MODULI SPACES OF FLAT MIXED FIELDS AND T-DUALITY

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Abstract. We study the geometry of mixed fields, consisting of a connection and a B-field on a principal circle bundle over a Riemann surface, from the perspective of gauge theory and T-duality. Motivated by the foundational work of Atiyah–Bott and Segal, we introduce a twisted Yang–Mills functional whose critical locus, in the flat case, is governed by the simultaneous vanishing of the curvature and the H-flux. We show that the gauge group is a semi-direct product of abelian groups parametrised by an integer λ. The moduli spaces are constructed by presymplectic reduction and shown to be Heisenberg contact manifolds for $\lambda \neq 0$, whose topology we characterise completely. We show that T-duality preserves the twisted Yang–Mills functional and acts on the configuration space of flat mixed fields. We identify the subgroups of gauge transformations that are compatible with the T-duality map and describe the induced action on the singular quotient of T-dualizable flat mixed fields. Precisely at $\lambda = 1$ does T-duality descend to an involutive contactomorphism of the moduli space.

1. Introduction

1.1. **Background and motivation.** T-duality in string theory can be understood as a transformation acting on the worldsheet fields of a two-dimensional nonlinear sigma model [25]. In the presence of supersymmetry, this duality has been studied in [33, 27]. When the target manifold admits an abelian isometry which preserves the worldsheet action, the construction becomes particularly transparent: one obtains not only an explicit transformation of the worldsheet fields, but also the Buscher rules [13], which determine the dual background. The procedure is symmetric in nature, in the sense that starting from either the original theory or its dual, the same gauging argument reproduces the other side of the correspondence [2].

From a geometric point of view, T-duality for principal circle bundles with flux was formulated in [7, 8]. Related perspectives involving twisted K-theory, bundle gerbes and D-branes may be found in [5, 6, 35]. Topological and noncommutative formulations were developed in [12, 34], while the relation with generalized geometry is discussed in [14]. Let Z be a principal $\mathbb{T}$-bundle over a manifold X

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & Z \\ & & \downarrow{\scriptstyle \pi} \\ & & X, \end{array}$$

equipped with an H-flux, that is, a closed 3-form $H \in \Omega^3(Z)$ with integral periods (we suppress the factor $\frac{1}{2\pi i}$ for simplicity). Let $\{U_\alpha\}$ be a good cover of X, and let A be a connection 1-form on Z. We choose local 2-forms $B = \{B_\alpha\}$ such that

$$H|_{\pi^{-1}(U_\alpha)} = dB_\alpha, \qquad B_\alpha \in \Omega^2(\pi^{-1}(U_\alpha)).$$

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Associated to this data (Z, A, B, H), there is a canonical T-dual quadruple $(\widehat{Z}, \widehat{A}, \widehat{B}, \widehat{H})$, where

$$\begin{tikzcd} \widehat{\mathbb{T}} \arrow[r] & \widehat{Z} \arrow[d, "\widehat{\pi}"'] \\ & X \end{tikzcd}$$

is a principal $\widehat{\mathbb{T}}$-bundle over X, equipped with the dual connection $\widehat{A}$, dual B-fields $\{\widehat{B}_\alpha\}$, and dual flux $\widehat{H}$. The dual pair is characterized by the relations

$$\pi_*(H) = F_{\widehat{A}}, \qquad \widehat{\pi}_*(\widehat{H}) = F_A,$$

where $F_A = dA$ and $F_{\widehat{A}} = d\widehat{A}$ denote the curvatures. In this way, one obtains a T-duality transformation

$$\mathcal{T} \colon (Z, A, B) \longmapsto (\widehat{Z}, \widehat{A}, \widehat{B}).$$

A fundamental consequence of this correspondence is that it induces isomorphisms in twisted cohomology and twisted K-theory between the dual spaces [7, 8]. A more detailed review will be given in Section 5.1.

From the viewpoint of gauge theory, one is naturally led to consider abelian gauge fields on Z together with B-fields encoding gerbe-theoretic data. The work of Atiyah and Bott [3] revealed that the space of connections carries a natural symplectic form, and the gauge group acts in a Hamiltonian fashion. This fits into the broader framework of Hamiltonian group actions, moment maps and symplectic reduction developed in [36, 18, 15, 16].

The present paper extends this Atiyah–Bott framework by incorporating B-fields into the configuration space. We therefore consider

$$\mathcal{C} = \{(A, B)\}, \tag{1.1}$$

consisting of a connection A on Z together with a B-field as in the T-duality picture. We shall refer to such pairs as *mixed fields.*

Fix a Riemannian metric h on X. The connection A determines a $\mathbb{T}$-invariant metric on Z by

$$h_A := \pi^* h + A \odot A,$$

where $A \odot A$ denotes the symmetric product. Motivated by the seminal work of Segal [37] and Brylinski [11], we introduce the *twisted Yang–Mills functional*

$$\mathrm{YM}(A, B) := \int_X |F_A|^2 \, d\,\mathrm{vol}_h + \int_Z H \wedge *_{h_A} H,$$

which extends the classical Yang–Mills functional by the additional term involving the H-flux. The second term measures the L^2-energy of the H-flux with respect to the metric h_A determined by the connection A.

We will see that when $X = \Sigma$ is a closed Riemannian surface of genus g and $\langle c_1(Z), [\Sigma] \rangle = 0$, the minima of this functional are precisely those mixed fields satisfying

$$F_A = 0, \qquad H = 0,$$

which we call *flat mixed fields.* Moreover, the two-dimensionality of the base allows the gauge group of the circle bundle and the symmetry group of the B-field to couple, so rather than acting independently, they combine into a semi-direct product indexed by a *level* $\lambda \in \mathbb{Z}$. The resulting moduli spaces of flat mixed fields are shown to exhibit a Heisenberg-type geometry, that is a principal circle bundle over the torus $T^{2g} \times T^{2g}$ whose topology is governed by the standard symplectic form multiplied by λ.

We then turn to T-duality, showing that it preserves the twisted Yang–Mills functional and gives rise to a natural involution on a suitably defined T-dualizable moduli stack of flat mixed fields. At $\lambda = 1$, the stack collapses onto the moduli space and T-duality descends to an involutive contactomorphism of the moduli space itself.

Two natural directions remain: higher-rank torus fibrations, and the extension from $U(1)$ to $SU(N)$ twisted Yang–Mills theory, where spherical T-duality should provide the appropriate counterpart. Both will be pursued elsewhere.

1.2. **Main results.** In the following, we describe the main results of this paper in more detail.

1.2.1. *Presymplectic reduction and the moduli space of flat mixed fields.*

A first basic result (Proposition 2.5) is

$$\begin{aligned}
&\mathrm{YM}(A,B)\\
&= \int_\Sigma |F_A|^2 \, d\,\mathrm{vol}_h + \int_Z H \wedge *_{h_A} H \\
&= \int_\Sigma \left| F_A - \frac{2\pi i \, \langle c_1(Z), [\Sigma]\rangle}{\mathrm{Area}_h(\Sigma)} \, d\,\mathrm{vol}_h \right|^2 d\,\mathrm{vol}_h + \frac{4\pi^2}{\mathrm{Area}_h(\Sigma)} \langle c_1(Z), [\Sigma]\rangle^2 + \int_Z f^2 \, d\,\mathrm{vol}_{h_A},
\end{aligned} \tag{1.2}$$

where

$$H = f \, d\,\mathrm{vol}_{h_A} .$$

The corresponding Euler–Lagrange equations are

$$\begin{cases} d_h^* F_A = 0, \\ d_{h_A}^* H = 0. \end{cases}$$

It follows from (1.2) that when

$$\langle c_1(Z), [\Sigma]\rangle = 0,$$

the absolute minima are precisely the flat mixed fields satisfying

$$F_A = 0, \qquad H = 0.$$

The case of non-trivial degree,

$$\langle c_1(Z), [\Sigma]\rangle \neq 0$$

is also of interest, corresponding to *projectively flat mixed fields*, but we defer this to future study.

The configuration space $\mathcal{C}$ in (1.1) carries, for each integer $\lambda \in \mathbb{Z}$, a natural *level-λ action* of the group

$$\mathbb{G}_\lambda = \mathbb{G}_1 \ltimes_{\rho_\lambda} \mathbb{G}_2, \qquad \mathbb{G}_1 = C^\infty(\Sigma, \mathbb{T}), \quad \mathbb{G}_2 = \mathrm{Pic}_{\mathbb{T}}^{\nabla}(Z),$$

where $C^\infty(\Sigma, \mathbb{T})$ denotes the group of smooth circle-valued functions on Σ, while $\mathrm{Pic}_{\mathbb{T}}^{\nabla}(Z)$ is the Picard group of $\mathbb{T}$-equivariant line bundles over Z equipped with $\mathbb{T}$-invariant connections. The integrality of λ is exactly what makes the semi-direct group law close (Lemma 2.3), and we shall refer to $\mathbb{G}$ as the *gauge group of mixed fields.* The corresponding Lie algebras are

$$\mathfrak{g}_1 := \mathrm{Lie}(\mathbb{G}_1) = C^\infty(\Sigma, \mathbb{R}),$$

where the infinitesimal action of $f \in C^\infty(\Sigma, \mathbb{R})$ on $\Omega^1(\Sigma)$ is given by df, and by Section 5 of [28],

$$\mathfrak{g}_2 := \mathrm{Lie}(\mathbb{G}_2) = \Omega^1(Z)^{\mathbb{T}},$$

the space of $\mathbb{T}$-invariant 1-forms on Z. For $\omega \in \Omega^1(Z)^{\mathbb{T}}$, the induced infinitesimal action on $\Omega^2(Z)^{\mathbb{T}}$ is given by $d\omega$.

A first indication that $\mathbb{G}_\lambda$ is the correct symmetry group is provided by Proposition 2.5, where we prove that the twisted Yang–Mills functional is invariant under its action. The terminology "gauge group" is further justified in Section 3. There we show that the action of $\mathrm{Pic}^{\nabla}_{\mathbb{T}}(Z)$ on B-fields admits a natural interpretation in loop space geometry. More precisely, it corresponds to gauge transformations of the connections on the holonomy line bundles over the free loop space LZ associated to the B-fields on Z.

We next return to the geometry of the configuration space itself. In Section 2, we construct a natural presymplectic form Ω on $\mathcal{C}$ which is degenerate, but nevertheless retains enough structure to support a Hamiltonian description of the gauge action. Indeed, the action of $\mathbb{G}_\lambda$ is Hamiltonian, with moment map

$$\mu(A, B) = \int_\Sigma F_A(\cdot) + \int_\Sigma \iota_v H \wedge (\iota_v \cdot) \in \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*,$$

as shown in Proposition 2.6. The twisted Yang–Mills functional may be recovered directly from the moment map, namely with respect to the natural L^2-metric, Proposition 2.7 establishes the identity

$$\mathrm{YM}(A, B) = |\mu(A, B)|^2.$$

The *moduli space of flat mixed fields* is defined by the presymplectic reduction

$$\mu^{-1}(0)/\mathbb{G}_\lambda,$$

where $\mu^{-1}(0)$ consists precisely of those configurations satisfying

$$F_A = 0, \qquad H = 0.$$

Theorem A (Theorems 4.2, 4.3 and 4.5). Let $\lambda \in \mathbb{Z} \setminus \{0\}$.

(*i*) The moduli space $\mu^{-1}(0)/\mathbb{G}_\lambda$ is a Heisenberg contact manifold of dimension $4g+1$, that is a principal circle bundle over the torus $T^{2g} \times T^{2g}$ with first Chern class

$$c_1\big(\mu^{-1}(0)/\mathbb{G}_\lambda\big) = -\lambda \sum_{j,k} (\Theta_g)_{jk}\, \alpha_j \cup \beta_k,$$

where

$$\Theta_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

is the standard symplectic matrix in $Sp(2g, \mathbb{R})$, $\alpha_j \in H^1(T^{2g}, \mathbb{Z})$ is the dual basis to the coordinate x_j and $\beta_k \in H^1(T^{2g}, \mathbb{Z})$ is the dual basis to the coordinate y_k.

(*ii*) The fundamental group of $\mu^{-1}(0)/\mathbb{G}_\lambda$ is a $\mathbb{Z}$ central extension of $\mathbb{Z}^{2g} \times \mathbb{Z}^{2g}$,

$$\pi_1\big(\mu^{-1}(0)/\mathbb{G}_\lambda\big) = (\mathbb{Z}^{2g} \times \mathbb{Z}^{2g}) \times_{c_\lambda} \mathbb{Z},$$

with the group law

$$(\mathbf{m}, \mathbf{n}, \ell)(\mathbf{m}', \mathbf{n}', \ell') = \big(\mathbf{m} + \mathbf{m}', \mathbf{n} + \mathbf{n}', \ell + \ell' + c_\lambda((\mathbf{m}, \mathbf{n}), (\mathbf{m}', \mathbf{n}'))\big)$$

and 2-cocycle

$$c_\lambda((\mathbf{m},\mathbf{n}),(\mathbf{m}',\mathbf{n}')) = \mathbf{n}'\Theta_g\mathbf{m}^T + (\lambda-1)\,\mathbf{m}'\Theta_g\mathbf{n}^T.$$

(iii) The integral cohomology and K-theory groups are given by

$$H^k\big(\mu^{-1}(0)/\mathbb{G}_\lambda,\mathbb{Z}\big) \cong \mathbb{Z}^{b_k} \oplus \bigoplus_{j\geq 1} \mathbb{Z}_{|\lambda|j}^{\varepsilon_{k-2}(j)},$$

and

$$K^n(\mu^{-1}(0)/\mathbb{G}_\lambda) \cong \mathbb{Z}^{\binom{4g+1}{2g}} \oplus \bigoplus_{\substack{k\equiv_2 n\\ j\geq 1}} \mathbb{Z}_{|\lambda|j}^{\varepsilon_{k-2}(j)},$$

with the Betti numbers

$$b_k = \begin{cases} \binom{4g}{k} - \binom{4g}{k-2}, & 0\leq k\leq 2g,\\ \binom{4g}{k-1} - \binom{4g}{k+1}, & 2g+1\leq k\leq 4g+1,\end{cases}$$

and torsion multiplicities

$$\varepsilon_r(j) = \binom{4g}{r+2-2j} - \binom{4g}{r-2j}$$

for $0\leq r\leq 2g-1$, and extended to $2g\leq r\leq 4g-2$ by $\varepsilon_r(j) = \varepsilon_{4g-2-r}(j)$.

The cohomology of Heisenberg manifolds was computed by Lee and Packer in [31, Thm. 2.1] and the K-theory at $\lambda=1$ is [30, Thm. 3.9]. What the present paper contributes is the K-theory for $|\lambda|\geq 2$, which to our knowledge does not appear in the literature and which follows from the λ-uniform integral intertwiner of Lemma 4.4.

At $\lambda=0$, the gauge group becomes a product $\mathbb{G}_1\times\mathbb{G}_2$ and each factor acts independently on the mixed fields, so the moduli space degenerates to a torus T^{4g+1}.

1.2.2. *T-duality of the twisted Yang–Mills functional and the T-dualizable moduli stack.* In T-duality, the radius of the circle fibre is transformed under the Buscher rules. Hence, we consider the family of metrics

$$h_{A,R} = \pi^*h_X + R^2\,A\odot A, \qquad \widehat{h}_{\widehat{A},1/R} = \widehat{\pi}^*h_X + \frac{1}{R^2}\,\widehat{A}\odot\widehat{A},$$

where h_X is a Riemannian metric on X. Under T-duality, the fibre radius is exchanged according to

$$R \Longleftrightarrow 1/R.$$

It is therefore natural to ask whether the twisted Yang–Mills functional introduced above is compatible with this symmetry. To incorporate the radius parameter into the picture, we consider the rescaled functional

$$\mathrm{YM}(A,B)_R := \sqrt{R}\left(\int_X |F_A|^2 d\,\mathrm{vol}_h + \int_Z H\wedge *_{h_{A,R}} H\right).$$

The volume of the circle fibre scales linearly with R, while the Hodge star operator acting on forms with a vertical component introduces an additional R-dependence. The factor $\sqrt{R}$ precisely compensates for these scaling effects. The resulting functional enjoys the expected transformation law under T-duality.

Theorem B (Theorem 5.2).

$$\mathrm{YM}(A,B)_R = \mathrm{YM}(\widehat{A},\widehat{B})_{1/R}.$$

From the discussion in Section 5.1, it follows that the T-duality transformation restricts to a self-map

$$\mathcal{T}\colon \mu^{-1}(0) \longrightarrow \mu^{-1}(0).$$

This suggests that one should look for gauge transformations that are compatible with T-duality. Accordingly, we make the following definition. A gauge transformation

$$(g,(L,\nabla^L)) \in \mathbb{G}_\lambda$$

is said to be *T-dualizable* at

$$(A,B) \in \mu^{-1}(0)$$

if there exists a dual pair

$$(\widehat{g},(\widehat{L},\nabla^{\widehat{L}}))$$

such that

$$\mathcal{T}\big((g,(L,\nabla^L))\cdot(A,B)\big) = (\widehat{g},(\widehat{L},\nabla^{\widehat{L}}))\cdot\mathcal{T}(A,B).$$

The collection of all T-dualizable gauge transformations at (A,B) forms a subgroup

$$\mathbb{G}^T_{(A,B)} := \left\{(g,(L,\nabla^L)) \in \mathbb{G}_\lambda \;\middle|\; (g,(L,\nabla^L)) \text{ is T-dualizable at } (A,B)\right\} < \mathbb{G}_\lambda.$$

Let $h \in \mathbb{G}^T_{(A,B)}$, then $h\cdot(A,B) \in \mu^{-1}(0)$ and $\mathbb{G}^T_{h\cdot(A,B)} = \mathbb{G}^T_{(A,B)}$. This observation allows us to introduce an equivalence relation on $\mu^{-1}(0)$ by declaring

$$(A,B) \underset{T}{\sim} (A',B') \quad\Longleftrightarrow\quad (A',B') = h\cdot(A,B) \text{ for some } h \in \mathbb{G}^T_{(A,B)}.$$

The quotient

$$\mu^{-1}(0)\big/\underset{T}{\sim}$$

will be called the *T-dualizable moduli stack of flat mixed fields.*

We can describe $\mathbb{G}^T_{(A,B)}$ and the moduli stack more explicitly. Let $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2g}\times\mathbb{R}^{2g}$ and define

$$G_{(\mathbf{x},\mathbf{y})} := \left\{(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^{2g}\times\mathbb{Z}^{2g} \;\middle|\; (\lambda-1)\big(\mathbf{n}\Theta_g\mathbf{x}^T - \mathbf{m}\Theta_g\mathbf{y}^T\big) \in \mathbb{Z}\right\}.$$

A direct verification shows that

$$G_{(\mathbf{x},\mathbf{y})} = G_{(\mathbf{x}+\mathbf{a},\mathbf{y}+\mathbf{b})}, \qquad (\mathbf{a},\mathbf{b}) \in \mathbb{Z}^{2g}\times\mathbb{Z}^{2g}.$$

Using $G_{(\mathbf{x},\mathbf{y})}$ we introduce a relation $\approx_T$ on $\mathbb{R}^{2g}\times\mathbb{R}^{2g}\times\mathbb{R}/\mathbb{Z}$ by declaring

$$(\mathbf{x},\mathbf{y},[t]) \approx_T (\mathbf{x}',\mathbf{y}',[t']) \quad\Longleftrightarrow\quad \mathbf{x}'=\mathbf{x}+\mathbf{m},\;\; \mathbf{y}'=\mathbf{y}+\mathbf{n},\;\; [t'] = \big[t+\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y})\big]$$

for some $(\mathbf{m},\mathbf{n}) \in G_{(\mathbf{x},\mathbf{y})}$, where $\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) = \mathbf{n}\Theta_g\mathbf{x}^T + (\lambda-1)\,\mathbf{m}\Theta_g\mathbf{y}^T$ is the level-λ fibre shift.

