EQUIVARIANT INDEX ON TORIC CONTACT MANIFOLDS

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ABSTRACT. We compute the equivariant index of the twisted horizontal Dolbeault operator on compact toric contact manifolds of Reeb type. The operator is elliptic transverse to the Reeb foliation and its equivariant index defines a distribution on the torus. Using the good cone condition, we show that the symbol localises to the closed Reeb orbits corresponding to the edges of the moment cone and obtain an Atiyah-Bott-Lefschetz type formula for the index. For the horizontal Dolbeault operator, we obtain an expression for the index as a sum over the lattice points of the moment cone, by applying an adaptation of the Lawrence-Varchenko polytope decomposition to rational polyhedral cones.

1. INTRODUCTION

There has been considerable interest in K-contact and Sasaki manifolds in recent years, in part due to their role in theoretical physics as backgrounds in supersymmetric gauge theories [11, 25] and the AdS/CFT correspondence [12, 13, 22, 23]. Let (M, H) be a $(2n + 1)$-dimensional compact co-oriented contact manifold with contact form α and associated Reeb vector field R_α. We say that (M, H) is toric if it carries an effective action by a torus G of dimension $n + 1$ preserving the contact structure. Recall that (M, H) is of Reeb type if the Reeb vector field is generated by a one-parameter subgroup of G. Toric contact manifolds of Reeb type carry an invariant Sasakian structure [4] and as shown by Lerman [17], they admit a combinatorial description in terms of their moment map images, which are strictly convex rational polyhedral cones. This structure has been exploited to compute various invariants of toric contact manifolds, such as the volume [14, 22], the first and second homotopy groups [18], the equivariant cohomology ring [21] and the cylindrical contact homology [2].

In this paper, we consider the index of the horizontal Dolbeault operator on compact toric contact manifolds of Reeb type endowed with an invariant Sasakian structure. This operator is the odd dimensional analogue of the Dolbeault operator in Kähler geometry. It appears for instance in [11, 25] in the calculation of perturbative partition functions of certain supersymmetric field theories, in [26] in relation to deformations of Sasakian structures and in [5] to compute the dimension of moduli spaces of instantons on contact 5-manifolds. The operator is elliptic transverse to the Reeb foliation and on toric Sasaki manifolds, it is elliptic in directions transversal to the G-orbits.

In [3], Atiyah-Singer proved that a pseudodifferential operator A that is G-transversally elliptic may have an infinite-dimensional kernel and cokernel, but they define a virtual trace-class representation of G. The index of A can therefore

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be defined as a generalised function on G by

$$\operatorname{ind}_G^M(A)(t) = \operatorname{tr}(t|_{\ker A}) - \operatorname{tr}(t|_{\operatorname{coker}A}).$$

Decomposing $\ker A$ and $\operatorname{coker}A$ into isotypical components, we have

$$\operatorname{ind}_G^M(A)(t) = \sum_{\mu \in \hat{G}} m(\mu)\chi_\mu,$$

where $m \colon \hat{G} \to \mathbb{Z}$ encodes the multiplicities of the irreducible G-representations appearing in the index character. We will compute explicitly the function m when A is the horizontal Dolbeault operator $\overline{\partial}_H$ and more generally, we derive a localisation theorem for the index when $\overline{\partial}_H$ is coupled to a holomorphic bundle.

Our first main result is an Atiyah-Bott-Lefschetz type formula for the twisted horizontal Dolbeault operator:

Theorem 1.1. *Let $\overline{\partial}_H^E$ be the horizontal Dolbeault operator on a compact toric Sasaki $(2n+1)$-manifold M twisted by a G-equivariant transversally holomorphic bundle E over M. For any $t \in G$,*

$$ind_G^M(\overline{\partial}_H^E)(t) = \sum_{L \in E(C)} \chi_{E|_L}(t) \prod_{i=1}^{n} \left(\frac{1}{1 - t^{-w_L^i}} \right)^{\pm} \delta(1 - t^{\mu_L}),$$

where $E(C)$ is the set of edges of the moment cone C, $\{w_L^1, \ldots, w_L^n\}$ are the isotropy weights and μ_L is the weight of the action of G on the closed Reeb orbit corresponding to L.

The signs $\pm$ dictate whether the denominator is expanded about $t = 0$ or $t = \infty$ and are fixed by the pairing of the isotropy weights with a polarizing vector, see Section 4 for detailed explanations.

Our method is based on Atiyah's algorithm outlined in [3] to stratify M using the torus action and reduce the index calculation to computations on lower dimensional submanifolds. Using the good cone property of toric contact manifolds of Reeb type, we construct a deformation vector field that in fact localises the index to contributions from a finite number of closed Reeb orbits corresponding to the edges of the moment cone. We note that a general cohomological formula for the index of G-transversally elliptic operators was obtained in [6, 7] and more specifically for contact manifolds in [10]. These formulas are however not well-adapted to computing the multiplicities since they provide an expression for the index that is valid only on a neighbourhood of each $t \in G$. Even in the elliptic case, deducing the function m from the Atiyah-Segal-Singer fixed point theorem is not easy. Our approach is to exploit the combinatorial structure of the manifold M to determine the function m as explicitly as possible. For instance, Theorem 1.1 can be readily applied to compute the dimensions of moduli spaces of instantons [5] and transverse Seiberg-Witten monopoles [16] on toric Sasaki manifolds.

We further remark that in Theorem 1.1 the Sasakian structure is not assumed to be quasi-regular and the formula applies in particular to the irregular $Y^{p,q}$ spaces [13]. When the Sasakian structure is quasi-regular, the manifold is an orbifold circle bundle over a toric Kähler orbifold X. In this case, one could also apply the

equivariant orbifold index theorem to the Dolbeault operator on X twisted by the character line bundles of the circle.

Our second result is an expression for the index of $\overline{\partial}_H$ in terms of the integral points of the moment cone:

Theorem 1.2. *The index of the horizontal Dolbeault operator $\overline{\partial}_H$ is given by*

$$ind_G^M(\overline{\partial}_H)(t) = (-1)^n \sum_{\mu \in C^\circ \cap \mathbb{Z}_G^*} t^\mu + \sum_{\mu \in (-C) \cap \mathbb{Z}_G^*} t^\mu,$$

where $\mathbb{Z}_G^*$ denotes the dual integral lattice of $\mathfrak{g}$, C° the interior of the moment cone and $-C$ the negative cone.

This follows by adapting and applying a version of the Lawrence-Varchenko formula to the polar decomposition of cones over a polytope (Proposition 5.14). Another key ingredient in the proof is Lerman's local description of toric contact manifolds [17], that allows us to relate the weights of the torus action to the moment cone and identify the localisation formula in Theorem 1.1 with the Lawrence-Varchenko formula for rational polyhedral cones.

In [23], Martelli-Sparks-Yau considered the Dolbeault operator on an orbifold resolution of the non-compact Kähler cone of a Sasaki manifold and showed that the equivariant index equals the integral points of the moment cone. This corresponds essentially to the second term of the index of $\overline{\partial}_H$ in Theorem 1.2. A similar limiting argument as in [23] applied to this term would compute the volume of the momentum polytope, which up to a constant equals the volume of the toric Sasaki manifold. Another related recent work is by Lin-Loizides-Sjamaar-Song [20], where they study the equivariant index of the basic Dirac operator on Riemannian foliations whose leaf space is symplectic and establish a quantization commutes with reduction theorem. This includes toric K-contact manifolds as a special case, however their setup is complementary to ours as their operator only acts on basic sections, corresponding to the invariant part of our index.

The paper is structured as follows. In Section 2 we provide a brief review of contact and Sasakian structures, recall some results from [17] including a local normal form for toric contact manifolds and Lerman's construction of a toric contact manifold from a good cone. We also introduce the main object of this paper, the horizontal Dolbeault operator. Section 3 introduces the fundamental concepts in the theory of G-transversally elliptic operators and our main computational tool, Atiyah's algorithm for localising the index. In Section 4 we introduce a deformation vector field and apply the localisation argument to derive Theorem 1.1. Finally, in Section 5, we prove a cone version of the Lawrence-Varchenko formula and apply it to the index of the horizontal Dolbeault operator to obtain an explicit lattice point formula.

2. Toric contact manifolds

Let M be a smooth compact manifold of dimension $2n+1$. Recall that a contact structure on M is a hyperplane distribution $H \subset TM$ defined globally by $H = \ker \alpha$

for a 1-form α such that $d\alpha|_H$ is non-degenerate. The contact form α defines a volume form $\alpha \wedge (d\alpha)^n$ on M and its conformal class determines a co-orientation of the pair (M, H). Associated to every co-oriented contact manifold is a symplectic cone $(C(M), \omega)$ defined by

$$C(M) = M \times \mathbb{R} \ \text{ and } \ \omega = d(e^r \alpha),$$

where r is the coordinate in $\mathbb{R}$ and $2\frac{\partial}{\partial r}$ is the Liouville vector field. The Reeb vector field associated to α is the unique vector field R_α satisfying $\iota_{R_\alpha} \alpha = 1$ and $\iota_{R_\alpha} d\alpha = 0$. We let $V \subset TM$ denote the rank one sub-bundle spanned by R_α.

A *contact metric structure* on (M, α, R_α) is a reduction of structure of the tangent bundle to $U(n) \subset GL(2n + 1, \mathbb{R})$. Alternatively, it consists of an endomorphism $\Phi \colon TM \to TM$ and a Riemannian metric g such that $\Phi^2 = -I + \alpha \otimes R_\alpha$ and $g(\Phi X, \Phi Y) = g(X, Y) - \alpha(X)\alpha(Y)$ for all vector fields X, Y. This yields an orthogonal decomposition $TM = V \oplus H$ together with a unitary structure on H. The restriction $J = \Phi|_H$ of Φ to H defines the complex structure on H and $d\alpha(X, Y) = g(X, \Phi Y)$ restricted to H is the Hermitian 2-form associated to J.

We say that M is a *K-contact manifold* if R_α is a Killing vector field with respect to g. This is equivalent to the characteristic foliation generated by R_α being a Riemannian foliation. Extending g to the symplectic cone, we obtain a metric $h = dr^2 + r^2 g$ on $C(M)$ and an associated almost complex structure J_C defined by $h(X, J_C Y) = \omega(X, Y)$. A contact metric structure $(\alpha, R_\alpha, \Phi, g)$ on M is called *Sasakian* if (h, J_C, ω) is a Kähler structure on $C(M)$. Sasaki manifolds constitute the most important class of K-contact manifolds and are the odd dimensional counterparts to Kähler manifolds.

Example 2.1. Geometric quantisation provides examples of *quasi-regular* K-contact manifolds, that is when all leaves of the characteristic foliation are circles. Let (B, ω) be a symplectic manifold such that $[\omega] \in H^2(B, \mathbb{Z})$. Let M denote the principal S^1-bundle over B with Chern class equal to $[\omega]$. There is a connection form α on M such that $d\alpha = \pi^*\omega$. Since ω is symplectic, we have $\alpha \wedge (d\alpha)^n = \alpha \wedge \pi^*\omega^n \neq 0$, so α is a contact form and its Reeb vector field R_α is the generator of the free S^1-action on M. Such contact manifolds are regular and the projection $\pi \colon M \to B$ is known as the Boothby-Wang fibration [8]. This construction generalises to symplectic orbifolds (B, ω) such that $[\omega] \in H^2(B, \mathbb{R})$ admits a lift to a class $c \in H^2_{\mathrm{orb}}(B, \mathbb{Z})$, the degree 2 orbifold cohomology of B. Then c defines a Seifert fibration $\pi \colon M \to B$ carrying a pseudo-free S^1-action and M admits the structure of a quasi-regular K-contact manifold. When B is a Kähler orbifold, M acquires a Sasakian structure.

Let G be a torus of dimension $n+1$, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ its dual Lie algebra. We will denote by $\mathbb{Z}_G = \ker(\exp \colon \mathfrak{g} \to G)$ the integral lattice of $\mathfrak{g}$. Suppose that G acts on a manifold M. If $v \in \mathfrak{g}$, we denote by $v(p) \in T_p M$ the tangent vector induced by the action of G on M.

Definition 2.2. A contact manifold (M, H) of dimension $2n + 1$ is called *toric* if there is an effective action by an $(n + 1)$-dimensional torus G on M preserving the

contact form. The α-*moment map* $\phi_\alpha \colon M \to \mathfrak{g}^*$ is defined by

$$\langle \phi_\alpha(p), v \rangle = \alpha_p(v(p))$$

for all $p \in M$ and $v \in \mathfrak{g}$. The *moment cone* associated with ϕ_α is defined by

$$C = \{ t\phi_\alpha(p) \mid t \geqslant 0, p \in M \},$$

and can be identified with the union of $\{0\}$ with the image of the moment map $e^r \phi_\alpha$ of the lifted Hamiltonian G-action on $C(M)$, where G acts trivially on $\mathbb{R}$.

