THE UNIVERSITY of ADELAIDE

Undergraduate Seminar

School of Mathematical Sciences

Tuesday, 4th September, 1.10pm

Seminar room 7.15 of Ingkarni Wardli

30-minute seminar, followed by sandwiches and juice

Dr Pedram Hekmati

Examples of counterexamples

Abstract: This aims to be an example of an exemplary talk on examples of celebrated counterexamples in mathematics. A famous example, for example, is Euler's counterexample to Fermat's conjecture in number theory.

http://www.maths.adelaide.edu.au/news/undergraduate.html

Examples of counterexamples

Undergraduate Seminar

Pedram Hekmati

A counterexample is a specific instance of the falsity of a proposed conjecture.

Importance

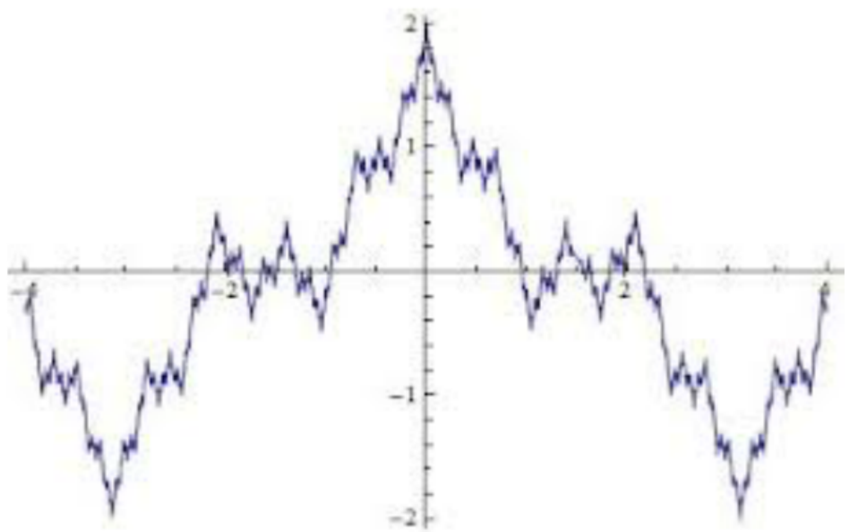
- intuition for mathematical concepts

- probe boundaries of theorems and conjectures

- hints on how to improve a conjecture

Is every continuous function is differentiable?

No! The Weierstrass function:



Original Conjecture:	All prime numbers are odd
Counterexample:	2
New conjecture:	All prime numbers greater than 2 are odd

Lack of counterexamples?

Fermat's last theorem

Arithmeticorum Liber II. 61

QVÆSTIO VIII.

OBSERVATIO DOMINI PETRI DE FERMAT.

QVÆSTIO IX.

Conjecture (1637): No positive integers x, y, z can satisfy

$$x^n + y^n = z^n$$

for any natural number n greater than 2.

(1601-1665)

(b. 1953)

RESOLVED!