Theorem C (Theorem 5.7).

(*i*) For every $(A,B) \in \mu^{-1}(0)$,

$$\mathbb{G}^T_{(A,B)} \;\cong\; \mathbb{G}_0 \oplus \ker l_{(A,B)},$$

where

$$\mathbb{G}_0 := \Big\{ (g,(L,\nabla^L)) \in \mathbb{G} \;\Big|\; g = e^{2\pi i f_1},\ \mathrm{hol}^v(\nabla^L) = e^{2\pi i f_2},\ f_1, f_2 \in C^\infty(\Sigma,\mathbb{R}) \Big\} < \mathbb{G}_\lambda,$$

and

$$l_{(A,B)} : \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \longrightarrow \mathbb{R}/\mathbb{Z}, \qquad (\mathbf{m},\mathbf{n}) \longmapsto \big\{ (\lambda-1)\big(\mathbf{n}\Theta_g[a_0]^T - \mathbf{m}\Theta_g[b_0]^T\big) \big\},$$

is a homomorphism. Here $a_0 \in \Omega^1_{\mathrm{cl}}(\Sigma)$ and $b_0 \in \Omega^1_{\mathrm{cl}}(\Sigma)$ are differential forms canonically determined by A and B (see (4.3)), while $[a_0]$ and $[b_0]$ denote the corresponding vectors in $H^1(\Sigma,\mathbb{R}) \cong \mathbb{R}^{2g}$ after fixing a symplectic basis.

(*ii*) The relation $\approx_T$ is an equivalence relation, and there is a natural bijection

$$\mu^{-1}(0)/\underset{T}{\sim} \cong \big(\mathbb{R}^{2g} \times \mathbb{R}^{2g} \times \mathbb{R}/\mathbb{Z}\big)/\approx_T .$$

(*iii*) The T-duality map $\mathcal{T} : \mu^{-1}(0) \longrightarrow \mu^{-1}(0)$ induces a well-defined involutive transformation

$$\mathcal{T}_* : \mu^{-1}(0)/\underset{T}{\sim} \longrightarrow \mu^{-1}(0)/\underset{T}{\sim},$$

which, under the identification in (*ii*), reads

$$\mathcal{T}_*\big[(\mathbf{x},\mathbf{y}),[t]\big] = \big[(-\mathbf{y},-\mathbf{x}),\ [t + \mathbf{x}\Theta_g\mathbf{y}^T]\big].$$

There is a canonical surjection from the T-dualizable moduli stack to the moduli space of flat mixed fields,

$$p\colon \mu^{-1}(0)/\underset{T}{\sim} \to\ \mu^{-1}(0)/\mathbb{G}_\lambda$$

where the pre-image

$$p^{-1}(A,B) \cong \mathbb{G}_\lambda/\mathbb{G}^T_{(A,B)} \cong \mathrm{im}(l_{(A,B)})$$

is determined by the action of the full gauge group $\mathbb{G}_\lambda$ relative to the T-dualizable subgroup $\mathbb{G}^T_{(A,B)}$. Hence, along p the $\mathbb{R}/\mathbb{Z}$ fibers of the two quotients are related by a fiber-wise map whose kernel is precisely the finitely generated subgroup $\mathrm{im}(l_{(A,B)}) \subset \mathbb{R}/\mathbb{Z}$. Over generic points in the base $T^{2g} \times T^{2g}$, $\mathrm{im}(l_{(A,B)})$ is a dense subgroup of rank $4g$, making $p^{-1}(A,B)$ countably infinite, while over rational points in $T^{2g} \times T^{2g}$, $\mathrm{im}(l_{(A,B)})$ is a finite cyclic group, and the fiber collapses.

We conclude the paper by presenting the moduli space as an action Lie groupoid $\mathcal{H}$ carrying a canonical $\mathbb{R}/\mathbb{Z}$-valued cocycle Φ. The kernel of Φ is a Lie groupoid $\mathcal{K}$ that presents the moduli stack. T-duality acts on $\mathcal{H}$, but descends to a genuine groupoid automorphism only on $\mathcal{K}$.

Organization of the paper. In Section 2, we introduce the configuration space of mixed fields and its gauge symmetry, define the twisted Yang–Mills functional, and establish its basic geometric properties. In particular, we construct a natural presymplectic structure on the configuration space and identify the corresponding moment map.

Section 3 provides an interpretation of the B-field gauge symmetry in terms of holonomy line bundles over the free loop space.

In Section 4, we perform the presymplectic reduction of the configuration space and determine the topology and geometry of the moduli space of flat mixed fields, proving Theorem A.

The second part of the paper is devoted to the interaction of these constructions with T-duality. Section 5.1 reviews the geometric framework of T-duality for principal circle bundles with flux. In Section 5.2, we establish the T-duality invariance of the suitably rescaled twisted Yang–Mills functional, proving Theorem B. Section 5.3 introduces the T-dualizable moduli stack and studies the induced action of T-duality on this quotient, leading to Theorem C. Finally, Section 5.4 gives a groupoid presentation of the moduli stack and describes the action of T-duality.

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2. The configuration space of mixed fields

2.1. **The configuration space of mixed fields and gauge transformations.**

Let Σ be a closed oriented 2-manifold of genus g. Fix a Riemannian metric h on Σ together with a good open cover $\mathcal{U} = \{U_\alpha\}$. Let Z be a principal $\mathbb{T}$-bundle over Σ,

$$\begin{tikzcd} \mathbb{T} \arrow[r] & Z \arrow[d, "\pi"'] \\ & \Sigma \end{tikzcd}$$

The metric h on the base, together with the standard metric on the circle, induces a natural metric on the total space Z. Let A be a connection on Z and let B be a $\mathbb{T}$-invariant B-field on Z. Motivated by topological T-duality reviewed in Section 5.1, we consider the *configuration space*

$$\mathcal{C} = \{\, (A, B) \,\} \,.$$

Let us first clarify what is meant by B, since the notation (A, B) suppresses some data. A $\mathbb{T}$-invariant B-field is a *curving* for a $\mathbb{T}$-invariant gerbe on Z. Relative to the cover $\{U_\alpha \times \mathbb{T}\}$ it consists of local 2-forms B_α subject to

$$B_\beta - B_\alpha = dA_{\alpha\beta}, \tag{2.1}$$

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma}, \tag{2.2}$$

where $(A_{\alpha\beta}, g_{\alpha\beta\gamma})$ is part of the connective structure on the gerbe, see Section 3. The class of the $\mathbb{T}$-invariant Čech 2-cocycle $g_{\alpha\beta\gamma}$ in $H^2(Z; \underline{U(1)}) \cong H^3(Z, \mathbb{Z})$ is the Dixmier–Douady class of the gerbe. Throughout we write simply B for the curving and suppress the cochain

$(A_{\alpha\beta}, g_{\alpha\beta\gamma})$ from the notation, except at places where it matters. This is harmless since the flux $H = dB$ is a global 3-form and both the presymplectic form (2.18) and the moment map (2.19) are built from H and from deformations δB, which by the discussion below are global.

With this understood, we fix the cohomology class of the cocycle $g_{\alpha\beta\gamma}$ (i.e. the H-flux), and the configuration space is the affine space

$$\mathcal{C} = \left\{ (A, B) \;:\; B = \{B_\alpha, A_{\alpha\beta}\} \text{ a } \mathbb{T}\text{-invariant cochain satisfying (2.1) and (2.2)} \right\}. \tag{2.3}$$

Consider the vector space of $\mathbb{T}$-invariant cochains,

$$V := \left\{ (b_\alpha, a_{\alpha\beta}) \;:\; b_\beta - b_\alpha = da_{\alpha\beta}, \quad a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = 0 \right\}.$$

Then $\mathcal{C}$ is a torsor over $\Omega^1(\Sigma) \times V$, the first factor acting on A and the second on $(B_\alpha, A_{\alpha\beta})$. Indeed, if $(B_\alpha, A_{\alpha\beta})$ and $(B'_\alpha, A'_{\alpha\beta})$ are two configurations with the same A, their difference $(b_\alpha, a_{\alpha\beta}) = (B'_\alpha - B_\alpha, A'_{\alpha\beta} - A_{\alpha\beta})$ satisfies $b_\beta - b_\alpha = dA'_{\alpha\beta} - dA_{\alpha\beta} = da_{\alpha\beta}$ by (2.1), and $\sum_{\text{cyc}} a_{\alpha\beta} = 0$ by (2.2) with the term $g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$ cancelling because $g_{\alpha\beta\gamma}$ is the same for both, so the difference lies in V. Conversely, adding $(b_\alpha, a_{\alpha\beta}) \in V$ to a configuration preserves (2.1) and (2.2), so the action is free and transitive. We also note that the sub-torsor at fixed gluing data, on which $a_{\alpha\beta} = 0$ and b_α is a global form, is a torsor over the space of $\mathbb{T}$-invariant 2-forms $\Omega^2(Z)^{\mathbb{T}}$.

It will be convenient to record the local decomposition

$$B_\alpha = b_{1,\alpha} + A \wedge b_{0,\alpha}, \qquad b_{1,\alpha} \in \Omega^2(U_\alpha), \quad b_{0,\alpha} \in \Omega^1(U_\alpha), \tag{2.4}$$

which separates the part of B_α with a vertical leg from the part without. On a single chart the decomposition is a canonical isomorphism $\Omega^2(U_\alpha \times \mathbb{T})^{\mathbb{T}} \cong \Omega^2(U_\alpha) \oplus \Omega^1(U_\alpha)$, where $b_{0,\alpha} = \iota_v B_\alpha$ and $b_{1,\alpha} = B_\alpha - A \wedge b_{0,\alpha}$, the latter being basic since $\iota_v b_{1,\alpha} = \iota_v B_\alpha - b_{0,\alpha} = 0$, and ι_v denoting the contraction with the fibre-generating vector field. The decomposition is however not a product $\prod_\alpha \Omega^2(U_\alpha) \times \prod_\alpha \Omega^1(U_\alpha)$ since $\{B_\alpha\}$ is subject to the gluing (2.1), which by (2.4) reads

$$b_{0,\beta} - b_{0,\alpha} = -d(\iota_v A_{\alpha\beta}), \qquad b_{1,\beta} - b_{1,\alpha} = dA_{\alpha\beta} + A \wedge d(\iota_v A_{\alpha\beta}),$$

so the local pieces are tied together across overlaps. On the flat locus of Section 4, where Z is trivialized and the gerbe data may be taken to vanish, the decomposition becomes global and will be used in that form in Lemma 4.1.

The deformation of a connection A does not change the cohomology class represented by its curvature. *For the B-field, we restrict to deformations that preserve the gluing data $A_{\alpha\beta}$, or equivalently, we deform the curving only.* Such a deformation satisfies $\delta B_\beta - \delta B_\alpha = 0$ by (2.1), and is therefore a global $\mathbb{T}$-invariant 2-form on Z. The transgression of such a 2-form gives the corresponding deformation of the connection on the holonomy line bundle over the loop space associated to the B-field; see Remark 3.1. Therefore the tangent space at a point (A, B) is

$$T_{(A,B)}\mathcal{C} \cong \Omega^1(\Sigma) \times \Omega^2(Z)^{\mathbb{T}}.$$

Let $\mathbb{G}_1 := C^\infty(\Sigma, \mathbb{T})$ be the gauge group for connections. For $g \in \mathbb{G}_1$,

$$g \cdot A \;=\; A + d\ln g.$$

Let $\mathbb{G}_2 := \mathrm{Pic}^{\nabla}_{\mathbb{T}}(Z)$ be the Picard group of $\mathbb{T}$-equivariant line bundles with $\mathbb{T}$-invariant connections over Z. Its action on B-fields is

$$(L, \nabla^L) \cdot B \ = \ \{ \, B_\alpha + (\nabla^L)^2 \, \}.$$

We call $\mathbb{G}_2$ the *gauge group of the B-fields.* In Section 3 we justify this terminology by relating the Picard group action on B-fields to gauge transformations of line bundle connections over free loop spaces (see Proposition 3.2). The relation between gerbes, holonomy and transgression can be found in [10].

Throughout, we use the notation

$$u := g^{-1} dg, \qquad \eta := F_{\nabla^L} = (\nabla^L)^2,$$

for $g \in \mathbb{G}_1$ and $\mathcal{L} = (L, \nabla^L) \in \mathbb{G}_2$. We shall make no distinction between a form on Σ and its pullback to Z. Thus u is a closed 1-form on Σ with integral periods, while η is a closed $\mathbb{T}$-invariant 2-form on Z, again with integral periods.

The vertical holonomy of ∇^L is a $\mathbb{T}$-valued function

$$\mathrm{hol}^v(\nabla^L) \colon \Sigma \longrightarrow \mathbb{T},$$

and one has

$$\iota_v \eta = - \, d \log \mathrm{hol}^v(\nabla^L). \tag{2.5}$$

In particular, $\iota_v \eta$ is basic and descends to a closed 1-form on Σ with integral periods.

Interestingly, a $\mathbb{T}$-invariant gerbe over a circle bundle over a Riemann surface admits a natural action of the gauge group $\mathbb{G}_1$ by gerbe automorphisms, indexed by an integer $\lambda \in \mathbb{Z}$. These combine into an action by a semi-direct product $\mathbb{G}_1 \ltimes_{\rho_\lambda} \mathbb{G}_2$ on the configuration space as explained below.

Definition 2.1. Let $\lambda \in \mathbb{Z}$. For $(g, \mathcal{L} = (L, \nabla^L)) \in \mathbb{G}_1 \times \mathbb{G}_2$, the associated *level-$\lambda$ transformation* of $(A, B) \in \mathcal{C}$ is defined by

$$\begin{aligned} \widetilde{A} &= A + u, \\ \widetilde{B}_\alpha &= B_\alpha + \eta + \lambda \, u \wedge \iota_v B_\alpha, \\ \widetilde{A}_{\alpha\beta} &= A_{\alpha\beta} + \lambda \, (\iota_v A_{\alpha\beta}) \, u. \end{aligned} \tag{2.6}$$

We write this transformation as

$$(g, \mathcal{L}) \cdot (A, B) = (\widetilde{A}, \widetilde{B}).$$

There are two elementary identities behind these formulas. Since u is basic and $\iota_v^2 = 0$,

$$\iota_v \big(u \wedge \iota_v B_\alpha \big) = (\iota_v u) \, \iota_v B_\alpha - u \wedge \iota_v^2 B_\alpha = 0, \tag{2.7}$$

and

$$\iota_v \widetilde{A}_{\alpha\beta} = \iota_v \big(A_{\alpha\beta} + \lambda (\iota_v A_{\alpha\beta}) u \big) = \iota_v A_{\alpha\beta}. \tag{2.8}$$

Thus the λ-dependent terms are nilpotent in the simplest possible sense, namely once inserted, they do not produce a second correction of the same kind.

There is also a dimensional fact which will be used repeatedly. Since the gerbe data is $\mathbb{T}$-invariant and $\mathcal{L}_v = d\iota_v + \iota_v d = 0$, we have

$$d \iota_v B_\alpha = - \iota_v H.$$

It follows that

$$d\big(u \wedge \iota_v B_\alpha\big) = -u \wedge d(\iota_v B_\alpha) = u \wedge \iota_v H = 0. \tag{2.9}$$

Indeed, both u and $\iota_v H$ are basic, so their wedge product is the pullback of a 3-form on the surface Σ. This is the only point at which the assumption $\dim \Sigma = 2$ enters, but in an essential way.

Lemma 2.2. *Let $(A, B) \in \mathcal{C}$, with gerbe data*

$$(B_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma}),$$

and let $(\widetilde{A}, \widetilde{B})$ be its transform under (2.6). Then

$$\widetilde{B}_\beta - \widetilde{B}_\alpha = d\widetilde{A}_{\alpha\beta},$$

and

$$\widetilde{A}_{\alpha\beta} + \widetilde{A}_{\beta\gamma} + \widetilde{A}_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma}.$$

Thus the transformed local data again defines a B-field on Z. Moreover,

$$\widetilde{H} = H, \qquad F_{\widetilde{A}} = F_A.$$

In particular, the Dixmier–Douady class of the gerbe is unchanged.

Proof. Since all the local data is $\mathbb{T}$-invariant, applying ι_v to the descent equation

$$B_\beta - B_\alpha = dA_{\alpha\beta}$$

yields

$$\iota_v B_\beta - \iota_v B_\alpha = -d(\iota_v A_{\alpha\beta}).$$

Put $h_{\alpha\beta} := \iota_v A_{\alpha\beta}$. Since $du = 0$, we have

$$d(h_{\alpha\beta} u) = dh_{\alpha\beta} \wedge u = -u \wedge dh_{\alpha\beta},$$

and it follows that

$$\lambda u \wedge \iota_v B_\beta - \lambda u \wedge \iota_v B_\alpha = -\lambda u \wedge dh_{\alpha\beta} = d(\lambda h_{\alpha\beta} u).$$

The form η is global and disappears on taking differences, so

$$\begin{aligned} \widetilde{B}_\beta - \widetilde{B}_\alpha &= dA_{\alpha\beta} + d(\lambda h_{\alpha\beta} u) \\ &= d\big(A_{\alpha\beta} + \lambda(\iota_v A_{\alpha\beta}) u\big) \\ &= d\widetilde{A}_{\alpha\beta}. \end{aligned}$$

The correction in $\widetilde{A}_{\alpha\beta}$ is thus precisely the term required to repair the descent equation.