The classification of compact toric contact manifold was completed by Lerman [17]. In dimensions greater than three and when the G-action is not free, the contact toric manifolds are classified by their moment cones, which are good cones:

Definition 2.3. A cone $C \subset \mathfrak{g}^*$ is *good* if there exists a minimal set of primitive vectors $v_1, \ldots, v_d \in \mathbb{Z}_G$, with $d \geqslant n + 1$, such that:

(i) $C = \bigcap\limits_{j=1}^{d} \{ y \in \mathfrak{g}^* \mid \langle y, v_j \rangle \geqslant 0 \}$,

(ii) Any codimension-k face of C, $1 \leqslant k \leqslant n$, is the intersection of exactly k facets whose set of normals can be completed to an integral base of $\mathbb{Z}_G$.

Remark 2.4. Good cones are rational polyhedral, meaning that the normals to the facets are integral vectors.

Toric contact manifolds can be further divided into Reeb and non-Reeb types. We say that M is of *Reeb type* if R_α is generated by an element $R \in \mathfrak{g}$.

Theorem 2.5 ([4], [17]). *If (M, α) is a toric contact manifold of Reeb type, then its moment cone C is a strictly convex good cone. The image of the α-moment map ϕ_α is a compact convex simple polytope P given by the intersection of the characteristic hyperplane*

$$\mathcal{H} = \{ \eta \in \mathfrak{g}^* \mid \eta(R) = 1 \},$$

determined by the vector R, with the moment cone C.

Remark 2.6. Strictly convex means that C contains no linear subspaces of positive dimension, so it is a cone over a polytope. Toric contact manifolds with good moment cones C that are not strictly convex are diffeomorphic to $\mathbb{T}^k \times S^{k+2l-1}$, for some $k > 1, l \geqslant 0$ [19].

Toric contact manifolds with an invariant K-contact structure must be of Reeb type [18] and they always admit an invariant Sasakian structure [4]. In the sequel, we will therefore assume that our toric contact manifolds are of Reeb type of dimension greater than three and equipped with an invariant Sasakian structure. We will need the following result characterising the Reeb vector fields associated to a Sasakian structure:

Theorem 2.7 ([22]). *Let $v_1, \ldots, v_d \in \mathfrak{g}$ be the defining integral normals of the moment cone $C \in \mathfrak{g}^*$ associated with a toric contact manifold of Reeb type (M, H).*

A vector $R \in \mathfrak{g}$ generates the Reeb vector field of an invariant Sasakian 1-form α such that $H = \ker \alpha$ if and only if

$$R = \sum_{j=1}^{d} a_j v_j, \ \text{with } a_j \in \mathbb{R}^+ \text{ for all } j = 1, \ldots, d.$$

Example 2.8. Returning to Example 2.1, if (B, ω) is an integral toric symplectic manifold, then the principal S^1-bundle over B is a good toric contact manifold with moment cone

$$C = \{r(x, 1) \in \mathbb{R}^n \times \mathbb{R} \mid x \in P, r \geq 0\},$$

where $P \subset \mathbb{R}^n$ is the integral Delzant polytope associated with B.

Example 2.9. For each pair of coprime integers p, q with $0 < q < p$, the $Y^{p,q}$ spaces are toric Sasaki-Einstein metrics on $S^2 \times S^3$. The Sasakian structure is irregular of rank 2 whenever $4p^2 - 3q^2$ is a not perfect square [13]. In higher dimensions they generalise to the family of toric contact manifolds $N_{k,m}^{2n+1}$, $n \geq 2$, $k \geq 1$ and $0 \leq m < kn$, associated to the good cones $C(k, m) \subset (\mathbb{R}^{n+1})^*$ defined by the normals

$$v_i = e_i + e_{n+1}, \ v_n = -\sum_{i=0}^{n-1} e_i + m e_n + e_{n+1}, \ v_- = k e_n + e_{n+1}, \ v_+ = -e_n + e_{n+1},$$

where $e_i \in (\mathbb{R}^{n+1})$, $i = 1, \ldots, n$, are the canonical basis vectors of $\mathbb{R}^{n+1}$. Unlike the $Y^{p,q}$ spaces, $N_{k,m}^{2n+1}$ are not all diffeomorphic [1].

2.1. **Lerman's construction.** The classification of toric contact manifolds of Reeb type is analogous to Delzant's classification of toric symplectic manifolds [4], [17]. In this section, we briefly recall the construction of a toric contact manifold from its moment cone and elucidate the relation with the isotropy weights.

Let $C \subset \mathfrak{g}^*$ be a strictly convex good cone given by

$$C = \bigcap_{i=1}^{d} \{\eta \in \mathfrak{g}^* \mid \eta(v_i) \geq 0\},$$

where $v_i \in \mathbb{Z}_G$, $i = 1, \ldots, d$, are the inward pointing normals of C and $\dim \mathfrak{g}^* > 2$.

Let $\{e_1, \ldots, e_d\}$ denote the standard basis of $\mathbb{R}^d$ and define the map $\beta \colon \mathbb{R}^d \to \mathfrak{g}$ by $\beta(e_i) = v_i$. Denote by $\mathfrak{k}$ the kernel of β. Since C is strictly convex, β is surjective and we have the short exact sequences

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathfrak{g} \to 0 \ \text{ and } \ 0 \to \mathfrak{g}^* \xrightarrow{\beta^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} \mathfrak{k}^* \to 0.$$

Since $\beta(\mathbb{Z}^d) \subset \mathbb{Z}_G$, β induces a map $\tilde{\beta} \colon \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \to \mathfrak{g}/\mathbb{Z}_G = G$. Let

$$K = \left\{ [t] \in \mathbb{T}^d \mid \sum_{i=1}^{d} t_i v_i \in \mathbb{Z}_G \right\}$$

denote the kernel of $\tilde{\beta}$. It is a compact abelian subgroup with Lie algebra $\mathfrak{k} = \ker(\beta)$. Consider the standard action of $\mathbb{T}^d$ on $(\mathbb{C}^d, \omega_{st} = i/2\pi \sum_{j=1}^{d} dz_j \wedge d\bar{z}_j)$ given by

$$[t] \cdot (z_1, \ldots, z_d) = (e^{2\pi i t_1} z_1, \ldots, e^{2\pi i t_d} z_d).$$

The corresponding moment map $\phi\colon \mathbb{C}^d \to (\mathbb{R}^d)^*$ is given by

$$\phi(z_1, \ldots, z_d) = \sum_{j=1}^{d} |z_j|^2 e_j^*,$$

where $\{e_j^*\}$ is the basis dual to the canonical basis $\{e_j\}$. Since K is a subgroup of $\mathbb{T}^d$, it acts on $\mathbb{C}^d$ with moment map

$$\phi_K(z_1, \ldots, z_d) = \sum_{j=1}^{d} |z_j|^2 \iota^*(e_j^*) \in \mathfrak{k}^*.$$

The reduced space $W_C = \frac{(\phi_K^{-1}(0)\setminus\{0\})}{K}$ is a toric symplectic cone with symplectic form ω_C induced by ω_{st}. It carries an action of $G = \mathbb{T}^d/K$ induced by the $\mathbb{T}^d$-action and an action of $\mathbb{R}$ induced by the standard radial $\mathbb{R}$-action on $\mathbb{C}^d$.

Let σ be a section of $\beta\colon \mathbb{R}^d \to \mathfrak{g}$, giving a splitting $\mathbb{R}^d \cong \iota(\mathfrak{k}) \oplus \sigma(\mathfrak{g})$. Since σ is injective, its image defines an n-torus $\sigma(G) \subset \mathbb{T}^d$. The action of G on $(\phi_K^{-1}(0)\setminus\{0\})$ via $\sigma(G) \subset \mathbb{T}^d$ is Hamiltonian with moment map $\tilde{\phi} = \sigma^* \circ \phi\colon (\phi_K^{-1}(0)\setminus\{0\}) \to \mathfrak{g}^*$. The G-action and the moment map $\tilde{\phi}$ descend to the quotient W_C making it a Hamiltonian G-space with moment map

$$\phi_G\colon W_C \to \mathfrak{g}^*$$
$$[z_1, \ldots, z_d] \mapsto \sigma^*(\phi(z_1, \ldots, z_d)),$$

where we denote by $[z_1, \ldots, z_d] \in W_C$ the class of $(z_1, \ldots, z_d) \in (\phi_K^{-1}(0)\setminus\{0\})$ in the quotient. The image of ϕ_G is the cone $C\setminus\{0\}$. The sphere $S^{2d-1} = \{z \in \mathbb{C}^d; |z|\}$ is a $\mathbb{T}^d$-invariant hypersurface of contact type in $\mathbb{C}^d$ and

$$M_C = \frac{(\phi_K^{-1}(0) \bigcap S^{2d-1})}{K}$$

is a G-invariant hypersurface of contact type in W_C. Therefore it has a toric contact structure induced by the G-invariant contact form $\alpha = i_{X_C}\omega_C$, where X_C is the Liouville vector field induced by the $\mathbb{R}$-action on W_C. The moment cone of (M_C, α) is C.

Lemma 2.10. $\phi_K^{-1}(0) = \phi^{-1}(\beta^*(C))$

Proof. Since $0 \to \mathfrak{g}^* \xrightarrow{\beta^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} \mathfrak{k}^* \to 0$ is exact, we have $(\iota^*)^{-1}(0) = \beta^*(\mathfrak{g}^*)$. Therefore $\phi_K^{-1}(0) = (\iota^* \circ \phi)^{-1}(0) = \phi^{-1}((\iota^*)^{-1}(0)) = \phi^{-1}(\beta^*(\mathfrak{g}^*)) = \phi^{-1}(\beta^*(\mathfrak{g}^*) \cap \phi(\mathbb{C}^d))$. It follows from

$$\beta^*(\mathfrak{g}^*) \cap \phi(\mathbb{C}^d) = \{\beta^*(\eta) \mid \eta \in \mathfrak{g}^* \, and \, \langle \beta^*(\eta), e_i \rangle \geqslant 0 \text{ for all } i\}$$
$$= \{\beta^*(\eta) \mid \eta \in \mathfrak{g}^* \, and \, \langle \eta, \beta(e_i) \rangle \geqslant 0 \text{ for all } i\}$$
$$= \{\beta^*(\eta) \mid \eta \in \mathfrak{g}^* \, and \, \langle \eta, v_i \rangle \geqslant 0 \text{ for all } i\}$$
$$= \{\beta^*(\eta) \mid \eta \in C\}$$

that $\phi_K^{-1}(0) = \phi^{-1}(\beta^*(C))$. $\qquad\square$

The following lemma informs us how to read the isotropy groups from the moment cone.

Lemma 2.11. *Let (M, α) be a toric contact manifold of Reeb type with moment cone C. Let $p \in M$ and $\eta = \phi_\alpha(p)$ the image of p under the α-moment map. If $\eta(v_i) = 0$, for a subset of indices $i \in I \subset \{1, \ldots, d\}$, then its isotropy Lie algebra $\mathfrak{g}_p$ is generated by the vectors v_i, $i \in I$.*

Proof. Lerman's construction implies that $M \cong M_C$ and that $C = \phi_G(W_C)$. Therefore $\eta(v_i) = 0$ if and only if $\phi_G(p)(v_i) = 0$, where we are considering p as an element of W_C via the inclusion $M \cong M_C \subset W_C$. From Lemma 2.10 we have that $\phi_K^{-1}(0) = \phi^{-1}(\beta^*(C))$. Let $z = (z_1, \ldots, z_d) \in \phi_K^{-1}(0)$ be such that $\beta^*(\eta) = \phi(z)$. Then

$$|z_j|^2 = \langle \phi(z), e_j \rangle = \langle \beta^*(\eta), e_j \rangle = \langle \eta, \beta(e_j) \rangle = \langle \eta, v_j \rangle.$$

Therefore $z_j = 0$ if and only if $\eta(v_j) = 0$. The torus $\mathbb{T}^d$ acts on $\phi_K^{-1}(0) \subset \mathbb{C}^d$ via the standard action and $\eta_{(v_j)} = 0$ if and only if $e_j \in \mathfrak{t}_z^d$. The G-action on $\phi_K^{-1}(0)$ is given by a section σ of $\beta \colon \mathbb{T}^d \to G$. Since $K = \ker \beta$, we have $\sigma_*(v_j) = e_j + k$, where $k \in \mathfrak{k}$ and σ_* is the Lie algebra map induced by σ. It follows that $\mathfrak{g}_p = \mathfrak{t}_z^d/\mathfrak{k}$, therefore $[\sigma_*(v_j)] = [e_j] \in \mathfrak{t}_z^d/\mathfrak{k}$. Since the G-action is given by the section σ, we have that $v_j \in \mathfrak{g}_p$ is equivalent to $[\sigma_*(v_j)] \in \mathfrak{t}_z^d/\mathfrak{k}$, and therefore $v_j \in \mathfrak{g}_p$ if and only if $\eta(v_j) = 0$. Since C is a good cone, the v_j satisfying $\eta(v_j) = 0$ form an integral basis of $\mathfrak{g}_p$. $\qquad\square$

Next we consider the weights of the isotropy representations. First we need a lemma from [9]:

Lemma 2.12. *Let $\rho \colon G \to GL(V)$ be a faithful representation of a torus G preserving a symplectic form ω such that $\dim V = 2 \dim G$. If ρ is faithful, then its weights form a basis of the weight lattice $\mathbb{Z}_G^*$ of G.*

We have the following specialisation of Lerman's local normal form for the moment map [17], when restricted to a vertex.