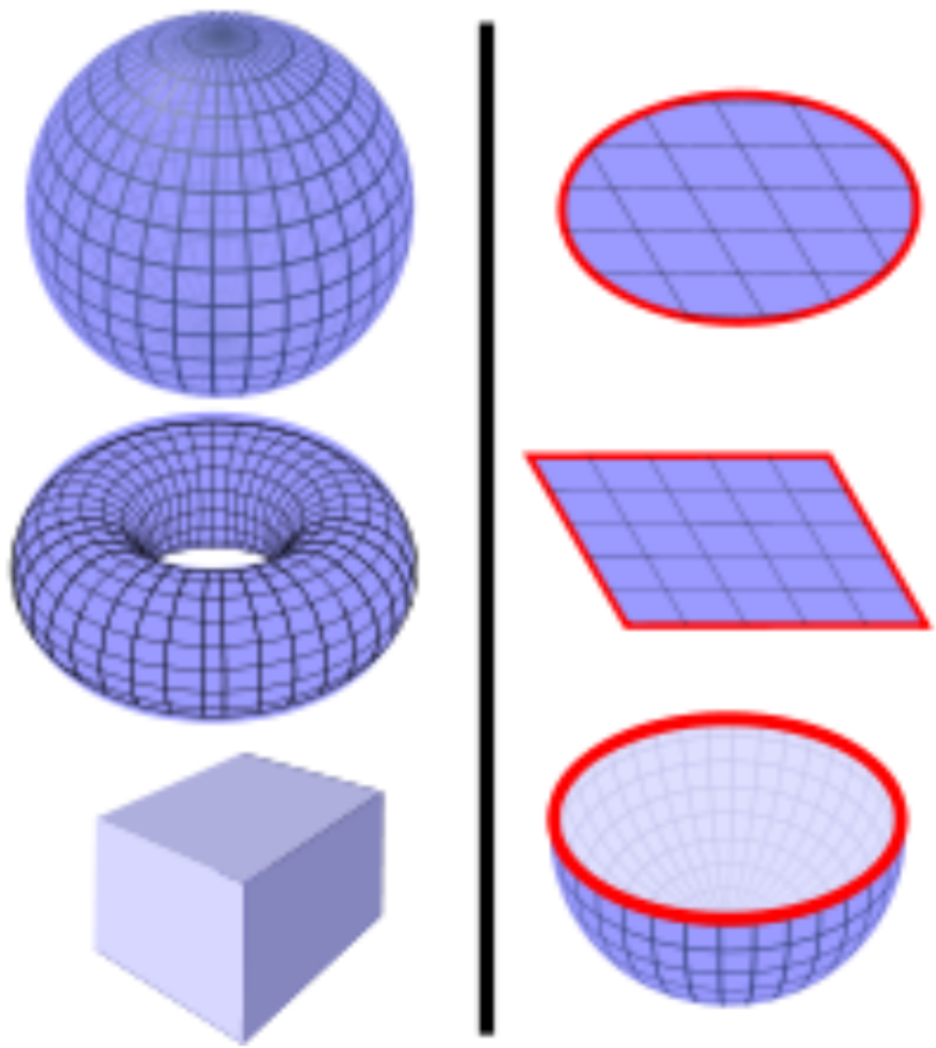
Complete proof by Andrew Wiles in 1995.

(1854-1912)

Poincaré conjecture

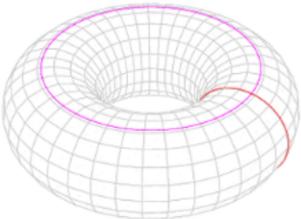
All closed simply connected 3-manifolds are homeomorphic to the 3-sphere.

Closed Not closed

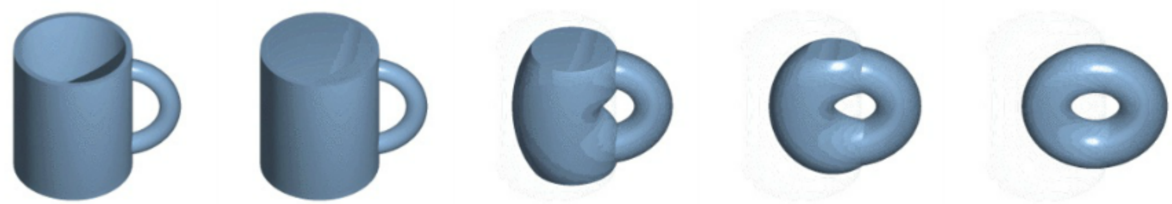


Simply connected:

Not simply connected:



Homeomorphism:



RESOLVED!

Complete proof in 2003 by Grigory Perelman, as a consequence of Thurston's geometrisation conjecture.

