_{\alpha\beta}) : U_\alpha \cap U_\beta \to \mathrm{U}(1), \tag{3.2.20}$$

then it follows that

$$h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} = \exp\big(i\pi(f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha})\big) = \exp(i\pi\tilde{\eta}_{\alpha\beta\gamma}) = \eta_{\alpha\beta\gamma}. \tag{3.2.21}$$

Thus, the maps

$$h_{\alpha\beta}\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}^c(4) \tag{3.2.22}$$

satisfy the cocycle condition, and therefore defines a Spin^c-structure on X.

Conversely, we have a geometric formulation of a Spin^c-structure as a complex line bundle L over X such that $TX \oplus L$ has a Spin-structure. So suppose X has a Spin^c-structure given by a line bundle L, such that $TX \oplus L$ is spin. Then by part (I) and the Whitney product formula,

$$0 = w_2(TX \oplus L) = w_2(TX) + w_1(TX)w_1(L) + w_2(L). \tag{3.2.23}$$

As both TX and L are orientable, $w_1(TX)w_1(L) = 0$ which implies $w_2(TX)+w_2(L) = 0$. As these are both $\mathbb{Z}_2$ classes, $w_2(TX) = w_2(L)$. As $w_2(L)$ has an integral lift given by the first Chern class $c_1(L)$, it follows that

$$w_2(TX) = c_1(L) \pmod 2. \tag{3.2.24}$$

□

Remark 5. As X is Spin if, and only if, $w_2(TX) = 0$; we can view the existence of Spin-structures as the obstruction of 'higher orientations' of X. ⧫

From Theorem 3.2.3 is clear that not every 4-manifold admits a Spin-structure. For example, $\mathbb{C}P^2$ has non-trivial second Stiefel-Whitney class and thus is not Spin. However, a result of Hirzebruch and Hopf in [HH58], shows that every compact, oriented 4-manifold has a Spin^c-structure.

Theorem 3.2.4. *Every compact, oriented 4-manifold X admits a* Spin^c*-structure.*

Proof. We only prove the case when X is simply-connected, as this is the only case needed for Donaldson's theorem. Consider the short exact sequence of groups:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}_2 \longrightarrow 1, \tag{3.2.25}$$

where the first arrow is multiplication by 2, and φ is reduction mod 2. This induces a long exact sequence in cohomology which includes

$$\cdots \longrightarrow H^2(X;\mathbb{Z}) \longrightarrow H^2(X;\mathbb{Z}) \xrightarrow{\varphi_*} H^2(X;\mathbb{Z}_2) \xrightarrow{\beta} H^3(X;\mathbb{Z}) \longrightarrow \cdots, \tag{3.2.26}$$

where β is the associated Bockstein, or connecting homomorphism. We show that $w_2(TX) \in \operatorname{im}(\varphi_*) \subset H^2(X;\mathbb{Z}_2)$, so that Theorem 3.2.3 implies X has a Spin^c-structure. By exactness of (3.2.26), $\operatorname{im}(\varphi_*) = \ker(\beta)$. However, X is simply-connected, and so $H^1(X;\mathbb{Z}) = 0$. As X is compact, Poincaré duality then implies $H^3(X;\mathbb{Z}) = 0$. Hence $\operatorname{im}(\varphi_*) = \ker(\beta) = H^2(X;\mathbb{Z}_2)$, and $w_2(TX) \in \operatorname{im}(\varphi_*)$ as required. □

Remark 6. A stronger result has been proved in [TV94], which shows that all 4-manifolds admit Spin^c-structures. ⧫

We now explain the local construction of making a Spin manifold Spin^c. Suppose that X is a Spin manifold, and so there is a collection of transition functions

$$\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(4). \tag{3.2.27}$$

Let L be a complex line bundle over X with a Hermitian metric. Then the transition functions for L are given by

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{U}(1). \tag{3.2.28}$$

We form the Spin^c-structure on X by taking the transition functions

$$h_{\alpha\beta}\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}^c(4). \tag{3.2.29}$$

Now suppose that X is a manifold with a Spin^c-structure. Through the map $\pi : \mathrm{Spin}^c(4) \to \mathrm{U}(1)$ we can form the complex line bundle with transition functions

$$\pi \circ \tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{U}(1). \tag{3.2.30}$$

This line bundle is precisely the determinant line bundle defined previously. The notation of L^2 for the determinant line bundle now makes sense, it is the square of line bundle used to construct a Spin^c-structure on a Spin manifold.

Let L^2 be the determinant line bundle of a Spin^c-structure P on X. Recall that we have the parametrisation of $H^2(X;\mathbb{Z})$ by complex line bundles over X. We define a group action on the space of Spin^c-structures on X by $H^2(X;\mathbb{Z})$ as follows. Let $\tilde{g}_{\alpha\beta}$ be the transition functions for the Spin^c-structure, and $h_{\alpha\beta}$ the transition functions for a complex line bundle E over X. We form the new Spin^c-structure over X given by the transition functions $h_{\alpha\beta}\tilde{g}_{\alpha\beta}$ which is equivalent to $P \otimes E$. Taking the determinant line bundle of $P \otimes E$ we see that L^2 is twisted by the cocycle $h^2_{\alpha\beta}$, which is the bundle

$$L^2 \otimes E^2. \tag{3.2.31}$$

Taking the first Chern class of this bundle, we get

$$c_1(L^2 \otimes E^2) = c_1(L^2) + 2c_1(E). \tag{3.2.32}$$

This shows that if $H^2(X,\mathbb{Z})$ is torsion-free, then Spin^c-structures are parametrised by $H^2(X;\mathbb{Z})$. Equivalently, we say that the space of Spin^c-structures on X is a **$H^2(X,\mathbb{Z})$-torsor**.

Recall that for the $\mathrm{Spin}^c(4)$ group we had the representations $\rho^c_\pm$ onto the complex vector spaces $W_\pm \cong \mathbb{C}^2$. Using these representations, we form two complex vector bundles with the transition functions

$$\rho^c_\pm \circ \tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{U}(2). \tag{3.2.33}$$

Definition 3.2.5. Let X be a 4-manifold with a Spin^c-structure.

(i) The bundle associated to the the transition functions $\rho^c_+ \circ \tilde{g}_{\alpha\beta}$ is denoted by S^+ and is called the **complex positive spinor bundle**.

(ii) The bundle associated to the the transition functions $\rho^c_- \circ \tilde{g}_{\alpha\beta}$ is denoted by S^- and is called the **complex negative spinor bundle**.

(iii) The **complex spinor bundle** S is the direct sum $S = S^+ \oplus S^-$.

Sections $\psi \in \Gamma(S^+)$ are referred to as **positive spinors**, while sections $\psi \in \Gamma(S^-)$ are **negative spinors**.

Note that both S^+ and S^- are equipped with Hermitian metrics. Similar to the case in section 3.1, we see that there is an isomorphism

$$TX \otimes \mathbb{C} \cong \mathrm{Hom}(S^+, S^-). \tag{3.2.34}$$

3.3 Clifford Multiplication

In this section X is a Riemannian 4-manifold with Riemannian metric g, and X has a Spin^c-structure. Let S be the corresponding spinor bundle.

Definition 3.3.1. Clifford multiplication is a linear map

$$\gamma : TX \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S), \tag{3.3.1}$$

such that for all $v \in TX \otimes \mathbb{C}$:

(i) $\gamma(v)$ maps $S^{\pm}$ to $S^{\mp}$.

(ii) $\gamma(v)^2 := \gamma(v)\gamma(v) = -|v|^2 \,\mathrm{Id}$.

(iii) $\gamma(v)^* = -\gamma(\bar{v})$.

The goal of this section is prove the existence of the map γ. This is done in a standard way: First we construct the map locally on a vector space, then globalise via an associated bundle construction.

Recall the representation defined in Section 3.1

$$\kappa : \mathbb{R}^4 \otimes \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(W_+, W_-), \tag{3.3.2}$$

where $W_{\pm}$ are two copies of $\mathbb{C}^2$ with the standard Hermitian inner product. By considering the adjoint of $\kappa(x)$, we have a map

$$-\kappa(x)^* = -\overline{\kappa(x)}^T \in \mathrm{Hom}_{\mathbb{C}}(W_-, W_+). \tag{3.3.3}$$

Let $W = W_+ \oplus W_-$ and define

$$\gamma : \mathbb{R}^4 \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(W), \tag{3.3.4}$$

by

$$\gamma(x) = \begin{bmatrix} 0 & -\overline{\kappa(x)}^T \\ \kappa(x) & 0 \end{bmatrix}. \tag{3.3.5}$$

It is clear that γ maps $W_\pm$ to $W_\mp$, and by the properties of block matrices γ is also linear. Note

$$\begin{aligned}\gamma(x)^2 &= \begin{bmatrix} 0 & -\overline{\kappa(x)}^T \\ \kappa(x) & 0 \end{bmatrix} \begin{bmatrix} 0 & -\overline{\kappa(x)}^T \\ \kappa(x) & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\overline{\kappa(x)}^T \kappa(x) & 0 \\ 0 & -\kappa(x)\overline{\kappa(x)}^T \end{bmatrix} = \big(-\det \kappa(x)\big) I \\ &= -|x|^2 I. \end{aligned} \tag{3.3.6}$$

We now investigate the action of γ on the standard basis of $\mathbb{R}^4$. If e_1, e_2, e_3, e_4 is the standard basis of $\mathbb{R}^4$, then the corresponding matrices $\kappa(e_i)$ were given in (3.1.3). Therefore, the corresponding γ matrices are

$$\begin{aligned} \gamma(e_1) &= \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \gamma(e_2) = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \\ \gamma(e_3) &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \gamma(e_4) = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{3.3.7}$$

It is easy to see that the matrices $\gamma(e_i)$ are skew-Hermitian, and satisfy the following property:

$$\gamma(e_i)\gamma(e_j) + \gamma(e_j)\gamma(e_i) = \delta_{ij} I, \tag{3.3.8}$$

known as the **Clifford relation**. Therefore, this leads to the following definition.

Definition 3.3.2. $\mathrm{End}_{\mathbb{C}}(W)$ is called the **complexified Clifford algebra** of $(\mathbb{R}^4, \langle\ ,\ \rangle)$. From now on, we denote $\mathbb{C}\mathrm{l}(\mathbb{R}^4) = \mathrm{End}_{\mathbb{C}}(W)$.

In fact, it is possible to define the map γ via the matrices $\gamma(e_i)$, as we now show. Let $x \in \mathbb{R}^4 \otimes \mathbb{C}$, then $x = \sum_{i=1}^4 x^i e_i$ for $x^i \in \mathbb{C}$. An alternate definition for γ is given by extending linearly:

$$\gamma(x) = \sum_{i=1}^{4} x^i \gamma(e_i). \tag{3.3.9}$$

From this it easy to prove that the map γ constructed satisfies the requirements stated in Definition 3.3.1.

Proposition 3.3.3. *The map* $\gamma : \mathbb{R}^4 \otimes \mathbb{C} \to \mathbb{C}\mathrm{l}(\mathbb{R}^4)$ *satisfies the following.*

(I) $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2\langle v, w\rangle I$, *for all* $v, w \in \mathbb{R}^4 \otimes \mathbb{C}$.

(II) $\gamma(v)^* = -\gamma(\bar{v})$, *for all* $v \in \mathbb{R}^4 \otimes \mathbb{C}$.

Proof. (I): Suppose $v = \sum_{i=1}^4 v^i e_i$ and $w = \sum_{j=1}^4 w^j e_j$ are two elements of $\mathbb{R}^4 \otimes \mathbb{C}$. Then

$$\begin{aligned}\gamma(v)\gamma(w) &= \Big(\sum_{i=1}^4 v^i \gamma(e_i)\Big)\Big(\sum_{j=1}^4 w^j \gamma(e_j)\Big) \\ &= \sum_{i=1}^4 \sum_{j=1}^4 v^i w^j \gamma(e_i)\gamma(e_j) \\ &= -\sum_{i=1}^4 v^i w^i \, I = -\langle v, w\rangle I\end{aligned} \tag{3.3.10}$$

A similar calculation gives $\gamma(w)\gamma(v) = -\langle w, v\rangle I = -\langle v, w\rangle I$, which implies

$$\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2\langle v, w\rangle I. \tag{3.3.11}$$

(II): If $v = \sum_{i=1}^4 v^i e_i$, then $\bar{v} = \sum_{i=1}^4 \bar{v}^i e_i$, and

$$\begin{aligned}\gamma(v)^* &= \Big(\sum_{i=1}^4 v^i \gamma(e_i)\Big)^* = \sum_{i=1}^4 \big(v^i \gamma(e_i)\big)^* \\ &= \sum_{i=1}^4 \overline{\big(v^i \gamma(e_i)\big)}^T = \sum_{i=1}^4 \bar{v}^i \overline{\gamma(e_i)}^T \\ &= -\sum_{i=1}^4 \bar{v}^i \gamma(e_i) = -\gamma(\bar{v}).\end{aligned} \tag{3.3.12}$$

□

Corollary 3.3.3.1. *If* $|v| = 1$*, then* $\gamma(v)$ *is a unitary transformation.*

We now investigate the algebraic structure of the complexified Clifford algebra. It follows that $\mathbb{C}l(\mathbb{R}^4)$ has a basis given by

$$\begin{gathered} I, \quad \gamma(e_i), \quad \gamma(e_i)\gamma(e_j) \quad \text{for } i < j, \quad \gamma(e_i)\gamma(e_j)\gamma(e_k) \quad \text{for } i < j < k, \\ \gamma(e_1)\gamma(e_2)\gamma(e_3)\gamma(e_4) \end{gathered} \tag{3.3.13}$$

Now, it follows from (3.3.8) that if $i \neq j$, then

$$\gamma(e_i)\gamma(e_j) = -\gamma(e_j)\gamma(e_i). \tag{3.3.14}$$

Therefore we can view $\bigwedge^2(\mathbb{R}^4 \otimes \mathbb{C})$ as the complex subspace of $\mathbb{C}\mathrm{l}(\mathbb{R}^4)$ generated by the products $\gamma(e_i)\gamma(e_j)$. More generally, (3.3.13) gives an isomorphism

$$\mathbb{C}\mathrm{l}(\mathbb{R}^4) = \bigoplus_{i=0}^{4} \bigwedge^i (\mathbb{R}^4 \otimes \mathbb{C}), \tag{3.3.15}$$

as vector spaces. Hence the complexified Clifford algebra can be viewed as the complexified exterior algebra, just with a different product.

Now, just like the situation in Hodge theory, the inner product $\bigwedge^2(\mathbb{R}^4 \otimes \mathbb{C})$ has a decomposition into self-dual and anti-self-dual parts,

$$\bigwedge^2(\mathbb{R}^4 \otimes \mathbb{C}) = \bigwedge^2_+(\mathbb{R}^4 \otimes \mathbb{C}) \oplus \bigwedge^2_-(\mathbb{R}^4 \otimes \mathbb{C}). \tag{3.3.16}$$

The self-dual part $\bigwedge^2_+(\mathbb{R}^4 \otimes \mathbb{C})$ is generated by

$$e_1 \wedge e_2 + e_3 \wedge e_4, \quad e_1 \wedge e_3 - e_2 \wedge e_4, \quad e_1 \wedge e_4 + e_2 \wedge e_3. \tag{3.3.17}$$

The corresponding elements in the Clifford algebra are

$$\begin{aligned}
\gamma(e_1)\gamma(e_2) + \gamma(e_3)\gamma(e_4) &= \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \\
&= \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -2i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{3.3.18}$$

Similarly,

$$\begin{aligned}\gamma(e_1)\gamma(e_3) - \gamma(e_2)\gamma(e_4) &= \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \gamma(e_1)\gamma(e_4) + \gamma(e_2)\gamma(e_3) &= \begin{bmatrix} 0 & -2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned} \tag{3.3.19}$$

Hence it follows that $\bigwedge^2_+(\mathbb{R}^4 \otimes \mathbb{C})$ can be identified with the space of trace-free endomorphisms of W_+, which is precisely the Lie algebra of $\mathrm{SU}_+(2)$. Similarly, $\bigwedge^2_-(\mathbb{R}^4 \otimes \mathbb{C})$ is the space of trace-free endomorphisms of W_-.

We define a quadratic map $q : W_+ \to \bigwedge^2_+(\mathbb{R}^4 \otimes \mathbb{C})$ as follows. Let $\psi \in W_+$. As W_+ is finite dimensional, there is an isomorphism $W_+ \cong (W_+)^*$; and by the Riesz representation theorem every $\psi^* \in (W_+)^*$ is defined by $\psi^* = \langle \psi, - \rangle$. We set

$$q(\psi) = \psi \otimes \psi^* - \frac{|\psi|^2}{2} \operatorname{Id}. \tag{3.3.20}$$

It is clear that $q(\psi) \in W_+ \otimes (W_+)^* \cong \operatorname{End}_{\mathbb{C}}(W_+)$, and we show $q(\psi)$ defines an element in $\bigwedge^2_+(\mathbb{R}^4 \otimes \mathbb{C})$.