On a triple overlap, we obtain from (2.2)

$$\widetilde{A}_{\alpha\beta} + \widetilde{A}_{\beta\gamma} + \widetilde{A}_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma} + \lambda\, \iota_v\big(g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma}\big) u.$$

The transition function $g_{\alpha\beta\gamma}$ is $\mathbb{T}$-invariant. Hence

$$\iota_v\big(g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma}\big) = g_{\alpha\beta\gamma}^{-1} v(g_{\alpha\beta\gamma}) = 0,$$

and the triple-overlap condition is unchanged. Finally, using (2.9),

$$\begin{aligned}\widetilde{H}|_{U_\alpha} &= d\widetilde{B}_\alpha \\ &= dB_\alpha + d\eta + \lambda\, d(u \wedge \iota_v B_\alpha) \\ &= H|_{U_\alpha}.\end{aligned}$$

Likewise, $F_{\widetilde{A}} = F_{A+u} = F_A$, as $du = 0$. $\square$

We now give a geometric interpretation of the group law encoded in (2.6). The mixed term which appears when two transformations are composed is the curvature of a natural line bundle on the base. In fact, let

$$p_i \colon \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T}, \qquad i = 1, 2,$$

denote the two projections. In our normalization, the curvature of the Poincaré line bundle $\mathcal{P}$ with standard connection over $\mathbb{T} \times \mathbb{T}$ is

$$F_{\nabla^{\mathcal{P}}} = p_1^*\vartheta \wedge p_2^*\vartheta.$$

For $\mathcal{L} = (L, \nabla^L)$, set

$$h_{\mathcal{L}} := \big(\mathrm{hol}^v(\nabla^L)\big)^{-1}.$$

By (2.5), we have $h_{\mathcal{L}}^{-1} dh_{\mathcal{L}} = \iota_v \eta$. Define the line bundle with connection

$$(\xi_{g,\mathcal{L}}, \nabla^{\xi_{g,\mathcal{L}}}) := (g, h_{\mathcal{L}})^*(\mathcal{P}, \nabla^{\mathcal{P}}) \longrightarrow \Sigma.$$

Its curvature is

$$F_{\nabla^{\xi_{g,\mathcal{L}}}} = u \wedge \iota_v \eta. \tag{2.10}$$

For $\lambda \in \mathbb{Z}$, define

$$\rho_\lambda(g)(\mathcal{L}) := \mathcal{L} \otimes \pi^*(\xi_{g,\mathcal{L}}, \nabla^{\xi_{g,\mathcal{L}}})^{\otimes\lambda}, \tag{2.11}$$

where negative tensor powers are understood as powers of the dual line bundle. At the level of curvature forms,

$$F_{\rho_\lambda(g)(\mathcal{L})} = \eta + \lambda\, u \wedge \iota_v \eta. \tag{2.12}$$

The Poincaré line bundle is multiplicative in each variable and gives canonical connection-preserving isomorphisms

$$\xi_{g,\mathcal{L}_1 \otimes \mathcal{L}_2} \cong \xi_{g,\mathcal{L}_1} \otimes \xi_{g,\mathcal{L}_2}$$

and

$$\xi_{g_2 g_1,\mathcal{L}} \cong \xi_{g_2,\mathcal{L}} \otimes \xi_{g_1,\mathcal{L}}.$$

Moreover, tensoring $\mathcal{L}$ with the pullback of a line bundle from Σ does not alter its vertical holonomy. Hence

$$\xi_{g_2,\rho_\lambda(g_1)(\mathcal{L})} \cong \xi_{g_2,\mathcal{L}}.$$

These identities show that

$$\rho_\lambda(g_2 g_1) = \rho_\lambda(g_2) \circ \rho_\lambda(g_1),$$

and therefore define a homomorphism

$$\rho_\lambda \colon \mathbb{G}_1 \longrightarrow \mathrm{Aut}(\mathbb{G}_2).$$

At the level of curvature forms, the same fact is reflected in the elementary identity

$$\begin{aligned}\eta + \lambda u_1 \wedge \iota_v\eta + \lambda u_2 \wedge \iota_v\big(\eta + \lambda u_1 \wedge \iota_v\eta\big)\\ = \eta + \lambda(u_1+u_2)\wedge \iota_v\eta,\end{aligned} \tag{2.13}$$

where we have used $\iota_v(u_1 \wedge \iota_v\eta) = 0$.

We may therefore form the semi-direct product

$$\mathbb{G}_\lambda := \mathbb{G}_1 \ltimes_{\rho_\lambda} \mathbb{G}_2.$$

Writing $\mathcal{L}_i = (L_i, \nabla^{L_i})$, its multiplication is

$$(g_2, \mathcal{L}_2)(g_1, \mathcal{L}_1) = \big(g_2g_1,\ \mathcal{L}_2 \otimes \rho_\lambda(g_2)(\mathcal{L}_1)\big). \tag{2.14}$$

Equivalently,

$$\begin{aligned}&(g_2, (L_2, \nabla^{L_2}))(g_1, (L_1, \nabla^{L_1}))\\ &\quad = \Big(g_2g_1,\ (L_2, \nabla^{L_2}) \otimes (L_1, \nabla^{L_1}) \otimes \pi^*(\xi_{g_2,\mathcal{L}_1}, \nabla^{\xi_{g_2,\mathcal{L}_1}})^{\otimes\lambda}\Big).\end{aligned}$$

Lemma 2.3. *For every $\lambda \in \mathbb{Z}$, the transformations* (2.6) *define an action of*

$$\mathbb{G}_\lambda = \mathbb{G}_1 \ltimes_{\rho_\lambda} \mathbb{G}_2$$

on $\mathcal{C}$.

Proof. Let

$$(g_i, \mathcal{L}_i) \in \mathbb{G}_\lambda, \qquad u_i = g_i^{-1}dg_i, \qquad \eta_i = F_{\nabla^{L_i}}, \quad i = 1, 2.$$

The line-bundle component of the product $(g_2, \mathcal{L}_2)(g_1, \mathcal{L}_1)$ has curvature

$$\eta_{21} = \eta_2 + \eta_1 + \lambda\, u_2 \wedge \iota_v\eta_1. \tag{2.15}$$

The $\mathbb{G}_1$-component has Maurer–Cartan form

$$(g_2g_1)^{-1}d(g_2g_1) = u_1 + u_2.$$

Apply first $(g_1, \mathcal{L}_1)$, and then $(g_2, \mathcal{L}_2)$. The ordinary connection transforms as

$$A \longmapsto A + u_1 + u_2.$$

For the local 2-forms, the first transformation gives

$$B'_\alpha = B_\alpha + \eta_1 + \lambda u_1 \wedge \iota_v B_\alpha.$$

By (2.7),

$$\iota_v B'_\alpha = \iota_v B_\alpha + \iota_v\eta_1.$$

The second transformation therefore gives

$$\begin{aligned}B''_\alpha &= B'_\alpha + \eta_2 + \lambda u_2 \wedge \iota_v B'_\alpha\\ &= B_\alpha + \eta_1 + \eta_2 + \lambda(u_1+u_2)\wedge\iota_v B_\alpha + \lambda u_2 \wedge \iota_v\eta_1\\ &= B_\alpha + \eta_{21} + \lambda(u_1+u_2)\wedge\iota_v B_\alpha.\end{aligned}$$

This is exactly the transformation associated with the product $(g_2, \mathcal{L}_2)(g_1, \mathcal{L}_1)$.

For the overlap 1-forms, (2.8) gives

$$\iota_v A'_{\alpha\beta} = \iota_v A_{\alpha\beta}.$$

Hence

$$\begin{aligned}A''_{\alpha\beta} &= A_{\alpha\beta} + \lambda(\iota_v A_{\alpha\beta})u_1 + \lambda(\iota_v A_{\alpha\beta})u_2\\ &= A_{\alpha\beta} + \lambda(\iota_v A_{\alpha\beta})(u_1+u_2),\end{aligned}$$

which again agrees with the transformation defined by the product element. Thus (2.6) respects the multiplication (2.14), and hence defines a group action.

For completeness, the inverse is

$$(g,\mathcal{L})^{-1} = \big(g^{-1}, \rho_\lambda(g^{-1})(\mathcal{L}^\vee)\big).$$

If $u = g^{-1}dg$ and $F_{\nabla^L} = \eta$, then the curvature of its line-bundle component is

$$-\eta + \lambda\, u \wedge \iota_v \eta.$$

□

We call $\mathbb{G}_\lambda$ *the gauge group of the mixed fields.* The integrality of λ is exactly what is needed for the group to close and is the source of the quantisation of the level. The action ρ_λ of $\mathbb{G}_1$ on $\mathbb{G}_2$ is infinite-dimensional origin of the Heisenberg extension found in Section 4.

The following proposition shows that, once one asks for a semi-direct product action built naturally from the fields, (2.6) is the only possibility.

Proposition 2.4. *Let $c : \mathbb{G}_1 \times \mathcal{C} \to \mathbb{G}_2$ be a map and suppose $c(g, A, B)$ depends on $g \in \mathbb{G}_1$ $\mathbb{R}$-linearly through $u = g^{-1}dg$, and is constructed from u, A and B using only the operations $\wedge$, d and ι_v. Suppose further that*

$$(A, B) \longmapsto \big(A + u,\ B + \eta + c(g, A, B)\big)$$

defines an action of a semi-direct product $\mathbb{G}_1 \ltimes \mathbb{G}_2$ on $\mathcal{C}$. Then

$$c(g, A, B) = \lambda\, u \wedge \iota_v B, \qquad \lambda \in \mathbb{Z}.$$

Proof. Linearity in u and the fact that c has degree 2 force $c(g, A, B) = u \wedge X$, with X a 1-form built from u, A and B using $\wedge$, d and ι_v. Since u is basic, $\iota_v u = 0$ and $u \wedge u = 0$, so any u-dependent term of X does not contribute to $u \wedge X$, and we may take X built from A and B alone. Applying the operations to A and B produces A, B, $H = dB$, $F_A = dA$, $\iota_v B$, $\iota_v H$, $d\iota_v B$, together with $\iota_v A = 1$ and the vanishing combinations $\iota_v \iota_v B = 0$ and $\iota_v F_A = \iota_v dA = -\,d(\iota_v A) = 0$. Contractions of mixed wedge products reduce to these, for instance $\iota_v(A \wedge \iota_v B) = (\iota_v A)\,\iota_v B - A \wedge \iota_v \iota_v B = \iota_v B$. Of the resulting nonzero forms, only $\iota_v B$ and A have degree 1. Hence $X = \lambda\, \iota_v B + \kappa\, A$ for scalars λ, κ, that is

$$c(g, A, B) = \lambda\, u \wedge \iota_v B + \kappa\, u \wedge A.$$

Composing two transformations, (g_1, η_1) and then (g_2, η_2), the second uses the already-transformed connection $A + u_1$, so the B-component acquires

$$\big(\eta_1 + \eta_2\big) + \lambda\big(u_1 + u_2\big) \wedge \iota_v B + \lambda\, u_2 \wedge \iota_v \eta_1 + \kappa\big(u_1 + u_2\big) \wedge A + \kappa\, u_2 \wedge u_1,$$

where we used $\iota_v(u_1 \wedge \iota_v B) = 0$ from (2.7). The terms proportional to $(u_1 + u_2)$ are precisely the correction for the composite parameter, and $\lambda\, u_2 \wedge \iota_v \eta_1$ is the twist of Lemma 2.3. The remaining term $\kappa\, u_2 \wedge u_1$ is a closed 2-form on Σ that pairs two elements of $\mathbb{G}_1$, so it is the 2-cocycle of a central extension of $\mathbb{G}_1$ by $\mathbb{G}_2$, rather than a term in an action of $\mathbb{G}_1$ on $\mathbb{G}_2$. To obtain a semi-direct product $\mathbb{G}_1 \ltimes \mathbb{G}_2$, we must therefore have $\kappa = 0$.

The term $\lambda\, u_2 \wedge \iota_v \eta_1$ is a closed basic 2-form whose period $\lambda \int_\Sigma u_2 \wedge \iota_v \eta_1$ must be integral for the composite to lie in $\mathbb{G}_2$. As u_2 and $\iota_v \eta_1$ have integral periods, this forces $\lambda \in \mathbb{Z}$. □

Clearly,

$$\mathfrak{g}_1 := \mathrm{Lie}(\mathbb{G}_1) = C^\infty(\Sigma, \mathbb{R}),$$

and for $f \in C^\infty(\Sigma, \mathbb{R})$ the associated Killing vector field on $\mathcal{C}$ is

$$\rho(f) = \big(df,\ \lambda\, df \wedge \iota_v B\big) \in \Omega^1(\Sigma) \times \Omega^2(Z)^{\mathbb{T}}, \tag{2.16}$$

together with the deformation $\lambda\,(\iota_v A_{\alpha\beta})\, df$ of the gluing data. Moreover (cf. Sec. 5 in [28]),

$$\mathfrak{g}_2 := \mathrm{Lie}(\mathbb{G}_2) = \Omega^1(Z)^{\mathbb{T}},$$

the space of $\mathbb{T}$-invariant 1-forms on Z. For $\omega \in \Omega^1(Z)^{\mathbb{T}}$, the induced Killing vector field on $\Omega^2(Z)^{\mathbb{T}}$ is $d\omega$.

2.2. **Twisted Yang-Mills functional.** Motivated by Segal's paper [37], we introduce the *twisted Yang-Mills functional*

$$\mathrm{YM}(A, B) := \int_\Sigma |F_A|^2\, d\,\mathrm{vol}_h + \int_Z H \wedge *_{h_A} H,$$

where the twisted term $\int_Z H \wedge *_{h_A} H$ is added into the usual Yang-Mills functional and measures the L^2-energy of the H-flux with respect to the metric h_A. Thus the functional couples the gauge field A and the B-field through the geometry of the total space Z.

Proposition 2.5. *(i) The twisted Yang-Mills functional* $\mathrm{YM}(A, B)$ *is gauge invariant.*
(ii) One has the identity

$$\begin{aligned} &\mathrm{YM}(A, B) \\ &= \int_\Sigma \left| F_A - \frac{2\pi i\, \langle c_1(Z), [\Sigma]\rangle}{\mathrm{Area}_h(\Sigma)}\, d\,\mathrm{vol}_h \right|^2 d\,\mathrm{vol}_h \\ &\quad + \frac{4\pi^2}{\mathrm{Area}_h(\Sigma)} \langle c_1(Z), [\Sigma]\rangle^2 + \int_Z f^2\, d\,\mathrm{vol}_{h_A}, \end{aligned} \tag{2.17}$$

where $H = f\, d\,\mathrm{vol}_{h_A}$.
(iii) The Euler–Lagrange equations of $\mathrm{YM}(A, B)$ *are*

$$\begin{cases} d_h^* F_A = 0, \\ d_{h_A}^* H = 0. \end{cases}$$

Proof. (i) The connection transforms by $A \mapsto A + u$ with u closed and basic, so $F_{g\cdot A} = F_A$ and

$$\mathrm{dvol}_{h_{g\cdot A}} = (A+u) \wedge \pi^*(\mathrm{dvol}_h) = A \wedge \pi^*(\mathrm{dvol}_h) = \mathrm{dvol}_{h_A},$$

since $u \wedge \pi^*(\mathrm{dvol}_h)$ is a basic 3-form on Σ. In particular the metric h_A, and hence the Hodge star $*_{h_A}$, are unchanged. By Lemma 2.2, the transformed data again patches to a B-field on Z with $\widetilde{H} = H$, thus both F_A and H are strictly invariant for every level λ, and therefore so is $\mathrm{YM}(A, B)$. Writing $H = f\, \mathrm{dvol}_{h_A}$, one has

$$\int_Z H \wedge *_{h_A} H = \int_Z f^2\, d\,\mathrm{vol}_{h_A}.$$

(ii) Since Σ is two-dimensional, any $i\mathbb{R}$-valued 2-form is proportional to the volume form. Thus

$$F_A = f_A\, d\,\mathrm{vol}_h, \qquad f_A \in C^\infty(\Sigma; i\mathbb{R}).$$

By Chern–Weil theory,
$$\int_\Sigma F_A = 2\pi i\,\langle c_1(Z), [\Sigma]\rangle,$$
and hence
$$\int_\Sigma f_A\, d\,\mathrm{vol}_h = 2\pi i\,\langle c_1(Z), [\Sigma]\rangle.$$

Let
$$\bar f := \frac{1}{\mathrm{Area}_h(\Sigma)} \int_\Sigma f_A\, d\,\mathrm{vol}_h = \frac{2\pi i\,\langle c_1(Z), [\Sigma]\rangle}{\mathrm{Area}_h(\Sigma)}.$$
Decomposing $f_A = (f_A - \bar f) + \bar f$ and integrating, the cross term vanishes, so
$$\int_\Sigma |F_A|^2\, d\,\mathrm{vol}_h = \int_\Sigma |f_A - \bar f|^2\, d\,\mathrm{vol}_h + |\bar f|^2\ \mathrm{Area}_h(\Sigma).$$
Substituting back yields exactly (2.17).

(iii) We compute the first variation. Note that
$$d\,\mathrm{vol}_{h_A} = A \wedge \pi^*(d\,\mathrm{vol}_h).$$
For a variation $A_t = A + t\,\pi^* a$, one checks that
$$d\,\mathrm{vol}_{h_{A_t}} = d\,\mathrm{vol}_{h_A},$$
since $\pi^* a \wedge \pi^*(d\,\mathrm{vol}_h) = 0$. In particular, the term
$$\int_Z H \wedge *_{h_A} H = \int_Z f^2\, d\,\mathrm{vol}_{h_A}$$
is independent of A.

Now let $A_t = A + t\,\pi^* a$ with $a \in \Omega^1(\Sigma)$, so that $F_{A_t} = F_A + t\,da$, and hence
$$\frac{d}{dt}\Big|_{t=0} \int_\Sigma |F_{A_t}|^2\, d\,\mathrm{vol}_h = 2\int_\Sigma \langle F_A, da\rangle\, d\,\mathrm{vol}_h = 2\int_\Sigma \langle d_h^* F_A, a\rangle\, d\,\mathrm{vol}_h\,.$$
Thus $d_h^* F_A = 0$.

Similarly, for $B_t = B + t\,\beta$, one has $H_t = H + t\,d\beta$, and therefore
$$\frac{d}{dt}\Big|_{t=0} \int_Z H_t \wedge *_{h_A} H_t = 2\int_Z \langle H, d\beta\rangle\, d\,\mathrm{vol}_{h_A} = 2\int_Z \langle d_{h_A}^* H, \beta\rangle\, d\,\mathrm{vol}_{h_A}\,.$$
Hence $d_{h_A}^* H = 0$.

Combining the two equations gives the result. $\square$

2.3. **Hamiltonian geometry on the configuration space.** On $\mathcal{C}$ define a 2-form
$$\Omega := \int_\Sigma \big[\ \delta A \wedge \delta A\ +\ \iota_v \delta B \wedge \iota_v \delta B\ \big], \tag{2.18}$$
with v the vector field generating the $\mathbb{T}$-action along the fibres of $Z \to \Sigma$. This 2-form is translation invariant on each model tangent space. It is also clear that Ω degenerates along the b_1-direction of (2.4), since $\iota_v B_\alpha = b_{0,\alpha}$, so that Ω sees only δA and δb_0, and not δb_1.