(1946-2012)

(b. 1943)

(b. 1966)

Riemann hypothesis

(1826-1866)

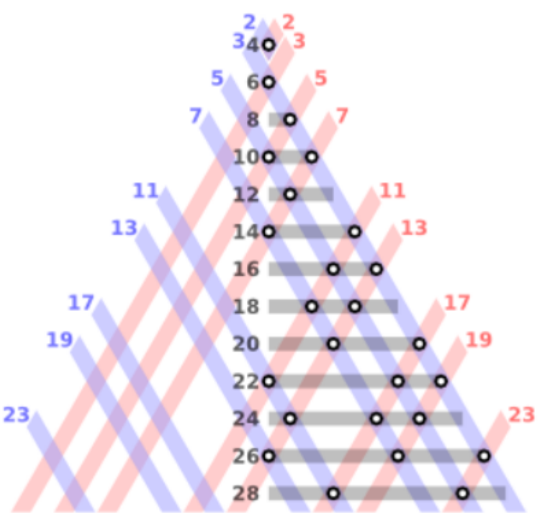
Riemann zeta function: $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Conjecture (1859): All non-trivial zeros of the Riemann zeta function have real part 1/2.

UNRESOLVED!

Goldbach's Conjecture

Conjecture (1742): Every even integer greater than 2 can be expressed as the sum of two primes.



UNRESOLVED!

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Mathematics > Algebraic Topology

The Goldbach's conjecture proved

Agostino Prástaro

(Submitted on 13 Aug 2012 (v1), last revised 19 Aug 2012 (this version, v3))

We give a direct proof of the Goldbach's conjecture in number theory, formulated in the Euler's form. The proof is also constructive, since it gives a criterion to find two prime numbers ≥ 1, such that their sum gives a fixed even number ≥ 2 (A prime number is an integer that can be divided only for itself other than for 1. In this paper we consider 1 as a prime number). The proof is obtained by recasting the problem in the framework of the Commutative Algebra and Algebraic Topology.

Comments:	15 pages
Subjects:	**Algebraic Topology (math.AT)**; Commutative Algebra (math.AC)
MSC classes:	11R04, 11T30, 11D99, 11U05, 81R50, 81T99, 20H15
Cite as:	arXiv:1208.2473v3 [math.AT]

Submission history

From: Agostino Prastaro [view email]
[v1] Mon, 13 Aug 2012 00:19:49 GMT (16kb)
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[v3] Sun, 19 Aug 2012 12:38:43 GMT (17kb)

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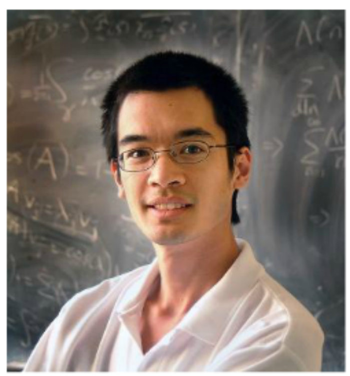
Goldbach's weak conjecture: Every odd number greater than 7 can be expressed as the sum of three odd primes.



(b. 1975)

arXiv:1201.6656v4 [math.NT] 3 Jul 2012

EVERY ODD NUMBER GREATER THAN 1 IS THE SUM OF AT MOST FIVE PRIMES

1. INTRODUCTION

EVERY ODD NUMBER GREATER THAN 1 IS THE SUM OF AT MOST FIVE PRIMES

TERENCE TAO

ABSTRACT. We prove that every odd number N greater than 1 can be expressed as the sum of at most five primes, improving the result of Ramaré that every even natural number can be expressed as the sum of at most six primes. We follow the circle method of Hardy-Littlewood and Vinogradov, together with Vaughan's identity; our additional techniques, which may be of interest for other Goldbach-type problems, include the use of smoothed exponential sums and optimisation of the Vaughan identity parameters to save or reduce some logarithmic losses, the use of multiple scales following some ideas of Bourgain, and the use of Montgomery's uncertainty principle and the large sieve to improve the L^2 estimates on major arcs. Our argument relies on some previous numerical work, namely the verification of Richstein of the even Goldbach conjecture up to 4×10^{14}, and the verification of van de Lune and (independently) of Wedeniwski of the Riemann hypothesis up to height 3.29×10^9.