Let $\psi \in W_+$ and consider its representation in components

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \tag{3.3.21}$$

Then

$$\psi \otimes \psi^* = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \begin{bmatrix} \bar{\psi}_1 & \bar{\psi}_2 \end{bmatrix} = \begin{bmatrix} |\psi_1|^2 & \psi_1 \bar{\psi}_2 \\ \bar{\psi}_1 \psi_2 & |\psi_2|^2 \end{bmatrix} \tag{3.3.22}$$

and so

$$q(\psi) = \frac{1}{2} \begin{bmatrix} |\psi_1|^2 - |\psi_2|^2 & 2\psi_1 \bar{\psi}_2 \\ 2\bar{\psi}_1 \psi_2 & |\psi_2|^2 - |\psi_1|^2 \end{bmatrix} \tag{3.3.23}$$

Hence $q(\psi)$ is a trace-free endomorphism of W_+, and thus by the previous discussion defines an element in $\bigwedge^2_+(\mathbb{R}^4 \otimes \mathbb{C})$.

Hence $q(\psi)$ defines an complex self-dual two-form. However, we strengthen this and show that $q(\psi)$ is a purely imaginary self-dual two-form.

Proposition 3.3.4. *$q(\psi)$ is a purely imaginary self-dual two form.*

Proof. To show that $q(\psi)$ is imaginary, it suffices to show that $iq(\psi)$ is real.

Recall that $\mathbb{R}^4$ is represented by complex 2×2 matrices of the form

$$\begin{bmatrix} u & v \\ -\overline{v} & \overline{u} \end{bmatrix}. \tag{3.3.24}$$

So $L \in \mathrm{End}(W_+)$ is the representative of an element of $\mathbb{R}^4$ if

$$\mathrm{tr}(L) \in \mathbb{R}, \quad \text{and} \quad L + L^* = \mathrm{tr}(L)I. \tag{3.3.25}$$

Now, by definition $\mathrm{tr}\, q(\psi) = 0$ for all $\psi \in W_+$. Hence, to show that $iq(\psi)$ represents a real-valued self-dual 2-form we need

$$0 = iq(\psi) + (iq(\psi))^* = iq(\psi) - i\overline{q(\psi)}^T \tag{3.3.26}$$

or equivalently

$$q(\psi) = \overline{q(\psi)}^T.$$

But this follows immediately as both I and $\psi \otimes \psi^*$ are Hermitian matrices, and $q(\psi)$ is also Hermitian. □

The groups Spin(4) and $\mathrm{Spin}^c(4)$ act on $\mathrm{End}(W)$ via the adjoint representation, denoted by Ad and Ad^c, which is just conjugation by an element of the respective groups. Explicitly, for $T \in \mathrm{End}(W)$ and $p = (A_+, A_-) \in \mathrm{Spin}(4)$,

$$\mathrm{Ad}(p)(T) = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} T \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}^{-1} = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} T \begin{bmatrix} A_+^{-1} & 0 \\ 0 & A_-^{-1} \end{bmatrix}. \tag{3.3.27}$$

Similarly, if $p = [A_+, A_-, \lambda] \in \mathrm{Spin}^c(4)$,

$$\mathrm{Ad}^c(p)(T) = \begin{bmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{bmatrix} T \begin{bmatrix} (\lambda A_+)^{-1} & 0 \\ 0 & (\lambda A_-)^{-1} \end{bmatrix}. \tag{3.3.28}$$

Note that as Spin(4) and $\mathrm{Spin}^c(4)$ act by conjugation, the action preserves the basis (3.3.13), and so preserves the direct sum decomposition given in (3.3.15).

Now suppose that (X, g) is a Riemannian 4-manifold with a Spin(4), or $\mathrm{Spin}^c(4)$-structure. We use the adjoint representations to construct a complex vector bundle over X, whose fiber is the complexified Clifford algebra $\mathrm{End}(W)$. This bundle is denoted as $\mathrm{Cl}(X)$, and is called the **Clifford bundle**.

As the adjoint action preserves the direct sum decomposition, we have the following decomposition of the Clifford bundle

$$\mathrm{Cl}(X) = \bigoplus_{i=0}^{4} \bigwedge^{k} TX \otimes \mathbb{C} \cong \bigoplus_{i=0}^{4} \bigwedge^{k} T^*X \otimes \mathbb{C}. \tag{3.3.29}$$

Hence, we can view the Clifford bundle as being the exterior algebra of the cotangent bundle, with a modified product.

We would like the maps γ and q defined previously to extend to bundle maps on $\mathrm{Cl}(X)$. To be able to do this, they need to be equivariant with respect to the relevant group actions. This turns out to be the case, as we now show.

Proposition 3.3.5. *If $v \in \mathbb{R}^4 \otimes \mathbb{C}$, $\psi \in W_+$, and $p = [A_+, A_-, \lambda] \in \mathrm{Spin}^c(4)$. Then*

(I) $\mathrm{Ad}^c(p)\gamma(v) = \gamma(\rho^c(v))$,

(II) $\mathrm{Ad}^c(p)q(\psi) = q(\rho^c_+(\psi))$,

where ρ^c and ρ^c_+ are the spin representations defined in Section 3.1.

Proof. (I): Using (3.3.5),

$$\begin{aligned} \mathrm{Ad}^c(p)\gamma(v) &= \begin{bmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{bmatrix} \begin{bmatrix} 0 & -\overline{\kappa(v)}^T \\ \kappa(v) & 0 \end{bmatrix} \begin{bmatrix} (\lambda A_+)^{-1} & 0 \\ 0 & (\lambda A_-)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{bmatrix} \begin{bmatrix} 0 & -\overline{\kappa(v)}^T (\lambda A_-)^{-1} \\ \kappa(v)(\lambda A_+)^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (\lambda A_+)(-\overline{\kappa(v)}^T)(\lambda A_-)^{-1} \\ (\lambda A_-)\kappa(v)(\lambda A_+)^{-1} & 0 \end{bmatrix} \end{aligned} \tag{3.3.30}$$

Now, as $\lambda A_\pm \in \mathrm{SU}(2)$, $(\lambda A_\pm)^{-1} = \overline{(\lambda A_\pm)}^T = \overline{\lambda A_\pm}^T$. Hence

$$\begin{aligned} (\lambda A_+)(-\overline{\kappa(v)}^T)(\lambda A_-)^{-1} &= (\lambda A_+)(\overline{-\kappa(v)}^T)\overline{\lambda A_-}^T \\ &= \overline{(\lambda A_+)^{-1}}^T \overline{\lambda A_-(-\kappa(v))}^T \\ &= \overline{(\lambda A_-)(-\kappa(v))(\lambda A_+)^{-1}}^T. \end{aligned} \tag{3.3.31}$$

Therefore

$$\begin{aligned} \mathrm{Ad}^c(p)\gamma(v) &= \begin{bmatrix} 0 & \overline{(\lambda A_-)(-\kappa(v))(\lambda A_+)^{-1}}^T \\ (\lambda A_-)\kappa(v)(\lambda A_+)^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \overline{\rho^c(p)(-\kappa(v))}^T \\ \rho^c(p)\kappa(v) & 0 \end{bmatrix} = \gamma(\rho^c(v)). \end{aligned} \tag{3.3.32}$$

(II): Identify $\mathrm{End}(W_+)$ with the space of trace-free endomorphisms of W lying in the upper 2×2 block. Then

$$\begin{bmatrix} q(\psi) & 0 \\ 0 & 0 \end{bmatrix} \tag{3.3.33}$$

is an element of $\mathrm{End}(W)$, and the action of $\mathrm{Spin}^c(4)$ on this element is then

$$\begin{aligned} \mathrm{Ad}^c(p)q(\psi) &= \begin{bmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{bmatrix} \begin{bmatrix} q(\psi) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\lambda A_+)^{-1} & 0 \\ 0 & (\lambda A_-)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{bmatrix} \begin{bmatrix} q(\psi)(\lambda A_+)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (\lambda A_+)q(\psi)(\lambda A_+)^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.3.34}$$

On the other hand, recall that the dual of element $\psi \in W_+$ is given by the conjugate transpose. Hence, as $\lambda A_+ \in \mathrm{SU}(2)$ we have

$$\begin{aligned} q(\rho_+^c \psi) &= (\lambda A_+ \psi) \otimes (\lambda A_+ \psi)^* - \frac{|\lambda A_+ \psi|^2}{2} I \\ &= \lambda A_+ \psi \overline{(\lambda A_+ \psi)}^T - \frac{|\psi|^2}{2} I \\ &= \lambda A_+ \psi \overline{\psi}^T \overline{(\lambda \psi)}^T - \frac{|\psi|^2}{2} I \\ &= (\lambda A_+)\psi \otimes \psi^* (\lambda A_+)^{-1} - (\lambda A_+) \frac{|\psi|^2}{2} I (\lambda A_+)^{-1} \\ &= \lambda A_+ \Big(\psi \otimes \psi^* - \frac{|\psi|^2}{2} I \Big) (\lambda A_+)^{-1} \\ &= (\lambda A_+) q(\psi) (\lambda A_+)^{-1}. \end{aligned} \tag{3.3.35}$$

Therefore

$$\mathrm{Ad}^c(p)q(\psi) = \begin{bmatrix} (\lambda A_+)q(\psi)(\lambda A_+)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} q(\rho_+^c(\psi)) & 0 \\ 0 & 0 \end{bmatrix} = q(\rho_+^c(\psi)).$$

□

Hence, we see that the maps γ and q extend to bundle maps. Explicitly we have the maps

$$\begin{aligned} \gamma &: TX \otimes \mathbb{C} \to \mathrm{End}(S), \\ q &: \Gamma(S^+) \to \Omega_+^2(X, \mathbb{C}). \end{aligned} \tag{3.3.36}$$

Further the maps inherit the same properties as shown in Proposition 3.3.3. Thus we have constructed the Clifford multiplication map defined in Definition 3.3.1.

Moreover, by Proposition 3.3.4, we see that q maps into the space of purely imaginary self-dual two forms, and so q actually defines a map

$$q : \Gamma(S^+) \to i\Omega^2_+(X). \tag{3.3.37}$$

We may also extend the Clifford multiplication map γ to differential forms via the Riemannian metric g. Suppose that $\{e_i\}_{i=1}^4$ is a local orthonormal frame for TX, and let $\{\theta^i\}_{i=1}^4$ be the corresponding dual frame. Then every 1-form $\omega \in \Omega^1(X)$, is a linear combination $\omega = \sum_{i=1}^4 \omega_i \theta^i$ which, by lowering the index, corresponds uniquely to the vector field $\omega^\# = \sum_{i=1}^4 \tilde{\omega}^i e_i$, where $\tilde{\omega}^i = g^{ij}\omega_j$. We define Clifford multiplication by the 1-form ω as

$$\gamma(\omega) \coloneqq \gamma(\omega^\#) = \sum_{i=1}^4 \tilde{\omega}^i \gamma(e_i). \tag{3.3.38}$$

3.4 Spin^c Connections

Let (X, g) be an oriented compact Riemannian 4-manifold and let $\mathrm{Fr}(X)$ be the orthonormal frame bundle of X. Let ∇ denote the Levi-Civita connection for TX. We can also view ∇ as a connection ω on the frame bundle with values in $\mathfrak{so}(4)$, the Lie algebra of SO(4), i.e. ω is a 1-form

$$\omega : T\,\mathrm{Fr}(X) \to \mathfrak{so}(4). \tag{3.4.1}$$

Let P be a Spin^c-structure on X, with a determinant line bundle L^2, and 2-sheeted covering map

$$\pi : P \to \mathrm{Fr}(X) \times L^2. \tag{3.4.2}$$

Finally, let A be a connection on the principal bundle L^2, i.e. A is the 1-form

$$A : TL^2 \to i\mathbb{R}, \tag{3.4.3}$$

taking values in the Lie algebra of U(1) $\mathfrak{u}(1) \cong i\mathbb{R}$.

The connections ω and A define a connection $\omega \times A$ on $\mathrm{Fr}(X) \times L^2$ with values in $\mathfrak{so}(4) \oplus i\mathbb{R}$. However, $\mathrm{Fr}(X) \times L^2$ is the quotient of P by $\mathbb{Z}_2$, and as $\mathfrak{spin}^c(4) = \mathfrak{so}(4) \oplus i\mathbb{R}$, the connection $\omega \times A$ lifts to a unique connection τ on P. In fact, we have the commutative diagram

$$\begin{array}{ccc} TP & \xrightarrow{\ \tau\ } & \mathfrak{spin}^c(4) \\ {\scriptstyle d\pi}\downarrow & & \downarrow{\scriptstyle p_*} \\ T(\mathrm{Fr}(X) \times L^2) & \xrightarrow[\omega \times A]{} & \mathfrak{so}(4) \oplus i\mathbb{R}, \end{array} \tag{3.4.4}$$

where p_* is the differential of the map

$$\begin{aligned} p : \mathrm{Spin}^c(4) &\to \mathrm{SO}(4) \times \mathrm{U}(1), \\ p([A_+, A_-, \lambda]) &= \big(\rho(A_+, A_-), \lambda^2\big). \end{aligned} \tag{3.4.5}$$

Consider the spinor bundle S associated to the Spin^c-structure P, and let $\psi \in \Gamma(S)$. Then the differential of ψ is given by

$$\begin{aligned} D^A\psi &= d\psi + \mathrm{Ad}^c_*(\tau)\psi \\ &= d\psi + \mathrm{ad}^c(\tau)\psi, \end{aligned} \tag{3.4.6}$$

which induces a covariant derivative

$$\nabla^A : \Gamma(S) \to \Gamma(T^*X \otimes S) \cong \Gamma\big(\,\mathrm{Hom}(TX, S)\big), \tag{3.4.7}$$

on the spinor bundle.

Recall that TX acts on S via Clifford multiplication, and so for $Y \in TX$ we have

$$\begin{aligned} D^A(\gamma(Y)\psi) &= d(\gamma(Y)\psi) + \mathrm{ad}^c(\tau)(\gamma(Y)\psi) \\ &= \gamma(dY)\psi + \gamma(Y)d\psi + \mathrm{ad}^c(\tau)(\gamma(Y)\psi). \end{aligned} \tag{3.4.8}$$

We now examine the term $\mathrm{ad}^c(\tau)(\gamma(Y)\psi)$ closely. Consider a tangent vector $v \in TP$, then $t(v) \in (y, s) \in \mathfrak{spin}^c(4) = \mathfrak{spin}(4) \oplus i\mathbb{R}$. Moreover,

$$\mathrm{ad}^c(\tau)(\gamma(Y)\psi) = (y + s)\gamma(Y)\psi = y\gamma(Y)\psi + \gamma(Y)(s\psi). \tag{3.4.9}$$

However, as $y \in \mathfrak{spin}(4)$ and Y is a vector field, it follows that

$$\rho_*(y)X = \mathrm{ad}_y(Y) = y\gamma(Y) - \gamma(Y)y, \tag{3.4.10}$$

where ρ_* is the differential of the representation $\rho : \mathrm{Spin}(4) \to \mathrm{SO}(4)$ defined in Section 3.1. Hence

$$y\gamma(Y) = \gamma(Y)y + \rho_*(y)(Y), \tag{3.4.11}$$

and as $\rho_*(y) = \omega(d\pi(v))$, we have

$$\mathrm{ad}^c(\tau)(\gamma(Y)\psi) = \gamma(Y)\big(\,\mathrm{ad}^c(\tau)\psi\big) + \gamma(\rho_*(\omega)Y)\psi. \tag{3.4.12}$$

Therefore

$$\begin{aligned} D^A(\gamma(Y)\psi) &= \gamma(dY)\psi + \gamma(Y)d\psi + \gamma(Y)\big(\,\mathrm{ad}^c(\tau)\psi\big) + \gamma(\rho_*(\omega)Y)\psi \\ &= \gamma\big(dY + \rho_*(\omega)Y\big)\psi + \gamma(Y)\big(d\psi + \mathrm{ad}^c(\tau)\psi\big) \\ &= \gamma(\nabla Y)\psi + \gamma(Y)D^A\psi, \end{aligned} \tag{3.4.13}$$

which shows compatibility between Clifford multiplication and D^A. We summarise the previous discussion in the following proposition.

Proposition 3.4.1. *Suppose that $\tilde{X}$ and $\tilde{Y}$ are vector fields on X, and let $\psi \in \Gamma(S)$ be a spinor. Then the spinor covariant derivative with respect to any unitary connection A on L^2 satisfies the following compatibility condition:*

$$\nabla^A_{\tilde{Y}}(\gamma(\tilde{X})\psi) = \gamma(\tilde{X})\nabla^A_{\tilde{Y}}\psi + \gamma(\nabla_{\tilde{Y}}\tilde{X})\psi. \tag{3.4.14}$$

Definition 3.4.2. The covariant derivative $\nabla^A : \Gamma(S) \to \Gamma(T^*X \otimes S)$ which satisfies the compatibility condition (3.4.14) is called a Spin^c**-connection**.