The form Ω is insensitive to the level $\lambda \in \mathbb{Z}$. Indeed, by (2.16) the twist contributes $\lambda\, df \wedge \iota_v B$ to the Killing vector field of $f \in \mathfrak{g}_1$, and by (2.7) this contribution satisfies $\iota_v(\lambda\, df \wedge \iota_v B) = 0$. It therefore drops out of the second term of (2.18) identically, while leaving the first term untouched. *Consequently* Ω*, its moment map, and the identity* $\mathrm{YM} = |\mu|^2$ *below are the same for every* λ*, so the level is invisible to the presymplectic geometry of* $\mathcal{C}$*.*

Proposition 2.6. *The action of* $\mathbb{G}_\lambda$ *on* $\mathcal{C}$ *given by* (2.6) *is Hamiltonian for the* 2*-form* (2.18)*, with moment map*

$$\mu(A,B) = \int_\Sigma F_A \,\cdot\, + \int_\Sigma \iota_v H \wedge (\iota_v \cdot) \;\in\; \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*, \tag{2.19}$$

where $F_A = dA$ *and* $H = dB$*.*

Proof. For any $f \in C^\infty(\Sigma, \mathbb{R})$, one has

$$\int_\Sigma \delta F_A \cdot f \;=\; \int_\Sigma d(\delta A)\, f \;=\; \int_\Sigma df \wedge \delta A.$$

For any $\omega \in \Omega^1(Z)^{\mathbb{T}}$, one has

$$\int_\Sigma \delta(\iota_v H) \wedge \iota_v \omega \;=\; \int_\Sigma d(\iota_v \delta B) \wedge \iota_v \omega \;=\; \int_\Sigma \iota_v(d\omega) \wedge \iota_v \delta B.$$

These identities are precisely the variational statements that identify the contraction of a vector field with $d\langle \mu, (\, f, \omega\,)\rangle$, yielding (2.19). ☐

Proposition 2.7. *With respect to the induced* L^2*-metric, one has*

$$\mathrm{YM}(A,B) = |\mu(A,B)|^2.$$

Proof. We compute the squared L^2-norm of each component of the moment map. For the curvature component, one has

$$\left| \int_\Sigma F_A \,\cdot \right|^2 = \int_\Sigma |F_A|^2 \, d\,\mathrm{vol}_h \,.$$

For the second component, write $H = f\, d\,\mathrm{vol}_{h_A}$ and since

$$d\,\mathrm{vol}_{h_A} = A \wedge \pi^*(d\,\mathrm{vol}_h), \quad \text{and} \quad A(v) = 1,$$

we obtain

$$\iota_v(d\,\mathrm{vol}_{h_A}) = \pi^*(d\,\mathrm{vol}_h), \qquad \iota_v H = f\, \pi^*(d\,\mathrm{vol}_h).$$

Now for any $\omega \in \Omega^1(Z)^{\mathbb{T}}$,

$$\int_\Sigma \iota_v H \wedge (\iota_v \omega) = \int_\Sigma f\, \pi^*(d\,\mathrm{vol}_h) \wedge \iota_v \omega = \int_Z f\, A \wedge \pi^*(d\,\mathrm{vol}_h) \wedge \omega = \int_Z f\, \omega \, d\,\mathrm{vol}_{h_A} \,.$$

It follows that the L^2-norm of the second component is

$$\left| \int_\Sigma \iota_v H \wedge (\iota_v \cdot) \right|^2 = \int_Z f^2 \, d\,\mathrm{vol}_{h_A} = \int_Z H \wedge *_{h_A} H.$$

Combining the two components, we obtain

$$|\mu(A,B)|^2 = \int_\Sigma |F_A|^2 \, d\,\mathrm{vol}_h + \int_Z H \wedge *_{h_A} H = \mathrm{YM}(A,B),$$

as claimed. ☐

3. Picard transformation and abelian gauge theory on free loop spaces

In this section we explain how the action of the Picard group on B-fields is reflected on the free loop space. For differential forms and the fundamental equivariant cohomology theory on loop spaces and relation to index theory, see [1, 4]. Related developments involving T-duality and loop space geometry may also be found in [19, 26, 17, 20, 21, 22, 23, 24].

Recall that a B-field on a smooth manifold M determines a holonomy line bundle

$$\mathcal{L}^B \longrightarrow LM$$

equipped with a natural connection $\nabla^{\mathcal{L}^B}$. Our goal is to identify the effect of the Picard action

$$(L, \nabla^L) \in \mathrm{Pic}^{\nabla}(M)$$

on the corresponding loop space data. More precisely, we show that replacing B by

$$(L, \nabla^L) \cdot B$$

does not change the underlying holonomy line bundle, but modifies the connection $\nabla^{\mathcal{L}^B}$ by the gauge transformation

$$\mathrm{hol}(\nabla^L) : LM \to \mathbb{T},$$

where $\mathrm{hol}(\nabla^L)$ denotes the holonomy of ∇^L around loops. Thus the Picard action on gerbes is transgressed to the ordinary gauge action on the loop space.

Throughout this subsection, M is an arbitrary smooth manifold. In the T-duality applications considered later, we specialize to $M = Z$ and assume all geometric data are $\mathbb{T}$-invariant.

Let M be a smooth manifold. We recall several standard constructions on the loop space LM. The evaluation map

$$ev : S^1 \times LM \longrightarrow M, \qquad (t, \gamma) \mapsto \gamma(t),$$

induces the *transgression map*

$$\tau : \Omega^\bullet(M) \to \Omega^{\bullet-1}(LM), \qquad \tau(\xi) = \int_{S^1} ev^*\xi.$$

Given $\omega \in \Omega^i(M)$, define for each $s \in [0,1]$ a form $\widehat{\omega}_s \in \Omega^i(LM)$ by

$$\widehat{\omega}_s(X_1, \ldots, X_i)(\gamma) = \omega\big(X_1|_{\gamma(s)}, \ldots, X_i|_{\gamma(s)}\big),$$

for vector fields $X_1, \ldots, X_i$ defined near the loop γ. Since $d\widehat{\omega}_s = \widehat{d\omega}_s$, the averaged form

$$\overline{\omega} := \int_0^1 \widehat{\omega}_s \, ds \in \Omega^i(LM)$$

is invariant under the rotation action of S^1 generated by the vector field K. Moreover,

$$\tau(\omega) = \iota_K \, \overline{\omega}.$$

Let $\{\mathsf{U}_\alpha\}$ be an open cover of M. To ensure that $\{L\mathsf{U}_\alpha\}$ covers LM, we choose a *Brylinski open cover*: a maximal family satisfying

$$H^2(\mathsf{U}_{\alpha_I}) = H^3(\mathsf{U}_{\alpha_I}) = 0 \quad \text{for all finite intersections } \mathsf{U}_{\alpha_I}.$$

Such covers always exist and include, for example, tubular neighbourhoods of individual loops.

Let H be a closed 3-form on M with integral periods. Since $H^3(\mathsf{U}_\alpha)=0$, we may write

$$H|_{\mathsf{U}_\alpha} = dB_\alpha,$$

for purely imaginary 2-forms B_α, and since $H^2(\mathsf{U}_{\alpha\beta})=0$,

$$B_\beta - B_\alpha = dA_{\alpha\beta},$$

for purely imaginary 1-forms $A_{\alpha\beta}$. The triple (H,B,A) defines a connective structure of a gerbe $\mathcal{G}_B$ on M.

A geometric realization of $\mathcal{G}_B$ is a family of line bundles

$$(L_{\alpha\beta}, \nabla^L_{\alpha\beta}) \longrightarrow \mathsf{U}_{\alpha\beta},$$

with the usual gerbe compatibility conditions. Since $H^2(\mathsf{U}_{\alpha\beta})=0$, each $L_{\alpha\beta}$ is trivial, and

$$\nabla^L_{\alpha\beta} = d + A_{\alpha\beta}, \qquad (\nabla^L_{\alpha\beta})^2 = B_\beta - B_\alpha.$$

The holonomy of $\mathcal{G}_B$ defines a line bundle

$$\mathcal{L}^B \longrightarrow LM,$$

equipped with Brylinski local sections $\{\sigma_\alpha\}$ over $\{L\mathsf{U}_\alpha\}$. On overlaps these satisfy

$$\sigma_\alpha = e^{-\int_0^1 \iota_K A_{\alpha\beta}}\sigma_\beta = e^{-\tau(A_{\alpha\beta})}\sigma_\beta.$$

The natural connection on $\mathcal{L}^B$ is described in this basis by

$$\nabla^{\mathcal{L}^B} = d - \iota_K \overline{B}_\alpha = d - \tau(B_\alpha), \tag{3.1}$$

with curvature

$$F_B = (\nabla^{\mathcal{L}^B})^2 = -\tau(H).$$

Remark 3.1. From the construction, we see that adding a global 2-form ω to the B-field does not change the holonomy line bundle $\mathcal{L}^B$. It only modifies the connection $\nabla^{\mathcal{L}^B}$ by the transgressed 1-form $\tau(\omega)$.

Let $\mathrm{Pic}^\nabla(M)$ denote the Picard group of line bundles with connection on M. The action $(L,\nabla^L)\cdot B$ on B-fields has the following interpretation on the loop space:

Proposition 3.2. *For $(L,\nabla^L) \in \mathrm{Pic}^\nabla(M)$,*

$$\nabla^{\mathcal{L}^{(L,\nabla^L)\cdot B}} = \nabla^{\mathcal{L}^B} + d\ln \mathrm{hol}(\nabla^L),$$

where $\mathrm{hol}(\nabla^L) : LM \to \mathbb{T}$ is the holonomy of the connection ∇^L around loops.

Proof. By Proposition 6.1.1 of [10],

$$-\iota_K\, \overline{(\nabla^L)^2} = \tau\big((\nabla^L)^2\big) = d\ln \mathrm{hol}(\nabla^L).$$

Using (3.1), we compute under the basis $\{\sigma_\alpha\}$:

$$\nabla^{\mathcal{L}^{(L,\nabla^L)\cdot B}} = d - \iota_K\big(\overline{B}_\alpha + \overline{(\nabla^L)^2}\big) = d - \iota_K\overline{B}_\alpha + d\ln\mathrm{hol}(\nabla^L) = \nabla^{\mathcal{L}^B} + d\ln\mathrm{hol}(\nabla^L).$$

The desired equality follows. ∎

Thus the Picard transformation $(L,\nabla^L)\cdot B$ induces the gauge transformation

$$\mathrm{hol}(\nabla^L) : LM \to \mathbb{T}$$

on the connection $\nabla^{\mathcal{L}^B}$ of the holonomy line bundle $\mathcal{L}^B \to LM$.

4. Moduli spaces of flat mixed fields

The aim of this section is to study the presymplectic quotient of the configuration space of mixed fields by the gauge group. We shall see that, although the defining equations are linear, the quotient exhibits a non-trivial geometry.

We begin by introducing the locus on which the reduction takes place. Denote by $\mathcal{C}^{flat}$ the subset of the configuration space $\mathcal{C}$ given by

$$\mathcal{C}^{flat} \;:=\; \Big\{\, (A,B) \;\Big|\; F_A \text{ exact on } \Sigma,\, H \text{ exact on } Z \Big\} \subset \mathcal{C}.$$

It follows immediately from (2.6) that $\mathcal{C}^{flat}$ is preserved by the action of the gauge group $\mathbb{G}$.

Associated to $\mathcal{C}^{flat}$ is a topologically trivial complex line bundle $\mathcal{L}$ equipped with a connection

$$\nabla^{\mathcal{L}} \;=\; d + \Lambda,$$

such that at a point (A,B) and for a tangent vector

$$(\delta A, \delta B) \in T_{(A,B)}\mathcal{C}^{flat} \cong \Omega^1(\Sigma) \times \Omega^2(Z)^{\mathbb{T}},$$

the connection 1-form is defined by

$$\Lambda_{(A,B)}\big(\delta A, \delta B\big) \;=\; \int_\Sigma \Big(A^{\text{bas}} \wedge \delta A \;+\; \iota_v B \wedge \iota_v \delta B \Big), \tag{4.1}$$

where A^{bas} denotes the basic form on Σ associated to A, and v is the vector field generating the $\mathbb{T}$-action along the fibres of $Z \to \Sigma$.

A direct computation shows that the curvature of $\nabla^{\mathcal{L}}$ is precisely

$$(\nabla^{\mathcal{L}})^2 \;=\; \Omega,$$

where Ω is the 2-form on the configuration space introduced in (2.18). Consequently, $(\mathcal{L}, \nabla^{\mathcal{L}})$ is a prequantum line bundle [29] over $\mathcal{C}^{flat}$:

$$\begin{array}{c} (\mathcal{L}, \nabla^{\mathcal{L}}) \\ {\scriptstyle \pi}\big\downarrow \\ \mathcal{C}^{flat}. \end{array}$$

The action of $\mathbb{G}_\lambda$ on $\mathcal{C}^{flat}$ lifts naturally to an action on $\mathcal{L}$, and the connection $\nabla^{\mathcal{L}}$ is invariant under this lifted action. Moreover, one verifies that

$$\mu(X) = L_{\rho(X)} - \nabla^{\mathcal{L}}_{\rho(X)},$$

where μ is the moment map of Proposition 2.6, $X \in \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$, and $\rho(X)$ denotes the corresponding Killing vector field on $\mathcal{C}^{flat}$. We therefore obtain the presymplectic reduction

$$\mathcal{C}^{flat}/\mathbb{G}_\lambda = \mu^{-1}(0)/\mathbb{G}_\lambda,$$

which we shall refer to as the *moduli space of flat mixed fields.* The quotient has the following simple finite-dimensional description.

Lemma 4.1. *Let $\lambda \in \mathbb{Z}$ be the level of the action* (2.6)*. The product $\mathbb{R}^{2g} \times \mathbb{R}^{2g} \times \mathbb{R}/\mathbb{Z}$ carries a natural $(\mathbb{Z}^{2g} \times \mathbb{Z}^{2g})$-action defined by*

$$(\mathbf{m},\mathbf{n}) \cdot (\mathbf{x},\mathbf{y},[t]) := \big(\mathbf{x}+\mathbf{m}, \mathbf{y}+\mathbf{n},\; [t + \phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y})]\big)\,, \tag{4.2}$$

$$\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) := \mathbf{n}\Theta_g\mathbf{x}^T + (\lambda-1)\,\mathbf{m}\Theta_g\mathbf{y}^T,$$

where

$$(\mathbf{m},\mathbf{n}) = (m_1,\cdots,m_{2g},n_1,\cdots,n_{2g}) \in \mathbb{Z}^{2g}\times\mathbb{Z}^{2g}$$

acts on

$$(\mathbf{x},\mathbf{y},[t]) = (x_1,\cdots,x_{2g},y_1,\cdots,y_{2g},[t]) \in \mathbb{R}^{2g}\times\mathbb{R}^{2g}\times\mathbb{R}/\mathbb{Z}.$$

There is a one-to-one correspondence between the orbit space of the action of $\mathbb{G}_\lambda$ on $\mu^{-1}(0)$ and the orbit space of the $(\mathbb{Z}^{2g}\times\mathbb{Z}^{2g})$-action on $\mathbb{R}^{2g}\times\mathbb{R}^{2g}\times\mathbb{R}/\mathbb{Z}$ described in (4.2).

Proof. Take an element $(A,B)\in\mu^{-1}(0)$. Fix a trivialization of Z, and let θ denote the resulting global vertical coordinate. Then we may write

$$\begin{cases} A &= a_0 + d\theta, \\ B &= b_1 + A\wedge b_0, \end{cases} \tag{4.3}$$

where $a_0, b_0 \in \Omega^1(\Sigma)$ and $b_1\in\Omega^2(\Sigma)$. Note that, since $\iota_v A = 1$ and b_0, b_1 are basic,

$$\iota_v B = (\iota_v A)\, b_0 - A\wedge(\iota_v b_0) = b_0. \tag{4.4}$$

In this description the datum (A,B) is encoded by the triple (a_0,b_0,b_1). Moreover, the condition $(A,B)\in\mu^{-1}(0)$ forces a_0 and b_0 to be closed:

$$a_0\in\Omega^1_{\mathrm{cl}}(\Sigma), \qquad b_0\in\Omega^1_{\mathrm{cl}}(\Sigma).$$

Indeed $F_A = da_0$, while $H = db_1 + da_0\wedge b_0 - a_0\wedge db_0 - d\theta\wedge db_0$, whose first three terms are basic forms of degree 3 on the surface Σ and hence vanish. Thus, $H=0$ is equivalent to $db_0 = 0$, and $b_1\in\Omega^2(\Sigma)$ is unconstrained.

Now choose an element $(g,(L,\nabla^L))\in\mathbb{G}_\lambda$, and recall the notation $u = g^{-1}dg$, $\eta = (\nabla^L)^2$. By (2.6) and (4.4), the transformed pair $(\widetilde{A},\widetilde{B})$ is given by

$$\widetilde{A} = A + u, \qquad \widetilde{B} = B + \eta + \lambda\, u\wedge b_0.$$

By (2.7), $\iota_v(u\wedge b_0) = 0$, hence

$$\widetilde{b}_0 = \iota_v\widetilde{B} = b_0 + \iota_v\eta.$$

Expressing $(\widetilde{A},\widetilde{B})$ again in the form (4.3) and subtracting,

$$\widetilde{b}_1 = \widetilde{B} - \widetilde{A}\wedge\widetilde{b}_0 = \big(b_1 + A\wedge b_0 + \eta + \lambda\, u\wedge b_0\big) - (A+u)\wedge(b_0+\iota_v\eta),$$

so that the triple (a_0,b_0,b_1) is carried to $(\widetilde{a}_0,\widetilde{b}_0,\widetilde{b}_1)$ with

$$\begin{cases} \widetilde{a}_0 &= a_0 + u, \\ \widetilde{b}_0 &= b_0 + \iota_v\eta, \\ \widetilde{b}_1 &= b_1 + \big(\eta - d\theta\wedge\iota_v\eta - u\wedge\iota_v\eta\big) + (\iota_v\eta)\wedge a_0 + (\lambda-1)\, u\wedge b_0. \end{cases} \tag{4.5}$$

Here we used $A\wedge\iota_v\eta = d\theta\wedge\iota_v\eta + a_0\wedge\iota_v\eta$ and $-a_0\wedge\iota_v\eta = (\iota_v\eta)\wedge a_0$, and the λ-dependence enters solely through the cancellation of $\lambda\, u\wedge b_0$ against the term $-u\wedge b_0$ coming from $-\widetilde{A}\wedge\widetilde{b}_0$. By (2.5), $\iota_v\eta$ is a closed 1-form with integral periods. Likewise u has integral periods, and the 2-form

$$\eta - d\theta\wedge\iota_v\eta - u\wedge\iota_v\eta$$

descends to Σ and represents an integral cohomology class.

Finally, fix a symplectic basis $\{\theta_1, \ldots, \theta_g, \eta_1, \ldots, \eta_g\}$ of $H^1(\Sigma;\mathbb{R})$, normalised so that $\int_\Sigma \alpha \wedge \beta = [\alpha]\,\Theta_g\,[\beta]^T$. Write the cohomology classes as

$$[a_0] = \sum_{i=1}^{g}\big(x_i\theta_i + x_{i+g}\eta_i\big), \qquad [b_0] = \sum_{i=1}^{g}\big(y_i\theta_i + y_{i+g}\eta_i\big), \tag{4.6}$$

and similarly

$$[u] = \sum_{i=1}^{g}\big(m_i\theta_i + m_{i+g}\eta_i\big), \qquad [\iota_v\eta] = \sum_{i=1}^{g}\big(n_i\theta_i + n_{i+g}\eta_i\big), \tag{4.7}$$

and set $[t] := [\int_\Sigma b_1] \in \mathbb{R}/\mathbb{Z}$.