1. INTRODUCTION

Two of most well-known conjectures in additive number theory are the even and odd Goldbach conjectures, which we formulate as follows[1]:

Conjecture 1.1 (Even Goldbach conjecture). *Every even natural number x can be expressed as the sum of at most two primes.*

Conjecture 1.2 (Odd Goldbach conjecture). *Every odd number x larger than 1 can be expressed as the sum of at most three primes.*

It was famously established by Vinogradov [47], using the Hardy-Littlewood circle method, that the odd Goldbach conjecture holds for all sufficiently large odd x. Vinogradov's argument can be made effective, and various explicit thresholds for "sufficiently large" have been given in the literature; in particular, Chen and Wang [5] established the odd Goldbach conjecture for all $x \geqslant \exp(\exp(11.503)) \approx \exp(99012)$, and Liu & Wang [24] subsequently extended this result to the range $x \geqslant \exp(3100)$. At the other extreme, by combining Richstein's numerical verification [41] of the even Goldbach conjecture for

How to come up with a conjecture?

- accumulated evidence

- generalisation to a larger set

- the 3 I's: inspiration - intuition - insight

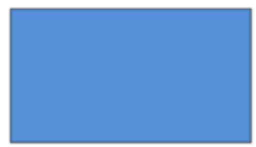
Consequences of counterexamples

- need to weaken the conjecture

- need to strengthen the conjecture

- vacuous statement

Original Conjecture: All rectangles are squares.

Counterexample: 

Weaker conjecture: All rectangles have four sides.

Original Conjecture:	All rectangles are squares.
Strengthened conjecture:	All shapes that are rectangles and have four sides of equal length are squares.

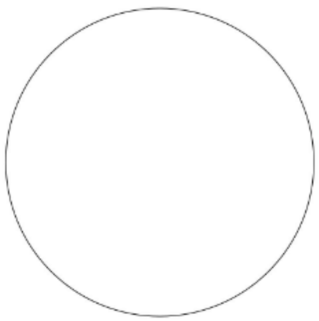
Original Conjecture: For any set, every infinite subset has precisely seven elements.

Counterexample: Natural numbers is an infinite subset of the integers.

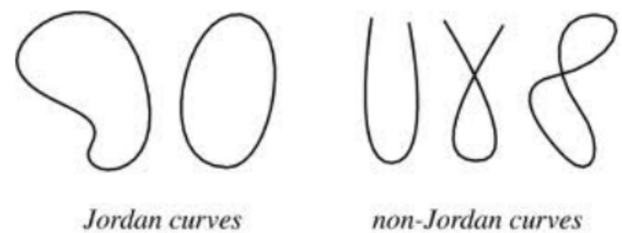
Problem: Statement about the empty set.

Geometry

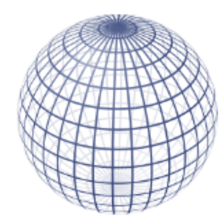
Circle in the plane



In 1906, Arthur Schönflies showed that for every non-intersecting closed curve, the interior and exterior regions are homeomorphic to the interior and exterior regions defined by the circle.



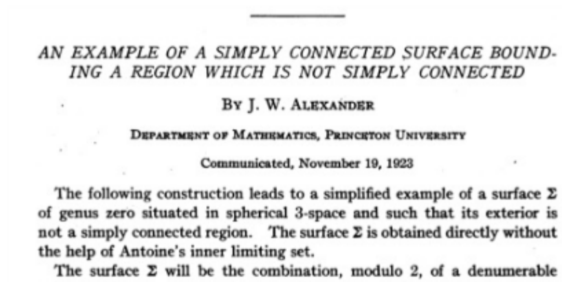
True in higher dimensions?



Counterexample: Alexander Horned Sphere

(1888-1971)

In 1923, James Waddell Alexander II constructed a surface that is homeomorphic to the 2-sphere. But, the exterior region is not simply connected!