We provide a local description of a Spin^c-connection ∇^A. Let $e : U \to \mathrm{Fr}(X)$ be a local section of the frame bundle. Then e is a local orthonormal frame of vector fields on $U \subset X$. The local representation of the Levi-Civita connection $\omega^e = e^*(\omega) : TU \to \mathfrak{so}(4)$ is then given by

$$\omega^e = \sum_{i<j} \omega_{ij} E_{ij}, \tag{3.4.15}$$

with E_{ij} the basis matrices of $\mathfrak{so}(4)$; and ω_{ij} are the 1-forms defining the Levi-Civita connection $\omega_{ij} = g(\nabla e_i, e_j)$. Similarly, fix a section $s : U \to L^2$, and take the local form

$$A^s = s^*(A) : TU \to i\mathbb{R}. \tag{3.4.16}$$

Now, $e \times s : U \to \mathrm{Fr}(X) \times L^2$ is a local section of the principal bundle $\mathrm{Fr}(X) \times L^2$, and let $\widetilde{e \times s}$ denote the lifting of this section to the principal bundle P. As

$$\begin{aligned} (\widetilde{e \times s})^*(p_*(\tau)) &= (\widetilde{e \times s})^*\pi^*(\omega \times A) \\ &= (\pi \circ \widetilde{e \times s})^*(\omega \times A) \\ &= (e \times s)^*(\omega \times A) \\ &= (\omega^e, A^s) \\ &= \Big(\sum_{i<j} \omega_{ij} E_{ij}, A^s\Big), \end{aligned} \tag{3.4.17}$$

by the commutativity of (3.4.4). We have an extension of (3.4.4) given by

$$\begin{array}{ccccc} & & TP & \xrightarrow{\ \tau\ } & \mathfrak{spin}^c(4) \\ & \nearrow^{d\widetilde{e\times s}} & \big\downarrow{\scriptstyle d\pi} & & \big\downarrow{\scriptstyle p_*} \\ TU & \xrightarrow[d(e\times s)]{} & T(\mathrm{Fr}(X) \times L) & \xrightarrow[\omega\times A]{} & \mathfrak{so}(4) \oplus i\mathbb{R}. \end{array} \tag{3.4.18}$$

We wish to find a local representative of $\tau^{\widetilde{e\times s}}$. In order to obtain this, we need a relation between E_{ij} and $\gamma(e_i)\gamma(e_j)$.

Proposition 3.4.3. *If* $\rho : \mathrm{Spin}(4) \to \mathrm{SO}(4)$ *is the spin representation, then its differential is given by*

$$\begin{aligned} \rho_* &: \mathfrak{spin}(4) \to \mathfrak{so}(4) \\ \rho_*(\gamma(e_i)\gamma(e_j)) &= 2E_{ij}, \end{aligned} \tag{3.4.19}$$

for all $i < j$.

Proof. [Fri00] □

Therefore, E_{ij} corresponds to the element $\frac{1}{2}\gamma(e_i)\gamma(e_j)$, and so the local description of the connection is given by

$$\tau^{\widetilde{e \times s}} = \Big(\frac{1}{2}\sum_{i<j}\omega_{ij}\gamma(e_i)\gamma(e_j), \frac{1}{2}A^s\Big). \tag{3.4.20}$$

From this, it follows that the covariant derivative of $\psi \in \Gamma(S)$ is given by

$$\nabla^A\psi = d\psi + \frac{1}{2}\sum_{i<j}\omega_{ij}\gamma(e_i)\gamma(e_j) + \frac{1}{2}A^s\psi. \tag{3.4.21}$$

From (3.4.21), it is clear that ∇^A is invariant on the bundles $\Gamma(S^\pm)$; as $\gamma(e_j)$ will map $\Gamma(S^\pm)$ to $\Gamma(S^\mp)$, and following with $\gamma(e_i)$ maps back to $\Gamma(S^\pm)$.

Recall that the space of connections on L^2 is affine over $i\Omega^1(X)$, and so for two connections A and A', the difference $A' - A$ is a $i\mathbb{R}$-valued one-form a. Using (3.4.21), we have

$$\begin{aligned} \nabla^A_X\psi - \nabla^{A'}_X\psi &= \frac{1}{2}A(X)\psi - \frac{1}{2}A'(X)\psi \\ &= \frac{1}{2}(A - A')(X)\psi \\ &= \frac{1}{2}a(X)\psi. \end{aligned} \tag{3.4.22}$$

We summarise the previous results in the following proposition.

Proposition 3.4.4. *Let* ∇^A *be a* Spin^c *connection on* X. *Then we have the following:*

(I) *For any* $X \in TX$, *we have* $\nabla^A_X : \Gamma(S^\pm) \to \Gamma(S^\pm)$.

(II) *If* A *and* A' *are two unitary connections on the determinant line bundle* L^2*; then* $A' = A + a$ *for some* $a \in i\Omega(X)$, *and*

$$\nabla^{A'} = \nabla^A + \frac{1}{2}a. \tag{3.4.23}$$

We now look at the curvature of the spin connection ∇^A. Let $\Omega : T\,\mathrm{Fr}(X) \times T\,\mathrm{Fr}(X) \to \mathfrak{so}(4)$ denote the curvature form of the Levi-Civita connection, given locally by

$$\Omega = \sum_{i<j} \Omega_{ij} E_{ij}, \tag{3.4.24}$$

where $\Omega_{ij} \in \Omega^2(X)$. Let F_A denote the curvature of connection A on the determinant line bundle, i.e. $F_A = dA$. From the commutativity of (3.4.18) we see that

$$\Omega^\tau = \frac{1}{2} \sum_{i<j} \pi^*(\Omega_{ij})\gamma(e_i)\gamma(e_j) \oplus \frac{1}{2}\pi^*(F_A). \tag{3.4.25}$$

Hence it follows that

$$\Omega^A \psi = (\nabla^A \circ \nabla^A)\psi = \frac{1}{2} \sum_{i<j} \Omega_{ij}\gamma(e_i)\gamma(e_j)\psi + \frac{1}{2} F_A \psi. \tag{3.4.26}$$

Suppose e_1, e_2, e_3, e_4 is a local orthonormal frame and set

$$R_{ijkl} = \Omega_{ij}(e_k, e_l), \tag{3.4.27}$$

so that

$$\Omega_{ij} = \sum_{k<l} R_{ijkl}\theta^k \wedge \theta^l = \frac{1}{2} \sum_{k,l=1}^{4} R_{ijkl}\theta^k \wedge \theta^l, \tag{3.4.28}$$

where $\theta^1, \theta^2, \theta^3, \theta^4$ is the coframe dual to e_1, e_2, e_3, e_4. The R_{ijkl} are the components of the **Riemann-Christoffel curvature tensor**.

Proposition 3.4.5. *The components of the Riemann-Christoffel tensor satisfy the following identities:*

(I) (Symmetry): $R_{ijkl} = R_{klij}$.

(II) (Skew-Symmetry): $R_{ijkl} = -R_{jikl}$, *and* $R_{ijkl} = -R_{ijlk}$.

(III) (Algebraic Bianchi): $R_{ijkl} + R_{iklj} + R_{iljk} = 0$.

Proof. A proof can be found in any book on Riemannian geometry, e.g. [Lee18] or [Jos17]. □

Hence we can rewrite (3.4.21) as

$$\begin{aligned} \Omega^A \psi &= \frac{1}{2} \sum_{i<j} \Big(\sum_{k<l} R_{ijkl}\theta^k \wedge \theta^l \Big) \gamma(e_i)\gamma(e_j)\psi + \frac{1}{2} F_A \psi, \\ &= \frac{1}{8} \sum_{i,j} \Big(\sum_{k,l} R_{ijkl}\theta^k \wedge \theta^l \Big) \gamma(e_i)\gamma(e_j)\psi + \frac{1}{2} F_A \psi. \end{aligned} \tag{3.4.29}$$

Define

$$s = \sum_{i,j=1}^{4} R_{ijji}, \tag{3.4.30}$$

to be the **scalar curvature** of X, and we claim that

$$\sum_{i,j=1}^{4} \gamma(e_i)\gamma(e_j)\Omega^A(e_i, e_j) = \frac{s}{2} + \sum_{i<j} F_A(e_i, e_j)\gamma(e_i)\gamma(e_j). \tag{3.4.31}$$

To see this: substitute the local formula for ∇^A into the left hand side of (3.4.31) to get

$$\begin{aligned}
\sum_{i,j=1}^{4} \gamma(e_i)\gamma(e_j)\Omega^A(e_i, e_j) &= \sum_{i,j=1}^{4} \gamma(e_i)\gamma(e_j) \Bigg(\frac{1}{2} \sum_{k<l} \Omega_{kl}(e_i, e_j)\gamma(e_k)\gamma(e_l) \\
&\qquad + \frac{1}{2} F_A(e_i, e_j) \Bigg) \\
&= \frac{1}{4} \sum_{i,j,k,l=1}^{4} R_{klij}\gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l) \\
&\qquad + \frac{1}{2} \sum_{i,j=1}^{4} F_A(e_i, e_j)\gamma(e_i)\gamma(e_j) \\
&= \frac{1}{4} \sum_{i,j,k,l=1}^{4} R_{ijkl}\gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l) \\
&\qquad + \sum_{i<j} F_A(e_i, e_j)\gamma(e_i)\gamma(e_j),
\end{aligned} \tag{3.4.32}$$

by the symmetry of R_{ijkl}. It remains to look at the term

$$\frac{1}{4} \sum_{i,j,k,l=1}^{4} R_{ijkl}\gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l). \tag{3.4.33}$$

If i, j, k, and l are all distinct, then

$$\begin{aligned}
R_{ijkl}\gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l) + R_{iklj}\gamma(e_i)\gamma(e_k)\gamma(e_l)\gamma(e_j) \\
+R_{iljk}\gamma(e_i)\gamma(e_l)\gamma(e_j)\gamma(e_k) = \gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l)\big(R_{ijkl} + R_{iklj} + R_{iljk}\big)
\end{aligned} \tag{3.4.34}$$

which is 0 by the algebraic Bianchi identity. Similarly, if i, j, and l are distinct

$$\begin{aligned}
&R_{ijil}\gamma(e_i)\gamma(e_j)\gamma(e_i)\gamma(e_l) \\
&\quad +R_{ilij}\gamma(e_i)\gamma(e_l)\gamma(e_i)\gamma(e_j) = \gamma(e_i)\gamma(e_j)\gamma(e_i)\gamma(e_l)\big(R_{ijil} - R_{ilij}\big) \\
&\qquad = \gamma(e_i)\gamma(e_j)\gamma(e_i)\gamma(e_l)\big(R_{ijil} - R_{ijil}\big) \\
&\qquad = 0,
\end{aligned} \tag{3.4.35}$$

by symmetry of curvature components. Therefore the only terms that are non-zero in (3.4.33) are those which contain R_{ijij} or R_{ijji}. Hence

$$\begin{aligned}\sum_{i,j,k,l=1}^{4} & R_{ijkl}\gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l) \\ &= \sum_{i,j=1}^{4} R_{ijij}\gamma(e_i)\gamma(e_j)\gamma(e_i)\gamma(e_j) \\ &\quad + \sum_{i,j=1}^{4} R_{ijji}\gamma(e_i)\gamma(e_j)\gamma(e_j)\gamma(e_i) \\ &= -\sum_{i,j=1}^{4} R_{ijij} + \sum_{i,j=1}^{4} R_{ijji} \\ &= 2\sum_{i,j=1}^{4} R_{ijji} = 2s,\end{aligned} \tag{3.4.36}$$

by skew-symmetry of the curvature components, proving (3.4.31).

3.5 Dirac Operators

Suppose X is a 4-dimensional Riemannian manifold with a Spin^c-structure. Let ∇^A denote the Spin$^c(4)$-connection, and let S be the corresponding spinor bundle.

Definition 3.5.1. The **Dirac operator** $\not{D}_A$ is the first order differential operator given by the composition

$$\Gamma(S) \xrightarrow{\nabla^A} \Gamma(T^*X \otimes S) \xrightarrow{\cong} \Gamma(TX \otimes S) \xrightarrow{\gamma} \Gamma(S). \tag{3.5.1}$$

Note we have identified $T^*X \cong TX$ via the Riemannian metric.

Locally, the Dirac operator is given by

$$\not{D}_A : \Gamma(S) \to \Gamma(S) \tag{3.5.2}$$

given by

$$\not{D}_A \psi = \sum_{i=1}^{4} \gamma(e_i)\nabla^A_{e_i}\psi, \tag{3.5.3}$$

where $\{e_i\}_{i=1}^4$ is a local orthonormal frame for TX, and $\psi \in \Gamma(S)$. From this it is clear that $\not{D}_A$ is linear, and (3.5.1) implies that $\not{D}_A$ is independent of the choice of local frame.

Definition 3.5.2. A spinor $\psi \in \Gamma(S)$ is **harmonic** if $\not{D}_A\psi = 0$.

However, the spinor bundle S splits into the positive and negative spinor bundles $S^{\pm}$. Hence, the Dirac operator $\not{D}_A$ splits into a sum of the operators $\not{D}_A^{\pm} : \Gamma(S^{\pm}) \to \Gamma(S^{\mp})$, with

$$\not{D}_A = \begin{bmatrix} 0 & \not{D}_A^- \\ \not{D}_A^+ & 0 \end{bmatrix}. \tag{3.5.4}$$

Proposition 3.5.3. *The Dirac operator $\not{D}_A^{\pm}$ is well-defined. i.e. $\not{D}_A^{\pm}$ maps $\Gamma(S^{\pm})$ to $\Gamma(S^{\mp})$.*

Proof. By Proposition 3.4.4 the action of the connection ∇^A sends $\Gamma(S^{\pm})$ to $\Gamma(S^{\pm})$ while the action of Clifford multiplication sends $\Gamma(S^{\pm})$ to $\Gamma(S^{\mp})$. □

The Dirac operator also has the following properties.

Proposition 3.5.4. *Let $\psi \in \Gamma(S)$ be a spinor. Then*

(I) (Leibniz Rule)*: If $f : X \to \mathbb{R}$ is a smooth map, then*

$$\not{D}_A(f\psi) = \gamma(df)\psi + f\not{D}_A\psi \tag{3.5.5}$$

(II) *If $A' = A + a$ is another connection on L^2, then*

$$\not{D}_{A+a}\psi = \not{D}_A\psi + \frac{1}{2}\gamma(a)\psi. \tag{3.5.6}$$

Proof. Let $e_1, \ldots, e_4$ be an orthonormal basis for TX. Using the Leibniz rule, and (3.3.38) we have

$$\begin{aligned} \not{D}_A(f\psi) &= \sum_{i=1}^{4} \gamma(e_i)\nabla_{e_i}^A(f\psi) \\ &= \sum_{i=1}^{4} \gamma(e_i)\big(df(e_i)\psi + f\nabla_{e_i}^A\psi\big) \\ &= \sum_{i=1}^{4} df(e_i)\gamma(e_i)\psi + f\sum_{i=1}^{4} \gamma(e_i)\nabla_{e_i}^A\psi \\ &= \gamma(df)\psi + f\not{D}_A\psi, \end{aligned} \tag{3.5.7}$$

which proves (I). Similarly

$$\begin{aligned}
\not{D}_{A+a}\psi &= \sum_{i=1}^{4} \gamma(e_i)\nabla_{e_i}^{A+a}\psi \\
&= \sum_{i=1}^{4} \gamma(e_i)\Big(\nabla_{e_i}^{A}\psi + \frac{1}{2}a(e_i)\psi\Big) \\
&= \sum_{i=1}^{4} \gamma(e_i)\nabla_{e_i}^{A}\psi + \frac{1}{2}\sum_{i=1}^{4} a(e_i)\gamma(e_i)\psi \\
&= \not{D}_A\psi + \frac{1}{2}\gamma(a)\psi,
\end{aligned} \tag{3.5.8}$$

proving (II). □

The following is an important property of the Dirac operator.

Theorem 3.5.5. *Let X be a compact 4-dimensional* Spin^c *manifold. Then the Dirac operator $\not{D}_A : \Gamma(S) \to \Gamma(S)$ is formally self-adjoint. i.e. for any $\phi, \psi \in \Gamma(S)$*

$$\int_X \left\langle \not{D}_A\psi, \varphi\right\rangle dV = \int_X \left\langle \psi, \not{D}_A\varphi\right\rangle dV, \tag{3.5.9}$$

where dV is the volume form on X.