The subgroup $\mathbb{G}^0 < \mathbb{G}_\lambda$ of elements with $[u] = 0$ and $[\iota_v\eta] = 0$ acts on (a_0, b_0, b_1) by shifting a_0 and b_0 by exact 1-forms and b_1 by an exact 2-form together with an integral class. Note that for such elements $u = df_1$ is exact and, on the flat locus, $\lambda\, u \wedge b_0 = \lambda\, d(f_1 b_0)$ is exact as well, so that $\mathbb{G}^0$ is independent of λ. Hence $\mathbb{G}^0$ acts trivially on $(\mathbf{x}, \mathbf{y}, [t])$, and the residual action is that of $\mathbb{G}_\lambda/\mathbb{G}^0 \cong \mathbb{Z}^{2g} \times \mathbb{Z}^{2g}$. Passing to (4.5) and discarding the integral classes which vanish in $\mathbb{R}/\mathbb{Z}$, these being $[\eta - d\theta \wedge \iota_v\eta]$ and $[u \wedge \iota_v\eta]$, the latter with periods $\mathbf{m}\Theta_g\mathbf{n}^T \in \mathbb{Z}$, the surviving shift of the fibre coordinate is

$$\Delta t = \int_\Sigma \Big((\iota_v\eta) \wedge a_0 + (\lambda - 1)\, u \wedge b_0\Big) = \mathbf{n}\Theta_g\mathbf{x}^T + (\lambda - 1)\,\mathbf{m}\Theta_g\mathbf{y}^T = \phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x}, \mathbf{y}).$$

That (4.2) is indeed an action is verified directly: composing $(\mathbf{m}, \mathbf{n})$ and then $(\mathbf{m}', \mathbf{n}')$ shifts the fibre by $\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x}, \mathbf{y}) + \phi^\lambda_{(\mathbf{m}',\mathbf{n}')}(\mathbf{x} + \mathbf{m}, \mathbf{y} + \mathbf{n})$, which differs from $\phi^\lambda_{(\mathbf{m}+\mathbf{m}',\mathbf{n}+\mathbf{n}')}(\mathbf{x}, \mathbf{y})$ by $\mathbf{n}'\Theta_g\mathbf{m}^T + (\lambda - 1)\mathbf{m}'\Theta_g\mathbf{n}^T \in \mathbb{Z}$, using the integrality of Θ_g. □

Theorem 4.2. *Let $\lambda \in \mathbb{Z}$ and let Θ_g be the standard symplectic matrix*

$$\Theta_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \in Sp(2g, \mathbb{R}).$$

(i) For $\lambda \neq 0$, the moduli space $\mu^{-1}(0)/\mathbb{G}_\lambda$ is a Heisenberg contact manifold of dimension $4g+1$, that is a principal circle bundle over the torus $T^{2g} \times T^{2g}$ with first Chern class

$$c_1(\mu^{-1}(0)/\mathbb{G}_\lambda) = -\,\lambda \sum_{j,k} (\Theta_g)_{jk}\, \alpha_j \cup \beta_k,$$

where $\alpha_j \in H^1(T^{2g}, \mathbb{Z})$ is the dual basis to the coordinate x_j and $\beta_k \in H^1(T^{2g}, \mathbb{Z})$ is the dual basis to the coordinate y_k.

(ii) For $\lambda = 0$, the moduli space is a trivial circle bundle over $T^{2g} \times T^{2g}$,

$$\mu^{-1}(0)/\mathbb{G}_\lambda \;\cong\; T^{2g} \times T^{2g} \times \mathbb{R}/\mathbb{Z} \;=\; T^{4g+1}.$$

Proof. (i) By Lemma 4.1 the moduli space is the quotient of $\mathbb{R}^{2g} \times \mathbb{R}^{2g} \times \mathbb{R}/\mathbb{Z}$ by the action (4.2). That action is free, since $(\mathbf{m}, \mathbf{n}) \cdot (\mathbf{x}, \mathbf{y}, [t]) = (\mathbf{x}, \mathbf{y}, [t])$ already forces $\mathbf{m} = \mathbf{n} = \mathbf{0}$ from the first two components, and it is properly discontinuous, hence

$$\mu^{-1}(0)/\mathbb{G}_\lambda = (\mathbb{R}^{2g} \times \mathbb{R}^{2g} \times \mathbb{R}/\mathbb{Z})/(\mathbb{Z}^{2g} \times \mathbb{Z}^{2g})$$

is a smooth principal circle bundle over $T^{2g} \times T^{2g}$.

To determine its Chern class we construct an invariant connection on $\mathbb{R}^{2g} \times \mathbb{R}^{2g} \times \mathbb{R}/\mathbb{Z}$. Let

$$\alpha = dt + (1 - \lambda)\,\mathbf{x}\Theta_g d\mathbf{y}^T - \mathbf{y}\Theta_g d\mathbf{x}^T.$$

Since $(\mathbf{m},\mathbf{n})^*dt = dt + d\phi^\lambda_{(\mathbf{m},\mathbf{n})}$ with

$$d\phi^\lambda_{(\mathbf{m},\mathbf{n})} = \mathbf{n}\Theta_g d\mathbf{x}^T + (\lambda-1)\,\mathbf{m}\Theta_g d\mathbf{y}^T,$$

one computes

$$(\mathbf{m},\mathbf{n})^*\alpha = dt + \mathbf{n}\Theta_g d\mathbf{x}^T + (\lambda-1)\mathbf{m}\Theta_g d\mathbf{y}^T + (1-\lambda)(\mathbf{x}+\mathbf{m})\Theta_g d\mathbf{y}^T - (\mathbf{y}+\mathbf{n})\Theta_g d\mathbf{x}^T = \alpha,$$

the terms in $\mathbf{m}$ and $\mathbf{n}$ cancelling in pairs. Hence α descends to a connection 1-form on $\mu^{-1}(0)/\mathbb{G}_\lambda$. Its curvature is, using $\Theta_g^T = -\Theta_g$,

$$d\big(\mathbf{y}\Theta_g d\mathbf{x}^T\big) = \sum_{j,k}(\Theta_g)_{jk}\, dy_j \wedge dx_k = -\sum_{j,k}(\Theta_g)_{jk}\, dx_k \wedge dy_j = d\mathbf{x}\Theta_g d\mathbf{y}^T,$$

hence

$$d\alpha = (1-\lambda)\, d\mathbf{x}\Theta_g d\mathbf{y}^T - d\mathbf{x}\Theta_g d\mathbf{y}^T = -\lambda \sum_{j,k}(\Theta_g)_{jk}\, dx_j \wedge dy_k.$$

Therefore $c_1(\mu^{-1}(0)/\mathbb{G}_\lambda) = [d\alpha]$, the fibre coordinate $t \in \mathbb{R}/\mathbb{Z}$ having period 1. With $\alpha_j, \beta_k \in H^1(T^{2g},\mathbb{Z})$ dual to x_j and y_k,

$$c_1(\mu^{-1}(0)/\mathbb{G}_\lambda) = -\lambda \sum_{j,k}(\Theta_g)_{jk}\, \alpha_j \cup \beta_k = -\lambda \sum_{i=1}^{g}\big(\alpha_i \cup \beta_{i+g} - \alpha_{i+g} \cup \beta_i\big).$$

Introducing a symplectic basis

$$e_i = \alpha_i, \qquad f_i = \beta_{i+g}, \qquad (1 \le i \le g),$$

and

$$e_{g+i} = -\alpha_{g+i}, \qquad f_{g+i} = \beta_i, \qquad (1 \le i \le g),$$

one has

$$c_1(\mu^{-1}(0)/\mathbb{G}_\lambda) = -\lambda \sum_{j=1}^{2g} e_j \wedge f_j.$$

For $\lambda \neq 0$ the 2-form $d\alpha$ is nondegenerate on $\mathbb{R}^{4g}$, so $\alpha \wedge (d\alpha)^{2g} \neq 0$ and $\mu^{-1}(0)/\mathbb{G}_\lambda$ is a Heisenberg contact manifold of dimension $4g+1$, with contact 1-form α.

(ii) For $\lambda = 0$, the Chern class $c_1(\mu^{-1}(0)/\mathbb{G}_\lambda)$ is zero and the circle bundle trivialises,

$$\mu^{-1}(0)/\mathbb{G}_\lambda \;\cong\; T^{2g} \times T^{2g} \times \mathbb{R}/\mathbb{Z} \;=\; T^{4g+1}.$$

In particular $\mu^{-1}(0)/\mathbb{G}_\lambda$ carries no contact structure. $\square$

4.1. **The topology of the moduli space.** Throughout this subsection $\lambda \in \mathbb{Z}\setminus\{0\}$ is fixed and $M_{|\lambda|} \xrightarrow{\pi} T^{4g}$ denotes the principal circle bundle underlying $\mu^{-1}(0)/\mathbb{G}_\lambda$ at level λ, with Euler class

$$c_1(M_{|\lambda|}) = \pm\lambda\,\omega \in H^2(T^{4g};\mathbb{Z}), \qquad \omega = \sum_{j=1}^{2g} e_j \wedge f_j,$$

We write $V := H^1(T^{4g};\mathbb{Z}) \cong \mathbb{Z}^{4g}$, so that $H^r(T^{4g};\mathbb{Z}) \cong \Lambda^r V$ is free of rank $\binom{4g}{r}$, and introduce the integral Lefschetz operator

$$L_r : \Lambda^r V \longrightarrow \Lambda^{r+2} V, \qquad L_r(\xi) = \omega \wedge \xi,$$

with $L = \bigoplus_r L_r$. Note that the sign of the Euler class is immaterial for the discussion below, since $\operatorname{coker}(-\lambda L_r) = \operatorname{coker}(\lambda L_r)$ and $\ker(-\lambda L_r) = \ker(\lambda L_r)$.

The fundamental group of $M_{|\lambda|}$ is the integer Heisenberg lattice $H(|\lambda|,\dots,|\lambda|)$ of [32], namely a central extension of $\mathbb{Z}^{2g}\times\mathbb{Z}^{2g}$ by $\mathbb{Z}$ twisted by the 2-cocycle

$$c_\lambda((\mathbf{m},\mathbf{n}),(\mathbf{m}',\mathbf{n}')) = \mathbf{n}'\Theta_g\mathbf{m}^T + (\lambda-1)\,\mathbf{m}'\Theta_g\mathbf{n}^T. \tag{4.8}$$

The integral cohomology of $M_{|\lambda|}$ was computed by Lee and Packer in [31].

Theorem 4.3 ([31, Thm. 2.1])**.** *Let $\lambda\in\mathbb{Z}\setminus\{0\}$ and $k\in\mathbb{N}$, then*

$$H^k(M_{|\lambda|};\mathbb{Z}) \;\cong\; \operatorname{coker}\big(\lambda L_{k-2}\big) \,\oplus\, \ker\big(\lambda L_{k-1}\big) \;\cong\; \mathbb{Z}^{b_k} \,\oplus\, \bigoplus_{j\geq 1}\mathbb{Z}_{|\lambda|j}^{\varepsilon_{k-2}(j)},$$

with rank

$$b_k = \begin{cases} \binom{4g}{k}-\binom{4g}{k-2}, & 0\leq k\leq 2g,\\ \binom{4g}{k-1}-\binom{4g}{k+1}, & 2g+1\leq k\leq 4g+1,\end{cases}$$

and torsion multiplicities

$$\varepsilon_r(j) \;=\; \binom{4g}{r+2-2j} - \binom{4g}{r-2j} \tag{4.9}$$

for $0\leq r\leq 2g-1$, and extended to $2g\leq r\leq 4g-2$ by $\varepsilon_r(j)=\varepsilon_{4g-2-r}(j)$.

The K-theory of $M_{|\lambda|}$ has been computed by Aslaksen–Lee–Packer [30], but only for $\lambda=1$. Their strategy is to run the K-theoretic Gysin sequence of the circle bundle $M_{|\lambda|}\to T^{4g}$ in parallel with its cohomological counterpart. In the cohomological Gysin sequence the connecting map is the Lefschetz operator λL, and passing to K-theory replaces the integral Euler class $\lambda\omega$ by the K-theoretic Euler class $\Theta_\lambda := 1-e^{\lambda\omega}$. Since $K^\bullet(T^{4g})$ is torsion-free and the Chern character restricts to an isomorphism of integral lattices $K^\bullet(T^{4g})\xrightarrow{\cong} H^\bullet(T^{4g};\mathbb{Z})\cong\Lambda^\bullet V$, the entire computation takes place in $\Lambda^\bullet V$ and the K-groups of $M_{|\lambda|}$ are read off from the kernel and cokernel of multiplication by Θ_λ. The key point is to compare the connecting map Θ_λ with the cohomological one λL *over* $\mathbb{Z}$, then $\operatorname{coker}(\lambda L)$, whose torsion is the cohomology torsion of Theorem 4.3, controls the torsion of the K-groups as well.

Aslaksen–Lee–Packer establish such an integral comparison at $\lambda=1$ by decomposing $\Lambda^\bullet V$ into subspaces on which the Lefschetz operator acts, and exhibiting on each block a unimodular change of basis conjugating the K-theoretic Euler class $e^\omega-1$ to the Lefschetz operator L [30, Prop. 2.3, Prop. 2.6]. The construction proceeds by induction over the blocks and relies on an operator identity of the form $[X,L]=L$ that is homogeneous of degree one in L [30, Lem. 2.5]. This identity is scale-invariant and for that reason cannot relate λL to $e^{\lambda\omega}-1$ when $\lambda\neq 1$. Namely, since $e^{\lambda\omega}-1=\lambda L+\frac{\lambda^2}{2}L^2+\cdots$, the ratio of the L^2-coefficient to the L-coefficient is $\lambda/2$, which varies with the level, whereas a conjugation built from $[X,L]=L$ can only reproduce the fixed value $\frac{1}{2}$ of the case $\lambda=1$.

To remove this restriction, we construct a single $\mathbb{Z}$-linear automorphism Ψ_λ of the whole exterior algebra $\Lambda^\bullet V$ intertwining λL with $e^{\lambda\omega}-1$. In place of the blockwise conjugation at $\lambda=1$ [30, Prop. 2.6], which it recovers in an equivalent form, one obtains a single intertwiner valid at every level.

Lemma 4.4. *Let $\lambda\in\mathbb{Z}$. There is a $\mathbb{Z}$-linear automorphism $\Psi=\Psi_\lambda$ of $\Lambda^\bullet V$ that preserves $\Lambda^{\mathrm{ev}}V$ and $\Lambda^{\mathrm{odd}}V$, preserves the degree filtration and induces the identity on its associated graded, and satisfies*

$$\Psi\circ(\lambda L) = \big(e^{\lambda\omega}-1\big)\circ\Psi.$$

Proof. For $1 \le i \le 2g$ write $x_i := e_i \wedge f_i \in \Lambda^2 V$, so that $\omega = \sum_{i=1}^{2g} x_i$. Because each x_i is a wedge of two distinct generators, the x_i commute and satisfy $x_i^2 = 0$. Let

$$R := \mathbb{Z}[x_1, \dots, x_{2g}] / \big(x_1^2, \dots, x_{2g}^2\big)$$

act on $\Lambda^\bullet V$ by wedge multiplication. The operator λL is precisely multiplication by the element $\lambda\omega = \lambda \sum_i x_i \in R$, and since the x_i square to zero the exponential is a finite product,

$$e^{\lambda\omega} = \prod_{i=1}^{2g} e^{\lambda x_i} = \prod_{i=1}^{2g} \big(1 + \lambda x_i\big) \in R, \tag{4.10}$$

the middle equality using that the x_i commute and $e^{\lambda x_i} = 1 + \lambda x_i$ because $x_i^2 = 0$. Thus the whole comparison of λL with $e^{\lambda\omega} - 1$ takes place inside the commutative square-zero algebra R, and reduces to producing a ring automorphism of R carrying $\lambda\omega$ to $e^{\lambda\omega} - 1$.

The ring R is free over $\mathbb{Z}$ on the square-free monomials $x_C := \prod_{i \in C} x_i$, $C \subseteq \{1, \dots, 2g\}$. Define

$$\psi(x_i) := x_i \prod_{j<i} \big(1 + \lambda x_j\big).$$

Since $x_i \cdot \psi(x_i) = x_i^2 \prod_{j<i}(1 + \lambda x_j) = 0$, the elements $\psi(x_i)$ again satisfy $\psi(x_i)^2 = 0$, so by the universal property of R this assignment extends to a ring endomorphism $\psi \colon R \to R$.

For every $1 \le m \le 2g$ we have the elementary factorisation,

$$\prod_{i \le m} \big(1 + \lambda x_i\big) - 1 = \Big[\prod_{i \le m-1} \big(1 + \lambda x_i\big) - 1\Big] + \lambda x_m \prod_{j \le m-1} \big(1 + \lambda x_j\big). \tag{4.11}$$

Set $P_m := \prod_{i \le m} \big(1 + \lambda x_i\big) - 1$, with the convention $P_0 = \prod_\varnothing - 1 = 0$. Then (4.11) reads $P_m = P_{m-1} + \lambda\, \psi(x_m)$, and because ψ is $\mathbb{Z}$-linear and a ring homomorphism, we get

$$\psi\big(\lambda \sum_i x_i\big) = \sum_i \lambda\, \psi(x_i) = \sum_{m=1}^{2g} \big(P_m - P_{m-1}\big) = P_{2g} - P_0 = \prod_{i=1}^{2g} \big(1 + \lambda x_i\big) - 1 = e^{\lambda\omega} - 1. \tag{4.12}$$

Grade R by monomial length. From the definition, $\psi(x_C) = x_C +$(strictly longer monomials), so in the monomial basis $\psi = \mathrm{id} + N$ with N strictly length-raising, hence nilpotent. Therefore ψ is unitriangular and hence a $\mathbb{Z}$-linear automorphism of R.

We can lift ψ uniformly from R to $\Lambda^\bullet V$ via the pair-free decomposition [31, §1]. The monomial basis of $\Lambda^\bullet V$ consists, up to sign, of the elements $\mu_S \wedge x_C$, where μ_S is a pair-free monomial (at most one factor from each pair $\{e_i, f_i\}$), $\operatorname{supp} S \subseteq \{1, \dots, 2g\}$ records the occupied pairs, and $C \subseteq \{1, \dots, 2g\} \setminus \operatorname{supp} S$. Since $x_i \wedge \mu_S = 0$ for $i \in \operatorname{supp} S$, one has an R-module decomposition

$$\Lambda^\bullet V = \bigoplus_S \mu_S \wedge R, \qquad \mu_S \wedge R \cong R/I_S, \quad I_S := \big(x_i : i \in \operatorname{supp} S\big),$$

the isomorphism being $r + I_S \mapsto \mu_S \wedge r$. Under $L = \omega \wedge (-)$ each block R/I_S is carried to itself by multiplication by ω. For $i \in \operatorname{supp} S$ we have $\psi(x_i) \in (x_i) \subseteq I_S$, so ψ preserves I_S and induces a unitriangular ring automorphism ψ_S of R/I_S. Define

$$\Psi\big(\mu_S \wedge r\big) := \mu_S \wedge \psi_S(r),$$

extended additively over the blocks. Being blockwise unitriangular, Ψ is a $\mathbb{Z}$-linear automorphism of $\Lambda^\bullet V$. Moreover, for $h \in R$,

$$\Psi\big(h \cdot (\mu_S \wedge r)\big) = \mu_S \wedge \psi_S(hr) = \mu_S \wedge \psi_S(h)\,\psi_S(r) = \psi(h) \cdot \Psi\big(\mu_S \wedge r\big), \tag{4.13}$$

where the middle equality uses that ψ_S is a ring homomorphism and the last uses $\psi_S(h) = \psi(h) \bmod I_S$. Since $\psi_S(r) - r$ raises monomial length and multiplication by x_C raises the exterior degree by $2|C|$, the operator $\Psi - \mathrm{id}$ strictly raises the exterior degree by even amounts. Consequently Ψ preserves the parity splitting $\Lambda^\bullet V = \Lambda^{\mathrm{ev}} V \oplus \Lambda^{\mathrm{odd}} V$, preserves the degree filtration, and induces the identity on the associated graded. Finally, combining (4.13) (with $h = \lambda \sum_i x_i$) and (4.12), for every $v \in \Lambda^\bullet V$

$$\Psi(\lambda L\, v) = \Psi\Big(\big(\lambda \textstyle\sum_i x_i\big) \cdot v\Big) = \psi\Big(\lambda \textstyle\sum_i x_i\Big) \cdot \Psi(v) = \big(e^{\lambda\omega} - 1\big) \cdot \Psi(v).$$

□

We can now state the K-theory of $M_{|\lambda|}$ for all levels λ.