AN EXAMPLE OF A SIMPLY CONNECTED SURFACE BOUNDING A REGION WHICH IS NOT SIMPLY CONNECTED

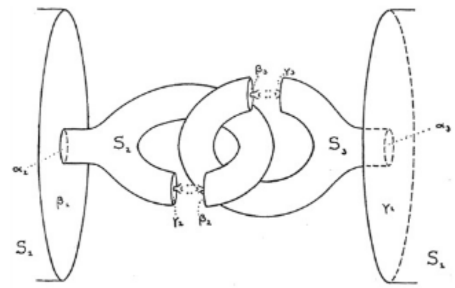
By J. W. Alexander

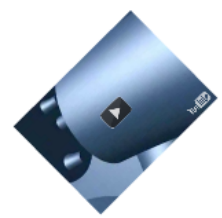
Department of Mathematics, Princeton University

Communicated, November 19, 1923

The following construction leads to a simplified example of a surface Σ of genus zero situated in spherical 3-space and such that its exterior is not a simply connected region. The surface Σ is obtained directly without the help of Antoine's inner limiting set.

The surface Σ will be the combination, modulo 2, of a denumerable





Algebra

(1707-1783)

In 1729, Christian Goldbach wrote a letter to Leonhard Euler:

"Note the observation by Fermat that all numbers of the form 2^(2^r)+1, that is 3,5,17 etc., are primes, which he himself admits that he was not able to prove , and as far as I know, nobody else has proved it either."

Today, numbers of the form 2^(2^r)+1 are called Fermat numbers, and the statement is called Fermat's conjecture.

In 1732, Euler published a 5-page paper with a counterexample and six additional conjectures of his own, the first of which is Fermat's little theorem:

For any prime number p and any integer x, p divides the integer x^p-x.

Euler gave the first published proof of this conjecture in 1736, but the result was already known to Fermat (without proof) and Leibniz (unpublished proof, 1683).

Counterexample: Euler's strategy was to determine the possible forms of the primes p dividing the numbers $2^{(2^r)}+1$, e.g.

- if p divides 2^4+1, then $p=4k+1$
- if p divides $2^{32}+1$, then $p=64k+1$

Now, $2^{32}+1$ = 4,294,967,297. By checking only 5 cases (k=3,4,7,9,10), he concluded that 4,294,967,297 = 641 x 6,700,417.

Analysis

Problem with infinite sums: $$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots = \infty$$

We are interested in convergent sums: $$\left| \sum_{n=1}^{\infty} a_n \right| < \infty$$

Now, clearly $$\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} b_n < \infty$$

Question

$$b_n \le a_n \text{ for all } n \text{ and } \left|\sum_{n=1}^{\infty} a_n\right| < \infty \quad \Rightarrow \quad \left|\sum_{n=1}^{\infty} b_n\right| < \infty \text{ ?}$$

Accumulate evidence

$$a_n = \frac{1}{n^2}, \quad b_n = \frac{1}{n^3} \quad \Rightarrow \quad b_n \le a_n$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \approx 1.645 \qquad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3) \approx 1.206$$

Counterexample

$$a_n = 0,\ b_n = -\frac{1}{n} \quad \Rightarrow \quad b_n \leq a_n \text{ for all } n$$

$$\left| \sum_{n=1}^{\infty} a_n \right| = 0 < \infty \ \text{ but } \ \left| \sum_{n=1}^{\infty} b_n \right| = \infty$$

Strengthened Conjecture

$$|b_n| \leq |a_n| \text{ for all } n \text{ and } \left|\sum_{n=1}^{\infty} a_n\right| < \infty \quad \Rightarrow \quad \left|\sum_{n=1}^{\infty} b_n\right| < \infty$$

Counterexample

$$a_n = \frac{(-1)^{n+1}}{n}, \ b_n = \frac{1}{n} \ \Rightarrow \ |b_n| \leq |a_n|$$

$$\sum_{n=1}^{\infty} a_n = \ln(2) < \infty, \quad \sum_{n=1}^{\infty} b_n = \infty$$

Sums that converge, but not absolutely,

$$\sum_{n=1}^{\infty} |a_n| = \infty$$

are called conditionally convergent.

In 1854, Riemann proved that a conditionally convergent sum can take any value by rearrangement of the terms in the sum!

Even if this wasn't an example of an exemplary talk, I sure hope it wasn't a counterexample.

Thank you!