Proof. Let $x \in X$, and choose a local orthonormal frame $\{e_i\}_{i=1}^4$ for TX such that $\nabla_{e_i}e_j = 0$, for all i, j at x. As Clifford multiplication by unit vectors is a unitary transformation, it follows that

$$\left\langle \not{D}_A\psi, \phi\right\rangle = \sum_{i=1}^{4}\left\langle \gamma(e_i)\nabla_{e_i}^{A}\psi, \phi\right\rangle = \sum_{i=1}^{4}\left\langle \gamma(e_i)\gamma(e_i)\nabla_{e_i}^{A}\psi, \gamma(e_i)\phi\right\rangle = \sum_{i=1}^{4} -\left\langle \nabla_{e_i}^{A}\psi, \gamma(e_i)\phi\right\rangle, \tag{3.5.10}$$

due to $\gamma(e_i)\gamma(e_i) = -|e_i|^2\,\mathrm{Id}$. Further, ∇^A is a unitary connection,

$$e_i\left\langle \psi, \gamma(e_i)\phi\right\rangle = \left\langle \nabla_{e_i}^{A}\psi, \gamma(e_i)\phi\right\rangle + \left\langle \psi, \nabla_{e_i}^{A}\big(\gamma(e_i)\phi\big)\right\rangle. \tag{3.5.11}$$

Using this, and the Leibniz rule for a spin connection, we have

$$\begin{aligned}
\left\langle \not{D}_A\psi,\phi\right\rangle &= -\sum_{i=1}^{4}\left\langle \nabla^A_{e_i}\psi,\gamma(e_i)\phi\right\rangle \\
&= \sum_{i=1}^{4}\Big(\left\langle \psi,\nabla^A_{e_i}\big(\gamma(e_i)\phi\big)\right\rangle - e_i\left\langle \psi,\gamma(e_i)\phi\right\rangle\Big) \\
&= \sum_{i=1}^{4}\Big(\left\langle \psi,\gamma\big(\nabla_{e_i}e_i\big)\phi\right\rangle + \left\langle \psi,\gamma(e_i)\nabla^A_{e_i}\phi\right\rangle - e_i\left\langle \psi,\gamma(e_i)\phi\right\rangle\Big) \qquad (3.5.12)\\
&= \sum_{i=1}^{4}\Big(\left\langle \psi,\gamma(e_i)\nabla^A_{e_i}\phi\right\rangle - e_i\left\langle \psi,\gamma(e_i)\phi\right\rangle\Big) \\
&= \left\langle \psi,\not{D}_A\phi\right\rangle - \sum_{i=1}^{4} e_i\left\langle \psi,\gamma(e_i)\phi\right\rangle .
\end{aligned}$$

Now, we can view $\langle\psi,\gamma(e_i)\phi\rangle$ as the ith component of some vector field Y. Then the previous equation becomes

$$\left\langle \not{D}_A\psi,\phi\right\rangle = \left\langle \psi,\not{D}_A\phi\right\rangle + \operatorname{div} Y \qquad (3.5.13)$$

Now since all the terms in (3.5.13) are independent of the choice of coordinates, the equality holds even if $\nabla_{e_i}e_j \neq 0$. As

$$\int_X \operatorname{div} Y\; dV = 0 \qquad (3.5.14)$$

by Stokes' theorem (as X has empty boundary). The result now follows from integrating (3.5.13). □

Another important property of the Dirac operator is that it satisfies an associated **Linchnerowicz**, or **Weitzenböck** formula, which gives a relationship between a Dirac operator and the vector bundle Laplacian. But first, we need the definition.

Definition 3.5.6. The **vector bundle Laplacian** $\Delta^A : \Gamma(S) \to \Gamma(S)$ is defined by

$$\Delta^A\psi = -\sum_{i=1}^{4}\big(\nabla^A_{e_i}\nabla^A_{e_i}\psi - \nabla^A_{\nabla^A_{e_i}e_i}\psi\big). \qquad (3.5.15)$$

where e_1, e_2, e_3, e_4 is a local orthonormal frame for TX.

Remark 7. The term $\nabla^A_{\nabla^A_{e_i} e_i}\psi$ is to make the vector bundle Laplacian independent of the choice of frame. ♦

Theorem 3.5.7 (Lichnerowicz Formula)**.** *Let X be a 4-manifold with a* Spin^c*-structure, and A a connection on the determinant line bundle L^2. Then*

$$(\not{D}_A)^*\not{D}_A\psi = \Delta^A\psi + \frac{s}{4}\psi + \frac{1}{2}\sum_{i<j} F_A(e_i, e_j)\gamma(e_i)\gamma(e_j)\psi, \tag{3.5.16}$$

where $(\not{D}_A)^*$ and $(\nabla^A)^*$ are the formal adjoints of the operators $\not{D}_A$ and ∇^A repsectively; s is the scalar curvature of X; and F_A is the curvature of A.

Proof. Let e_1, e_2, e_3, e_4 be a local frame for TX such that $\nabla_{e_i} e_j = 0$ for all i, j at $x \in X$. Using that $\not{D}_A$ is formally self-adjoint, we have

$$\begin{aligned}
(\not{D}_A)^*(\not{D}_A)\psi &= \Big(\sum_{i=1}^4 \gamma(e_i)\nabla^A_{e_i}\Big)\Big(\sum_{j=1}^4 \gamma(e_j)\nabla^A_{e_j}\Big)\psi \\
&= \sum_{i,j=1}^4 \gamma(e_i)\nabla^A_{e_i}\big(\gamma(e_j)\nabla^A_{e_j}\big)\psi \\
&= \sum_{i,j=1}^4 \gamma(e_i)\big(\gamma(\nabla^A_{e_i}e_j)\nabla^A_{e_j} + \gamma(e_j)\nabla^A_{e_i}\nabla^A_{e_j}\big)\psi \\
&= \sum_{i,j=1}^4 \gamma(e_i)\gamma(e_j)\nabla^A_{e_i}\nabla^A_{e_j}\psi \\
&= -\sum_{i=1}^4 \nabla^A_{e_i}\nabla^A_{e_i}\psi + \frac{1}{2}\sum_{i,j=1}^4 \gamma(e_i)\gamma(e_j)\big(\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i}\big)\psi \\
&= -\Delta^A\psi + \frac{1}{2}\sum_{i,j=1}^4 \gamma(e_i)\gamma(e_j)\Omega^A(e_i, e_j)\psi.
\end{aligned} \tag{3.5.17}$$

Substituting (3.4.31), immediately gives (3.5.16). □

3.5.1 The Atiyah-Singer Index Theorem

One can show that the Dirac operators $\not{D}_A$ defines a first-order **elliptic differential operator**. As $\not{D}_A$ is an elliptic operator on a compact manifold, it is a **Fredholm** operator, and $\ker \not{D}_A$ and $\operatorname{coker} \not{D}_A$ are both finite-dimensional. We define the **index** of $\not{D}_A$ to be the integer

$$\operatorname{ind}(\not{D}_A) = \dim\big(\ker \not{D}_A\big) - \dim\big(\operatorname{coker} \not{D}_A\big), \tag{3.5.18}$$

The index is actually a topological invariant, and can be computed in terms of certain characteristic classes. This result is known as the **Atiyah-Singer index theorem**, and is one of the major results of the 20th century. For the special case of a Dirac operator on a 4-manifold we have the following.

Theorem 3.5.8 (Atiyah-Singer)**.** *Suppose X is a compact, oriented* Spin^c *4-manifold with the correspond spinor bundle S. If $\not{D}_A$ is a Dirac operator on the spinor bundle, then*

$$\begin{aligned} \operatorname{ind}(\not{D}_A) &= -\frac{1}{8}\tau(X) + \frac{1}{8}\int_X c_1(L^2) \smile c_1(L^2) \\ &= \frac{-\tau(X) + Q\big(c_1(L^2), c_1(L^2)\big)}{8}, \end{aligned} \tag{3.5.19}$$

where $\tau(X)$ is the signature of X.

The Atiyah-Singer index theorem yields the following as a corollary.

Theorem 3.5.9 (Rokhlin)**.** *If X is a compact, oriented, smooth 4-manifold which is spin, then the signature $\tau(X)$ is divisible by 16.*

Proof. See [Fri00, Chapter 4] □

Chapter 4

The Seiberg-Witten Equations

In this chapter we introduce the Seiberg-Witten equations. The solutions of said equations have a natural action of a gauge group, which leads to the construction of a moduli space of solutions. This moduli space will provide the basis for a proof of Donaldson's theorem.

4.1 The Seiberg-Witten Equations

Let (X, g) be an oriented Riemannian 4-manifold with a Spin^c-structure and corresponding spinor bundles S^+, S^-.

Definition 4.1.1. We define the **configuration space** of X to be

$$\mathcal{C}(X) = \mathcal{A} \times \Gamma(S^+), \tag{4.1.1}$$

where $\mathcal{A}$ is the space of U(1)-connections on the determinant line bundle L^2.

We are now ready to define the Seiberg-Witten equations.

Definition 4.1.2. The **Seiberg-Witten equations** are

$$\begin{aligned} \not{D}_A^+ \psi &= 0, \\ F_A^+ &= q(\psi), \end{aligned} \tag{4.1.2}$$

for $(A, \psi) \in \mathcal{C}(X)$; where F_A^+ is the self-dual part of the curvature of A. We also define the **perturbed Seiberg-Witten equations** as

$$\begin{aligned} \not{D}_A^+ \psi &= 0, \\ F_A^+ &= q(\psi) + \phi, \end{aligned} \tag{4.1.3}$$

where ϕ is an imaginary self-dual two form, i.e. $\phi \in i\Omega^2_+(X)$.

We note that the equations are non-linear due to the $q(\psi)$ terms, which are quadratic in ψ.

To study the solutions of the Seiberg-Witten equations, it will be useful to redefine the equations as the level-set of a smooth map.

Definition 4.1.3. The **perturbed Seiberg-Witten map** is the map

$$\begin{aligned} F_\phi : \mathcal{C}(X) &\to \Gamma(S^-) \times i\Omega^2_+(X), \\ (A, \psi) &\mapsto (\not{D}^+_A \psi, F^+_A - q(\psi) - \phi). \end{aligned} \tag{4.1.4}$$

It is clear that if (A, ψ) is a solution to (4.1.3), then $(A, \psi) \in F_\phi^{-1}(0, 0)$.

4.2 The Moduli Space

We wish to study the space of solutions to the Seiberg-Witten equations given a 4-manifold with a Spin^c-structure. Recall that the bundles S^+ and S^- have an action by the group of bundle morphisms of L^2 which cover the identity, i.e. the gauge group of L^2.

The group of gauge transformations is

$$\mathcal{G} = C^\infty(X, \mathrm{U}(1)), \tag{4.2.1}$$

smooth functions from X to the circle group, by Corollary $A.1.3.1$. Note that $g \in \mathcal{G}$ acts on $\mathcal{C}(X)$ by

$$g \cdot (A, \psi) = (g^* A, g^{-1}\psi) = (A + 2g^{-1}dg, g^{-1}\psi). \tag{4.2.2}$$

Definition 4.2.1. Suppose $(A, \psi), (A', \psi') \in \mathcal{C}(X)$, then we say that (A, ψ) and (A', ψ') are **gauge equivalent** if there exists $g \in \mathcal{G}$ such that $g \cdot (A, \psi) = (A', \psi')$. It is clear that this defines a equivalence relation on $\mathcal{C}(X)$. We denote the equivalence class of (A, ψ) under gauge equivalence by $[A, \psi]$.

We would want the solutions to the Seiberg-Witten equation to be invariant under the action of $\mathcal{G}$, which is equivalent to asking whether the Seiberg-Witten map is $\mathcal{G}$-equivariant. This is indeed the case; to See why, define an action of $g \in \mathcal{G}$ on $\Gamma(S^-) \times i\Omega^2_+(X)$ by

$$g \cdot (\psi, \eta) = (g^{-1}\psi, \eta). \tag{4.2.3}$$

Then we have the following.

Lemma 4.2.2. *The Seiberg-Witten map is equivariant with respect to the action of $\mathcal{G}$, i.e., for all $g \in \mathcal{G}$,*

$$F_\phi\big(g \cdot (A, \psi)\big) = g \cdot F_\phi(A, \psi). \tag{4.2.4}$$

Proof. Let $g \in \mathcal{G}$, and set $g(A) = A + 2g^{-1}dg$. By the action of $\mathcal{G}$ on $\mathcal{C}(X)$ we have

$$F_\phi\big(g \cdot (A, \psi)\big) = \big(\not{D}^+_{g(A)}(g^{-1}\psi), F^+_{g(A)} - q(g^{-1}\psi) - \phi\big). \tag{4.2.5}$$

However, expanding out (4.2.5), and noting that g maps into U(1), we see that

$$\begin{aligned} F_{g(A)} &= F_{A+2g^{-1}dg} \\ &= F_A + 2d\big(g^{-1}dg\big) \\ &= F_A, \\ q(g^{-1}\psi) &= (g^{-1}\psi) \otimes (g^{-1}\psi)^* - \frac{|g^{-1}\psi|^2}{2}\,\mathrm{Id} \\ &= \big(g^{-1}\overline{g^{-1}}\big)\psi \otimes \psi^* - |g^{-1}|^2 \frac{|\psi|^2}{2}\,\mathrm{Id} \\ &= \psi \otimes \psi^* - \frac{|\psi|^2}{2}\,\mathrm{Id} \\ &= q(\psi). \end{aligned} \tag{4.2.6}$$

For the term involving the Dirac operator, Proposition 3.5.4 gives

$$\begin{aligned} \not{D}^+_{g(A)}\psi &= \not{D}^+_A\psi + \gamma(g^{-1}dg)\psi \\ &= g^{-1}g\not{D}^+_A\psi + g^{-1}\gamma(dg)\psi \\ &= g^{-1}\not{D}^+_A(g\psi). \end{aligned} \tag{4.2.7}$$

Thus

$$\not{D}^+_{g(A)}(g^{-1}\psi) = g^{-1}\not{D}^+_A(gg^{-1}\psi) = g^{-1}\not{D}^+_A\psi, \tag{4.2.8}$$

and so

$$F_\phi\big(g \cdot (A, \psi)\big) = \big(g^{-1}\not{D}^+_A\psi, F^+_A - q(\psi) - \phi\big) = g \cdot F_\phi(A, \psi). \tag{4.2.9}$$

□

Corollary 4.2.2.1. *The Seiberg-Witten equations are invariant under gauge transformation.*

We now investigate the action of $\mathcal{G}$ further, and in particular, ask whether the action is free. Suppose $g \in \mathcal{G}$ fixes (A, ψ), i.e.

$$(A + 2g^{-1}dg, g^{-1}\psi) = (A, \psi). \tag{4.2.10}$$

This occurs only if g is constant, and $\psi = 0$; and the stabiliser group at $(A, 0)$ is S^1.

Definition 4.2.3. We say that $(A, \psi) \in \mathcal{C}(X)$ is **reducible** if $\psi = 0$, and **irreducible** otherwise. i.e., (A, ψ) is reducible if it is fixed by the action of $\mathcal{G}$.

Proposition 4.2.4. *If X is simply-connected, then a reducible solution to the perturbed Seiberg-Witten equations is unique up to gauge equivalence.*

Proof. Suppose that $(A, 0)$ and $(A + a, 0)$ are two reducible solutions, i.e.

$$F_A^+ = \phi, \quad \text{and} \quad F_{A+a}^+ = F_A^+ + d^+ a = \phi. \tag{4.2.11}$$

Hence $d^+ a = 0$. However, we have the following short exact sequence

$$0 \longrightarrow i\Omega^0(X) \xrightarrow{\ d\ } i\Omega^1(X) \xrightarrow{\ d^+\ } i\Omega^2_+(X) \longrightarrow 0, \tag{4.2.12}$$

and by Hodge theory $\ker d^+ = \ker d$. Thus

$$\dim \left(\frac{\ker d^+}{\operatorname{im} d} \right) = b^1(X) = 0, \tag{4.2.13}$$

as X is simply-connected. Therefore $d^+ a = 0$ implies that $a = df$ for some function $f : X \to i\mathbb{R}$, and hence $A + a = A + df$. However, note that $e^{f/2} \in \mathcal{G}$, and

$$A + df = A + 2\big(e^{f/2}\big)^{-1} d\big(e^{f/2}\big). \tag{4.2.14}$$

Therefore $(A, 0)$ is gauge equivalent to $(A + a, 0)$. □

Definition 4.2.5.

$$\mathcal{G}_0 = \{ g \in \mathcal{G} \mid g(x_0) = 1 \} \tag{4.2.15}$$

for some chosen base point $x_0 \in X$. Further, we set $\widetilde{\mathcal{B}} = \mathcal{C}(X)/\mathcal{G}_0$.

We see that $\mathcal{G}_0$ fits into the short exact sequence

$$1 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G} \longrightarrow S^1 \longrightarrow 1 \tag{4.2.16}$$

which splits by the map

$$\begin{aligned} &\mathcal{G} \to \mathcal{G}_0 \times S^1, \\ &g \mapsto \big(g(x_0)^{-1} g, g(x_0)\big). \end{aligned} \tag{4.2.17}$$

The importance of $\mathcal{G}_0$ is that it is the non-constant elements of $\mathcal{G}$ (by (4.2.17)) and thus acts freely on $\mathcal{C}(X)$. Hence, once $\mathcal{C}(X)$ given a topology, $\widetilde{\mathcal{B}}$ will be a smooth manifold.

However, we wish to quotient $\mathcal{C}(X)$ by the full gauge group $\mathcal{G}$. As we have shown that the action is not free, it follows that $\mathcal{B} = \mathcal{C}(X)/\mathcal{G}$ will have singularities at the reducible points $(A, 0)$. We set

$$\mathcal{C}^* = \{(A, \psi) \in \mathcal{C}(X) \mid \psi \neq 0\}, \tag{4.2.18}$$

$$\mathcal{B}^* = \mathcal{C}^*/\mathcal{G}, \tag{4.2.19}$$

$$\widetilde{\mathcal{B}}^* = \mathcal{C}^*/\mathcal{G}_0, \tag{4.2.20}$$

which are the original spaces with these reducible points removed.