Theorem 4.5. *Let $\lambda \in \mathbb{Z} \setminus \{0\}$ and $M_{|\lambda|} = \mu^{-1}(0)/\mathbb{G}_\lambda$.*

$$K^0(M_{|\lambda|}) \;\cong\; \mathbb{Z}^{\binom{4g+1}{2g}} \oplus \bigoplus_{\substack{k \text{ even} \\ j \ge 1}} \mathbb{Z}_{|\lambda| j}^{\varepsilon_{k-2}(j)}, \qquad K^1(M_{|\lambda|}) \;\cong\; \mathbb{Z}^{\binom{4g+1}{2g}} \oplus \bigoplus_{\substack{k \text{ odd} \\ j \ge 1}} \mathbb{Z}_{|\lambda| j}^{\varepsilon_{k-2}(j)},$$

where $\varepsilon_r(j)$ is the torsion multiplicity (4.9).

Proof. The K-theoretic Gysin sequence for the circle bundle $M_{|\lambda|} \to T^{4g}$ reads

$$\cdots \to K^i(T^{4g}) \xrightarrow{\cdot(1-e^{\lambda\omega})} K^i(T^{4g}) \xrightarrow{\pi^*} K^i(M_{|\lambda|}) \xrightarrow{\pi_!} K^{i+1}(T^{4g}) \to \cdots.$$

Transporting the computation to $\Lambda^\bullet V$, the connecting map becomes wedge product by $\Theta_\lambda = 1 - e^{\lambda\omega}$. By Lemma 4.4 there is a parity and filtration preserving $\mathbb{Z}$-linear automorphism Ψ of $\Lambda^\bullet V$ with

$$\Theta_\lambda = -\,\Psi \circ (\lambda L) \circ \Psi^{-1}, \tag{4.14}$$

so on each parity $\ker \Theta_\lambda = \Psi(\ker \lambda L) \cong \ker(\lambda L)$ and $\operatorname{im} \Theta_\lambda = \Psi(\operatorname{im} \lambda L)$, and Ψ induces an isomorphism $\operatorname{coker} \Theta_\lambda \cong \operatorname{coker}(\lambda L)$ compatible with the degree filtration. Since $\ker \Theta_\lambda \cong \ker(\lambda L)$ is free, being the kernel of a homomorphism of free abelian groups, splitting the Gysin sequence by parity gives

$$K^0(M_{|\lambda|}) \cong \operatorname{coker}\big(\Theta_\lambda|_{\Lambda^{\mathrm{ev}}}\big) \oplus \ker\big(\Theta_\lambda|_{\Lambda^{\mathrm{odd}}}\big), \quad K^1(M_{|\lambda|}) \cong \operatorname{coker}\big(\Theta_\lambda|_{\Lambda^{\mathrm{odd}}}\big) \oplus \ker\big(\Theta_\lambda|_{\Lambda^{\mathrm{ev}}}\big).$$

Using (4.14) to replace Θ_λ by λL, the degree r summand of $\operatorname{coker}(\lambda L)$ is $\operatorname{coker}(\lambda L_r)$, whose torsion is $\bigoplus_j \mathbb{Z}_{|\lambda| j}^{\varepsilon_r(j)}$ by Theorem 4.3. Summing over r of each parity identifies $\operatorname{Tors} K^i$ with $\bigoplus_{k \equiv i} \operatorname{Tors} H^k(M_{|\lambda|})$.

For the free part, by the Hard Lefschetz theorem over $\mathbb{Q}$, L_r is injective for $r < 2g$ and surjective for $r \ge 2g$. Since $\Lambda^\bullet V$ is torsion-free, the ranks are the same over $\mathbb{Z}$,

$$\operatorname{rank} L_r = \begin{cases} \binom{4g}{r}, & 0 \le r < 2g, \\ \binom{4g}{r+2}, & 2g \le r \le 4g-2. \end{cases} \tag{4.15}$$

These contributions telescope and using $\binom{4g}{k} = \binom{4g}{4g-k}$ over a fixed parity, we get

$$\operatorname{rank} K^0(M_{|\lambda|}) = \operatorname{rank} K^1(M_{|\lambda|}) = \binom{4g}{2g} + \binom{4g}{2g-1} = \binom{4g+1}{2g}.$$

□

5. Interaction with T-duality

5.1. **Review of T-duality for principal circle bundles in background flux.** In [7, 8], spacetime Z was compactified in one direction. More precisely, Z is a principal $\mathbb{T}$-bundle over X

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & Z \\ & & \downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

classified up to isomorphism by its first Chern class $c_1(Z) \in H^2(X, \mathbb{Z})$. Assume that spacetime Z is endowed with an H-flux which is a representative in the degree 3 Deligne cohomology of Z, that is $H \in \Omega^3(Z)$ with integral periods (for simplicity, we drop factors of $\frac{1}{2\pi i}$), together with the following data. Consider a local trivialization $U_\alpha \times \mathbb{T}$ of $Z \to X$, where $\{U_\alpha\}$ is a good cover of X. Let $H_\alpha = H\Big|_{U_\alpha \times \mathbb{T}} = dB_\alpha$, where $B_\alpha \in \Omega^2(U_\alpha \times \mathbb{T})$ and finally, $B_\alpha - B_\beta = F_{\alpha\beta} \in \Omega^1(U_{\alpha\beta} \times \mathbb{T})$. Then the choice of H-flux entails that we are given a local trivialization as above and locally defined 2-forms B_α on it, together with closed 2-forms $F_{\alpha\beta}$ defined on double overlaps, that is, $(H, B_\alpha, F_{\alpha\beta})$. Also the first Chern class of $Z \to X$ is represented in integral cohomology by (F, A_α) where $\{A_\alpha\}$ is a connection 1-form on $Z \to X$ and $F = dA_\alpha$ is the curvature 2-form of $\{A_\alpha\}$.

The T-dual is another principal $\mathbb{T}$-bundle over M, denoted by $\widehat{Z}$,

$$\begin{array}{ccc} \widehat{\mathbb{T}} & \longrightarrow & \widehat{Z} \\ & & \downarrow{\scriptstyle \widehat{\pi}} \\ & & X \end{array}$$

To define it, we see that $\pi_*(H_\alpha) = d\pi_*(B_\alpha) = d\widehat{A}_\alpha$, so that $\{\widehat{A}_\alpha\}$ is a connection 1-form whose curvature $d\widehat{A}_\alpha = \widehat{F}_\alpha = \pi_*(H_\alpha)$ that is, $\widehat{F} = \pi_* H$. So let $\widehat{Z}$ denote the principal $\mathbb{T}$-bundle over M whose first Chern class is $c_1(\widehat{Z}) = [\pi_* H, \pi_*(B_\alpha)] \in H^2(X; \mathbb{Z})$.

The Gysin sequence for Z enables us to define a T-dual H-flux $[\widehat{H}] \in H^3(\widehat{Z}, \mathbb{Z})$, satisfying

$$c_1(Z) = \widehat{\pi}_* \widehat{H}\,,$$

where π_* and similarly $\widehat{\pi}_*$, denote the pushforward maps. Note that $\widehat{H}$ is not fixed by this data, since any integer degree 3 cohomology class on X that is pulled back to $\widehat{Z}$ also satisfies the requirements. However, $\widehat{H}$ is determined uniquely (up to cohomology) upon imposing the condition $[H] = [\widehat{H}]$ on the correspondence space $Z \times_X \widehat{Z}$ as will be explained now.

The *correspondence space* (sometimes called the doubled space) is defined as

$$Z \times_X \widehat{Z} = \{(x, \widehat{x}) \in Z \times \widehat{Z} : \pi(x) = \widehat{\pi}(\widehat{x})\}.$$

Then we have the following commutative diagram,

$$\begin{tikzcd}
 & (Z \times_X \widehat{Z}, [H] = [\widehat{H}]) \arrow[dl, "p"] \arrow[dr, "\widehat{p}"] & \\
(Z, [H]) \arrow[dr, "\pi"'] & & (\widehat{Z}, [\widehat{H}]) \arrow[dl, "\widehat{\pi}"] \\
 & X &
\end{tikzcd}$$

By requiring that

$$p^*[H] = \widehat{p}^*[\widehat{H}] \in H^3(Z \times_X \widehat{Z}, \mathbb{Z}),$$

determines $[\widehat{H}] \in H^3(\widehat{Z}, \mathbb{Z})$ uniquely, via an application of the Gysin sequence. An alternate way to see this is is explained below.

Let $(H, B_\alpha, F_{\alpha\beta}, L_{\alpha\beta})$ denote a gerbe with connection on Z. We also choose a connection 1-form A on Z. Then define

$$\widehat{A}_\alpha = -\iota_v B_\alpha \tag{5.1}$$

on the chart U_α and the connection 1-form $\widehat{A} = \widehat{A}_\alpha + d\widehat{\theta}_\alpha$ on the chart $U_\alpha \times \widehat{\mathbb{T}}$, where θ_α is the coordinate on $\widehat{\mathbb{T}}$. In this way we get a T-dual circle bundle $\widehat{Z} \to X$ with connection 1-form $\widehat{A}$.

Without loss of generality, we can assume that H is $\mathbb{T}$-invariant. Consider

$$\Omega = H - A \wedge F_{\widehat{A}}$$

where $F_{\widehat{A}} = d\widehat{A}$ and $F_A = dA$ are the curvatures of A and $\widehat{A}$ respectively. One checks that the contraction $i_v(\Omega) = 0$ and the Lie derivative $L_v(\Omega) = 0$ so that Ω is a basic 3-form on Z, that is Ω comes from the base X.

Setting

$$\widehat{H} = F_A \wedge \widehat{A} + \Omega \tag{5.2}$$

this defines the T-dual flux 3-form. One verifies that $\widehat{H}$ is a closed 3-form on $\widehat{Z}$. It follows that on the correspondence space, one has as desired,

$$\widehat{H} = H + d(A \wedge \widehat{A}).$$

Our next goal is to determine the T-dual curving or B-field. The Buscher rules imply that on the open sets $U_\alpha \times \mathbb{T} \times \widehat{\mathbb{T}}$ of the correspondence space $Z \times_X \widehat{Z}$, one has

$$\widehat{B}_\alpha = B_\alpha + A \wedge \widehat{A} - d\theta_\alpha \wedge d\widehat{\theta}_\alpha\,, \tag{5.3}$$

Note that

$$\iota_v \widehat{B}_\alpha = \iota_v \left(B_\alpha + A \wedge \widehat{A} - d\theta_\alpha \wedge d\widehat{\theta}_\alpha \right) = -\widehat{A}_\alpha + \widehat{A} - d\widehat{\theta}_\alpha = 0$$

so that $\widehat{B}_\alpha$ is indeed a 2-form on $\widehat{Z}$ and not just on the correspondence space. Obviously, $d\widehat{B}_\alpha = \widehat{H}$. Following the descent equations one arrives at the complete T-dual gerbe with connection, $(\widehat{H}, \widehat{B}_\alpha, \widehat{F}_{\alpha\beta}, \widehat{L}_{\alpha\beta})$. cf. [9].

The rules for transforming the Ramond-Ramond (RR) fields can be encoded in the [7, 8] generalization of *Hori's formula*

$$T_* G = \int_{\mathbb{T}} e^{-A\wedge\widehat{A}}\, G\,, \tag{5.4}$$

where $G \in \Omega^\bullet(Z)^{\mathbb{T}}$ is the total RR fieldstrength,

$$\begin{array}{ccc} G \in \Omega^{even}(Z)^{\mathbb{T}} & \text{for} & \text{Type IIA;} \\ G \in \Omega^{odd}(Z)^{\mathbb{T}} & \text{for} & \text{Type IIB,} \end{array}$$

and where the right hand side of equation (5.4) is an invariant differential form on $Z \times_X \widehat{Z}$, and the integration is along the $\mathbb{T}$-fiber of Z.

Recall that the twisted cohomology is defined as the cohomology of the complex

$$H^\bullet(Z, H) = H^\bullet(\Omega^\bullet(Z), d_H = d + H\wedge).$$

By the identity (5.4), T_* maps d_H-closed forms G to $d_{\widehat{H}}$-closed forms T_*G. So T-duality T_* induces a map on twisted cohomologies,

$$T : H^\bullet(Z, H) \to H^{\bullet+1}(\widehat{Z}, \widehat{H}).$$

Define the Riemannian metrics on Z and $\widehat{Z}$ respectively by

$$h_{A,R} = \pi^* g_X + R^2\, A \odot A, \qquad \widehat{h}_{\widehat{A},1/R} = \widehat{\pi}^* g_X + 1/R^2\, \widehat{A} \odot \widehat{A}. \tag{5.5}$$

where h_X is a Riemannian metric on X. Then $h_{A,R}$ is $\mathbb{T}$-invariant and the length of each circle fibre is R; $\widehat{h}_{\widehat{A},1/R}$ is $\widehat{\mathbb{T}}$-invariant and the length of each circle fibre is $1/R$.

The following theorem summarizes the main consequence of T-duality for principal circle bundles in a background flux.

Theorem 5.1 (T-duality isomorphism [7, 8])**.** *In the notation above, and with the above choices of Riemannian metrics and flux forms, the map* (5.4)

$$T \colon \Omega^{\overline{k}}(Z)^{\mathbb{T}} \to \Omega^{\overline{k+1}}(\widehat{Z})^{\widehat{\mathbb{T}}},$$

for $k = 0, 1$, (where $\overline{k}$ denotes the parity of k) are isometries, inducing isomorphisms on twisted cohomology groups,

$$T : H^\bullet(Z, H) \xrightarrow{\cong} H^{\bullet+1}(\widehat{Z}, \widehat{H}).$$

Therefore under T-duality one has the exchange,

$$R \Longleftrightarrow 1/R \quad \text{and} \qquad \text{background } H\text{-flux} \Longleftrightarrow \text{Chern class}$$

Moreover there is also an isomorphism of twisted K-theories,

$$T : K^\bullet(Z, H) \to K^{\bullet+1}(\widehat{Z}, \widehat{H}),$$

such that the following diagram commutes,

$$\begin{tikzcd} K^\bullet(Z,H) \arrow[r, "T"] \arrow[d, "Ch_H"'] & K^{\bullet+1}(\widehat{Z},\widehat{H}) \arrow[d, "Ch_{\widehat{H}}"] \\ H^\bullet(Z,H) \arrow[r, "T"'] & H^{\bullet+1}(\widehat{Z},\widehat{H}) \end{tikzcd}$$

5.2. **T-duality of the twisted Yang-Mills functional.** Recall that the metrics on the T-dual spaces are rescaled as in (5.5):

$$h_{A,R} = \pi^* h_X + R^2 A \odot A, \qquad \widehat{h}_{\widehat{A},1/R} = \widehat{\pi}^* h_X + \frac{1}{R^2} \widehat{A} \odot \widehat{A}.$$

To incorporate the radius parameter R, we introduce the rescaled twisted Yang–Mills functional

$$\mathrm{YM}(A,B)_R := \sqrt{R}\left(\int_\Sigma |F_A|^2 \, d\,\mathrm{vol}_h + \int_Z H \wedge *_{h_{A,R}} H\right),$$

and define $\mathrm{YM}(\widehat{A},\widehat{B})_{1/R}$ analogously on the dual side.

The normalization by $\sqrt{R}$ is chosen so that the functional transforms symmetrically under the exchange

$$R \Longleftrightarrow 1/R.$$

The following result shows that this normalization is precisely the one compatible with T-duality.

Theorem 5.2. *One has*

$$\mathrm{YM}(A,B)_R = \mathrm{YM}(\widehat{A},\widehat{B})_{1/R}.$$

Proof. Since Σ is two-dimensional, we may write

$$F_A = f_A \, d\,\mathrm{vol}_h \, .$$

Using the decomposition (5.2), together with the fact that $\widehat{H}$ is basic, we obtain

$$\widehat{H} = \widehat{A} \wedge F_A = \widehat{A} \wedge f_A \, d\,\mathrm{vol}_h \, .$$

By the definition of the rescaled metric $\widehat{h}_{\widehat{A},1/R}$, the corresponding volume form satisfies

$$\widehat{A} \wedge d\,\mathrm{vol}_h = R \, d\,\mathrm{vol}_{\widehat{h}_{\widehat{A},1/R}},$$

and therefore

$$\widehat{H} = R \, f_A \, d\,\mathrm{vol}_{\widehat{h}_{\widehat{A},1/R}} \, .$$

It follows that

$$\int_{\widehat{Z}} \widehat{H} \wedge *_{\widehat{h}_{\widehat{A},1/R}} \widehat{H} = R \int_\Sigma |F_A|^2 \, d\,\mathrm{vol}_{h_{\widehat{A},1/R}} \, .$$

Interchanging the roles of the dual spaces yields

$$\int_Z H \wedge *_{h_{A,R}} H = \frac{1}{R} \int_\Sigma |F_{\widehat{A}}|^2 \, d\,\mathrm{vol}_h \, .$$

Substituting these identities into the definition of $\mathrm{YM}(\widehat{A},\widehat{B})_{1/R}$, we obtain

$$\begin{aligned}
\mathrm{YM}(\widehat{A},\widehat{B})_{1/R} &= \frac{1}{\sqrt{R}}\left(\int_\Sigma |F_{\widehat{A}}|^2\, d\,\mathrm{vol}_h + \int_{\widehat{Z}} \widehat{H}\wedge *_{\widehat{h}_{\widehat{A},1/R}}\widehat{H}\right)\\
&= \frac{1}{\sqrt{R}}\left(R\int_Z H\wedge *_{h_{A,R}}H + R\int_\Sigma |F_A|^2\, d\,\mathrm{vol}_h\right)\\
&= \sqrt{R}\left(\int_\Sigma |F_A|^2\, d\,\mathrm{vol}_h + \int_Z H\wedge *_{h_{A,R}}H\right)\\
&= \mathrm{YM}(A,B)_R.
\end{aligned}$$

□

5.3. **T-dualizable moduli stack of flat mixed fields.** In this section we examine the action of T-duality on the space of flat mixed fields. As recalled in Section 5.1, the T-duality correspondence carries a triple (Z,A,B) to its dual $(\widehat{Z},\widehat{A},\widehat{B})$. We first show that it defines a self-map on $\mu^{-1}(0)$.

Lemma 5.3. *The Buscher rules* (5.1)–(5.3) *induce a self-map*

$$\mathcal{T}:\mu^{-1}(0)\longrightarrow\mu^{-1}(0),\qquad \mathcal{T}(a_0,b_0,b_1)=(-b_0,\,-a_0,\,b_1+a_0\wedge b_0),$$

that is independent of the level λ and satisfies $\mathcal{T}^2=\mathrm{id}$.