Theorem 2.13. *Let $p \in M$ be such that $\phi_\alpha(p)$ is a vertex of the convex polytope $P = \phi_\alpha(M)$ and let $V = H_p$ be the fibre of the contact distribution on p. The isotropy group G_p acts on V preserving the symplectic form $d\alpha|_V$. Then*

$$\mathfrak{g}_p^\circ = \mathbb{R}\phi_\alpha(p)$$

and we can choose a splitting

$$\mathfrak{g}^* = \mathfrak{g}_p^\circ \oplus \mathfrak{g}_p^* = \mathbb{R}\phi_\alpha(p) \oplus \mathfrak{g}_p^*.$$

Let $i \colon \mathfrak{g}_p^* \to \mathfrak{g}^*$ be the corresponding embedding. Then there exists a G-invariant neighbourhood U of the zero section $G \cdot [1, 0]$ in

$$N = G \times_{G_p} V$$

and an open G-equivariant embedding $\varphi \colon U \to M$ with $\varphi([1, 0]) = p$ and a G-invariant 1-form α_N on N such that

(1) $\varphi^*\alpha = e^f \alpha_N$ form some function $f \in C^\infty(U)$;

(2) *the* α_N-*moment map* ϕ_{α_N} *is given by*

$$\phi_{\alpha_N}([a,v]) = \phi_\alpha(p) + i(\phi_V(v)),$$

where $\phi_V : V \to \mathfrak{g}^*$ *is the moment map for the representation of* G_p *on* V.

Consequently,

$$\phi_\alpha \circ \varphi([a,v]) = (e^f \phi_{\alpha_N})([a,v]) = e^{f([a,v])}(\phi_\alpha(p) + i(\phi_V(v)))$$

for some G-*invariant function* f *on* N.

The following corollary shows how the isotropy weights relate to the moment cone.

Corollary 2.14. *Let* $p \in M$ *be such that* $\phi_\alpha(p)$ *is a vertex of the convex polytope* $P = \phi_\alpha(M)$. *Then the representation* $G_p \to GL(V)$ *is faithful, and the weights of the action of the isotropy group* G_p *on* $V = H_p$, *the fibre of the contact distribution at* p, *form a basis of the weight lattice* $\mathbb{Z}^*_{G_p}$ *of* $\mathfrak{g}^*_p$ *that is dual to the basis* $\{v_1^p, \dots, v_n^p\}$ *of* $\mathfrak{g}_p$, *where* $v_1^p, \dots, v_n^p$ *are the normals to the faces of the moment cone* $C(\phi_\alpha)$ *meeting at* $\phi_\alpha(p)$.

Proof. Theorem 2.13 ensures that there is a neighbourhood of $G \cdot p$ that is G-equivariantly diffeomorphic to a neighbourhood of the zero section of $N = G \times_{G_p} V$. Since the action of G on N is effective, the representation of G_p on V must be faithful. The image of the moment map ϕ_V is

$$\phi_V(V) = \left\{ \sum_{i=1}^n s_i w_p^i \mid s_i \geqslant 0 \right\} \subset \mathfrak{g}_p^*,$$

where the w_p^i are the weights of the isotropy action of G_p on $V = H_p$. Therefore the weights w_p^i generate the edges of the cone $\phi_V(V)$. Lemma 2.12 shows that the weights w_p^i form a basis of the integral lattice of $\mathfrak{g}_p^*$. It follows that the weights $i(w_p^j) \in \mathfrak{g}^*$ satisfy $i(w_p^j)(v_k^p) = \delta_{jk}$. This implies that $w_p^j(v_k^p) = \delta_{jk}$, where we are viewing the normals $v_1^p, \dots, v_n^p$ as elements of $\mathfrak{g}_p$. Thus $w_p^1, \dots, w_p^n \in \mathfrak{g}_p^*$ is the dual basis to $v_1^p, \dots, v_n^p$. $\qquad\square$

Remark 2.15. Corollary 2.14 allows us to read the weights of the isotropy representation $G_p \to GL(V)$ from the moment cone; one simply needs to choose a vector v_0^p completing the set $\{v_1^p, \dots, v_n^p\}$ to an integral basis of the lattice $\mathbb{Z}_G \subset \mathfrak{g}$.

2.2. **The horizontal Dolbeault operator.** Let M be a toric contact manifold of Reeb type with $\dim M > 3$ endowed with a Sasakian structure $(\alpha, R_\alpha, \Phi, g)$. The transverse complex structure $J = \Phi|_H$ allows us to introduce a horizontal Dolbeault operator $\bar{\partial}_H$, and more generally $\bar{\partial}_H^E$ twisted by a transverse holomorphic bundle E. We briefly recall the definitions.

Let $\Omega_H^k(M) = \{\omega \in \Omega^k(M) \mid \iota_{R_\alpha}\omega = 0\} = \Gamma(M, \bigwedge^k H^*)$ denote the space of *horizontal* k-forms. The projection operators $P_V = \alpha \wedge \iota_{R_\alpha}$ and $P_H = 1 - P_V$ determine a splitting

$$\Omega^*(M) = \Omega_V^*(M) \oplus \Omega_H^*(M) = P_V(\Omega^*(M)) \oplus P_H(\Omega^*(M))$$

into horizontal and vertical forms, and $d_H = P_H \circ d$ is a differential on $\Omega_H^k(M)$. The transverse complex structure gives the usual decomposition of $\Omega_H^*(M) \otimes \mathbb{C}$ into horizontal (p, q)-forms and defines the horizontal Dolbeault complex:

$$0 \to \Omega_H^{0,0}(M) \xrightarrow{\overline{\partial}_H} \Omega_H^{0,1}(M) \xrightarrow{\overline{\partial}_H} \cdots \xrightarrow{\overline{\partial}_H} \Omega_H^{0,n}(M) \to 0,$$

with the associated symbol complex

$$0 \to \pi^*(\textstyle\bigwedge^{0,0} H^*) \xrightarrow{\sigma(\overline{\partial}_H)} \pi^*(\textstyle\bigwedge^{0,1} H^*) \xrightarrow{\sigma(\overline{\partial}_H)} \cdots \xrightarrow{\sigma(\overline{\partial}_H)} \pi^*(\textstyle\bigwedge^{0,n} H^*) \to 0,$$

where $\pi \colon T^*M \to M$ is the projection and $\sigma(\overline{\partial}_H)_{(p,\xi)}(\bar{w}) = \xi_H^{0,1} \wedge \bar{w}$, where $(p, \xi) \in T^*M$, $\bar{w} \in \bigwedge^{0,k} H^*|_p$ and $\xi_H^{0,1} = P_H(\xi)^{0,1}$ is the $(0,1)$-component of the horizontal projection of ξ.

A G-equivariant vector bundle $\pi_E \colon E \to M$ is transversally holomorphic if with respect to an open cover $\{U_\alpha\}$ of M, it is determined by transition functions $g_{\alpha\beta}$ satisfying $\overline{\partial}_H(g_{\alpha\beta}) = 0$. The twisted horizontal Dolbeault operator $\overline{\partial}_H^E$ is given locally by

$$\overline{\partial}_H^E \colon \Omega_H^{0,k}(M, E) \to \Omega_H^{0,k+1}(M, E)$$

$$w \otimes u_\alpha \mapsto \overline{\partial}_H(w) \otimes u_\alpha.$$

and we have the twisted horizontal Dolbeault complex

$$0 \to \Omega_H^{0,0}(M, E) \xrightarrow{\overline{\partial}_H^E} \Omega_H^{0,1}(M, E) \xrightarrow{\overline{\partial}_H^E} \cdots \xrightarrow{\overline{\partial}_H^E} \Omega_H^{0,n}(M, E) \to 0,$$

with the symbol $\sigma(\overline{\partial}_H^E)_{(p,\xi)}(\bar{w} \otimes \bar{e}) = \xi_H^{0,1} \wedge \bar{w} \otimes \bar{e}$, where $\bar{w} \otimes \bar{e} \in \bigwedge^{0,k} H|_p \otimes E|_p$.

3. Equivariant Index

3.1. Transversally elliptic operators.
In this section, we collect relevant facts about transversally elliptic operators [3, 24].

Let G be a compact Lie group and M a compact G-manifold. We denote by $\pi \colon T^*M \to M$ the natural projection. A complex of G-invariant pseudodifferential operators is called transversally elliptic, if it is elliptic in the directions transversal to G-orbits. More precisely, let $T_G^*M \subset T^*M$ be the closed subset defined by the union of conormals to the G-orbits,

$$T_G^*M = \{(p, \xi) \in T^*M \mid \xi(v(p)) = 0 \text{ for all } v \in \mathfrak{g}\}.$$

A complex of G-invariant pseudodifferential operators P is a sequence

$$0 \to \Gamma(E^0) \xrightarrow{P_1} \Gamma(E^1) \xrightarrow{P_2} \cdots \xrightarrow{P_n} \Gamma(E^n) \to 0,$$

where the P_i are G-invariant pseudodifferential operators and $\Gamma(E^i)$ is the space of sections of the G-vector bundle E^i, $i = 1, \ldots, n$. Its symbol σ_P on T^*M is given by the complex

$$0 \to \pi^* E^0 \xrightarrow{\sigma_1} \pi^* E^1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} \pi^* E^n \to 0,$$

where $\sigma_i = \sigma(P_i)$ is the symbol of the pseudodifferential operator P_i. Let

$$\mathrm{Char}(\sigma_P) = \{(p, \xi) \in T^*M \mid \sigma_P(p, \xi) \text{ is not exact}\}$$

denote the characteristic set of the complex.

Definition 3.1. A complex of G-invariant pseudodifferential operators P is G-*transversally elliptic* if $\mathrm{Char}(\sigma_P) \bigcap T^*_G M$ is compact.

A G-transversally elliptic symbol σ_P defines an element $[\sigma_P] \in K^*_G(T^*_G M)$. Conversely, given a symbol class $[\sigma] \in K^*_G(T^*_G M)$, there is a G-transversally elliptic pseudodifferential operator $A \colon \Gamma(M, E) \to \Gamma(M, F)$ such that $\sigma(A) = \sigma$. Let $\hat{G}$ be the set of isomorphism classes of irreducible complex representations of G and $\hat{R}(G)$ be the space of $\mathbb{Z}$-valued functions on $\hat{G}$. The elements $V \in \hat{R}(G)$ are thus infinite series

$$V = \sum_{\mu \in \hat{G}} m(\mu) \chi_{V_\mu}$$

with $m(\mu) \in \mathbb{Z}$. The kernel of A is not finite-dimensional, but it is shown in [3] that for every $\mu \in \hat{G}$ the space $\mathrm{Hom}_G(V_\mu, \ker(A))$ is a finite dimensional vector space of dimension $m(V_\mu, A)$. The integer $m(V_\mu, A) - m(V_\mu, A^*)$ depends only on the class of the symbol $\sigma(A)$ in $K^*_G(T^*_G M)$ and the index of σ is defined by

$$\mathrm{ind}^M_G(\sigma) = \sum_{\mu \in \hat{G}} (m(V_\mu, A) - m(V_\mu, A^*)) \chi_{V_\mu},$$

where the adjoint A^* of A is also a G-transversally elliptic pseudo-differential operator and defined by choosing a G-invariant metric. Atiyah showed in [3] that $\mathrm{ind}_G(\sigma)$ defines a distribution on G. The index depends only on the symbol class $[\sigma] \in K^*_G(T^*_G M)$ and descends to a map

$$\mathrm{ind}^M_G \colon K^*_G(T^*_G M) \to \hat{R}(G).$$

The following basic example will be important in the sequel.

Example 3.2. Let $M = S^1$ with S^1 acting on M by left translation. Let $E = M \times \mathbb{C}$ be the trivial line bundle and $F = M \times \{0\}$. Then $\Gamma(M, E) = C^\infty(S^1)$ and $\Gamma(M, F) = 0$. The operator $D = 0 \colon \Gamma(M, E) \to \Gamma(M, F)$ has symbol

$$\sigma_D \colon \pi^* E \to \pi^* F$$
$$(\xi, e) \mapsto (\xi, 0).$$

Since $T^*_{S^1} M = M \times \{0\}$, the operator D is S^1-transversally elliptic. The index of σ_D is

$$\mathrm{ind}^M_{S^1}(\sigma_D)(t) = \sum_{n \in \mathbb{Z}} \chi_{V_n}(t) = \sum_{n \in \mathbb{Z}} t^n,$$

where $V_n = \mathbb{C}$ is the S^1-module with the action $t \cdot z = t^n z$.

Let M be a $(2n + 1)$-dimensional toric contact manifold of Reeb type, $n > 1$, with an invariant Sasakian structure. Let $\overline{\partial}^E_H$ be the twisted horizontal Dolbeault complex on M as defined in Section 2.2.