Definition 4.2.6. We define the **moduli space** of solutions to the perturbed Seiberg-Witten equations to be

$$\begin{aligned} \mathcal{M}_\phi &= \{[A, \psi] \in \mathcal{B} \mid F_\phi(A, \psi) = 0\} \\ &= F_\phi^{-1}(0, 0)/\mathcal{G}. \end{aligned} \tag{4.2.21}$$

We also set

$$\begin{aligned} \widetilde{\mathcal{M}}_\phi &= \{[A, \psi] \in \widetilde{\mathcal{B}} \mid F_\phi(A, \psi) = 0\} \\ &= F_\phi^{-1}(0, 0)/\mathcal{G}_0, \end{aligned} \tag{4.2.22}$$

which is the space of solutions to the perturbed Seiberg-Witten equations, modulo based gauge transformations.

As stated before, we wish to view $\mathcal{B}^*$ as an infinite dimensional manifold. This is possible by modelling $\mathcal{B}^*$ on an arbitrary Hilbert or Banach space, rather than $\mathbb{R}^n$ or $\mathbb{C}^n$. In order to do this, we need to complete our spaces of sections with respect to the Sobolev norms.

Let E be a smooth $\mathrm{O}(n)$, or $\mathrm{U}(n)$-bundle over a compact Riemannian manifold (X, g) with connection ∇; and extend ∇ to $\Omega^k(X, E)$ by (A.2.7). Hence if $\sigma \in \Gamma(E)$, we set $\nabla_A^k \sigma = (\nabla_A \circ \cdots \circ \nabla_A)\sigma \in \Omega^k(X, E)$.

Definition 4.2.7. The k**th Sobolev norm** $\|\cdot\|_{p,k}$ on $\Gamma(E)$ for $p > 1$ is defined as

$$\|\sigma\|_{p,k} := \left(\sum_{i=0}^{k} \int_X |\nabla_A^i \sigma|^p \, dV \right)^{1/p}, \tag{4.2.23}$$

for $\sigma \in \Gamma(E)$, and where ∇_A^0 is the identity.

Definition 4.2.8. We denote by $L_k^p(E)$, the completion of $\Gamma(E)$ with respect to Sobolev norm $\|\cdot\|_{p,k}$.

Changing the Riemannian metric g on X changes the resulting Sobolev norm on $\Gamma(E)$. Likewise, changing the choice of metric on E or the connection A on E also change the norms. However, all these norms can be shown to be equivalent, and therefore do not affect $L^p_k(E)$. More details can be found in [DK90].

Proposition 4.2.9. *$L^p_k(E)$ is a Banach space for all p and k, and a Hilbert space for $p = 2$.*

We now use the Sobolev norms to topologise the configuration space $\mathcal{C}(X)$ which will allow us to view $\mathcal{B}^*$ as a manifold. From the definition, we see that there are many possible choices for a Sobolev norm. However, we choose to use L^2-spaces so that the resulting spaces are Hilbert manifolds.

Suppose X has a Spin^c-structure with line bundle L^2. We denote the space of unitary L^2_k connections on L^2 by $\mathcal{A}_{L^2_k}$. Similarly, we denote the space of L^2_k configurations by

$$\mathcal{C}_k(X) = \mathcal{A}_{L^2_k} \times L^2_k(S^+). \tag{4.2.24}$$

If $\mathbb{U}(1)$ is the trivial U(1)-bundle over X, then we define $\mathcal{G}_k = L^2_k(\mathbb{U}(1))$ to be the gauge group of all L^2_k maps from X to U(1). Finally $\mathcal{P}_k = L^2_k(\bigwedge^2_+ T^*X)$ is the space of L^2_k perturbations.

Now we let $\mathcal{B}$ have the quotient topology, and give $\mathcal{M}_\phi$ the subspace topology induced from $\mathcal{B}$. For all $k \geq 2$, we define $\mathcal{M}_{\phi,k}$ to be the L^2_k solutions to the perturbed Seiberg-Witten equations modulo L^2_{k+1} gauge transformations.

Proposition 4.2.10. *The configuration space $\mathcal{C}_k(X)$ is a smooth Hilbert manifold, and $\mathcal{G}_{k+1}$ is an infinite dimensional Lie group which acts smoothly on $\mathcal{C}_k(X)$. Moreover, $\mathcal{M}_{\phi,2}$ is diffeomorphic to $\mathcal{M}_{\phi,k}$ for all $k \geq 2$.*

Proof. [Nic00] □

Therefore, the moduli space of solutions is independent of the choice of Sobolev norm. We only consider the case when $k = 2$, and when we write $\mathcal{C}(X)$, $\mathcal{G}$, and $\mathcal{M}_\phi$ the appropriate Sobolev completion is understood.

4.3 Properties of the Moduli Space

The reason for studying the moduli space is because $\mathcal{M}_\phi$ has a rich geometry. We state the following properties of $\mathcal{M}_\phi$ without proof, as they require technical results from the theory of elliptic operators which is beyond the scope of this thesis. Proofs can be found in [Moo01].

The first of the properties is that $\mathcal{M}_\phi$ is a compact topological space. More precisely:

Theorem 4.3.1 (Compactness)**.** *Let X be a smooth 4-manifold with a* Spin^c*-structure. Then for every choice of $\phi \in i\Omega^2_+(X)$, the moduli space $\widetilde{\mathcal{M}}_\phi$ of solutions to the perturbed Seiberg-Witten equations is compact.*

The second property concerns the smooth structure of $\widetilde{\mathcal{M}}_\phi$.

Theorem 4.3.2 (Transversality)**.** *Let X be a compact smooth 4-manifold with a* Spin^c*-structure L^2. Then*

(I) *If $b^2_+(X) > 0$, we can choose $\phi \in i\Omega^2_+(X)$, such that there are no reducible solutions to the perturbed Seiberg-Witten equations, and $\widetilde{\mathcal{M}}_\phi$ is an oriented smooth manifold.*

(II) *If $b^2_+(X) = 0$ and $c_1(L^2)$ is such that*

$$\operatorname{ind}(\not{D}^+_A) = \frac{-\tau(X) + Q\big(c_1(L^2).c_1(L^2)\big)}{8} \geq 0,$$

then we can choose $\phi \in i\Omega^2_+(X)$ such that $\widetilde{\mathcal{M}}_\phi$ is an oriented smooth manifold.

In either case, for a generic choice of $\phi \in i\Omega^2_+(X)$, $\widetilde{\mathcal{M}}_\phi$ is an oriented smooth manifold of dimension

$$\dim(\widetilde{\mathcal{M}}_\phi) = 2\operatorname{ind}(\not{D}^+_A) - b^2_+(X) + b^1(X). \tag{4.3.1}$$

Strictly speaking the dimension given in Theorem 4.3.2 is a 'formal' dimension, due to the fact that $\dim(\widetilde{\mathcal{M}}_\phi)$ could be negative. If the formal dimension $\dim(\widetilde{\mathcal{M}}_\phi)$ is negative, then $\widetilde{\mathcal{M}}_\phi = \emptyset$.

By combining Theorem 4.3.1 and Theorem 4.3.2, we see that if X is a compact smooth 4-manifold with a Spin^c-structure and $b^2_+(X) > 0$. Then for a generic choice of $\phi \in i\Omega^2_+(X)$, the smooth locus of $\mathcal{M}_\phi$ (the restriction to the irreducible connections) is a compact, oriented smooth manifold of dimension

$$\dim(\mathcal{M}_\phi) = 2\operatorname{ind}(\not{D}_A) - b^2_+(X) + b^1(X) - 1. \tag{4.3.2}$$

This follows as quotienting by the residual S^1 action lowers the dimension by 1.

4.4 Tangent Space of $\mathcal{M}_\phi$

In this section, we consider the tangent space $T_{[A,\psi]}\mathcal{M}$ at a Seiberg-Witten solution (A, ψ).

Recall the Seiberg-Witten map defined in (4.1.4):

$$\begin{aligned} F : \mathcal{A} \times \Gamma(S^+) &\to \Gamma(S^-) \times i\Omega^2_+(X) \\ (A, \psi) &\mapsto (\not{D}^+_A \psi, F^+_A - q(\psi)), \end{aligned} \tag{4.4.1}$$

which was defined such that solution space to the Seiberg-Witten equations is the level set $F^{-1}(0,0)$, and the moduli space is $\mathcal{M} = F^{-1}(0,0)/\mathcal{G}$. For simplicity, we write

$$\mathcal{F} = \Gamma(S^-) \times i\Omega^2_+(X), \tag{4.4.2}$$

and compute the derivative

$$dF|_{(A,\psi)} : T_{(A,\psi)}\mathcal{C} \to T_{(0,0)}\mathcal{F} \tag{4.4.3}$$

of F at a Seiberg-Witten solution (A, ψ).

Since $\Gamma(S^{\pm})$ and $i\Omega^2_+(X)$ are vector spaces, they are their own tangent spaces. Since $\mathcal{A}$ is an affine space over $i\Omega^1(X)$ it follows that $i\Omega^1(X)$ is its tangent space. Hence we compute dF using curves. So let $\varphi \in \Gamma(S^+)$ and $\theta \in \mathcal{A}$, and consider the curve

$$t \mapsto F(A + t\theta, \psi + t\varphi). \tag{4.4.4}$$

Then

$$dF|_{(A,\psi)}(\theta, \varphi) = \left.\frac{d}{dt}\right|_{t=0} F(A + t\theta, \psi + t\varphi). \tag{4.4.5}$$

Now

$$\begin{aligned} F(A + t\theta, \psi + t\theta) &= \big(\not{D}^+_{A+t\theta}(\psi + t\varphi), F^+_{A+t\theta} - q(\psi + t\varphi)\big) \\ &= \big(\not{D}^+_{A+t\theta}(\psi + t\varphi), F^+_A + td^+\theta - q(\psi + t\varphi)\big). \end{aligned} \tag{4.4.6}$$

By expanding the first term, we obtain

$$\begin{aligned} \not{D}^+_{A+t\theta}(\psi + t\varphi) &= \not{D}^+_A(\psi + t\varphi) + \frac{1}{2}t\gamma(\theta)(\psi + t\varphi) \\ &= \not{D}^+_A\psi + t\not{D}^+_A\varphi + \frac{1}{2}t\gamma(\theta)\psi + \frac{1}{2}t^2\gamma(\theta)\varphi \\ &= t\Big(\not{D}^+_A\varphi + \frac{1}{2}\gamma(\theta)\psi\Big) + \frac{1}{2}t^2\gamma(\theta)\varphi, \end{aligned} \tag{4.4.7}$$

by Proposition 3.5.4, the fact that $\not{D}^+_A$ is linear, and ψ is a harmonic spinor. Also

$$\begin{aligned} q(\psi + t\varphi) &= (\psi + t\varphi) \otimes (\psi + t\varphi)^* - \frac{1}{2}|\psi + t\varphi|^2 \operatorname{Id} \\ &= \psi \otimes \psi^* + t\psi \otimes \varphi^* + t\varphi \otimes \psi^* + t^2\varphi \otimes \varphi^* \\ &\quad - \frac{1}{2}\big(|\psi|^2 + t\langle\psi, \varphi\rangle + t\langle\varphi, \psi\rangle + t^2|\varphi|^2\big) \operatorname{Id} \\ &= q(\psi) + t^2 q(\varphi) + t\Big(\psi \otimes \varphi^* + \varphi \otimes \psi^* - \frac{1}{2}\langle\psi, \varphi\rangle \operatorname{Id} - \frac{1}{2}\overline{\langle\psi, \varphi\rangle} \operatorname{Id}\Big). \end{aligned} \tag{4.4.8}$$

However, $F_A^+ = q(\psi)$, and so

$$F_A^+ + td^+\theta - q(\psi + t\varphi) = t\Big(d^+\theta - \psi\otimes\varphi^* - \varphi\otimes\psi^* + \frac{1}{2}\langle\psi,\varphi\rangle \operatorname{Id} + \frac{1}{2}\overline{\langle\psi,\varphi\rangle}\operatorname{Id}\Big) - t^2 q(\varphi). \tag{4.4.9}$$

Therefore

$$dF|_{(A,\psi)}(\theta,\varphi) = \big(\not{D}_A^+\varphi + \tfrac{1}{2}\gamma(\theta)\psi, d^+\theta - \psi\otimes\varphi^* - \varphi\otimes\psi^* + \tfrac{1}{2}\langle\psi,\varphi\rangle\operatorname{Id} + \tfrac{1}{2}\overline{\langle\psi,\varphi\rangle}\operatorname{Id}\big) \tag{4.4.10}$$

We also calculate the differential of the gauge action $\mathcal{G}$. Consider the gauge action as a map

$$\begin{aligned} \mathfrak{g} &: \mathcal{G} \to \mathcal{C} \\ \mathfrak{g}(g) &= (A + 2g^{-1}dg, g^{-1}\psi). \end{aligned} \tag{4.4.11}$$

Since X is simply-connected, we can write $g = e^f$ for some smooth function $f : X \to i\mathbb{R}$, so that

$$\mathfrak{g}(g) = (A + 2df, e^{-f}\psi). \tag{4.4.12}$$

The tangent space of $\mathcal{G}$ at 1 is the space of all functions $f : X \to i\mathbb{R}$, which is $i\Gamma(X\times\mathbb{R})$; i times sections of the trivial bundle. Now consider the curve

$$t \mapsto \mathfrak{g}(e^{tf}). \tag{4.4.13}$$

We compute the derivative of $\mathfrak{g}$

$$d\mathfrak{g}|_1 : T_1\mathcal{G} \to T_{(A,\psi)}\mathcal{C} \tag{4.4.14}$$

using this curve. As

$$\mathfrak{g}(e^{tf}) = (A + 2tdf, e^{-tf}\psi), \tag{4.4.15}$$

we have

$$d\mathfrak{g}|_1(f) = \left.\frac{d}{dt}\right|_{t=0} \mathfrak{g}(e^{tf}) = (2df, -f\psi). \tag{4.4.16}$$

Consider now the following composition

$$0 \longrightarrow T_1\mathcal{G} \xrightarrow{d\mathfrak{g}|_1} T_{(A,\psi)}\mathcal{C} \xrightarrow{dF|_{(A,\psi)}} T_{(0,0)}\mathcal{F} \longrightarrow 0. \tag{4.4.17}$$

Proposition 4.4.1. *At a solution (A,ψ) to the Seiberg-Witten equations,* (4.4.17) *is a differential complex.*

Proof. We need to show that $dF|_{(A,\psi)} \circ d\mathfrak{g}|_1 = 0$. A direct calculation gives

$$\begin{aligned} dF|_{(A,\psi)}\big(d\mathfrak{g}|_1(f)\big) &= dF|_{(A,\psi)}(2df, -f\psi) \\ &= \Big(\not{D}_A^+(-f\psi) + \tfrac{1}{2}\gamma(2df)\psi,\ d^+(2df) - (-f\psi)\otimes\psi^* \\ &\qquad - \psi\otimes(-f\psi)^* + \tfrac{1}{2}\langle\psi, -f\psi\rangle \operatorname{Id} + \tfrac{1}{2}\overline{\langle\psi,-f\psi\rangle}\operatorname{Id}\Big) \\ &= \Big(\not{D}_A^+(-f\psi) + \gamma(df)\psi,\ f\psi\otimes\psi^* + \overline{f}\psi\otimes\psi^* \\ &\qquad - \tfrac{1}{2}\overline{f}\langle\psi,\psi\rangle\operatorname{Id} - \tfrac{1}{2}f\langle\psi,\psi\rangle\operatorname{Id}\Big) \\ &= \big(\not{D}_A^+(-f\psi) + \gamma(df)\psi,\ (f+\overline{f})\psi\otimes\psi^* - \tfrac{1}{2}(f+\overline{f})|\psi|^2\operatorname{Id}\big). \end{aligned} \tag{4.4.18}$$

However, recall that f is purely imaginary and so $\overline{f} = -f$. Moreover, by Proposition 3.5.4 we have

$$\not{D}_A^+(-f\psi) + \gamma(df)\psi = -\gamma(df)\psi + f\not{D}_A^+\psi + \gamma(df)\psi = 0, \tag{4.4.19}$$

as ψ is a harmonic spinor. Therefore,

$$dF|_{(A,\psi)}\big(d\mathfrak{g}|_1(f)\big) = (0,0) \tag{4.4.20}$$

as required. □

Definition 4.4.2. The differential complex (4.4.17) is called the **Seiberg-Witten complex**.

The Seiberg-Witten complex is intimately related to the tangent space $T_{[A,\psi]}\mathcal{M}$. Suppose that (A,ψ) is a regular value of F, i.e. $dF|_{(A,\psi)}$ is surjective. Then $\ker dF|_{(A,\psi)}$ is the tangent space to the space of Seiberg-Witten solutions, i.e. $\ker dF|_{(A,\psi)} = T_{(A,\psi)}F^{-1}(0,0)$. Now consider the case that (A,ψ) is an irreducible solution. Then $\mathcal{G}$ embeds onto the orbit of (A,ψ), and so $\operatorname{im} d\mathfrak{g}|_1$ represents the tangent space to this orbit at (A,ψ). i.e. we have $\operatorname{im} d\mathfrak{g}|_1 = T_{(A,\psi)}\big(\mathcal{G}\cdot(A,\psi)\big)$.