Proof. On the flat locus, we trivialize Z and write $A=a_0+d\theta$, $B=b_1+A\wedge b_0$ as in (4.3), so that $\iota_v B=b_0$ by (4.4). Since $H=0$ and $F_A=0$, by (5.1) the dual connection is $\widehat{A}=-b_0+d\widehat{\theta}$, hence $\widehat{a}_0=-b_0$, and $\Omega=H-A\wedge F_{\widehat{A}}=0$, so $\widehat{H}=0$ by (5.2). The dual curving (5.3) is

$$\widehat{B}=B+A\wedge\widehat{A}-d\theta\wedge d\widehat{\theta}=b_1+A\wedge b_0+A\wedge(-b_0+d\widehat{\theta})-d\theta\wedge d\widehat{\theta}=b_1+a_0\wedge d\widehat{\theta},$$

using $A=a_0+d\theta$. Contracting with the dual vertical field $\widehat{v}=\partial_{\widehat{\theta}}$ gives $\widehat{b}_0=\iota_{\widehat{v}}\widehat{B}=-a_0$, and therefore

$$\widehat{b}_1=\widehat{B}-\widehat{A}\wedge\widehat{b}_0=\big(b_1+a_0\wedge d\widehat{\theta}\big)-(-b_0+d\widehat{\theta})\wedge(-a_0)=b_1+a_0\wedge b_0,$$

the terms $a_0\wedge d\widehat{\theta}$ cancelling. Hence $\mathcal{T}(a_0,b_0,b_1)=(-b_0,-a_0,b_1+a_0\wedge b_0)$.

By Proposition 2.5(i), together with (2.7), (2.9) and Lemma 2.2, both F_A and H are strictly invariant under the level-λ action, so the flat locus $\mu^{-1}(0)$ is the same subset of $\mathcal{C}$ for every λ. Under a level-λ gauge transformation the dual connection changes by

$$\widehat{A}=-\iota_v B\;\longmapsto\;-\iota_v\big(B+\eta+\lambda\,u\wedge\iota_v B\big)=\widehat{A}-\iota_v\eta,$$

again by (2.7). Since $\iota_v\eta$ is closed with integral periods (2.5), this is an ordinary gauge transformation of the dual bundle, $F_{\widehat{A}}$ is unchanged. Thus $\mathcal{T}$ maps $\mu^{-1}(0)$ to itself at every level.

Applying $\mathcal{T}$ twice,

$$\mathcal{T}^2(a_0,b_0,b_1)=\mathcal{T}(-b_0,-a_0,b_1+a_0\wedge b_0)=\big(a_0,\;b_0,\;b_1+a_0\wedge b_0+(-b_0)\wedge(-a_0)\big)=(a_0,b_0,b_1),$$

since $(-b_0)\wedge(-a_0)=b_0\wedge a_0=-a_0\wedge b_0$. Hence $\mathcal{T}^2=\mathrm{id}$. □

What the level λ does change is the *equivariance* of $\mathcal{T}$, which is precisely the content of the following definition. We allow, a priori, the two sides of the correspondence to be twisted at different levels λ on Z and λ' on $\widehat{Z}$; the analysis below shows that consistency forces $\lambda' = \lambda$.

Definition 5.4. We say that a gauge transformation $(g, (L, \nabla^L)) \in \mathbb{G}_\lambda$ is *T-dualizable at* (A, B) if there exists a gauge transformation $(\widehat{g}, (\widehat{L}, \nabla^{\widehat{L}})) \in \mathbb{G}_{\lambda'}$ on the dual side such that

$$\mathcal{T}\big((g, (L, \nabla^L)) \cdot (A, B)\big) = (\widehat{g}, (\widehat{L}, \nabla^{\widehat{L}})) \cdot \mathcal{T}(A, B). \tag{5.6}$$

Denote by $\mathbb{G}^T_{(A,B)} < \mathbb{G}_\lambda$ the subgroup consisting of all T-dualizable gauge transformations at (A, B), i.e.

$$\mathbb{G}^T_{(A,B)} := \Big\{(g, (L, \nabla^L)) \in \mathbb{G}_\lambda \ \Big|\ (g, (L, \nabla^L)) \text{ is T-dualizable at } (A, B)\Big\}.$$

Introduce the equivalence relation $\underset{T}{\sim}$ on $\mu^{-1}(0)$ by declaring

$$(A, B) \underset{T}{\sim} (A', B') \quad \Longleftrightarrow \quad (A', B') = h \cdot (A, B) \text{ for some } h \in \mathbb{G}^T_{(A,B)}.$$

Recall from Lemma 4.1 the level-λ cocycle $\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x}, \mathbf{y}) = \mathbf{n}\Theta_g\mathbf{x}^T + (\lambda - 1)\mathbf{m}\Theta_g\mathbf{y}^T$, and

$$\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x}, \mathbf{y}) = \mathbf{n}\Theta_g\mathbf{x}^T - \mathbf{m}\Theta_g\mathbf{y}^T, \tag{5.7}$$

is the cocycle of the untwisted action.

Proposition 5.5. *Let* $(A, B) \in \mu^{-1}(0)$ *have classes* $[a_0] = \mathbf{x}$, $[b_0] = \mathbf{y}$, *and let* $h = (g, (L, \nabla^L)) \in \mathbb{G}_\lambda$ *have* $[u] = \mathbf{m}$, $[\iota_v\eta] = \mathbf{n}$*. Then* (5.6) *forces*

$$[\widehat{u}] = -\mathbf{n}, \qquad [\iota_{\widehat{v}}\widehat{\eta}] = -\mathbf{m},$$

independently of λ *and* λ'*, and the residual obstruction to* (5.6) *in* $\mathbb{R}/\mathbb{Z}$ *is*

$$(\lambda - 1)\,\mathbf{m}\Theta_g\mathbf{y}^T - (\lambda' - 1)\,\mathbf{n}\Theta_g\mathbf{x}^T. \tag{5.8}$$

Hence,

(i) the obstruction is proportional to ϕ^0*, uniformly in* $(\mathbf{m}, \mathbf{n})$*, if and only if* $\lambda' = \lambda$*, in which case* (5.8) *equals* $-(\lambda - 1)\,\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x}, \mathbf{y})$ *modulo* $\mathbb{Z}$*,*

(ii) the obstruction vanishes identically if and only if $\lambda = \lambda' = 1$*.*

Proof. Write both sides of (5.6) in the coordinates of Lemma 4.1, using Lemma 5.3 for $\mathcal{T}$. The left-hand side is $\mathcal{T}$ applied to $(\mathbf{x} + \mathbf{m},\ \mathbf{y} + \mathbf{n},\ [t + \phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x}, \mathbf{y})])$, namely

$$\Big(-(\mathbf{y} + \mathbf{n}),\ -(\mathbf{x} + \mathbf{m}),\ \big[t + \phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x}, \mathbf{y}) + (\mathbf{x} + \mathbf{m})\Theta_g(\mathbf{y} + \mathbf{n})^T\big]\Big).$$

The right-hand side is $\widehat{h}$ applied to $\mathcal{T}(\mathbf{x}, \mathbf{y}, [t]) = (-\mathbf{y}, -\mathbf{x}, [t + \mathbf{x}\Theta_g\mathbf{y}^T])$, namely

$$\Big(-\mathbf{y} + \widehat{\mathbf{m}},\ -\mathbf{x} + \widehat{\mathbf{n}},\ \big[t + \mathbf{x}\Theta_g\mathbf{y}^T + \phi^{\lambda'}_{(\widehat{\mathbf{m}},\widehat{\mathbf{n}})}(-\mathbf{y}, -\mathbf{x})\big]\Big).$$

Matching the first two components gives $\widehat{\mathbf{m}} = -\mathbf{n}$ and $\widehat{\mathbf{n}} = -\mathbf{m}$, as claimed. Substituting and subtracting, the fibre components differ by

$$\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x}, \mathbf{y}) + (\mathbf{x} + \mathbf{m})\Theta_g(\mathbf{y} + \mathbf{n})^T - \mathbf{x}\Theta_g\mathbf{y}^T - \phi^{\lambda'}_{(-\mathbf{n},-\mathbf{m})}(-\mathbf{y}, -\mathbf{x}).$$

Expanding both cocycles and using $\mathbf{a}\Theta_g\mathbf{b}^T = -\mathbf{b}\Theta_g\mathbf{a}^T$ throughout, the terms $\mathbf{n}\Theta_g\mathbf{x}^T$ and $\mathbf{x}\Theta_g\mathbf{n}^T$ cancel in pairs and $\mathbf{m}\Theta_g\mathbf{n}^T$ is an integer since Θ_g is integral, so one is left with

(5.8). For (i), the expression (5.8) is a multiple of $\phi^0 = \mathbf{n}\Theta_g\mathbf{x}^T - \mathbf{m}\Theta_g\mathbf{y}^T$ for all $(\mathbf{m},\mathbf{n},\mathbf{x},\mathbf{y})$ precisely when the coefficients of $\mathbf{m}\Theta_g\mathbf{y}^T$ and $\mathbf{n}\Theta_g\mathbf{x}^T$ are opposite, i.e. $\lambda - 1 = \lambda' - 1$, and then equal to $-(\lambda-1)\phi^0$ modulo $\mathbb{Z}$. For (ii), the $(\mathbf{x},\mathbf{y})$-dependent part of (5.8) vanishes identically iff $\lambda = 1$ and $\lambda' = 1$. $\square$

Remark 5.6. Since $\mathcal{T}$ exchanges A with $\widehat{A}$, and hence the roles of $\mathbb{G}_1$ and $\mathbb{G}_2$, requiring $\mathcal{T}$ to relate the level-λ-theory to a dual theory of the same form forces the two sides to be twisted at the same level. For $\lambda' \neq \lambda$ the $\mathbf{x}$- and $\mathbf{y}$-obstructions in (5.8) decouple and one obtains a coherent but finer stratification with no description of the quotient in terms of the single homomorphism (5.9) below.

Assuming $\lambda' = \lambda$ henceforth, Proposition 5.5 shows that the T-dualizability condition at (A,B) reads

$$(\lambda-1)\,\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) \in \mathbb{Z}.$$

Accordingly we introduce, for $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2g}\times\mathbb{R}^{2g}$,

$$G_{(\mathbf{x},\mathbf{y})} := \left\{(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^{2g}\times\mathbb{Z}^{2g} \;\middle|\; (\lambda-1)\,\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) \in \mathbb{Z}\right\},$$

and note that $G_{(\mathbf{x},\mathbf{y})} = G_{(\mathbf{x}+\mathbf{a},\mathbf{y}+\mathbf{b})}$ for $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}^{2g}\times\mathbb{Z}^{2g}$.

Let $\mathbb{G}_0 < \mathbb{G}_\lambda$ be the subgroup

$$\mathbb{G}_0 := \left\{(g,(L,\nabla^L)) \in \mathbb{G}_\lambda \;\middle|\; g = e^{2\pi i f_1},\ \mathrm{hol}^v(\nabla^L) = e^{2\pi i f_2},\ \text{for some } f_1, f_2 \in C^\infty(\Sigma,\mathbb{R})\right\},$$

which by the proof of Lemma 4.1 is independent of λ. There is an exact sequence

$$1 \longrightarrow \mathbb{G}_0 \xrightarrow{i} \mathbb{G}_\lambda \xrightarrow{h} \mathbb{Z}^{2g}\times\mathbb{Z}^{2g} \longrightarrow 0,$$

where

$$h : \mathbb{G}_\lambda \longrightarrow \mathbb{Z}^{2g}\times\mathbb{Z}^{2g}, \qquad (g,(L,\nabla^L)) \longmapsto (\mathbf{m},\mathbf{n}) = (m_1,\dots,m_{2g},\, n_1,\dots,n_{2g}),$$

is an epimorphism, with the integers m_i, n_i defined as in (4.7) after fixing a symplectic basis $\{\theta_1,\dots,\theta_g,\eta_1,\dots,\eta_g\}$ of $H^1(\Sigma;\mathbb{R})$. Note that h is a homomorphism onto an abelian group even though $\mathbb{G}_\lambda$ is a semi-direct product, since by Lemma 2.3 the twist alters η by $\lambda\, u \wedge \iota_v\eta$, whose class $\lambda\,\mathbf{m}\Theta_g\mathbf{n}^T$ is integral and therefore invisible to $[\iota_v\eta]$.

Given $(A,B) \in \mu^{-1}(0)$, express A and B as in (4.3), Write the cohomology classes as

$$[a_0] = \sum_{i=1}^g \big(x_i\theta_i + x_{i+g}\eta_i\big) = (x_1,\cdots,x_{2g}), \qquad [b_0] = \sum_{i=1}^g \big(y_i\theta_i + y_{i+g}\eta_i\big) = (y_1,\cdots,y_{2g}).$$

We then introduce the homomorphism

$$l_{(A,B)} :\ \mathbb{Z}^{2g}\times\mathbb{Z}^{2g} \longrightarrow \mathbb{R}/\mathbb{Z}, \qquad (\mathbf{m},\mathbf{n}) \longmapsto \big\{(\lambda-1)\big(\mathbf{n}\,\Theta_g[a_0]^T - \mathbf{m}\,\Theta_g[b_0]^T\big)\big\}, \tag{5.9}$$

where the notation $\{\cdot\}$ denotes the class modulo $\mathbb{Z}$, and Θ_g, a_0, b_0 are as in Section 5.1. Thus $l_{(A,B)} = (\lambda-1)\cdot l^0_{(A,B)}$, where $l^0_{(A,B)}$ is the homomorphism of the untwisted theory and $G_{(\mathbf{x},\mathbf{y})} = \ker l_{(A,B)}$.

For $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2g}\times\mathbb{R}^{2g}$, define the relation $\approx_T$ on $\mathbb{R}^{2g}\times\mathbb{R}^{2g}\times\mathbb{R}/\mathbb{Z}$ by

$$(\mathbf{x},\mathbf{y},[t]) \approx_T (\mathbf{x}',\mathbf{y}',[t']) \iff \exists\,(\mathbf{m},\mathbf{n}) \in G_{(\mathbf{x},\mathbf{y})} :\ \mathbf{x}' = \mathbf{x}+\mathbf{m},\ \ \mathbf{y}' = \mathbf{y}+\mathbf{n},\ \ [t'] = \big[t+\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y})\big].$$

Theorem 5.7. *(i) For every $(A,B) \in \mu^{-1}(0)$,*

$$\mathbb{G}^T_{(A,B)} \;\cong\; \mathbb{G}_0 \oplus \ker l_{(A,B)}.$$

(ii) The relation $\approx_T$ is an equivalence relation, and the map $\Phi(A,B) = \big([a_0],[b_0],[\int_\Sigma b_1]\big)$ descends to a bijection

$$\mu^{-1}(0)/\underset{T}{\sim} \cong \; \big(\mathbb{R}^{2g} \times \mathbb{R}^{2g} \times \mathbb{R}/\mathbb{Z}\big)/\approx_T .$$

(iii) The T-duality map $\mathcal{T} : \mu^{-1}(0) \to \mu^{-1}(0)$ induces a well-defined involution $\mathcal{T}_*$ on the quotient stack, which under the identification in (ii) is

$$\mathcal{T}_*\big[(\mathbf{x},\mathbf{y}),[t]\big] = \big[(-\mathbf{y},-\mathbf{x}),\; [t + \mathbf{x}\Theta_g\mathbf{y}^T]\big].$$

Proof. (i) As in the proof of Lemma 4.1, write a flat mixed field as $A = a_0 + d\theta$, $B = b_1 + A\wedge b_0$ and identify (A,B) with the triple (a_0,b_0,b_1). By Lemma 5.3, $\mathcal{T}(a_0,b_0,b_1) = (-b_0,-a_0,b_1 + a_0\wedge b_0)$, and by (4.5) the level-λ action of $h = (g,(L,\nabla^L))$ is

$$h\cdot(a_0,b_0,b_1) = \Big(a_0 + u,\; b_0 + \iota_v\eta,\; b_1 + \big(\eta - A\wedge\iota_v\eta - u\wedge\iota_v\eta\big) + (\lambda-1)\,u\wedge b_0\Big).$$

Imposing (5.6) and comparing the first two components gives, as in Proposition 5.5, $\widehat{g}^{-1}d\widehat{g} = -\iota_v\eta$ and $\iota_{\widehat{v}}\widehat{\eta} = -u$, the exchange of the two gauge parameters; this imposes the integrality conditions on the corresponding closed 1-forms and is independent of the level. The remaining equation is an equality of 2-forms on Σ modulo integral classes, whose obstruction is, by Proposition 5.5,

$$(\lambda-1)\big(\mathbf{n}\Theta_g[a_0]^T - \mathbf{m}\Theta_g[b_0]^T\big) \in \mathbb{Z},$$

that is $(\mathbf{m},\mathbf{n}) \in \ker l_{(A,B)}$ in the notation of (5.9). The subgroup of gauge transformations with trivial integral part is $\mathbb{G}_0$, whence $\mathbb{G}^T_{(A,B)} \cong \mathbb{G}_0 \oplus \ker l_{(A,B)}$.

(ii) We first check $\approx_T$ is an equivalence relation. Reflexivity holds since $(\mathbf{0},\mathbf{0}) \in G_{(\mathbf{x},\mathbf{y})}$ and $\phi^\lambda_{(\mathbf{0},\mathbf{0})} = 0$; the admissible set is leaf-invariant, $G_{(\mathbf{x},\mathbf{y})} = G_{(\mathbf{x}+\mathbf{a},\mathbf{y}+\mathbf{b})}$ for $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}^{2g}\times\mathbb{Z}^{2g}$, since the defining quantity changes by $(\lambda-1)(\mathbf{n}\Theta_g\mathbf{a}^T - \mathbf{m}\Theta_g\mathbf{b}^T) \in \mathbb{Z}$; and composing $(\mathbf{m},\mathbf{n})$ then $(\mathbf{m}',\mathbf{n}')$ shifts the fibre by $\phi^\lambda_{(\mathbf{m}+\mathbf{m}',\mathbf{n}+\mathbf{n}')}(\mathbf{x},\mathbf{y}) + (\lambda-1)\mathbf{m}'\Theta_g\mathbf{n}^T - \mathbf{m}\Theta_g\mathbf{n}'^T$, whose last two terms lie in $\mathbb{Z}$, giving transitivity and (with $(\mathbf{m}',\mathbf{n}') = (-\mathbf{m},-\mathbf{n})$) symmetry.