Proposition 3.3. $\sigma(\overline{\partial}^E_H)$ *is a G-transversally elliptic symbol.*

Proof. Recall that the symbol $\sigma_k(\bar{\partial}_H^E)\colon \pi^*(\bigwedge^{0,k} H^* \otimes E) \to \pi^*(\bigwedge^{0,k+1} H^* \otimes E)$ is given by

$$\sigma_k(\bar{\partial}_H^E)_{(p,\xi)}(\bar{w} \otimes \bar{e}) = \xi_H^{0,1} \wedge \bar{w} \otimes \bar{e},$$

where $(p,\xi) \in T^*M$, $\bar{w} \otimes \bar{e} \in \bigwedge^{0,k} H|_p \otimes E|_p$. The complex $\sigma(\bar{\partial}_H^E)_{(p,\xi)}$ is exact so long as $\xi_H^{0,1} \neq 0$, that is when ξ is not a multiple of the contact form α. Since M is of Reeb type, a vector $R \in \mathfrak{g}$ generates the Reeb vector field. We have

$$(p,\xi) \in T_G^*M \implies \xi(v(p)) = 0 \text{ for all } v \in \mathfrak{g}.$$

Since the action of G generates the Reeb vector field, we have $\xi(R(p)) = 0$. By $\alpha_p(R(p)) = 1$, it follows that ξ is not a multiple of α_p and therefore $\sigma(\bar{\partial}_H^E)_{(p,\xi)}$ is exact.

$\square$

We conclude by recalling the multiplicative and excision properties of the index [3]. Consider a compact Lie group G_2 acting on two manifolds M_1 and M_2 and assume that another compact Lie group G_1 acts on M_1 commuting with the action of G_2. The exterior product of vector bundles induces a multiplication map

$$(3.1) \qquad \boxtimes\colon K_{G_1 \times G_2}(T_{G_1}^*M_1) \otimes K_{G_2}(T_{G_2}^*M_2) \to K_{G_1 \times G_2}(T_{G_1 \times G_2}^*(M_1 \times M_2)).$$

Theorem 3.4 (Multiplicative Property)**.** *For any $\sigma_1 \in K_{G_1 \times G_2}(T_{G_1}^*M_1)$ and any $\sigma_2 \in K_{G_2}(T_{G_2}^*M_2)$, we have*

$$ind_{G_1 \times G_2}^{M_1 \times M_2}(\sigma_1 \boxtimes \sigma_2) = ind_{G_1 \times G_2}^{M_1}(\sigma_1)\, ind_{G_2}^{M_2}(\sigma_2).$$

Theorem 3.5 (Excision Property)**.** *Let $j\colon U \to M$ be an open G-embedding into a compact G-manifold M. We have a pushforward map $j_*\colon K_G(T_G^*U) \to K_G(T_G^*M)$ and the composition*

$$K_G(T_G^*U) \xrightarrow{j_*} K_G(T_G^*M) \xrightarrow{ind_G^M} \hat{R}(G)$$

is independent of $j\colon U \to M$.

The product of a symbol σ by a G-equivariant vector bundle E is the symbol given by

$$(\sigma \otimes E)(p,\xi) = \sigma(p,\xi) \otimes \mathrm{Id}_E.$$

Note that the symbol $\sigma(\bar{\partial}_H^E)$ is of this form.

Proposition 3.6. *Let $\sigma \in K_G(T_G^*M)$, E a G-module and $\underline{E}$ the corresponding trivial G-equivariant bundle over M, then*

$$ind_G^M(\sigma \otimes \underline{E}) = ind_G^M(\sigma) \otimes \chi_E \in \hat{R}(G),$$

where χ_E is the character of the G-module E.

3.2. **Localisation.** In this section, we review the K-theoretic localisation method for computing the index of transversally elliptic operators developed in [3]. The core idea is to choose a filtration of the manifold that allows one to decompose the symbol into contributions from lower dimensional spaces, essentially reducing the problem to a computation on vector spaces. The main ingredient involved in the computations is Atiyah's pushed symbol σ^ϵ.

Let G be an $(n+1)$-dimensional torus acting on $\mathbb{C}^m$ with no fixed vector and weights $w^i \in \mathfrak{g}^*$, $i = 1, \ldots, m$. A G-invariant Riemannian metric h on $\mathbb{C}^m$ induces an isomorphism

$$T\mathbb{C}^m \to T^*\mathbb{C}^m$$
$$v \mapsto \tilde{v} = h(v, \cdot).$$

Given $\epsilon \in \mathfrak{g}$ we will denote by $\epsilon(p) \in T_p\mathbb{C}^m$ the vector generated by ϵ at p and by $\widetilde{\epsilon(p)} \in T_p^*\mathbb{C}^m$ its image under the isomorphism defined by h. Let $\sigma(\bar{\partial})$ be the symbol

$$0 \to \pi^*(\textstyle\bigwedge^{0,0}T^*\mathbb{C}^m) \xrightarrow{\sigma(\bar{\partial})} \pi^*(\textstyle\bigwedge^{0,1}T^*\mathbb{C}^m) \xrightarrow{\sigma(\bar{\partial})} \cdots \xrightarrow{\sigma(\bar{\partial})} \pi^*(\textstyle\bigwedge^{0,m}T^*\mathbb{C}^m) \to 0,$$

where $\pi \colon T^*\mathbb{C}^m \to \mathbb{C}^m$ is the projection and $\sigma_{(p,\xi)}(\bar{\partial})(w) = \xi^{0,1} \wedge w$. The symbol $\sigma(\bar{\partial})$ is exact away from the zero section of $T^*\mathbb{C}^m$, in fact, $\mathrm{Char}(\sigma(\bar{\partial})) = \mathbb{C} \times \{0\}$. Since $\mathrm{Char}(\sigma(\bar{\partial})) \cap T_G^*\mathbb{C}^m = \mathrm{Char}(\sigma(\bar{\partial}))$ is non-compact, it is not a G-transversally elliptic symbol. Atiyah shows in [3] how to obtain a G-transversally elliptic symbol σ^ϵ by deforming $\sigma(\bar{\partial})$ using the G-action. Namely, let $H_i = \{\epsilon \in \mathfrak{g} \mid w^i(\epsilon) = 0\}$ be the hyperplane in $\mathfrak{g}$ determined by the weight w^i and pick a vector $\epsilon \in \mathfrak{g}$ away from the hyperplanes H_i.

Definition 3.7. Atiyah's pushed symbol σ^ϵ is defined by

$$\sigma^\epsilon_{(p,\xi)}(w) = \sigma(\bar{\partial})_{(p,\xi+g(|\xi|)\widetilde{\epsilon(p)})}(w) = (\xi + g(|\,\xi\,|)\widetilde{\epsilon(p)})^{0,1} \wedge w,$$

where $\epsilon(p) \in T_p\mathbb{C}^m$ is the tangent vector generated by ϵ at p, $(p,\xi) \in T^*\mathbb{C}^m$, $w \in \bigwedge^{0,k}T^*\mathbb{C}^m$ and g is a bump function supported in a small neighbourhood of 0, so that σ^ϵ coincides with σ away from the zero section of $T^*\mathbb{C}^m$.

Since σ is an isomorphism away from the zero section of $T^*\mathbb{C}^m$, we know that $\sigma^\epsilon(p,\xi)$ fails to be an isomorphism if and only if $\xi + g(|\,\xi\,|)\widetilde{\epsilon(p)} = 0$. Therefore,

$$\mathrm{Char}(\sigma^\epsilon) = \{(p,\xi) \in T^*\mathbb{C}^m \mid \xi + g(|\,\xi\,|)\widetilde{\epsilon(p)} = 0\}$$

is still non-compact. If $(p,\xi) \in T_G^*\mathbb{C}^m|_p$, let $t \in \mathbb{R}_+$ be such that $g(|\,t\xi\,|) > 0$. Then

$$t\xi + g(|\,t\xi\,|)\widetilde{\epsilon(p)} = 0 \implies t\xi(\epsilon(p)) + g(|\,t\xi\,|)\widetilde{\epsilon(p)}(\epsilon(p)) = 0$$
$$\implies g(|\,t\xi\,|)h(\epsilon(p), \epsilon(p)) = 0$$
$$\implies \epsilon(p) = 0 \implies \widetilde{\epsilon(p)} = 0$$
$$\implies \xi = 0 \text{ and } p = 0,$$

where we have used the facts that $\xi(v(p)) = 0$ for all $v \in \mathfrak{g}$ if $(p,\xi) \in T_G^*\mathbb{C}^m|_p$, and $\epsilon(p) = 0$ if and only if $p = 0$ since the torus acts with no fixed vectors. Therefore $\mathrm{Char}(\sigma^\epsilon) \cap T_G^*\mathbb{C}^m = \{0\} \times \{0\}$ and σ^ϵ is a G-transversally elliptic symbol.

Definition 3.8. Let $t \in G$ and $\alpha \in \mathfrak{g}^*$, then

$$\left(\frac{1}{1-t^{-\alpha}} \right)^+ = - \sum_{k=1}^{\infty} t^{k\alpha} \text{ and } \left(\frac{1}{1-t^{-\alpha}} \right)^- = \sum_{k=0}^{\infty} t^{-k\alpha},$$

are the expansions in positive and negative powers of α. In other words, $(\cdot)^{\pm}$ are the Laurent expansions of a rational function around $t = \infty$ and $t = 0$, respectively.

Theorem 3.9. *Let σ^{ϵ} be Atiyah's pushed symbol, then*

$$ind_G^{\mathbb{C}^m}(\sigma^{\epsilon})(t) = \prod_{j=1}^{m} \left(\frac{1}{1-t^{\alpha_j}} \right)^{s_j} \in \hat{R}(G),$$

where $s_j = +$ if $\alpha_j(\epsilon) > 0$ and $s_j = -$ if $\alpha_j(\epsilon) < 0$.

Proof. See [3] (Theorem 8.1) or [7]. □

Atiyah shows in [3] how to extend the ideas above to evaluate the index for symbol classes in $K_G(T_G^* M)$, when M is a compact manifold with an action of a $(n+1)$-dimensional torus G. The G-action provides M with a filtration by closed subsets

$$M = M_0 \supset M_1 \supset \cdots \supset M_{n+1} \supset M_{n+2} = \varnothing$$

where $M_i = \{p \in M \mid \dim G_p \geqslant i\}$. This filtration determines a split exact sequence for each i,

$$0 \longrightarrow K_G(T_G^*(M - M_i)) \longrightarrow K_G(T_G^*(M - M_{i+1})) \xrightarrow{\theta_i} K_G(T_G^* M|_{(M_i - M_{i+1})}) \longrightarrow 0,$$

and a decomposition

$$K_G(T_G^* M) \cong \bigoplus_{i=0}^{n+1} \theta_i K_G(T_G^* M|_{(M_i - M_{i+1})}).$$

Note that $T_G^* M|_{(M_i - M_{i+1})}$ is a complex vector bundle over $T_G^*(M_i - M_{i+1})$, therefore we can compose the splittings θ_i with the Thom isomorphism, which we denote by ϕ_i, and obtain

$$K_G(T_G^* M) \cong \bigoplus_{i=0}^{n+1} \phi_i K_G(T_G^*(M_i - M_{i+1})).$$

This decomposition allows us to break up a symbol and evaluate its index on each piece separately. For instance, if $[\sigma] \in K_G(T_G^* M)$, then

$$\text{ind}_G^M(\sigma) = \sum_{i=0}^{n+1} \text{ind}_G^M(\phi_i(\sigma|_{M_i - M_{i+1}})).$$

The Thom isomorphism is well-understood, so in order to evaluate the maps ϕ_i, let us recall the definition of the splitting maps θ_i from [3].

To define θ_i it suffices to consider each connected component Y of $M_i - M_{i+1}$ separately. These components are open submanifolds of a unique fixed point set M^{T^i} of a subtorus $T^i \subset G$. The key ingredient in the construction of θ_i is a vector field v on a neighbourhood U of Y, such that v vanishes only on Y. One can

then proceed as in the discussion following Definition 3.7 and obtain an element of $K_G(T^*_G U)$, which is mapped into $K_G(T^*_G M)$ via the natural map $K_G(T^*_G U) \rightarrow K_G(T^*_G M)$ induced by the open inclusion $T^*_G U \rightarrow T^*_G M$. Such vector fields exist in general, as Atiyah shows with the following argument. Since Y is an open subset of M^{T^i}, T^i acts on the normal bundle N of Y leaving no non-zero vector fixed. Identifying N with a tubular neighbourhood U of Y, each vector $\epsilon \in \mathrm{Lie}(T^i)$ will generate a vector field on U that vanishes only on Y.

This approach to evaluating the index has some practical limitations, as it utilises different vector fields around the connected components and it requires computing the contributions from all levels of the filtration. The following proposition provides criteria for a single vector field on M to satisfy the requirement for the construction of θ_i around each connected component of $M_i - M_{i+1}$, and to localise away the contributions from θ_i below a certain filtration level. This differs from Atiyah's approach in that it uses a global vector field rather than multiple vector fields defined locally, and it renders this method computationally feasible.