Therefore, in $\mathcal{M} = F^{-1}(0,0)/\mathcal{G}$, the tangent space to $[A,\psi]$ is exactly

$$T_{[A,\psi]}\mathcal{M} = T_{(A,\psi)}F^{-1}(0,0)/T_{(A,\psi)}\big(\mathcal{G}\cdot(A,\psi)\big) = \ker dF|_{(A,\psi)}/\operatorname{im} d\mathfrak{g}|_1. \tag{4.4.21}$$

Notice that this is precisely the first cohomology group of the Seiberg-Witten complex. We investigate this relationship further.

Let SW denote the Seiberg-Witten complex. Then the zeroth cohomology group is

$$H^0_{(A,\psi)}(\mathrm{SW}) = \ker d\mathfrak{g}|_1. \tag{4.4.22}$$

Note that $H^0_{(A,\psi)}(\mathrm{SW})$ is trivial if, and only if, $\mathcal{G}$ acts freely at (A,ψ). The first cohomology group is

$$H^1_{(A,\psi)}(\mathrm{SW}) = \ker dF|_{(A,\psi)} / \operatorname{im} d\mathfrak{g}|_1, \tag{4.4.23}$$

and is called the **Zariski tangent space** of $\mathcal{M}$. By the previous work, it is likely to be the true tangent space $T_{[A,\psi]}\mathcal{M}$. The second cohomology group is

$$H^2_{(A,\psi)}(\mathrm{SW}) = \operatorname{coker} dF|_{(A,\psi)}, \tag{4.4.24}$$

and it measure the failure of (A,ψ) being a regular value of F. Hence, we call $H^2_{(A,\psi)}(\mathrm{SW})$ the **obstruction space** at (A,ψ).

Lemma 4.4.3. *If for all Seiberg-Witten solutions (A,ψ) we have $H^0_{(A,\psi)}(\mathrm{SW}) = 0$ and $H^2_{(A,\psi)}(\mathrm{SW}) = 0$, then $\mathcal{M}$ is either empty or is a smooth compact submanifold of $\mathcal{B}^*$. Moreover, if this is the case, we have*

$$T_{[A,\psi]}\mathcal{M} = H^1_{(A,\psi)}(\mathrm{SW}). \tag{4.4.25}$$

Proof. The vanishing of $H^2_{(A,\psi)}(\mathrm{SW})$ for all (A,ψ) implies that $(0,0)$ is a regular value of F. Therefore the solution space $F^{-1}(0,0)$ is either empty, or is a smooth submanifold of $\mathcal{C}$ by the inverse function theorem. The vanishing of $H^0_{(A,\psi)}(\mathrm{SW})$ for all (A,ψ) implies that there are no reducible solutions. Hence the gauge group $\mathcal{G}$ acts freely on $F^{-1}(0,0)$, and so $\mathcal{M} = F^{-1}(0,0)/\mathcal{G}$ is a smooth submanifold of $\mathcal{B}^*$.

Compactness follows from Theorem 4.3.1. □

Using Theorem 4.3.2, we see that the obstruction space can always be made to vanish by perturbing the Seiberg-Witten equations. In fact, the Seiberg-Witten complex is used extensively to prove the statements of Theorem 4.3.2.

Recall that $\mathcal{G}$ has a subgroup of based gauge transformations $\mathcal{G}_0$ which acts freely. We defined the space $\widetilde{\mathcal{M}} = F^{-1}(0,0)/\mathcal{G}_0$, and by the transversality theorem, Theorem 4.3.2, we can always take a perturbation such that $\widetilde{\mathcal{M}}_\phi$ is a smooth submanifold.

We wish to find an explicit representation of $T_{[A,0]}\widetilde{\mathcal{M}}_\phi$ where $[A,0]$ is the unique reducible point for the $\mathcal{G}$ action, as this is pivotal in the proof of Donaldson's theorem. Now, by Lemma 4.4.3 we see that

$$T_r\widetilde{\mathcal{M}}_\phi = H^1_{(A,0)}(\mathrm{SW}) \tag{4.4.26}$$

as $\mathcal{G}_0$ acts freely at r so that $H^0_{(A,0)}(\mathrm{SW})$ vanishes. However, computing the derivatives of the Seiberg-Witten map and the gauge action at the reducible point gives

$$dF|_{(A,0)}(\theta,\varphi) = (\not{D}^+_A\varphi, d^+\theta), \tag{4.4.27}$$

and

$$d\mathfrak{g}|_1(f) = (2df, 0). \tag{4.4.28}$$

Thus the Seiberg-Witten complex at the reducible point is given by

$$0 \longrightarrow i\Gamma(X \times \mathbb{R}) \xrightarrow{(2d,0)} i\Omega(X) \times \Gamma(W_+ \otimes L) \xrightarrow{d^+ \oplus \not{D}_A^+} i\Omega^2_+(X) \times \Gamma(W_- \otimes L) \longrightarrow 0 \tag{4.4.29}$$

The first cohomology group of is therefore

$$H^1_{(A,0)}(\mathrm{SW}) = \frac{\ker(d^+ \oplus \not{D}_A^+)}{\mathrm{im}(2d, 0)} = \ker \not{D}_A^+ \oplus \frac{\ker d^+}{\mathrm{im}\, d} = \ker \not{D}_A^+ \oplus H^1(X; \mathbb{R}), \tag{4.4.30}$$

as $\ker d^+ = \ker d$ by Hodge theory. Therefore, in the case that X is simply-connected, we have

$$T_{[A,0]}\widetilde{\mathcal{M}}_\phi = \ker \not{D}_A^+ \oplus H^1(X; \mathbb{R}) = \ker \not{D}_A^+. \tag{4.4.31}$$

Chapter 5

Donaldson's Theorem

Let X be a compact, oriented, simply connected 4-manifold (topological or smooth). Without a loss of generality (by passing to connected components) we may also assume that X is connected. We wish to provide a classification of X up to homeomorphism.

5.1 Characteristics

Suppose that Q is the intersection form of X. Then we have the following definition.

Definition 5.1.1. An element $x \in H^2(X; \mathbb{Z})$ is a **characteristic** if for all $y \in H^2(X; \mathbb{Z})$

$$Q(x, y) = Q(y, y) \pmod 2. \tag{5.1.1}$$

Hence we see that $0 \in H^2(X; \mathbb{Z})$ is a characteristic element if, and only if, Q is even. The reason we are interested in characteristics is because they correspond to Spin^c-structures on X. But first, we show the existence of such elements.

Lemma 5.1.2. *Suppose Q is a symmetric, bilinear, and unimodular form on a free $\mathbb{Z}$-module Z. Then there exists a characteristic element.*

Proof. Let $\widetilde{Z} = Z/2Z$ and $\widetilde{Q} = Q \pmod 2$. Then we have a symmetric, bilinear, unimodular $\mathbb{Z}_2$-form

$$\widetilde{Q} : \widetilde{Z} \times \widetilde{Z} \to \mathbb{Z}_2. \tag{5.1.2}$$

Now unimodularity implies that for every $\mathbb{Z}_2$-linear function $f : \widetilde{Z} \to \mathbb{Z}_2$, there exists $\widetilde{x}_f \in \widetilde{Z}$ such that $f(\cdot) = \widetilde{Q}(\widetilde{x}_f, \cdot)$. However, for all $a, b \in Z$

$$Q(a + b, a + b) = Q(a, a) + 2Q(a, b) + Q(b, b) = Q(a, a) + Q(b, b) \pmod 2, \tag{5.1.3}$$

and so it follows that the function $f : \widetilde{Z} \to \mathbb{Z}_2$ defined by $f(\widetilde{x}) = \widetilde{Q}(\widetilde{x}, \widetilde{x})$ is $\mathbb{Z}_2$-linear. Thus there exists $\widetilde{y} \in \widetilde{Z}$ such that $\widetilde{Q}(\widetilde{y}, \widetilde{x}) = \widetilde{Q}(\widetilde{x}, \widetilde{x})$, i.e.

$$Q(\widetilde{y}, \widetilde{x}) = Q(\widetilde{x}, \widetilde{x}) \pmod 2 \tag{5.1.4}$$

for all $\widetilde{x} \in \widetilde{Z}$. However, $\widetilde{y}$ and $\widetilde{x}$ represent cosets of Z, and so there exists $x, y \in Z$ such that $\widetilde{y} = y \pmod 2$ and $\widetilde{x} = x \pmod 2$. Therefore we have

$$Q(y, x) = Q(x, x) \pmod 2 \tag{5.1.5}$$

for all $x \in Z$, and so y is a characteristic. □

Corollary 5.1.2.1. *There exists a characteristic element for the intersection form.*

Before we show the correspondence of Spin^c-structures and characteristics, we state a result which relates characteristics to Spin-structures.

Proposition 5.1.3. *Let $w_2(TX)$ be the second Stiefel-Whitney class of X. Then the mod 2 reduction of a characteristic element c is $w_2(TX)$, i.e. $c = w_2(TX) \pmod 2$.*

Proof. We need to show that any characteristic is the integral lift of the second Stiefel-Whitney class. This follows from an alternate definition of $w_2(TX)$ via the Steenrod operations, along with Wu's formula. A detailed proof can be found in [MS74]. □

As a corollary to Proposition 5.1.3, we see if X has a Spin-structure then 0 is a characteristic element; i.e., the intersection form of X is even. Conversely, if $H^2(X; \mathbb{Z})$ is torsion-free (which is true by assumption as X is simply connected) then X admits Spin-structures if the intersection form is even. Explicitly, we have the following.

Corollary 5.1.3.1. *A compact, oriented, simply-connected smooth 4-manifold X has a* Spin*-structure if, and only if, its intersection form Q is even.*

Example 5.1.4. We now finally show that the M_{E_8} manifold is not smoothable. Recall that M_{E_8} is the manifold whose intersection form is

$$E_8 = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & 0 & -1 \\ & & & & -1 & 2 & -1 & 0 \\ & & & & 0 & -1 & 2 & 0 \\ & & & & -1 & 0 & 0 & 2 \end{bmatrix} \tag{5.1.6}$$

which is even. Hence by Proposition 5.1.3, if M_{E_8} is smooth then it is Spin. However Rokhlin's theorem, Theorem 3.5.9, implies that $\tau(M_{E_8}) \equiv 0 \pmod{16}$; which is a contradiction as $\tau(M_{E_8}) = 8$. Therefore, M_{E_8} is not smoothable as claimed. ◀

Theorem 5.1.5. *Suppose that L^2 is a* Spin^c*-structure on a compact, oriented, and smooth 4-manifold; then $c_1(L^2)$ is a characteristic element. Conversely, any characteristic element is $c_1(L^2)$ for some* Spin^c*-structure L^2 on X.*

Proof. Suppose that L^2 is a Spin^c-structure on X, and let E be a complex line bundle over X. Then $L^2 \otimes E$ is a second Spin^c-structure on X, with Chern class

$$c_1(L^2 \otimes E) = c_1(L^2) + 2c_1(E).$$

Using the Atiyah-Singer index theorem, we compute the indicies of the Dirac operator relative to the Spin^c-structures L^2, $L^2 \otimes E$, and consider the difference

$$\frac{1}{8}Q(c_1(L^2), c_1(L^2)) - \frac{1}{8}Q(c_1(L^2 \otimes E), c_1(L^2 \otimes E)) \in \mathbb{Z}. \tag{5.1.7}$$

However, this implies that

$$\frac{1}{2}Q(c_1(L^2), c_1(E)) + \frac{1}{2}Q(c_1(E), c_1(E)) \in \mathbb{Z}, \tag{5.1.8}$$

and so

$$Q(c_1(L^2), c_1(E)) = Q(c_1(E), c_1(E)) \pmod 2. \tag{5.1.9}$$

As every element $x \in H^2(X; \mathbb{Z})$ is the first Chern class $c_1(E)$ for some complex line bundle E over X, it follows that $c_1(L^2)$ is a characteristic.

By Theorem 3.2.3, X has a Spin^c-structure L^2 and $c_1(L^2)$ is a characteristic by the previous work. Suppose that x is another characteristic. Then for all $y \in H^2(X; \mathbb{Z})$,

$$Q(y - c_1(L^2), y) = 0 \pmod 2, \tag{5.1.10}$$

and so there exists a line bundle E over X such that

$$c_1(E^2) = 2c_1(E) = y - c_1(L^2). \tag{5.1.11}$$

Therefore, $L^2 \otimes E$ is another Spin^c-structure on X with

$$c_1(L^2 \otimes E) = c_1(L^2) + 2c_1(E) = y. \tag{5.1.12}$$

□

5.2 Donaldson's Theorem

Theorem 5.2.1 (Donaldson, [Don83]). *Let X be a compact, oriented, simply-connected smooth 4-manifold with negative definite intersection form $\big(H^2(X,\mathbb{Z}),Q\big)$. Then Q is $\mathbb{Z}$-diagonalisable, i.e. $Q=-I=\operatorname{diag}(-1,-1,\dots,-1)$.*

Proof. We proceed by cases.

Case 1: Suppose, for a contradiction, that $\big(H^2(X,\mathbb{Z}),Q\big)$ is even, and $b^2(X)>0$. Since is Q is even, it is spin by Corollary 5.1.3.1. Thus X has a Spin^c-structure with $L\cong X\times\mathbb{C}$, the trivial line bundle ($c_1(L)=0$). By the Atiyah-Singer index theorem Theorem 3.5.8

$$\begin{aligned}\operatorname{ind}(\not{D}_A^+) &= \frac{-\tau(X)+Q\big(c_1(L),c_1(L)\big)}{8}\\ &= -\frac{\tau(X)}{8}\\ &= \frac{b^2_-(X)}{8}>0.\end{aligned}\tag{5.2.1}$$

So by the transversality theorem, Theorem 4.3.2, there exists a pertubation ϕ such that $\widetilde{\mathcal{M}}_\phi$ is smooth and of dimension

$$\dim(\widetilde{\mathcal{M}}_\phi)=2\operatorname{ind}(\not{D}_A^+)-b^2_+(X)+b^1(X)=\frac{b^2_-(X)}{4}>0,\tag{5.2.2}$$

and there is exactly one reducible point $[A_0,0]\in\widetilde{\mathcal{M}}_\phi$ by Proposition 4.2.4. As $S^1=\mathrm{U}(1)$ is a Lie group which acts freely on $\widetilde{\mathcal{M}}_\phi$ except at the reducible point $[A_0,0]$, it follows that $\mathcal{M}_\phi=\widetilde{\mathcal{M}}_\phi/S^1$ is a smooth manifold except at $[A_0,0]\in\widetilde{\mathcal{M}}_\phi$. The question now is: what is the topology around this reducible point?

By Section 4.4 we know

$$T_{[A_0,0]}\widetilde{\mathcal{M}}_\phi=\ker(\not{D}_{A_0}^+),\tag{5.2.3}$$

and we claim the induced action of $\mathrm{U}(1)$ on $T_{[A_0,0]}\widetilde{\mathcal{M}}_\phi$ is free. To see this, suppose that g is a Riemannian metric for X. Firstly, by averaging over the $\mathrm{U}(1)$-action, we may assume that $\mathrm{U}(1)$ acts by isometries. Now suppose, for a contradiction, that $v\in T_{[A_0,0]}\widetilde{\mathcal{M}}_\phi$ is fixed by some group element $a\in\mathrm{U}(1)$; and consider the curve $\gamma(t)=e^{tv}$, i.e. $\gamma(0)=[A_0,0]$ and $\gamma'(0)=v$. This is a geodesic through $[A_0,0]$ with tangent vector v. We claim that $a\cdot\gamma(t)=\gamma(t)$ for all $t\in\mathbb{R}$, where $\gamma(t)$ is defined. To see this, recall that a geodesic is uniquely determined by the point it goes through, and the tangent vector at that point. Thus it suffices to show

$$a\cdot\exp(0)=\exp(0),\qquad\text{and}\qquad a_*\cdot\exp'(0)=\exp'(0).\tag{5.2.4}$$

However, this true by assumption; as a fixes both $[A_0, 0]$ and v. Hence, $a \cdot \gamma(t) = \gamma(t)$ in a small neighbourhood around $0 \in \mathbb{R}$. This, however, is a contradiction as we know that $[A_0, 0]$ is the only fixed point of $\mathrm{U}(1)$ in $\widetilde{\mathcal{M}}_\phi$. Therefore the induced $\mathrm{U}(1)$ action on $T_{[A_0,0]}\widetilde{\mathcal{M}}_\phi$ is free as claimed.

However, $\ker(\not{D}^+_{A_0})$ is naturally a complex vector space $\mathbb{C}^m$, and so the free S^1 action is just scalar multiplication by $\mathrm{U}(1) \subseteq \mathbb{C}$. Therefore the topology around $[A_0, 0] \in \mathcal{M}_\phi$ is $\mathbb{C}^m / \mathrm{U}(1)$. Now, thinking of $\mathbb{C}^m$ as a cone over S^{2m-1} (via polar decomposition), quotienting out by the circle action gives a cone over

$$S^{2m-1}/S^1 \cong \mathbb{C}P^{m-1}. \tag{5.2.5}$$

Thus $\mathcal{M}_\phi$ is smooth, except for one point where it looks like a cone over $\mathbb{C}P^{m-1}$.