Now represent every element of $\mu^{-1}(0)$ by a triple (a_0,b_0,b_1) with a_0,b_0 closed. Fixing a symplectic basis and setting $[a_0] = \mathbf{x}$, $[b_0] = \mathbf{y}$, $[t] = [\int_\Sigma b_1] \in \mathbb{R}/\mathbb{Z}$ defines

$$\Phi : \mu^{-1}(0) \longrightarrow \mathbb{R}^{2g}\times\mathbb{R}^{2g}\times\mathbb{R}/\mathbb{Z}, \qquad (A,B) \longmapsto (\mathbf{x},\mathbf{y},[t]).$$

Two triples have the same image under Φ iff they differ by an element of $\mathbb{G}_0$. Indeed, $\mathbb{G}_0$ consists exactly of the transformations with $[u] = [\iota_v\eta] = 0$, which by (4.5) fix $(\mathbf{x},\mathbf{y})$ and shift $[\int_\Sigma b_1]$ by an integer, hence fix $[t]$. Conversely a transformation fixing $(\mathbf{x},\mathbf{y},[t])$ has $[u] = [\iota_v\eta] = 0$ and so lies in $\mathbb{G}_0$. As $b_1 \in \Omega^2(\Sigma)$ is unconstrained on $\mu^{-1}(0)$, every value of $[t]$ is attained, so Φ is a bijection

$$\mu^{-1}(0)/\mathbb{G}_0 \;\xrightarrow{\cong}\; \mathbb{R}^{2g}\times\mathbb{R}^{2g}\times\mathbb{R}/\mathbb{Z}.$$

It remains to pass from $\mathbb{G}_0$ to the full T-dualizable group. By part (i) the residual group $\mathbb{G}^T_{(A,B)}/\mathbb{G}_0 \cong \ker l_{(A,B)} = G_{(\mathbf{x},\mathbf{y})}$ acts on the image through its integral part $(\mathbf{m},\mathbf{n})$, sending, by (4.5), $(\mathbf{x},\mathbf{y})$ to $(\mathbf{x}+\mathbf{m},\mathbf{y}+\mathbf{n})$ and shifting $[t]$ by $\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y})$; this is precisely $\approx_T$. Since

$\underset{T}{\sim}$ is generated by $\mathbb{G}_0$ together with these integral-part transformations, Φ carries it exactly to $\approx_T$, and therefore descends to a bijection

$$\mu^{-1}(0)/\underset{T}{\sim}\;\cong\;\big(\mathbb{R}^{2g}\times\mathbb{R}^{2g}\times\mathbb{R}/\mathbb{Z}\big)/\approx_T .$$

(iii) Suppose $(A,B)\underset{T}{\sim}(A',B')$, say $(A',B')=h\cdot(A,B)$ with $h\in\mathbb{G}^T_{(A,B)}$, and let $\widehat{h}$ realise (5.6). Then $\mathcal{T}(A',B')=\widehat{h}\cdot\mathcal{T}(A,B)$, and it remains to check $\widehat{h}\in\mathbb{G}^T_{\mathcal{T}(A,B)}$. By Proposition 5.5 the obstruction for $\widehat{h}$ at $\mathcal{T}(A,B)$, whose classes are $[\widehat{a}_0]=-\mathbf{y}$, $[\widehat{b}_0]=-\mathbf{x}$ and whose integral part is $(\widehat{\mathbf{m}},\widehat{\mathbf{n}})=(-\mathbf{n},-\mathbf{m})$, is

$$(\lambda-1)\big(\widehat{\mathbf{n}}\Theta_g[\widehat{a}_0]^T-\widehat{\mathbf{m}}\Theta_g[\widehat{b}_0]^T\big)=-(\lambda-1)\,\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}),$$

which differs by a sign from the integral quantity of part (i) and so also lies in $\mathbb{Z}$. Hence $\widehat{h}\in\mathbb{G}^T_{\mathcal{T}(A,B)}$, so $\mathcal{T}(A',B')\underset{T}{\sim}\mathcal{T}(A,B)$ and $\mathcal{T}$ induces a well-defined map $\mathcal{T}_*$ on the quotient. In the coordinates of part (ii), $\mathcal{T}$ sends $(\mathbf{x},\mathbf{y},[t])$ to $(-\mathbf{y},-\mathbf{x},[t+\mathbf{x}\Theta_g\mathbf{y}^T])$ by Lemma 5.3, which is the stated formula, and it respects $\approx_T$ by the computation just given. Finally $\mathcal{T}_*^2=\mathrm{Id}$ follows from $\mathbf{x}\Theta_g\mathbf{y}^T+(-\mathbf{y})\Theta_g(-\mathbf{x})^T=0$ by skew-symmetry. □

The following theorem characterises how the structure of the T-dualizable subgroup, and hence of the quotient stack, varies with the level.

Theorem 5.8. *Let $\lambda\in\mathbb{Z}\setminus\{0\}$.*

(i) If $\lambda=1$, the relation $\underset{T}{\sim}$ becomes ordinary gauge equivalence, so

$$\mu^{-1}(0)/\underset{T}{\sim}\;=\;\mu^{-1}(0)/\mathbb{G}_\lambda$$

and T-duality descends to an involutive contactomorphism of the moduli space

$$\mathcal{T}_*\big[(\mathbf{x},\mathbf{y}),[t]\big]=\big[(-\mathbf{y},-\mathbf{x}),[t+\mathbf{x}\Theta_g\mathbf{y}^T]\big].$$

(ii) If $\lambda\neq1$, the rank of the isotropy is the same at every level λ,

$$\operatorname{rank}G_{(\mathbf{x},\mathbf{y})}=4g-\operatorname{rank}\operatorname{im}(l^0_{(A,B)})$$

and the level dependence is captured by the finite group $\mathbb{Z}^{4g}/G_{(\mathbf{x},\mathbf{y})}\cong\operatorname{im}(l_{(A,B)})$. At a point where $\operatorname{Tors}\operatorname{im}(l^0_{(A,B)})\cong\mathbb{Z}/N$, with N the smallest positive integer for which $N\,\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y})\in\mathbb{Z}$ for all $(\mathbf{m},\mathbf{n})\in\mathbb{Z}^{2g}\times\mathbb{Z}^{2g}$, we have

$$\operatorname{Tors}\operatorname{im}(l_{(A,B)})\;\cong\;\mathbb{Z}/\tfrac{N}{\gcd(N,\,\lambda-1)}.$$

Proof. (i) At $\lambda=1$ we have $l_{(A,B)}=(\lambda-1)\,l^0_{(A,B)}\equiv0$ by (5.9), so $G_{(\mathbf{x},\mathbf{y})}=\ker l_{(A,B)}=\mathbb{Z}^{2g}\times\mathbb{Z}^{2g}$ for every (A,B) and by Theorem 5.7(i), this gives that $\mathbb{G}^T_{(A,B)}=\mathbb{G}_0\oplus(\mathbb{Z}^{2g}\times\mathbb{Z}^{2g})=\mathbb{G}_\lambda$ is the full gauge group. Hence $\underset{T}{\sim}$ is ordinary gauge equivalence and $\mu^{-1}(0)/\underset{T}{\sim}=\mu^{-1}(0)/\mathbb{G}_\lambda$. Moreover $\phi^1_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y})=\mathbf{n}\Theta_g\mathbf{x}^T$ is the cocycle of Lemma 4.1 at $\lambda=1$, so under the identification of Theorem 5.7(ii) the relation $\approx_T$ is exactly the relation presenting $\mu^{-1}(0)/\mathbb{G}_\lambda$. Hence $\mathcal{T}_*$ descends to the moduli space. Its formula is that of Theorem 5.7(iii), and it preserves the contact form $\alpha=dt-\mathbf{y}\Theta_g\,d\mathbf{x}^T$, since

$$\sigma^*\alpha=d\big(t+\mathbf{x}\Theta_g\mathbf{y}^T\big)-(-\mathbf{x})\Theta_g\,d(-\mathbf{y})^T=dt+d\mathbf{x}\,\Theta_g\mathbf{y}^T+\mathbf{x}\Theta_g\,d\mathbf{y}^T-\mathbf{x}\Theta_g\,d\mathbf{y}^T=dt-\mathbf{y}\Theta_g\,d\mathbf{x}^T=\alpha,$$

using $d\mathbf{x}\,\Theta_g\mathbf{y}^T=-\mathbf{y}\Theta_g\,d\mathbf{x}^T$, so $\mathcal{T}_*$ is an involutive contactomorphism of the moduli space.

(ii) Let $k := \lambda - 1 \neq 0$. Since $l_{(A,B)} = k \cdot l^0_{(A,B)}$, its image is $\mathrm{im}(l_{(A,B)}) = k \cdot \mathrm{im}(l^0_{(A,B)})$, which gives the first assertion. For the rank, write $S := \mathrm{im}(l^0_{(A,B)}) \leq \mathbb{R}/\mathbb{Z}$, a finitely generated subgroup. Multiplication by k on $\mathbb{R}/\mathbb{Z}$ has kernel $\frac{1}{k}\mathbb{Z}/\mathbb{Z}$, a finite group of order $|k|$, so $S \cap \ker(k\cdot)$ is finite and

$$\mathrm{rank}\big(\mathrm{im}(l_{(A,B)})\big) = \mathrm{rank}(kS) = \mathrm{rank}(S) - \mathrm{rank}\big(S \cap \ker(k\cdot)\big) = \mathrm{rank}(S).$$

Thus $\mathrm{rank}\,\mathrm{im}(l_{(A,B)}) = \mathrm{rank}\,\mathrm{im}(l^0_{(A,B)})$, independently of $\lambda \neq 1$. Since $l_{(A,B)} \colon \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{R}/\mathbb{Z}$ has $G_{(\mathbf{x},\mathbf{y})} = \ker l_{(A,B)}$, this kernel is free. Hence $\mathbb{Z}^{4g}/G_{(\mathbf{x},\mathbf{y})} \cong \mathrm{im}(l_{(A,B)})$ and the rank–nullity identity over $\mathbb{Z}$ yields

$$\mathrm{rank}\, G_{(\mathbf{x},\mathbf{y})} = 4g - \mathrm{rank}\,\mathrm{im}(l_{(A,B)}) = 4g - \mathrm{rank}\,\mathrm{im}(l^0_{(A,B)}).$$

For the finite part, decompose $S = S_{\mathrm{free}} \oplus \mathrm{Tors}\, S$ with $\mathrm{Tors}\, S \cong \mathbb{Z}/N$ finite cyclic, a finite subgroup of $\mathbb{R}/\mathbb{Z}$ being cyclic. Here N is the order of $\mathrm{Tors}\, S$, that is the smallest positive integer with $N\, \phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) \in \mathbb{Z}$ for all $(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^{2g} \times \mathbb{Z}^{2g}$. Multiplication by k respects this decomposition; on S_{free} it is injective, so $k\, S_{\mathrm{free}}$ is again free of the same rank and contributes no torsion, while on $\mathbb{Z}/N$ its image is the subgroup generated by k, namely $\mathbb{Z}/(N/\gcd(N,k))$. Hence $\mathrm{Tors}\,\mathrm{im}(l_{(A,B)}) \cong \mathbb{Z}/(N/\gcd(N, \lambda - 1))$. □

5.4. **The action groupoid and the isotropy cocycle.** In this section we give a groupoid presentation of the T-dualizable moduli stack and describe the induced action of T-duality on it, realised as an automorphism of the associated groupoid. Throughout, $\lambda \in \mathbb{Z} \setminus \{0\}$ is the level of the action (2.6) and, as in Remark 5.6, both sides of the T-duality correspondence are twisted at the same level.

A feature of the twisted theory, invisible at level 0, is that two distinct $\mathbb{R}/\mathbb{Z}$-valued cocycles are in play; the cocycle ϕ^λ of Lemma 4.1, which governs the deck action on the fibre and hence the topology of the moduli space, and the cocycle $(\lambda-1)\phi^0$ measuring the failure of T-duality to be equivariant, with ϕ^0 as in (5.7).

The moduli space of flat mixed fields is presented by the action Lie groupoid $\mathcal{H}$ with object space

$$\mathcal{H}^{(0)} \;=\; (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \times \mathbb{R}/\mathbb{Z}$$

and morphism space

$$\mathcal{H}^{(1)} \;=\; (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \times (\mathbb{Z}^{2g} \times \mathbb{Z}^{2g}) \times \mathbb{R}/\mathbb{Z},$$

with structure maps

$$s\big((\mathbf{x},\mathbf{y}),(\mathbf{m},\mathbf{n}),[t]\big) = \big((\mathbf{x},\mathbf{y}),[t]\big), \qquad t\big((\mathbf{x},\mathbf{y}),(\mathbf{m},\mathbf{n}),[t]\big) = \big((\mathbf{x}+\mathbf{m},\mathbf{y}+\mathbf{n}),[t+\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y})]\big),$$

where, by Lemma 4.1,

$$\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) \;=\; \mathbf{n}\,\Theta_g\,\mathbf{x}^T + (\lambda-1)\,\mathbf{m}\,\Theta_g\,\mathbf{y}^T.$$

The action is free and proper, and the orbit space is the moduli space

$$\mathcal{H}^{(0)}/\mathcal{H} \;=\; \mu^{-1}(0)/\mathbb{G}_\lambda \;=\; M_{|\lambda|}.$$

The T-dualizability condition of Proposition 5.5, namely $(\lambda-1)\,\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) \in \mathbb{Z}$, is encoded by the continuous groupoid cocycle

$$\Phi \colon \mathcal{H} \longrightarrow \mathbb{R}/\mathbb{Z}, \qquad \Phi\big((\mathbf{x},\mathbf{y},[t]),(\mathbf{m},\mathbf{n})\big) = \big[(\lambda-1)\,\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y})\big].$$

A direct computation using $\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x}+\mathbf{m}',\mathbf{y}+\mathbf{n}') = \phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) + \big(\mathbf{n}\Theta_g\mathbf{m}'^T - \mathbf{m}\Theta_g\mathbf{n}'^T\big)$ and the integrality of Θ_g shows that Φ is multiplicative,

$$\Phi(\gamma_1\gamma_2) = \Phi(\gamma_1) + \Phi(\gamma_2)$$

in $\mathbb{R}/\mathbb{Z}$, so Φ is a well-defined $\mathbb{R}/\mathbb{Z}$-valued groupoid cocycle. Note that Φ is identically zero precisely when $\lambda = 1$.

Definition 5.9. The *T-dualizable groupoid* is the kernel subgroupoid of $\mathcal{H}$,

$$\mathcal{K} \;=\; \ker\Phi \;=\; \Big\{\big((\mathbf{x},\mathbf{y},[t]),(\mathbf{m},\mathbf{n})\big) \;\big|\; (\lambda-1)\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) \in \mathbb{Z}\Big\} \;\rightrightarrows\; \mathbb{R}^{2g}\times\mathbb{R}^{2g}\times\mathbb{R}/\mathbb{Z}.$$

The orbit space of $\mathcal{K}$ is the T-dualizable moduli stack of flat mixed fields, and the inclusion $\mathcal{K} \hookrightarrow \mathcal{H}$ induces a canonical surjection from the stack to the moduli space.

To isolate the base structure of $\mathcal{K}$, it is convenient to work over the torus $T^{2g}\times T^{2g}$. Writing $(\bar{\mathbf{x}},\bar{\mathbf{y}}) \in T^{2g}\times T^{2g}$ for the class of $(\mathbf{x},\mathbf{y})$, the defining condition $(\lambda-1)\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) \in \mathbb{Z}$ descends to the torus. For another lift $(\mathbf{x}+\mathbf{p},\mathbf{y}+\mathbf{q})$ with $(\mathbf{p},\mathbf{q}) \in \mathbb{Z}^{2g}\times\mathbb{Z}^{2g}$,

$$(\lambda-1)\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x}+\mathbf{p},\,\mathbf{y}+\mathbf{q}) = (\lambda-1)\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) + (\lambda-1)\big(\mathbf{n}\Theta_g\mathbf{p}^T - \mathbf{m}\Theta_g\mathbf{q}^T\big),$$

and the correction term lies in $\mathbb{Z}$ by integrality of Θ_g, so membership in $\mathcal{K}$ is independent of the chosen lift, and the isotropy group

$$G_{(\mathbf{x},\mathbf{y})} = \big\{(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^{2g}\times\mathbb{Z}^{2g} \;\big|\; (\lambda-1)\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) \in \mathbb{Z}\big\}$$

depends only on the class $(\bar{\mathbf{x}},\bar{\mathbf{y}})$. Since $(\mathbf{m},\mathbf{n}) \in \mathbb{Z}^{2g}\times\mathbb{Z}^{2g}$ acts trivially on the base, $\overline{\mathbf{x}+\mathbf{m}} = \bar{\mathbf{x}}$ and $\overline{\mathbf{y}+\mathbf{n}} = \bar{\mathbf{y}}$, every morphism of $\mathcal{K}$ is a loop at its base point $(\bar{\mathbf{x}},\bar{\mathbf{y}})$, carrying the fibre translation by $\phi^\lambda_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y})$. The isotropy bundle of $\mathcal{K}$ over $T^{2g}\times T^{2g}$ is the bundle of abelian groups

$$\widetilde{\mathcal{K}}^{(1)} = \coprod_{(\bar{\mathbf{x}},\bar{\mathbf{y}})\in T^{2g}\times T^{2g}} \{(\bar{\mathbf{x}},\bar{\mathbf{y}})\}\times G_{(\mathbf{x},\mathbf{y})}$$

whose fibre over $(\bar{\mathbf{x}},\bar{\mathbf{y}})$ is the isotropy group $G_{(\mathbf{x},\mathbf{y})} = \ker l_{(A,B)}$, which varies with the arithmetic of the base point. By Theorem 5.8(ii) its rank is $4g - \operatorname{rank}\big(\operatorname{im} l^0_{(A,B)}\big)$, independent of the level $\lambda \neq 1$, so the stratification described below is the same at every such level and only the finite parts of the fibres depend on λ. The rank of $G_{(\mathbf{x},\mathbf{y})}$ is controlled by the number of rationally independent coordinates of $(\bar{\mathbf{x}},\bar{\mathbf{y}})$. At a generic point, where the coordinates are independent over $\mathbb{Q}$, the image $\operatorname{im} l_{(A,B)}$ is dense of full rank $4g$ in $\mathbb{R}/\mathbb{Z}$, so $G_{(\mathbf{x},\mathbf{y})}$ has rank 0 and (for $\lambda \neq 1$) is finite. At a fully rational point the image is finite cyclic and $G_{(\mathbf{x},\mathbf{y})}$ is a finite-index sublattice of $\mathbb{Z}^{4g}$ of full rank $4g$. Across the intermediate strata the rank drops step by step from $4g$ to 0 as the number of independent coordinates decreases. The fibre therefore jumps in rank along the rational hyperplanes of the base, so $\widetilde{\mathcal{K}}^{(1)} \to T^{2g}\times T^{2g}$ is not locally trivial and its total space is non-Hausdorff.

The T-duality map $\mathcal{T}_*$ on the stack is covered on $\mathcal{H}$ by the map

$$\Psi\big((\mathbf{x},\mathbf{y},[t]),(\mathbf{m},\mathbf{n})\big) = \Big(\big(\sigma(\mathbf{x},\mathbf{y}),[t+\mathbf{x}\Theta_g\mathbf{y}^T]\big),\;\tau(\mathbf{m},\mathbf{n})\Big),$$

where $\sigma(\mathbf{x},\mathbf{y}) = (-\mathbf{y},-\mathbf{x})$ and $\tau(\mathbf{m},\mathbf{n}) = (-\mathbf{n},-\mathbf{m})$, a bijection of the underlying object and morphism sets. Its failure to intertwine the deck action is exactly the obstruction of Proposition 5.5: for the object map $\psi_0(\mathbf{x},\mathbf{y},[t]) = \big(\sigma(\mathbf{x},\mathbf{y}),[t+\mathbf{x}\Theta_g\mathbf{y}^T]\big)$, the same computation gives

$$\psi_0\big((\mathbf{m},\mathbf{n})\cdot(\mathbf{x},\mathbf{y},[t])\big) - \tau(\mathbf{m},\mathbf{n})\cdot\psi_0\big((\mathbf{x},\mathbf{y},[t])\big) \;\equiv\; -(\lambda-1)\,\phi^0_{(\mathbf{m},\mathbf{n})}(\mathbf{x},\mathbf{y}) \pmod{\mathbb{Z}}.$$

Hence Ψ is a groupoid automorphism of $\mathcal{H}$ precisely on the locus where this vanishes. For $\lambda = 1$ where stack and moduli space coincide, it is an automorphism of $\mathcal{H}$ itself and $\mathcal{K} = \mathcal{H}$, while for $\lambda \neq 1$ it is equivariant for the deck action exactly on $G_{(\mathbf{x},\mathbf{y})}$ and so descends to an automorphism of $\mathcal{K} = \ker\Phi$.

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