Proposition 3.10. *Let $[\sigma] \in K_G(T^*_G M)$ and let v be a vector field on M such that:*
- *$\sigma(p, \xi + \lambda \tilde{v}(p))$ is an isomorphism for every $p \in M \backslash M_j$, $\lambda \in \mathbb{R} \backslash \{0\}$, $\xi \in (T^*_G M)_p$ and some $j \in \{1, \ldots, n+1\}$;*
- *$v(p)$ is tangent to the orbit $G \cdot p$;*
- *$v(p) = 0$ if and only if $p \in M_j$;*

where $\tilde{v} = h(v, \cdot)$ for a G-invariant Riemannian metric h on M. Then for $i < j$, we can use $v(p)$ to construct the map θ_i and we have $\theta_i(\sigma) = 0$.

Proof. We may assume that σ is the symbol of a transversally elliptic operator, since the group $K_G(T^*_G M)$ is generated by those symbols (see [3]), hence $\sigma(p, \xi)$ is invertible when $\xi \in (T^*_G M)_p$ is non-zero. Let i be an integer such that $i < j$. Note that $M_j \subset M_i$ and the vector field v is non-vanishing on $M_i - M_{i+1}$. To decompose the symbol σ, we first restrict it to $M_i - M_{i+1}$ obtaining $\sigma|_{(M_i - M_{i+1})}$. We identify U with the normal bundle N of $M_i - M_{i+1}$ and denote by $p \colon U \rightarrow M_i - M_{i+1}$ the projection. The pullback $p^*(\sigma|_{(M_i - M_{i+1})})$ is the extension of the restriction of σ to U. We note that $p^*(\sigma|_{(M_i - M_{i+1})})$ is not equal to the restriction of σ to U.

Since $\sigma(p, \xi)$ is non-invertible when $\xi = 0$, the pullback $p^*(\sigma|_{(M_i - M_{i+1})})$ does not define an element of $K_G(T^*_G U)$ as its support U is not compact. However, the vector field v can be used to deform this map so that its support becomes compact. To avoid clutter, we will use σ below to denote the extension $p^*(\sigma|_{(M_i - M_{i+1})})$ to U. We define θ_i using v as follows. Let

$$\tilde{\sigma}(p, \xi) = \sigma(p, \xi + g(|\xi|)\tilde{v}(p)),$$

where $(p, \xi) \in (T^*_G U)_p$, g is a bump function supported on a small neighbourhood of the zero section of $T^*_G U$ and σ is denoting the extension to U of the restriction of σ. Since $\sigma(p, \xi)$ is an isomorphism for $\xi \neq 0$ it follows that $\tilde{\sigma}(p, \xi)$ fails to be an isomorphism exactly when $\xi + g(|\xi|)\tilde{v}(p) = 0$. Since $v(p)$ is tangent to $G \cdot p$ and ξ is orthogonal to $G \cdot p$, we have

$$\xi + g(|\xi|)\tilde{v}(p) = 0 \iff \xi = 0 \text{ and } v(p) = 0.$$

Therefore the support of $\tilde{\sigma}(p, \xi)$ is compact and it defines an element of $K_G(T^*_G U)$. The class $\theta_i(\sigma)$ is defined as the image of $\tilde{\sigma}$ under the natural homomorphism $K_G(T^*_G U) \to K_G(T^*_G(M - M_{i+1})$ induced by the open inclusion $T^*_G U \to T^*_G(M - M_{i+1})$. Furthermore, we have that $v(p) = 0$ if and only if $p \in M_j$. In particular, v does not vanish on $M_i - M_{i+1}$. Therefore

$$\xi + g(|\xi|)\tilde{v}(p) \neq 0$$

for every $(p, \xi) \in T^*_G U$ and $\tilde{\sigma}$ is an isomorphism everywhere. It follows that $\tilde{\sigma}$ represents the trivial class in $K_G(T^*_G U)$ and $\theta_i(\sigma) = 0$.

$\square$

Remark 3.11. Proposition 3.10 is particularly useful in the following situation. Suppose that there is a $k \in \{1, \ldots, n+1\}$ such that $M_j = \varnothing$ for all $j > k$. If there is a vector field $v(p)$ satisfying the above hypothesis for $j = k$, then only the level k of the filtration contributes to the index and we have

$$\mathrm{ind}^M_G(\sigma) = \mathrm{ind}^M_G \phi_k(\sigma|_{M_k}).$$

4. THE INDEX OF $\overline{\partial}^E_H$

Adopting the localisation technique outlined in Section 3.2, we will compute the index of the twisted Dolbeault operator $\overline{\partial}^E_H$ and show that the contributions to the index come from a finite number of closed Reeb orbits.

Let M be a $(2n + 1)$-dimensional toric contact manifold of Reeb type, $n > 1$, equipped with an invariant Sasakian structure. We proceed by constructing a vector field satisfying the hypothesis of Proposition 3.10 for $j = n$. Let us fix a vector $R \in \mathfrak{g}$ that generates the Reeb vector field. Given a closed Reeb orbit L, corresponding to an edge of the moment cone, the good cone condition (Definition 2.3(ii)) ensures the existence of a vector $v^L_0 \in \mathbb{Z}_G$ such that $v^L_0, v^L_1, \ldots, v^L_n$ forms an integral basis of $\mathbb{Z}_G$, where v^L_i, $i = 1, \ldots, n$, are the cone normals at L. There is a unique element $\det \in \bigwedge^{n+1} \mathfrak{g}^*$ such that

$$\det(v^L_0, v^L_1, \ldots, v^L_n) = 1.$$

Since $\{v^L_0, v^L_1, \ldots, v^L_n\}$ is an integral basis of $\mathbb{Z}_G$, its dual basis $\{\mu_L, w^1_L, \ldots, w^n_L\}$ is an integral basis of the integral weights lattice $\mathbb{Z}^*_G$. We expand

$$R = \mu_L(R) v^L_0 + w^1_L(R) v^L_1 + \cdots + w^n_L(R) v^L_n.$$

Since R generates the Reeb vector field, its infinitesimal action is non-zero everywhere. Therefore its v^L_0 component, $\mu_L(R)$, cannot be zero, since the infinitesimal action of v^L_i, $i = 1, \ldots, n$, on L is zero. It follows that

$$\det\left(R, v^L_1, \ldots, v^L_n\right) = \mu_L(R) \neq 0,$$

and $\{R, v^L_1, \ldots, v^L_n\}$ forms a basis of $\mathfrak{g}$ (not necessarily integral).

Given a vector $\epsilon \in \mathfrak{g}$, we will define the vector field $\epsilon^\perp$ as its orthogonal complement with respect to the Reeb vector field. More precisely, let $\epsilon \in \mathfrak{g}$ and L_i be a Reeb orbit corresponding to an edge of the moment cone. Write

$$\epsilon = \eta_{L_i}(\epsilon) R + \eta^1_{L_i}(\epsilon) v^{L_i}_1 + \cdots + \eta^n_{L_i}(\epsilon) v^{L_i}_n.$$

Let U_i be an open neighbourhood of L_i and V_i a closed neighbourhood such that $L_i \subset V_i \subset U_i$. We can assume that the U_i's are all disjoint. Define

$$\phi^{L_i}(p) = \begin{cases} 0 & \text{if } p \in M \backslash U_i, \\ -\eta_{L_i}(\epsilon) & \text{if } p \in V_i, \end{cases}$$

and extend it to smoothly interpolate between 0 and $-\eta_{L_i}(\epsilon)$ on $U_i \backslash V_i$, defining a smooth bump function. Projecting out the Reeb vector field component corresponds to shrinking the contribution of R to the vector field generated by ϵ. We define $\epsilon^\perp$ as

$$\epsilon^\perp(p) = \epsilon(p) + \sum_{i=1}^{N} \phi^{L_i}(p) R(p).$$

Definition 4.1. An element $\epsilon \in \mathfrak{g}$ is called a *polarizing vector* if $\eta_L^i(\epsilon) \neq 0$ for $i = 1, \ldots, n$ and for every $L \subset M_n$. We say that a vector field $\epsilon^\perp$ is a *good deformation vector field* for $[\sigma] \in K_G(T_G^* M)$ if it satisfies the hypothesis of Proposition 3.10 for $j = n$.

Proposition 4.2. *If $\epsilon \in \mathfrak{g}$ is a polarizing vector, then $\epsilon^\perp$ is a good deformation vector field for $\sigma(\overline{\partial}_H^E) \in K_G(T_G^* M)$.*

Proof. By construction $\epsilon^\perp(p)$ is tangent to the G-orbits. The symbol $\sigma(\overline{\partial}_H)(p, v(p))$ is invertible if $v(p)$ is not parallel to the Reeb vector field $R(p)$, so we only need to prove that $\epsilon^\perp(p) = 0$ if and only if $p \in M_n$. Let $p \in U_i$ and suppose that $\epsilon^\perp(p) = 0$. Let $L = L_i$ be the orbit corresponding to the normals $v_1^L, \ldots, v_n^L$ and write

$$\epsilon^\perp(p) = (\eta_L(\epsilon) + \sum_{j=1}^{N} \phi^{L_j}(p)) R(p) + \eta_L^1(\epsilon) v_1^L(p) + \cdots + \eta_L^n(\epsilon) v_n^L(p) = 0.$$

This implies that

$$(\eta_L(\epsilon) + \sum_{j=1}^{N} \phi^{L_j}(p)) R + \eta_L^1(\epsilon) v_1^L + \cdots + \eta_L^n(\epsilon) v_n^L \in \mathfrak{g}_p.$$

For $p \in U_i$, there is a finite number of possible isotropy algebras $\mathfrak{g}_x$; they are all generated by a subset of $\{v_1^L, \ldots, v_n^L\}$. This implies that

$$(\eta_L(\epsilon) + \sum_{j=1}^{N} \phi^{L_j}(p)) = 0,$$

and since $\{v_1^L, \ldots, v_n^L\}$ is a linearly independent set, we have $v_j^L \in \mathfrak{g}_x$ for all $j = 1, \ldots, n$. It follows that the image of p under the moment map lies in the intersection of the faces determined by the normals $v_1^L, \ldots, v_n^L$, so p is a point in the orbit L. Let $p \in M \backslash \bigcup_{j=1}^{N} U_j$, we have

$$\epsilon^\perp(p) = \epsilon(p) + \sum_{i=1}^{N} \phi^{L_i}(p) R(p) = \epsilon(p).$$

Let L be one of the orbits L_i and write

$$\epsilon = \eta_L(\epsilon) R + \eta_L^1(\epsilon) v_1^L + \cdots + \eta_L^n(\epsilon) v_n^L.$$

Since $R, v_1^L, \ldots, v_n^L$ is a basis of $\mathfrak{g}$ and G acts freely on $M \backslash \bigcup_{j=1}^N U_j$, we have $\eta_L^j(\epsilon) \neq 0$ and $\epsilon(p)$ must be non-zero. Hence, $\epsilon^\perp$ is a good deformation vector field for $\sigma(\overline{\partial}_H)$ and also for $\sigma(\overline{\partial}_H^E)$ as they have the same characteristic sets. $\qquad\square$

The level n filtration $M_n \subset M$ is a disjoint union of closed Reeb orbits L_e indexed by the set $E(C)$ of edges of the moment cone C,

$$M_n = \bigsqcup_{e \in E(C)} L_e.$$

Since $M_{n+1} = \varnothing$, it follows by Proposition 4.2 that

$$\mathrm{ind}_G^M(\sigma(\overline{\partial}_H)) = \mathrm{ind}_G^M \phi_n(\sigma(\overline{\partial}_H)|_{M_n}) = \sum_{e \in E(C)} \mathrm{ind}_G^M \phi_n(\sigma(\overline{\partial}_H)|_{L_e}).$$

Given an orbit $L \subset M_n$ and a vector $\epsilon \in \mathfrak{g}$, we denote by $\epsilon_L^\perp$ the vector

$$\eta_L^1(\epsilon) v_1^L + \cdots + \eta_L^n(\epsilon) v_n^L \in \mathfrak{g}.$$

In a neighbourhood of each closed orbit $L \subset M_n$, there is a vector $\epsilon_L^\perp \in \mathfrak{g}$ generating the vector field $\epsilon^\perp$.

Proposition 4.3. *Let $L \subset M_n$ be a closed Reeb orbit. For any $t \in G$, we have*

$$ind_G^M \phi_n(\sigma(\overline{\partial}_H)|_L)(t) = \left(\frac{1}{1 - t^{-w_L^1}} \right)^{s_L^1} \cdots \left(\frac{1}{1 - t^{-w_L^n}} \right)^{s_L^1} \delta(1 - t^{\mu_L}),$$

where $\{\mu_L, w_L^1, \ldots, w_L^n\}$ is a basis of the weight lattice $\mathbb{Z}_G^*$ dual to $\{v_0^L, \ldots, v_n^L\}$, $s_L^i = +$ if $w_L^i(\epsilon_L^\perp) > 0$ and $s_L^i = -$ if $w_L^i(\epsilon_L^\perp) < 0$.