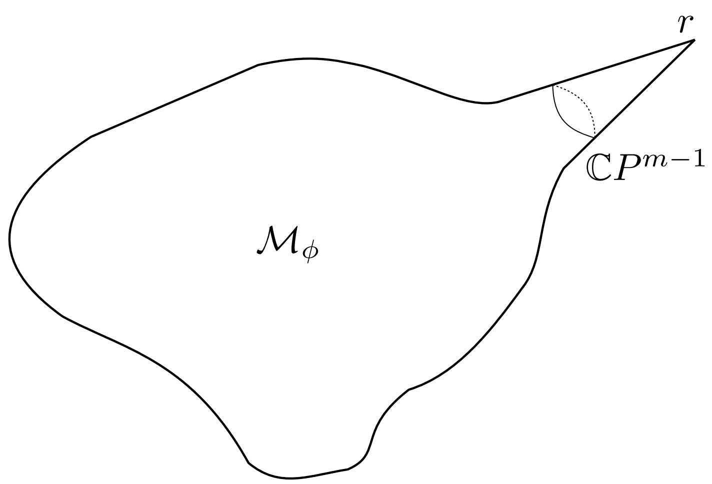

Figure 5.1: The moduli space $\mathcal{M}_\phi$, with $r = [A_0, 0]$ the vertex of the cone

Consider an open neighbourhood N of $[A_0, 0]$. Then $\mathcal{B}^* \supseteq \mathcal{M}^*_\phi = \mathcal{M}_\phi \setminus N$ is a compact smooth manifold with boundary $\mathbb{C}P^{m-1}$. (Compactness follows as $\widetilde{\mathcal{M}}_\phi$ is compact, see Theorem 4.3.1.) Further, as $\mathcal{M}_\phi$ is oriented by the transversality theorem, Theorem 4.3.2, it follows that $\mathcal{M}^*_\phi$ is oriented; and therefore, we can integrate over $\mathcal{M}^*_\phi$. Now by Stokes' theorem

$$\int_{\mathbb{C}P^{m-1}} c_1(E)^{m-1} = \int_{\partial \mathcal{M}^*_\phi} c_1(E)^{m-1} = \int_{\mathcal{M}^*_\phi} d\big(c_1(E)^{m-1}\big) = 0, \tag{5.2.6}$$

where $E \to \mathbb{C}P^{m-1}$ is the universal bundle. However, $E \to \mathbb{C}P^{m-1}$ is the Hopf bundle associated to the S^1-bundle $S^{2m-1} \to \mathbb{C}P^{m-1}$ with Chern class $1 \in H^2(\mathbb{C}P^{m-1}, \mathbb{Z}) \cong \mathbb{Z}$.

So

$$\int_{\mathbb{C}P^{m-1}} c_1(E)^{m-1} \neq 0,$$

which contradicts (5.2.6).

Therefore Q being negative-definite, and even cannot occur for compact, oriented, simply connected smooth 4-manifolds.

Case 2: Suppose now that Q is odd. The same argument presented in case 1 will yield a contradiction provided we can find a Spin^c-structure on X such that $\operatorname{ind}(\not{D}_A^+) > 0$.

Recall that Spin^c-structures are in a one-to-one correspondence with characteristics $c \in H^2(X, \mathbb{Z})$ by Theorem 5.1.5. Thus

$$\operatorname{ind}(\not{D}_A^+) = \frac{-\tau(X) + Q(c,c)}{8} = \frac{b_-^2(X) + Q(c,c)}{8}. \tag{5.2.7}$$

Now, (5.2.7) will be positive provided we can find a characteristic c such that

$$-Q(c,c) < b_-^2(X) = \operatorname{rk} Q. \tag{5.2.8}$$

A result of N. Elkies [Elk95] shows that if Q is not diagonal, then such a characteristic exists and will give a contradiction. Hence, Q must be diagonal. □

Corollary 5.2.1.1. *Let X be a compact, oriented, simply-connected smooth 4-manifold with positive definite intersection form $\big(H^2(X,\mathbb{Z}), Q\big)$. Then Q is $\mathbb{Z}$-diagonalisable, i.e. $Q = I$.*

When paired with Serre's classification of indefinite forms, Theorem 2.3.7, the theorems of Freedman and Donaldson yield the following remarkable corollary.

Corollary 5.2.1.2. *Let X be a compact, oriented, simply connected smooth 4-manifold. Then X is homeomorphic to one of the following:*

$$S^4, \qquad \#^n\mathbb{C}P^2\#^m\overline{\mathbb{C}P}^2, \qquad \#^n(S^2\times S^2)\#^m M_{E_8}.$$

Note that S^4 comes from the zero form.

5.3 The 11/8-Conjecture

We wish to realise indefinite intersection forms Q by smooth 4-manifolds. If Q is odd this is easy, it is realised by the connected sum $\#^n\mathbb{C}P^2\#^m\overline{\mathbb{C}P}^2$. However, if Q is even then it is of the form

$$Q = \oplus^n H \oplus^m E_8. \tag{5.3.1}$$

Taking $n = 0$ and $m = 1$ gives the form $Q = E_8$ which gives rise to a non-smoothable 4-manifold. Therefore there are obstructions to realising even indefinite intersection forms by smooth manifolds.

In fact, it is not immediate clear that (5.3.1) can be realised by a smooth 4-manifold. However, consider the $K3$ surface, which is defined as the hypersurface in $\mathbb{C}P^3$ given by the equation

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0. \tag{5.3.2}$$

The $K3$ surface is a smooth 4-manifold with the topological invariants $b^2(K3) = 22$ and $\tau(K3) = 16$. Hence, Theorem 2.3.7 yields

$$Q_{K3} = \oplus^3 H \oplus^2 E_8. \tag{5.3.3}$$

Therefore, for certain pairs of n and m, (5.3.1) can be realised by a smooth 4-manifold.

The goal of this section is to investigate the conditions on n and m so that (5.3.1) is realised by a smooth 4-manifold.

First, by Rokhlin's theorem, m must be even and so

$$Q = \oplus^n H \oplus^{2k} E_8. \tag{5.3.4}$$

Secondly, as Q is indefinite $n \geq 1$. Moreover, increasing n is not the obstruction to smoothness. For this is just the action of adding connected sums of $S^2 \times S^2$, which is smooth. Therefore the obstruction to smoothness is the minimum number of H's in Q.

It is conjectured by Y. Matsumoto that

Conjecture 5.3.1 (11/8-conjecture, [Mat82])**.** *Every smooth 4-manifold X with even intersection form satisfies the inequality*

$$b^2(X) \geq \frac{11}{8}|\tau(X)|. \tag{5.3.5}$$

The 11/8-conjecture can be rephrased in the following: The intersection form $Q = \oplus^n H \oplus^{2k} E_8$ is realised by a smooth 4-manifold if, and only if, $n \geq 3k$. Therefore it is conjectured that Q needs at least 3 H's for every pair of E_8's in Q to be realised by a smooth 4-manifold.

Note that only if direction of the 11/8-conjecture is true. As the form is realised by the smooth 4-manifold

$$\#^{n-3k}(S^2 \times S^2)\#^k K3. \tag{5.3.6}$$

One of the first steps towards a proof of the 11/8-conjecture was made by Donaldson.

Theorem 5.3.2 ([Don86], [Don87]). *The forms H and $H \oplus H$ are the only intersection forms that are realised by smooth 4-manifolds with $b^2_+ = 1$ or $b^2_+ = 2$.*

Equivalently, Theorem 5.3.2 states that if $k \geq 1$, then $n \geq 3$. Hence after S^4 and $\#^n(S^2 \times S^2)$, $K3$ has the simplest intersection form which is realised by a smooth 4-manifold.

By studying a finite-dimensional approximation to the Seiberg-Witten equations, M. Furuta was able to prove the following.

Theorem 5.3.3 (10/8-theorem, [Fur01]). *Every smooth 4-manifold X with even intersection form satisfies the following inequality*

$$b^2(X) \geq \frac{10}{8}|\tau(X)| + 2. \tag{5.3.7}$$

Equivalently, Furuta's 10/8 theorem states that $n \geq 2k + 1$. Hence for X to be smoothable, we need at least 2 copies of H for every pair of E_8 in Q_X.

In summary, suppose we have the following intersection form

$$Q = \oplus^n H \oplus^{2k} E_8. \tag{5.3.8}$$

Then

- If $n \leq 2k$, then Q is not realisable by a smooth 4-manifold.
- If $n \geq 3$, then Q is realised by the smooth 4-manifold $\#^{n-3k}H\#^k K3$.
- If $2k < n < 3k$, then it is unknown whether Q is realised by a smooth 4-manifold. However, it is conjectured that Q is not realised by a smooth 4-manifold.

Appendix A

Differential Geometry

In this appendix we present some basic results and constructions from differential geometry needed in this thesis.

A.1 Principal Bundles

We recall the basic theory of principal G-bundles needed for this thesis. A proof of the stated theorems can be found in [Ham17]. Throughout this section, G is a Lie group.

Definition A.1.1. A **principal G-bundle** P over X is a smooth submersion $\pi : P \to X$ onto a smooth manifold X, and a smooth right action $P \times G \to P$ such that:

(i) The action of G preserves the fibers of π, and acts freely and transitively on them.

(ii) For any $x \in X$, there is a neighbourhood $U \subset X$ of x and a fiber-preserving, G-equivariant diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times G$. Where G acts on $U \times G$ by $(x, h)g = (x, gh)$.

Note that the second condition is equivalent to the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad \phi_U \quad} & U \times G \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U. & \end{array}$$

Given two principal G-bundles $\pi_X : P \to X$ and $\pi_Y : P \to Y$, and a smooth map $f : X \to Y$, we say that a G-equivariant map $\varphi : P \to P$ is a **bundle map** covering f is the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P' \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

If $X = Y$, and f is a diffeomorphism, then a bundle map is called a **bundle morphism**. If a bundle map $\varphi : P \to P'$ is invertible, and its inverse $\varphi^{-1} : P' \to P$ is a bundle morhpism, then φ is called a **bundle isomorphism**.

A principal G-bundle is **trivial** if it is isomorphic to a the product bundle $X \times G \to X$. By definition, every principal G-bundle is locally trivial.

Let $\varphi : P \to P$ be a bundle isomorphism. Then φ is called a **bundle automorphism**. The set of bundle automorphisms forms a group under composition of function called the group of automorphisms of P. We denote by $\mathrm{Aut}(P)$ the group of bundle automorphisms.

Definition A.1.2. Let $\pi : P \to X$ be a principal G bundle. The subgroup $\mathcal{G}(P)$ of all bundle automorphisms which cover the identity map is called the **gauge group of** P.

We denote by $\mathrm{Ad}(G)$ the space G with the right adjoint action $x \cdot g = g^{-1}xg$ for all $x, g \in G$. Further, if X and Y are spaces with a G-action, then we denote by $C^\infty_G(X, Y)$ the space of smooth G-equivariant maps from X to Y.

Proposition A.1.3. *We have the following isomorphism of groups* $\mathcal{G}(P) \cong C^\infty_G(P, \mathrm{Ad}(G))$, *where the group structure on* $C^\infty_G(P, \mathrm{Ad}(G))$ *is given by pointwise multiplication.*

Proof. Suppose $f \in \mathcal{G}(P)$. Since $f(p)$ is in the same fiber as p, there exists a unique $\sigma_f(p) \in G$ such that

$$f(p) = p\sigma_f(p). \tag{A.1.1}$$

We claim that the map $f \mapsto \sigma_f$ defined by (A.1.1) is the desired isomorphism.

As f is smooth, it follows that σ_f is smooth, and by the G-equivariance of f

$$(pg)\sigma_f(pg) = f(pg) = f(p)g \tag{A.1.2}$$

which implies that

$$g\sigma_f(pg) = \sigma_f(p)g. \tag{A.1.3}$$

Hence $\sigma_f(pg) = g^{-1}\sigma(p)g$, and so $\sigma_f \in C^\infty_G(P, \mathrm{Ad}(G))$. The inverse of the map $f \mapsto \sigma_f$ is given by $\sigma \mapsto f_\sigma$, where f_σ is defined by

$$f_\sigma(p) = p\sigma(p). \tag{A.1.4}$$

As $f_\sigma(p)$ is in the same fiber as p, it is a bundle map. It is clear that $f_\sigma^{-1} = f_{\sigma^{-1}}$, and

$$f_\sigma(pg) = pg\sigma(pg) = pgg^{-1}\sigma(p)g = p\sigma(p)g = f_\sigma(p)g. \tag{A.1.5}$$

Hence f is a diffeomorphism, and $f \in \mathcal{G}(P)$.

To see that $f \mapsto \sigma_f$ is a group homomorphism, we have

$$(f' \circ f)(p) = f'(p\sigma_f(p)) = f'(p)\sigma_f(p) = p\sigma_{f'}(p)\sigma_f(p) = p(\sigma_{f'}\sigma_f)(p). \tag{A.1.6}$$

Therefore $\sigma_{f' \circ f} = \sigma_{f'}\sigma_f$. □

Corollary A.1.3.1. *If G is abelian, then $\mathcal{G}(P) \cong C^\infty(X, G)$.*

Proof. Let $\sigma \in C^\infty_G(P, \mathrm{Ad}(G))$. Then as G is abelian

$$\sigma(pg) = g^{-1}\sigma(p)g = g^{-1}g\sigma(p) = \sigma(p), \tag{A.1.7}$$

and σ is constant on the orbits. Thus, σ descends to a unique smooth map $\tilde{\sigma}$ defined on $P/G = X$. Therefore $\tilde{\sigma} \in C^\infty(X, G)$. □

As is the case with vector bundles, it is possible to give a local description of a principal G-bundle. Consider a local trivialisation $\{U_\alpha\}_{\alpha \in A}$ for a principal G-bundle $\pi : P \to X$. If the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ is non-empty, there are two trivialisations on $\pi^{-1}(U_{\alpha\beta})$ given by ϕ_α and ϕ_β. The composition

$$\phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G, \tag{A.1.8}$$

is fiber-preserving and G-equivariant. Therefore for $(x, h) \in U_{\alpha\beta} \times G$, we have

$$(\phi_\alpha \circ \phi_\beta^{-1})(x, h) = (x, g_{\alpha\beta}(x)h), \tag{A.1.9}$$

for some smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. The map $g_{\alpha\beta}$ is called a **transition function** for the principal G-bundle P.

Just like the situation for vector bundles, the transition functions satisfy the cocycle condition.

Lemma A.1.4. *Given a trivialisation $\{U_\alpha\}_{\alpha \in A}$ for a princpal G-bundle P. Then the transition functions satisfy*

$$g_{\alpha\alpha} = \mathrm{Id}, \quad g_{\alpha\beta}g_{\beta\alpha} = \mathrm{Id}, \quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \mathrm{Id}, \tag{A.1.10}$$

whenever the equations are defined.

In fact, the transition functions are sufficient to describe a principal G-bundle. Let $\{U_\alpha\}_{\alpha \in A}$ be a trivialisation for X with transition functions $\{g_{\alpha\beta}\}$ which satisfy the cocycle condition. For each $U_\alpha \subset X$, define the equivalence relation between elements $(x,h) \in U_\alpha \times G$ and $(x',h') \in U_\beta \times G$ by

$$(x,h) \sim (x',h'), \quad \text{if, and only if,} \quad x = x', \text{ and } h' = g_{\beta\alpha}(x)h \tag{A.1.11}$$

Then the disjoint union of quotient spaces

$$P = \coprod_{\alpha \in A} (U_\alpha \times G)/\sim, \tag{A.1.12}$$

defines a principal G-bundle.

Hence given a vector bundle E we can use the transitions functions to create a principal G-bundle. A canonical example is the construction of the Frame bundle $\mathrm{Fr}(TX)$ on a Riemannian manifold. Moreover, we can reverse the construction to generate a vector bundle from a principal G-bundle.

Definition A.1.5. Suppose $\pi : P \to X$ is a principal G-bundle and $\rho : G \to \mathrm{GL}(V)$ is a representation of G into a finite-dimensional vector space V. The **associated bundle** $E = P \times_\rho V$ is the quotient of $P \times V$ by the equivalence relation

$$(p,v) \sim (pg, \rho(g)^{-1}v), \tag{A.1.13}$$

for all $g \in G$ and $(p,v) \in P \times V$.

It follows that the associated bundle $E = P \times_\rho V$ is a vector bundle with fiber V.

A.2 Connections

In this section we provide a review of connections on principal G-bundles and vector bundles. A proof the stated assertions can be found in any differential geometry book, for example [Ham17] or [Tau11].

Definition A.2.1. A **connection** ∇ on a vector bundle $E \to X$ is a linear map

$$\nabla : \Omega^0(X,E) \to \Omega^1(X,E), \tag{A.2.1}$$

which satisfies the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s. \tag{A.2.2}$$

Given a connection ∇, and a vector field $Y \in \mathfrak{X}(X)$, we form the **covariant derivative** $\nabla X : \Gamma(E) \to \Gamma(E)$, which is denoted by ∇_X. It is $C^\infty(X)$-linear with respect to Y, and linear with respect to $s \in \Gamma(E)$. The covariant derivative satisfies the associated Leibniz rule

$$\nabla_X(fs) = (Xf)s + f\nabla_X s. \tag{A.2.3}$$

Note that, by a partition of unity, connections always exist on vector bundles.