Proof. To evaluate the index, we need to understand the symbols $\phi_n(\sigma(\overline{\partial}_H|)_{L_e})$. The map ϕ_n is a composition of the Thom isomorphism with the splitting homomorphism θ_n. Let L be a connected component of M_n and let N be its normal bundle in M. Since $L \subset M$ is an embedded circle, N is a trivial complex bundle $N \cong L \times \mathbb{C}^n$. Write $G = G_L \times S_0^1$, where G_L is the isotropy group associated with L and S_0^1 the circle generated by the vector $v_0^L \in \mathbb{Z}_G$. Taking $G_1 = G_L$, $G_2 = S_0^1$, $M_1 = L$ and $M_2 = \mathbb{C}^n$ in (3.1), we get

$$\boxtimes : K_G(T_G^* L) \otimes K_{G_L}(T_{G_L}^* \mathbb{C}^n) \to K_{G_L \times S_0^1}(T_{G_L \times S_0^1}^*(L \times \mathbb{C}^n)) \cong K_G(T_G^* N).$$

The map ϕ_n is given by taking the Bott element $[\sigma^{\epsilon_L^\perp}] \in K_{G_L}(T_{G_L}^* \mathbb{C}^n)$ in this product. That is, given $\sigma \in K_G(T_G^* L)$ we have

$$\phi_n(\sigma) = \sigma \boxtimes \sigma^{\epsilon_L^\perp} \in K_G(T_G^* N).$$

Identifying N with a tubular neighbourhood U of L in M and using excision to extend the symbol to M, we obtain $\phi_n(\sigma) = \sigma \boxtimes \sigma^{\epsilon_L^\perp} \in K_G(T_G^* M)$. It follows from Theorem 3.4 that

$$\mathrm{ind}_G^M \phi_n(\overline{\partial}_H|_L) = \mathrm{ind}_{G_L}^{\mathbb{C}^n}(\sigma^{\epsilon_L^\perp}) \mathrm{ind}_G^L(0),$$

where 0 is the zero operator on $L \cong S^1$ discussed in Example 3.2. According to Theorem 3.9,

$$\mathrm{ind}_{G_L}^{\mathbb{C}^n}(\sigma^{\epsilon_L^\perp})(g) = \left(\frac{1}{1 - g^{-\alpha_1}} \right)^{s_L^1} \cdots \left(\frac{1}{1 - g^{-\alpha_n}} \right)^{s_L^n},$$

where $\alpha_1, \ldots, \alpha_n$ are the weights of the G_L-action on $\mathbb{C}^n$, $g \in G_L$, $s_L^i = +$ if $\alpha_L^i(\epsilon_L^\perp) > 0$ and $s_L^i = -$ if $\alpha_L^i(\epsilon_L^\perp) < 0$. By Corollary 2.14, the weights $\alpha_1, \ldots, \alpha_n$ determine a basis of the weight lattice $\mathbb{Z}_G^* \subset \mathfrak{g}^*$ that is dual to $\{v_1^L, \ldots, v_n^L\} \subset \mathfrak{g}_L$.

Writing $G = G_L \times S_0^1$, the G-action on L is given by

$$t = (g, s) \cdot p = sp,$$

where $t = (g, s) \in G = G_L \times S_0^1$ and $p \in L$. We identify the subgroup $S_0^1 \subset G$ generated by v_0^L with S^1 via

$$e^{2\pi i s v_0^L} \mapsto e^{2\pi i s} \in S^1.$$

This identification is determined by the weight $\mu_L \in \mathbb{Z}_G^*$ defined by $\mu_L(v_0^L) = 1$ and $\mu_L(v_j^L) = 0$, $j \neq 0$. In fact, let $s = e^{2\pi i s v_0^L} \in S_0^1$. Then $s^{\mu_L} = e^{2\pi i s \mu_L(v_0^L)} = e^{2\pi i s} \in S^1$ and therefore

$$\operatorname{ind}_G^L(0)(g) = \operatorname{ind}_{G_L \times S_0^1}^L(0)(t, s) = \operatorname{ind}_{S^1}^L(0)(s^{\mu_L}).$$

Since S_0^1 acts freely and transitively on L we have

$$\operatorname{ind}_{S^1}^L(0)(s^{\mu_L}) = \sum_{k=-\infty}^{\infty} s^{k\mu_L} = \delta(1 - s^{\mu_L}).$$

We extend the weight vectors $\alpha_i \in \mathbb{Z}_{G_L}^*$, $i = 1, \ldots, n$ to $\mathbb{Z}_G^*$ by defining $\alpha_i(v_0^L) = 0$. Denote these extensions by $w_L^i \in \mathbb{Z}_G^*$, $i = 1, \ldots, n$. Note that $\{\mu_L, w_L^1, \ldots, w_L^n\}$ is a basis of $\mathbb{Z}_G^*$ dual to $\{v_0^L, v_1^L, \ldots, v_n^L\}$ and $w_L^i(\epsilon_L^\perp) = \alpha_i(\epsilon_L^\perp)$. Let $t = (g, s) \in G = G_L \times S_0^1$ and $\eta \in \mathfrak{g}^*$. We will write $t^\eta = (g, s)^\eta = g^\eta s^\eta$. Since G_L is generated by $\{v_1^L, \ldots, v_n^L\}$ and S_0^1 is generated by v_0^L, we have $t^{w_i} = (g, s)^{w_i} = g^{w_i} s^{w_i} = g^{\alpha_i}$ and $t^{\mu_L} = (g, s)^{\mu_L} = g^{\mu_L} s^{\mu_L} = s^{\mu_L}$. Given $t = (g, s) \in G = G_L \times S_0^1$, we have

$$\begin{aligned}
\operatorname{ind}_G^M \phi_n(\sigma(\overline{\partial}_H)|_L)(t) &= \operatorname{ind}_{G_L \times S_0^1}^M(\sigma(\overline{\partial}_H)|_L)(g, s) \\
&= \operatorname{ind}_{G_L}^{\mathbb{C}^n}(\overline{\partial}_{\epsilon^\perp})(g) \operatorname{ind}_{G_L \times S_0^1}^L(0)(g, s) \\
&= \left(\frac{1}{1 - t^{-w_L^1}}\right)^{s_L^1} \cdots \left(\frac{1}{1 - t^{-w_L^n}}\right)^{s_L^n} \delta(1 - t^{\mu_L}).
\end{aligned}$$

$\square$

Next we allow for twistings by an auxiliary bundle and derive the main result of this section, which is a Lefschetz type formula for the index of $\overline{\partial}_H^E$. Let L be a closed orbit corresponding to an edge of the moment cone C and let $E \to M$ be a G-equivariant transversally holomorphic bundle. Since $L = G/G_L$, the restriction $E|_L$ to L is a vector bundle of the form $G \times_{G_L} F_L$ for some G_L-module F_L. Since $G = L \times G_L$ we have

$$G \times_{G_L} F = (L \times G_L) \times_{G_L} F = L \times F.$$

Recall that M_n is the disjoint union of closed Reeb orbits L_e indexed by the edges of the moment cone C.

Theorem 4.4. *Let $\overline{\partial}_H^E$ be the horizontal Dolbeault operator on a toric compact Sasaki manifold twisted by a G-equivariant transversally holomorphic bundle E. For any $t \in G$,*

$$ind_G^M \sigma(\overline{\partial}_H^E)(t) = \sum_{L \subset M_n} \prod_{i=1}^n \chi_{E|_L}(t) \left(\frac{1}{1 - t^{-w_L^i}} \right)^{s_L^i} \delta(1 - t^{\mu_L}),$$

where $\chi_{E|_L}$ is the character of the G_L-module associated to the restriction $E|_L$, and

$$ s_L^i = \begin{cases} + & \text{if } w_L^i(\epsilon_L^\perp) > 0, \\ - & \text{if } w_L^i(\epsilon_L^\perp) < 0. \end{cases} $$

Proof. We have that

$$\operatorname{ind}_G^M \sigma(\overline{\partial}_H^E) = \sum_{L \in M_n} \operatorname{ind}_G^M \phi_n(\sigma(\overline{\partial}_H^E)|_L).$$

Restricting $\sigma(\overline{\partial}_H^E)$ to $L \subset M_n$ we get the symbol $\sigma(\overline{\partial}_H)|_L \otimes E|_L$. If follows that $E|_L = L \times F_L$, for some G_L-module F_L and $\chi_{E|_L} = \chi_{F_L}$. By the multiplicative property of the index, Proposition 3.6 and Proposition 4.3, we get

$$\begin{aligned}
\operatorname{ind}_G^M \phi_n(\sigma(\overline{\partial}_H^E)|_L(t) &= \operatorname{ind}_G^M (\sigma(\overline{\partial}_H)|_L) \chi_{F_L} \\
&= \operatorname{ind}_{G_L}^{\mathbb{C}^n} (\overline{\partial}_{\epsilon_L^\perp})(t) \operatorname{ind}_G^L(0)(t) \chi_{E|_L}(t) \\
&= \chi_{E|_L}(t) \left(\frac{1}{1 - t^{-w_L^1}} \right)^{s_L^1} \cdots \left(\frac{1}{1 - t^{-w_L^n}} \right)^{s_L^n} \delta(1 - t^{\mu_L}).
\end{aligned}$$

The result follows by summing over all the closed Reeb orbits in M_n . $\square$

5. A lattice point formula

In this section, we relate the index of the horizontal Dolbeault operator $\overline{\partial}_H$ to the lattice points of the moment cone.

5.1. **Polar decomposition of polytopes.** The Lawrence-Varchenko formula expresses the characteristic function of a polytope as an alternating sum of characteristic functions of certain cones associated to vertices of the polytope. By extending the formula to polyhedral rational cones, it will allow us to collect the multiplicities in the expression for the index in Theorem 4.4, once expanded into power series.

We begin by presenting the formula and relating the characteristic function of the interior of a polytope to the dual cones. Let P be a simple convex polytope in an n-dimensional vector space V^*. Let F be a face of P. The tangent cone to P at F is defined by

$$C_F = \{y + r(x - y) | r \geqslant 0, y \in F, x \in P\}.$$

Let $\sigma_1, \ldots, \sigma_d$ denote the facets of P. Since P is simple, exactly n facets intersect at each vertex. We will denote the set of vertices of P by $\operatorname{Vert}(P)$. For each face F of P, let $I_F \subset \{1, \ldots, d\}$ be the set of indices of the facets meeting at F so that

$$i \in I_F \text{ if and only if } F \subset \sigma_i.$$

In particular, if $F = p \in \operatorname{Vert}(P)$ we have $i \in I_v$ if and only if $v \in \sigma_i$.

Let $p \in \mathrm{Vert}(P)$ and denote by w_p^i, $i \in I_p$, the edge vector emanating from p that lies along the unique edge at p which is not contained on the facet σ_i. Notice that the w_p^i are only determined up to a positive scalar.

Definition 5.1. A vector $\xi \in V$ such that all the pairings $\langle w_p^i, \xi \rangle$ are non-zero is called a *polarizing vector* for P.

Let $H_1, \ldots, H_N$ be the hyperplanes in V determined by the edges of P under the pairing between V and V^*. A vector $\xi \in V$ is a polarizing vector for P if and only if it belongs to the complement

$$V_P = V \backslash (H_1 \cup \cdots \cup H_N).$$

The connected components of V_P are called chambers. The signs of the pairings $\langle w_p^i, \xi \rangle$ depend only on the chamber of V_P containing ξ.

Definition 5.2. Let $\xi \in V_P$ be a polarizing vector. For each vertex $p \in \mathrm{Vert}(P)$ and each edge vector w_p^i emanating from p, we define the corresponding polarized edge vector to be

$$w_p^{i\,\#} = \begin{cases} w_p^i & \text{if } \langle w_p^i, \xi \rangle > 0, \\ -w_p^i & \text{if } \langle w_p^i, \xi \rangle < 0. \end{cases}$$

Definition 5.3. Given a polarizing vector $\xi \in V_P$, the *polarized tangent cone* at $p \in \mathrm{Vert(P)}$ is defined by

$$C_p^{\#} = p + \sum_{w \in E_p^+(\xi)} \mathbb{R}_{<0} w + \sum_{w \in E_p^-(\xi)} \mathbb{R}_{\geqslant 0} w,$$

where

$$E_p^+(\xi) = \{w_p^i \mid \langle w_p^i, \xi \rangle > 0\} \text{ and } E_p^-(\xi) = \{w_p^i \mid \langle w_p^i, \xi \rangle < 0\}.$$

Theorem 5.4 (Lawrence-Varchenko)**.** *Let $P \subset V^*$ be a simple convex polytope and $\xi \in V_P$ a polarizing vector for P. Then for any $x \in V^*$, we have*

$$(5.1) \qquad \mathbf{1}_P(x) = \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{C_p^{\#}}(x),$$

where $\mathbf{1}_{C_p^{\#}}$ is the characteristic function of the polarized cone $C_p^{\#}$.