Suppose now that ∇ and ∇' are two connections on E. Then the Leibniz rule gives

$$\begin{aligned}(\nabla - \nabla')(fs) &= df \otimes s + f\nabla s - df \otimes s - f\nabla' s \\ &= f(\nabla - \nabla')s,\end{aligned} \tag{A.2.4}$$

which shows that difference is $C^\infty(X)$-linear. Therefore $\nabla - \nabla'$ is defined pointwise, and so is an $\mathrm{End}(E)$-valued 1-form.

Thus, given a connection ∇ on E, any other connection ∇' is given by $\nabla' = \nabla + a$ for some $a \in \Omega^1(X, \mathrm{End}(E))$. This implies that the space of all connections on E is an affine space over $\Omega^1(X, \mathrm{End}(E))$.

From this, there is a local description of a connection. Suppose $\{U_\alpha\}_{\alpha\in A}$ is a trivialising cover for E, with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$. A connection is then a collection of matrices $\{\omega_\alpha\}_{\alpha\in A}$ of E-valued 1-forms on U_α. Given a section $s \in \Gamma(E)$,

$$(\nabla s)_\alpha = ds_\alpha + \omega_\alpha s_\alpha, \tag{A.2.5}$$

and the matrices ω_α transform on double overlaps by

$$\omega_\alpha = g_{\beta\alpha}^{-1}\omega_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1} dg_{\beta\alpha}. \tag{A.2.6}$$

Therefore, we identify a connection ∇ with its connection matrices $\{\omega_\alpha\}_{\alpha\in A}$.

We may also extend a connection ∇ to a map on $\nabla : \Omega^k(X, E) \to \Omega^{k+1}(X, E)$ by setting

$$\nabla(\alpha s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s, \tag{A.2.7}$$

for $\alpha \in \Omega^k(X)$ and $s \in \Gamma(E)$.

The **curvature** of a connection ∇ is the map

$$F_\nabla = \nabla \circ \nabla : \Omega^0(X, E) \to \Omega^2(X, E). \tag{A.2.8}$$

The curvature is $C^\infty(X)$-linear, and therefore defines a $\mathrm{End}(E)$-valued 2-form. We also provide a local description of the curvature. Let $\{U_\alpha\}_{\alpha\in A}$ be the same trivialisation as above. The curvature is a collection of curvature matrices $\Omega = \{\Omega_\alpha\}_{\alpha\in A}$ given by

$$(\Omega s)_\alpha = (d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha)s_\alpha, \tag{A.2.9}$$

for a section $s \in \Gamma(E)$. The curvature matrices transform on double overlaps according to the rule

$$\Omega_\alpha = g_{\beta\alpha}^{-1} \Omega_\beta g_{\beta\alpha}. \tag{A.2.10}$$

It will also be useful to relate the curvature of two different connections. This is done in the following lemma.

Lemma A.2.2. *Given a connection* ∇ *and a 1-form* $a \in \Omega^1(X, \mathrm{End}(E))$*, then*

$$F_{\nabla+a} = F_\nabla + da + a \wedge a. \tag{A.2.11}$$

We now briefly discuss connections on a principal G-bundle. Suppose $\pi : P \to X$ is a principal G-bundle. Then the differential $d\pi : TP \to TX$ gives rise to a distribution $\ker(d\pi) \subset TP$ called the **vertical distribution**, whose fibers are $\ker(d\pi_p)$. The vertical distribution fits into the (pointwise) exact sequence

$$0 \longrightarrow \ker(d\pi_p) \hookrightarrow T_pP \longrightarrow T_{\pi(p)}X \longrightarrow 0.$$

Definition A.2.3. A **(Ehresmann) connection** on a principal G-bundle $\pi : P \to X$ is a distribution H such that

(i) $\ker(d\pi_p) \oplus H_p = T_pP$, for all $p \in P$.

(ii) H is right invariant under the action of G. i.e. if r_g denotes right multiplication in P by $g \in G$, then $dr_g(H_p) = H_{pg}$.

A distribution which satisfies condition (i) of Definition $A.2.3$ is called a **horizontal distribution**. Thus a Ehresmann connection is a right invariant horizontal connection.

Let $A \in \mathfrak{g}$, the Lie algebra of G. For $p \in P$, we define

$$\tilde{A}_p = \frac{d}{dt}\bigg|_{t=0} pe^{tA} \in T_pP. \tag{A.2.12}$$

The **fundamental vector field associated to** A is the vector field $\tilde{A}$ defined by $(\tilde{A})_p = \tilde{A}_p$. Note that $\tilde{A} \in \ker(d\pi)$.

Definition A.2.4. A **connection 1-form**, or **connection**, on a principal G-bundle $\pi : P \to X$ is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(X, \mathfrak{g})$ such that

(i) $dr_g(\omega) = (\mathrm{Ad}\, g^{-1})\omega$, where $\mathrm{Ad}\, g^{-1}$ is the differential of conjugation by $g^{-1} \in G$.

(ii) $\omega(\tilde{A}) = A$, for all $A \in \mathfrak{g}$.

It turns out that a connection and Ehresmann connection are equivalent objects. Explicitly

Theorem A.2.5. *Let $\pi : P \to X$ be a principal G-bundle.*

(I) *Let H be a Ehresmann connection on P. Then*

$$\omega_p(\tilde{A}_p + B_p) = A_p, \tag{A.2.13}$$

for all $p \in P$, $B_p \in H_p$, and $A_p \in \mathfrak{g}$ defines a connection 1-form ω.

(II) *Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form. Then*

$$H_p = \ker \omega_p \tag{A.2.14}$$

is an Ehresmann connection on P.

A.3 Hodge Theory

In this section we introduce the **Hodge star operator** on $\Omega^\bullet(X)$.

Let X be a compact, oriented, Riemannian 4-manifold. Suppose (e_1, e_2, e_3, e_4) is a positively oriented local orthonormal frame for TX, and let $\theta^1, \theta^2, \theta^3, \theta^4$ be dual coframe.

Define the **Hodge star operator**

$$* : \bigwedge^k T^*X \to \bigwedge^{4-k} T^*X, \tag{A.3.1}$$

by

$$*\,(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) = \theta^{j_1} \wedge \cdots \wedge \theta^{j_{4-k}}, \tag{A.3.2}$$

such that $\theta^{i_1}, \ldots, \theta^{i_k}, \theta^{j_1}, \ldots, \theta^{j_{4-k}})$ is a positively oriented frame; then extending linearly. For example,

$$*\,1 = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4, \quad *(\theta^1 \wedge \theta^2) = \theta^3 \wedge \theta^4. \tag{A.3.3}$$

From the properties of the wedge product

$$*\,(A\theta^1 \wedge \cdots \wedge \theta^k) = (\det A) * (\theta^1 \wedge \cdots \wedge \theta^k). \tag{A.3.4}$$

Hence, $*$ is invariant under orthogonal transformations.

Lemma A.3.1. $*^2 = (-1)^{k(4-k)} = (-1)^k : \bigwedge^k T^*X \to \bigwedge^k T^*X$. *Moreover, we have* $*^{-1} = (-1)^k *$.

Proof. It is clear that $*^2$ maps $\bigwedge^k T^*X$ to itself. Suppose that

$$*\,(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) = \theta^{j_1} \wedge \cdots \wedge \theta^{j_{4-k}}, \tag{A.3.5}$$

then

$$*^2 \left(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}\right) = \pm\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}, \tag{A.3.6}$$

depending on whether $\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}$ positively or negatively oriented. However,

$$\begin{aligned}\theta^{i_1} \wedge \cdots \wedge \theta^{i_k} &\wedge \theta^{j_1} \wedge \cdots \wedge \theta^{j_{4-k}} \\ &= (-1)^{k(4-k)}\theta^{j_1} \wedge \cdots \wedge \theta^{j_{4-k}} \wedge \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}\end{aligned} \tag{A.3.7}$$

which implies that $*^2 = (-1)^{k(4-k)} = (-1)^{-k^2} = (-1)^k$. □

One of the main reason's for considering the $*$ operator, is that it allows us define another inner product on $\Omega^k(X)$. For $\alpha, \beta \in \Omega^k(X)$, we define

$$(\alpha, \beta) = \int_X \alpha \wedge *\beta. \tag{A.3.8}$$

It follows that $(\ , \ )$ defines a symmetric, positive-definite, bilinear form on $\Omega^k(X)$, and hence is an inner product. (For a proof see [Wel08].) The $*$ operator further allows us to define a formal adjoint to the exterior derivative, with respect to the inner product $(\ , \ )$.

Definition A.3.2. The **codifferential** $d^* = \Omega^{k+1}(X) \to \Omega^k(X)$ is defined by

$$d^* = - * d * . \tag{A.3.9}$$

Proposition A.3.3. *The codifferential is the (formal) adjoint to the exterior derivative, with respect to* $(\ , \ )$*. i.e. for* $\alpha \in \Omega^k(X)$*, and* $\beta \in \Omega^{k+1}(X)$ *we have*

$$(d\alpha, \beta) = (\alpha, d^*\beta). \tag{A.3.10}$$

Proof. A direct calculation gives

$$\begin{aligned}(d\alpha, \beta) &= \int_X d\alpha \wedge *\beta \\ &= \int_X \big(d(\alpha \wedge *\beta) - (-1)^k \alpha \wedge d(*\beta)\big) \\ &= \int_X d(\alpha \wedge *\beta) - \int_X (-1)^k \alpha \wedge (-1)^k * * d(*\beta) \\ &= \int_X d(\alpha \wedge *\beta) + \int_X \alpha \wedge *(- * d*)\beta \\ &= \int_X d(\alpha \wedge *\beta) + (\alpha, d^*\beta).\end{aligned} \tag{A.3.11}$$

However, by Stokes' theorem,

$$\int_X d(\alpha \wedge *\beta) = \int_{\partial X} \alpha \wedge *\beta = 0. \tag{A.3.12}$$

□

As $*^2 = 1$ on $\bigwedge^2 T^*X$, we can decompose $\bigwedge^2 T^*X$ into the (± 1)-eigenspaces of $*$. i.e.

$$\bigwedge^2 T^*X = \bigwedge\nolimits_+^2 T^*X \oplus \bigwedge\nolimits_-^2 T^*X, \tag{A.3.13}$$

where

$$\begin{aligned} \bigwedge\nolimits_+^2 T^*X &= \{\omega \in \bigwedge^2 T^*X : *\omega = \omega\} \\ \bigwedge\nolimits_-^2 T^*X &= \{\omega \in \bigwedge^2 T^*X : *\omega = -\omega\}. \end{aligned} \tag{A.3.14}$$

As a vector space, a basis for $\bigwedge_+^2 T^*X$ is given by

$$\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4, \quad \theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4, \quad \theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3, \tag{A.3.15}$$

while a basis for $\bigwedge_-^2 T^*X$ is

$$\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4, \quad \theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4, \quad \theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3. \tag{A.3.16}$$

Elements in $\Omega_+^2 = \Gamma(\bigwedge_+^2 T^*X)$ and $\Omega_-^2(X) = \Gamma(\bigwedge_-^2 T^*X)$ are called **self-dual** and **anti-self-dual** 2-forms, respectively.

Let $\omega \in \Omega^2(X)$, then there is a decomposition

$$\omega = \omega_+ + \omega_- \in \Omega_+^2(X) \oplus \Omega_-^2(X), \tag{A.3.17}$$

where

$$\omega_+ = P_+(\omega) = \frac{1}{2}(\omega + *\omega), \quad \omega_- = P_-(\omega) = \frac{1}{2}(\omega - *\omega). \tag{A.3.18}$$

Define the operator $d^+ = P_+ \circ d : \Omega^1(X) \to \Omega_+^2(X)$. Then d^+ fits into the complex

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d^+} \Omega_+^2(X) \longrightarrow 0. \tag{A.3.19}$$

Proposition A.3.4. $\ker d = \ker d^+$.

Proof. It is clear that $\ker d \subseteq \ker d^+$. Now let $\omega \in \Omega^1(X)$, then

$$\begin{aligned} 2d^*d^+\omega &= d^*(d\omega + *d\omega) \\ &= d^*d\omega + d^*(*d\omega) \\ &= d^*d\omega - *d*(*d\omega), \\ &= d^*d\omega - (-1)^2 * d^2\omega, \\ &= d^*d\omega. \end{aligned} \tag{A.3.20}$$

Thus, $2d^*d^+ = d^*d$, and so if $d^+\omega = 0$ then $d^*d\omega = 0$. Hence

$$0 = (d^*d\omega, \omega) = (d\omega, d\omega), \tag{A.3.21}$$

and $d\omega = 0$. □

References

[BT82] Raoul Bott and Loring W Tu. *Differential Forms in Algebraic Topology.* Graduate texts in mathematics ; 82. Springer New York, New York, NY, 1982.

[DK90] S. K Donaldson and P.B. Kronheimer. *The geometry of four-manifolds.* Oxford mathematical monographs. Clarendon Press ; Oxford University Press, Oxford : New York, 1990.

[Don83] Simon K Donaldson. An application of gauge theory to four-dimensional topology. *Journal of Differential Geometry*, 18(2):279–315, 1983.

[Don86] Simon K Donaldson. Connections, cohomology and the intersection forms of 4-manifolds. *Journal of Differential Geometry*, 24(3):275–341, 1986.

[Don87] Simon K Donaldson. The orientation of yang-mills moduli spaces and 4-manifold topology. *Journal of Differential Geometry*, 26(3):397–428, 1987.

[Elk95] Noam D Elkies. A characterization of the Zˆn lattice. *Mathematical research letters*, 2(3):321–326, 1995.

[Fre82] Michael Hartley Freedman. The topology of four-dimensional manifolds. *Journal of Differential Geometry*, 17(3):357 – 453, 1982.

[Fri00] Thomas Friedrich. *Dirac operators in Riemannian geometry.* Graduate studies in mathematics, v. 25. American Mathematical Society, Providence, R.I., 2000.

[Fur01] Mikio Furuta. Monopole equation and the $\frac{11}{8}$-conjecture. *Mathematical Research Letters*, 8(3):279–291, 2001.

[Ham17] Mark J. D Hamilton. *Mathematical gauge theory: with applications to the standard model of particle physics.* Universitext. 2017.

[HH58] Friedrich Hirzebruch and Heinz Hopf. Felder von flächenelementen in 4-dimensionalen mannigfaltigkeiten. *Mathematische Annalen*, 136:156–172, 1958.

[Jos17] Jürgen Jost. *Riemannian geometry and geometric analysis*. Universitext. Seventh edition.. edition, 2017.

[KM94] P. B. Kronheimer and T. S Mrowka. The genus of embedded surfaces in the projective plane. *Mathematical research letters*, 1(6):797–808, 1994.

[Lee12] John M. Lee. *Introduction to smooth manifolds*. Graduate texts in mathematics ; 218. Springer, New York ; London, 2nd ed.. edition, 2012.

[Lee18] John M. Lee. *Introduction to Riemannian manifolds*. Graduate texts in mathematics ; 176. Second edition.. edition, 2018.

[LM89] H. Blaine Lawson and Marie Louise Michelsohn. *Spin geometry*. Princeton mathematical series, 38. Princeton University Press, Princeton, N.J., 1989.

[Mat82] Yukio Matsumoto. On the bounding genus of homology 3-spheres. *J. Fac. Sci. Univ. Tokyo Sect. IA Math*, 29(2):287–318, 1982.

[Mil58] John W. Milnor. On simply-connected 4-manifolds. In *Symp. Int. de Top. Alg. Mexico*, pages 122–128, 1958.

[Mil11] John W. Milnor. Differential topology forty-six years later. *Notices of the American Mathematical Society*, 58(6):804–809, 2011.

[Moo01] John Douglas Moore. *Lectures on Seiberg-Witten invariants*. Lecture notes in mathematics (Springer-Verlag) ; 1629. Springer, Berlin ; New York, 2nd ed.. edition, 2001.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes,*. Annals of mathematics studies, no. 76. Princeton University Press, Princeton, N.J., 1974.

[Nic00] Liviu I Nicolaescu. *Notes on Seiberg-Witten theory*. Graduate studies in mathematics ; v. 28. American Mathematical Society, Providence, R.I., 2000.

[Sco05] Alexandru Scorpan. *The wild world of 4-manifolds*. American Mathematical Society, Providence, R.I., 2005.

[Ser73] Jean Pierre Serre. *A course in arithmetic*. Graduate texts in mathematics. 7. Springer-Verlag, New York, 1973.

[Tau11] Clifford Taubes. *Differential geometry : bundles, connections, metrics and curvature*. Oxford mathematics. Oxford University Press, Oxford ; New York, 2011.

[TV94] Peter Teichener and Elmar Vogt. All 4-manifolds have spinc structures. *unpublished note, available from the authors' webpage*, 1994.

[Wel08] Raymond O. Wells. *Differential analysis on complex manifolds*. Graduate texts in mathematics ; 65. Springer-Verlag, New York, 3rd ed.. edition, 2008.

[Whi49] J. H. C Whitehead. On simply connected, 4-dimensional polyhedra. *Commentarii mathematici Helvetici*, 22(1):48–92, 1949.

[Wit88] Edward Witten. Topological quantum field theory. *Communications in mathematical physics*, 117(3):353–386, 1988.

[Wit94] Edward Witten. Monopoles and four-manifolds. *Mathematical research letters*, 1(6):769–796, 1994.