Proof. See Theorem 3.2 in [15]. $\qquad\square$

Next we show that by flipping the cones in (5.1) yields a cone decomposition of the interior of the polytope P.

Definition 5.5. Define the *dual polarized tangent cone* at $p \in \mathrm{Vert}(P)$ by

$$\check{C}_p^{\#} = p + \sum_{w \in E_p^+(\xi)} \mathbb{R}_{>0} w + \sum_{w \in E_p^-(\xi)} \mathbb{R}_{\leqslant 0} w$$

Suppose that $\xi \in V$ lies in one of the walls H_j separating the chambers of V_P. Let e be an edge of P perpendicular to this wall and let p be an endpoint of e. The edge vectors at p are w_p^j for $j \in I_e$, and an edge vector that lies along e is denoted by w_p^e.

Definition 5.6. The dual polarized tangent cone at the edge e is defined by

$$\check{C}_e^\# = p + \mathbb{R}w_p^e + \sum_{w \in E_p^+(\xi)} \mathbb{R}_{>0}w + \sum_{w \in E_p^-(\xi)} \mathbb{R}_{\leqslant 0}w.$$

One verifies that the cone $\check{C}_e^\#$ is independent of the choice of endpoint of the edge e. We also note that if $x \in \check{C}_p^\#$, then

$$(5.2) \qquad\qquad\qquad \langle \xi, x \rangle \geqslant \langle \xi, p \rangle .$$

Indeed, if $x \in \check{C}_p^\#$ we have

$$x = p + \sum_{w \in E_p^+(\xi)} a_w w + \sum_{w \in E_p^-(\xi)} b_w w,$$

where $a_w > 0$, $b_w \leqslant 0$. Therefore

$$\langle \xi, x \rangle = \langle \xi, p \rangle + \sum_{w \in E_p^+(\xi)} \overbrace{a_w \langle \xi, w \rangle}^{\geqslant 0} + \sum_{w \in E_p^-(\xi)} \overbrace{b_w \langle \xi, w \rangle}^{\geqslant 0} \geqslant \langle \xi, p \rangle$$

Theorem 5.7. *Let $P \subset V^*$ be a simple convex polytope and $\xi \in V_P$ a polarizing vector for P. Then for any $x \in V^*$, we have*

$$(5.3) \qquad\qquad (-1)^n \mathbf{1}_{P^\circ}(x) = \sum_{p \in \mathrm{Vert}(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{\check{C}_p^\#}(x),$$

where $n = \dim V$ and P° denotes the interior of P.

Proof. The proof proceeds along the same lines as that of Theorem 5.4 in [15] and comes down to verifying the identity (5.3) in three separate cases and proving independence of the choice of polarizing vector ξ.

Case 1: Suppose that $x \in P^\circ$. Pick any polarizing vector $\xi \in V_P$. Let $p \in \mathrm{Vert}(P)$ be the vertex for which $\langle \xi, p \rangle$ is minimal. Then $E_p^-(\xi) = \varnothing$ and we have $P^\circ \subset \check{C}_p^\#$. For any other vertex $q \in \mathrm{Vert}(P)$, at least one of the w_q^j's is flipped, and so $\check{C}_q^\# \cap P^\circ = \varnothing$. Hence P° is disjoint from the cones $\check{C}_q^\#$ for all other $q \neq p$ and (5.3), when evaluated at x, reads $(-1)^n = (-1)^n$.

Case 2: Suppose that $x \in \partial P$. Let σ be a facet that contains x and $p \in \mathrm{Vert}(P)$ be such that $p \in \sigma$. Assume that given another facet σ', we have $x \notin \sigma'$ if $p \notin \sigma'$. Choose a polarizing vector $\xi \in V_P$ such that

$$\langle \xi, p \rangle = \min_{y \in P} \langle \xi, y \rangle .$$

We have $E_p^-(\xi) = \varnothing$ and therefore $x \notin \check{C}_p^\#$ because $\check{C}_p^\# = C_p^\circ$ and $x \in \sigma \subset \partial C_p$. We show that $x \notin \check{C}_q^\#$ for any other $q \in \mathrm{Vert}(P)$. Suppose that $q - p$ is not an edge of P. Let $\sigma_q = \sigma \backslash \bigcup_{j \in I_q} \sigma_j$, then $\sigma_q \subset C_q^\circ$. Since $E_q^-(\xi) \neq \varnothing$ we have that $\check{C}_q^\# \cap C_q^\circ = \varnothing$ and therefore $\sigma_q \cap \check{C}_q^\# = \varnothing$. Thus if $x \in \sigma_q$ for some $q \in \mathrm{Vert}(P)$ we have $x \notin \check{C}_q^\#$. If $q - p$ is an edge of P, we have $\langle \xi, q - p \rangle > 0$. Any element y of $\check{C}_q^\#$ can be written uniquely as

$$(5.4) \qquad\qquad\qquad y = q + a(p - q) + r,$$

where $a \leqslant 0$ and r is a linear combination of the edges w_q^j that are not parallel to the edge $p - q$. Since $x \in \sigma$, we can write x uniquely as

$$x = q + b(p - q) + s,$$

where $b \geqslant 0$ and s is a linear combination of the edges w_q^j that are not parallel to the edge $p - q$. Since x does not belong to any facet that does not contain p, we have that $b \neq 0$, and it follows from (5.4) that $x \notin \check{C}_q^\#$. This proves that $x \notin \check{C}_q^\#$ for any vertex $q \in \mathrm{Vert}(P)$. Therefore (5.3), when evaluated at x, reads $0 = 0$.

Case 3: Suppose that $x \notin P$. Choose a polarizing vector $\xi \in V_P$ satisfying

$$\langle \xi, x \rangle < \min_{y \in P} \langle \xi, y \rangle .$$

It follows from (5.2) that x is not in $\check{C}_p^\#$ for any $p \in \mathrm{Vert}(P)$. Thus (5.3) for the polarizing vector ξ, when evaluated at x, reads $0 = 0$.

The final step is to show that the right-hand side of (5.3) is independent of ξ. More precisely, we prove that the right-hand side of (5.3) does not change when ξ crosses the walls H_j. Suppose H_j is not perpendicular to any edge vectors at p. The signs of $\langle \xi, w_p^j \rangle$ do not change, so the cone $\check{C}_p^\#$ does not change as ξ crosses the wall. The vertices whose contributions to the right-hand side of (5.3) change as ξ crosses H_j come in pairs because each edge of P that is perpendicular to H_j has two endpoints. For each such vertex p, denote by $Q_p(x)$ and $Q_p'(x)$ its contributions to the right-hand side of (5.3) before and after ξ crossed H_j. Let e be an edge perpendicular to H_j and p an endpoint of e. Let $Q_e(x)$ be the characteristic function of the cone $\check{C}_e^\#$ corresponding to the value of ξ as it crosses H_j. We have

$$Q_p(x) = (-1)^{|E_p^+(\xi)|} \mathbf{1}_{\check{C}_p^\#} \text{ and } Q_p'(x) = (-1)^{|E_p^+(\xi)|+1} \mathbf{1}_{\widetilde{C}'_p^\#},$$

therefore

$$Q_p(x) - S_p'(x) = (-1)^{|E_p^+(\xi)|} \mathbf{1}_{\check{C}_p^\#} - (-1)^{|E_p^+(\xi)|+1} \mathbf{1}_{\widetilde{C}'_p^\#}$$
$$= (-1)^{|E_p^+(\xi)|} (\mathbf{1}_{\check{C}_p^\#} + \mathbf{1}_{\widetilde{C}'_p^\#})$$
$$= (-1)^{|E_p^+(\xi)|} (\mathbf{1}_{\check{C}_e^\#}) = (-1)^{|E_p^+(\xi)|} Q_e(x)$$

If q is the other endpoint of e, then $|E_p^+(\xi)| = |E_q^+(\xi)| \pm 1$. Hence

$$Q_q(x) - Q_q'(x) = (-1)^{|E_q^+(\xi)|} Q_e(x) = (-1)^{|E_p^+(\xi)|+1} Q_e(x)$$

and

$$(Q_p(x) + Q_q(x)) - (Q_p'(x) + Q_q'(x)) = (-1)^{|E_p^+(\xi)|} Q_e(x) + (-1)^{|E_p^+(\xi)|+1} Q_e(x) = 0$$

Thus crossing H_j does not change the right-hand side of (5.3). $\square$

Formulas (5.1) and (5.3) also have an expression in terms of generating series.

Definition 5.8. Let V be a vector space with basis $e_1, \ldots, e_n$ and $\mathbb{Z}_V$ its integral lattice. If $A \subset V^*$ is a subset, we denote the generating series of A by

$$A(x) = \sum_{\mu \in A \cap \mathbb{Z}_V^*} x^\mu,$$

where $\mathbb{Z}_V^*$ is the dual of the integral lattice $\mathbb{Z}_V$, $x = (x_1, \ldots, x_n)$ and $x^\mu = (x_1^{\mu_1}, \ldots, x_n^{\mu_n})$.

Theorem 5.9. *Let $P \subset V^*$ be a simple convex polytope and $\xi \in V_P$ a polarizing vector for P. Then,*

$$P(x) = \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} C_p^\#(x)$$

Proof. The proof follows directly from Theorem 5.4. We have

$$\sum_{p \in \mathrm{Vert}(P)} (-1)^{|E_p^+(\xi)|} C_v^\#(x) = \sum_{p \in \mathrm{Vert}(P)} (-1)^{|E_p^+(\xi)|} \sum_{\mu \in C_v^\# \cap \mathbb{Z}_V^*} x^\mu$$

$$= \sum_{p \in \mathrm{Vert}(P)} (-1)^{|E_p^+(\xi)|} \sum_{\mu \in \mathbb{Z}_V^*} \mathbf{1}_{C_v^\#}(\mu) x^\mu$$

$$= \sum_{\mu \in \mathbb{Z}_V^*} \left(\sum_{p \in \mathrm{Vert}(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{C_v^\#}(\mu) \right) x^\mu$$

$$= \sum_{\mu \in \mathbb{Z}_V^*} \mathbf{1}_P(\mu) x^\mu = \sum_{\mu \in P \cap \mathbb{Z}_V^*} x^\mu = P(x).$$

$\square$

Similarly, for the dual Lawrence-Varchenko formula, we have:

Theorem 5.10.

$$(-1)^n P^\circ(x) = \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \check{C}_p^\#(x).$$

5.2. **Polar decomposition of cones.** In this section we explain how to adapt the Lawrence-Varchenko formula (5.1) to produce a polar decomposition of a rational polyhedral cone. More precisely, let $P \subset V^*$ be a simple polytope and let $C \subset V^* \times \mathbb{R}^*$ be the cone over P, i.e.

$$C = \{ r(\eta, 1^*) \in V^* \times \mathbb{R}^* \mid \eta \in P, r \geqslant 0 \}.$$

The cone C is the lift of the left-hand side of (5.1) from V^* to $V^* \times \mathbb{R}^*$. Lifting the right-hand side of (5.1), we can expect to obtain a polar decomposition of C. We will see that this is almost true; one must introduce an error term to obtain an identity.

Let $p \in P$ be a vertex of the polytope. We will denote by $\mu_p \in \mathbb{Z}_V^* \times \mathbb{Z}^*$ the primitive edge vector of C going through p. Given a polarizing vector $\xi \in V_P$ for P, let $C_p^\#$ be the polarized tangent cone of P at p and define

$$K_p^\# = C_p^\# + \mathbb{R}\mu_p.$$

Definition 5.11. Let $S(x)$ be the function defined as

$$S(x) = \sum_{p \in Vert(P)} (-1)^{\left|E_p^+(\xi)\right|} \mathbf{1}_{K_p^\#}(x).$$

The function $S(x)$ is the lift of the right-hand side of (5.1). Let $\mathcal{H} = \{(\eta, \frac{1}{2}^*) \in V^* \times \mathbb{R}^* \mid \eta \in V^*\}$ be the characteristic hyperplane and let

$$\mathcal{H}_\lambda = \{(\eta, \lambda^*) \in V^* \times \mathbb{R}^* \mid \eta \in V^*, \lambda^* \in \mathbb{R}^*\},$$

be its parallel shifts. The polytope P is the intersection of C with $\mathcal{H}$ and for $\lambda \geqslant 0$ the intersection of $\mathcal{H}_\lambda$ with C will be denoted P_λ. The projection $\pi \colon V^* \times \mathbb{R}^* \to V^*$ identifies the hyperplanes $\mathcal{H}_\lambda$ with the vector space V^* and the polytopes P_λ with $\lambda P \subset V^*$. When $\lambda < 0$, we define P_λ as the intersection $P_\lambda = \mathcal{H}_\lambda \cap (-C)$. In this case, π also identifies P_λ with $\lambda P \subset V^*$. If $\xi \in V_P$ is a polarizing vector for P, then ξ is a polarizing vector for every P_λ.

Let $p \in P$ be a vertex, then λp is a vertex of the polytope λP. The intersection of $\mathcal{H}_\lambda$ with $K_p^\#$ is equal to $C_{\lambda p}^\#$ for $\lambda \geqslant 0$, where $C_{\lambda p}^\#$ is the polarized polarized tangent cone cone at λp. When $\lambda < 0$ the intersection becomes $K_p^\# \cap \mathcal{H}_\lambda = \check{C}_{\lambda p}^\#$, the dual polarized tangent cone at λp. Therefore, restricting S to $\mathcal{H}_\lambda$ we get

$$S|_{\mathcal{H}_\lambda}(x) = \sum_{p \in Vert(P)} (-1)^{\left|E_v^+(\xi)\right|} \mathbf{1}_{K_p^\#}|_{\mathcal{H}_\lambda}(x) = \sum_{p \in Vert(P)} (-1)^{\left|E_v^+(\xi)\right|} \mathbf{1}_{K_p^\# \cap \mathcal{H}_\lambda}(x)$$

$$= \sum_{p \in Vert(P)} (-1)^{\left|E_v^+(\xi)\right|} \mathbf{1}_{C_{\lambda p}^\#}(x) = P_\lambda(x),$$

if $\lambda \geqslant 0$. Similarly, if $\lambda < 0$ we have

$$S|_{\mathcal{H}_\lambda}(x) = \sum_{p \in Vert(P)} (-1)^{\left|E_v^+(\xi)\right|} \mathbf{1}_{K_p^\#}|_{\mathcal{H}_\lambda}(x) = \sum_{p \in Vert(P)} (-1)^{\left|E_v^+(\xi)\right|} \mathbf{1}_{K_p^\# \cap \mathcal{H}_\lambda}(x)$$

$$= \sum_{p \in Vert(P)} (-1)^{\left|E_v^+(\xi)\right|} \mathbf{1}_{\check{C}_{\lambda p}^\#}(x) = (-1)^n P_\lambda^\circ(x).$$

It follows that

$$(5.5) \qquad\qquad\qquad S(x) = \mathbf{1}_C(x) + (-1)^n \mathbf{1}_{-C^\circ}(x).$$

In a similar manner for the dual polarized tangent cones, let

$$\check{K}_p^\# = \check{C}_p^\# + \mathbb{R}\mu_p.$$

We have $\check{K}_p^\# \cap \mathcal{H}_\lambda = \check{C}_{\lambda p}^\#$ for $\lambda \geqslant 0$ and $\check{K}_p^\# \cap \mathcal{H}_\lambda = C_{\lambda p}^\#$.

Definition 5.12. Let $\check{S}(x)$ be the function defined by

$$\check{S}(x) = \sum_{p \in Vert(P)} (-1)^{\left|E_p^+(\xi)\right|} \mathbf{1}_{\check{K}_p^\#}(x),$$

This is the lift of the right-hand side of (5.3).

Restricting to the hyperplanes $\mathcal{H}_\lambda$ we get

$$\check{S}|_{\mathcal{H}_\lambda}(x) = \sum_{p \in Vert(P)} (-1)^{|E_v^+(\xi)|} \mathbf{1}_{\widecheck{K}_p^\#}|_{\mathcal{H}_\lambda}(x) = \sum_{p \in Vert(P)} (-1)^{|E_v^+(\xi)|} \mathbf{1}_{\widecheck{K}_p^\# \cap \mathcal{H}_\lambda}(x)$$

$$= \sum_{p \in Vert(P)} (-1)^{|E_v^+(\xi)|} \mathbf{1}_{\check{C}_{\lambda p}^\#}(x) = (-1)^n P_\lambda^\circ(x),$$

when $\lambda \geqslant 0$, and

$$\check{S}|_{\mathcal{H}_\lambda}(x) = \sum_{p \in Vert(P)} (-1)^{|E_v^+(\xi)|} \mathbf{1}_{\widetilde{K}_p^\#}|_{\mathcal{H}_\lambda}(x) = \sum_{p \in Vert(P)} (-1)^{|E_v^+(\xi)|} \mathbf{1}_{\widetilde{K}_p^\# \cap \mathcal{H}_\lambda}(x)$$

$$= \sum_{p \in Vert(P)} (-1)^{|E_v^+(\xi)|} \mathbf{1}_{C_{\lambda p}^\#}(x) = P_\lambda(x),$$

when $\lambda < 0$. Therefore

(5.6) $$\check{S}(x) = (-1)^n \mathbf{1}_{C^\circ}(x) + \mathbf{1}_{-C}(x).$$

Formulas (5.5) and (5.6) can again be expressed in terms of generating series.

Proposition 5.13.

$$\sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} K_p^\#(x) = C(x) + (-1)^n(-C^\circ)(x)$$

$$= \sum_{\mu \in C \cap (\mathbb{Z}_V^* \times \mathbb{R}^*)} x^\mu + (-1)^n \sum_{\mu \in (-C^\circ) \cap (\mathbb{Z}_V^* \times \mathbb{R}^*)} x^\mu.$$

Proof. The proof is a straightforward application of (5.5).

$$\sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} K_p^\#(x) = \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \sum_{\mu \in K_p^\# \cap (\mathbb{Z}_V^* \times \mathbb{R}^*)} x^\mu$$

$$= \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \sum_{\mu \in (\mathbb{Z}_V^* \times \mathbb{R}^*)} \mathbf{1}_{K_p^\#}(\mu) x^\mu$$

$$= \sum_{\mu \in (\mathbb{Z}_V^* \times \mathbb{R}^*)} \left(\sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{K_p^\#}(\mu) \right) x^\mu$$

$$= \sum_{\mu \in (\mathbb{Z}_V^* \times \mathbb{R}^*)} \left(\mathbf{1}_C(\mu) + (-1)^n \mathbf{1}_{(-C^\circ)}(\mu) \right) x^\mu$$

$$= \sum_{\mu \in C \cap (\mathbb{Z}_V^* \times \mathbb{R}^*)} x^\mu + (-1)^n \sum_{\mu \in (-C^\circ) \cap (\mathbb{Z}_V^* \times \mathbb{R}^*)} x^\mu.$$

$\square$

Similarly, we have:

Proposition 5.14.

$$\sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \check{K}_p^{\#}(x) = (-1)^n C^{\circ}(x) + (-C)(x)$$

$$= (-1)^n \sum_{\mu \in C^{\circ} \cap (\mathbb{Z}_V^* \times \mathbb{R}*)} x^{\mu} + \sum_{\mu \in (-C) \cap (\mathbb{Z}_V^* \times \mathbb{R}*)} x^{\mu}.$$

These results can be slightly generalised as follows. Let W^* be a vector space and let $P \subset W^*$ be a simple convex polytope sitting on a hyperplane

$$\mathcal{H} = \{\eta \in W^* \mid \langle \eta, R \rangle = 1\}$$

determined by a vector $R \in W$. Let

$$C = \{r\eta \in W^* \mid \eta \in P, r \geqslant 0\}$$

be the cone over P and for each vertex $p \in P$ denote by μ_p the primitive edge vector of C going through p. Let $\xi \in H_P^*$ be a polarizing vector for P. As above, for each vertex $p \in P$ denote by $C_p^{\#} \subset H$ the polarized tangent cone of P at p and by $\check{C}_p^{\#} \subset H$ the dual polarized tangent cone of P at p.

Definition 5.15. Define the cones $K_p^{\#}$ and $\check{K}_p^{\#}$ by

$$K_p^{\#} = C_p^{\#} + \mathbb{R}\mu_p \text{ and } \check{K}_p^{\#} = \check{C}_p^{\#} + \mathbb{R}\mu_p.$$

Let $\{e_1, \ldots, e_{n+1}\}$ be a basis of W such that $e_{n+1} = R$. Then $\{e_1^*, \ldots, e_n^*\}$ is a basis of $\mathcal{H}$ and we have a linear isomorphism

$$T: W^* \to \mathcal{H} \times \mathbb{R}^*$$
$$e_i^* \mapsto (e_i^*, 0)$$
$$e_{n+1}^* \mapsto (0, 1^*).$$

The map T takes $\mathcal{H}$ to the hyperplane

$$T(\mathcal{H}) = \{(\eta, 1^*) \in \mathcal{H} \times \mathbb{R}^* \mid \eta \in \mathcal{H}\}$$

and P to a polytope $T(P) \subset T(\mathcal{H})$. Restricting T to $\mathcal{H}$, we get a linear automorphism $T: \mathcal{H} \to \mathcal{H}$ that we will also denote by T. Let T^{-1} be the inverse of T and $(T^{-1})^*$ its adjoint. Let $v \in \mathcal{H}^*$ and $\eta \in \mathcal{H}$, then

$$\langle \eta, v \rangle = \langle T(\eta), (T^{-1})^*(v) \rangle.$$

Since the edges of P are taken to the edges of $T(P)$, the vector $(T^{-1})^*(\xi)$ induces a polarization of $T(P)$ such that $T(C_p^{\#}) = C_{T(p)}^{\#}$ for every vertex $p \in P$. The identity (5.5) implies that

$$\sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{K_{T(p)}^{\#}}(x) = \mathbf{1}_{T(C)}(x) + (-1)^n \mathbf{1}_{(-T(C)^{\circ})}(x).$$

Let $S(x) = \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{K_p^\#}(x)$. Since $K_{T(p)}^\# = T(K_p^\#)$, we have $\mathbf{1}_{K_p^\#}(x) = \mathbf{1}_{T(K_p^\#)}(Tx)$ and therefore

$$S(x) = \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{K_p^\#}(x) = \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{T(K_p^\#)}(Tx)$$

$$= \mathbf{1}_{T(C)}(Tx) + (-1)^n \mathbf{1}_{(-T(C)^\circ)}(Tx)$$

$$= \mathbf{1}_C(x) + (-1)^n \mathbf{1}_{(-C^\circ)}(x),$$

In summary, we have:

Proposition 5.16. *Let W^* be a vector space and $P \subset W^*$ a simple convex polytope on a hyperplane H determined by a vector $R \in W$. Then, given a polarizing vector $\xi \in H_P^*$, we have*

$$S(x) = \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{K_p^\#}(x) = \mathbf{1}_C(x) + (-1)^n \mathbf{1}_{(-C^\circ)}(x).$$

A similar argument applied to the dual polarized tangent cones $\check{K}_p^\# = \check{C}_p^\# + \mathbb{R}\mu_p$ gives:

Proposition 5.17.

$$\check{S}(x) = \sum_{p \in Vert(P)} (-1)^{|E_p^+(\xi)|} \mathbf{1}_{\widetilde{K}_p^\#}(x) = (-1)^n \mathbf{1}_{C^\circ}(x) + \mathbf{1}_{-C}(x).$$

The identities involving generating series in Proposition 5.13 and 5.14 continue to hold, with the lattice $\mathbb{Z}_W^*$ in place of $\mathbb{Z}_V^*$.

5.3. **A formula for $\mathrm{ind}_G^M(\sigma(\overline{\partial}_H))$.** Applying the results in the previous section, we show next how to obtain explicitly the multiplicities $m(\mu)$ associated to the weights $\mu \in \mathfrak{g}^*$ appearing in the index

$$\mathrm{ind}_G^M(\sigma(\overline{\partial}_H))(t) = \sum_{\mu \in \mathbb{Z}_G^*} m(\mu) t^\mu.$$

Let $R \in \mathfrak{g}$ be the generator of the Reeb vector field, $\mathcal{H}$ the characteristic hyperplane determined by R and C the moment cone. The polytope $P = \mathcal{H} \cap C$ is the image of the α-moment map ϕ_α. Each vertex of P corresponds to an edge of C, corresponding to a connected component L of M_n.

Given a vertex $p \in P$, let $L \subset M_n$ be the closed Reeb orbit corresponding to p. Since C is a good cone there is a vector $v_0^L \in \mathfrak{g}$ such that $\{v_0^L, v_1^l, \ldots, v_n^L\}$ is an integral basis of $\mathbb{Z}_G$, where $\{v_1^l, \ldots, v_n^L\}$ is the set of normals to faces meeting at p. Let $\{\mu_L, w_L^1, \ldots, w_L^n\}$ be the dual basis of $\{v_0^L, v_1^l, \ldots, v_n^L\}$, Theorem 4.4 tells us that if $\epsilon \in \mathfrak{g}$ is a polarizing vector, as in Definition 4.1, then the index $\mathrm{ind}_G^M(\sigma(\overline{\partial}_H))$ is given by

$$\mathrm{ind}_G^M \sigma(\overline{\partial}_H)(t) = \sum_{L \subset M_n} \left(\frac{1}{1 - t^{-w_L^1}} \right)^{s_L^1} \cdots \left(\frac{1}{1 - t^{-w_L^n}} \right)^{s_L^n} \delta(1 - t^{\mu_L}),$$

where $s_L^i = +$ if $(\epsilon_L^\perp) > 0$ and $s_L^i = -$ if $(\epsilon_L^\perp) < 0$. Define the index sets

$$W_L^+(\epsilon_L^\perp) = \{i \in \{1, \ldots, n\} \mid w_L^i(\epsilon_L^\perp) > 0\}$$

and

$$W_L^-(\epsilon_L^\perp) = \{i \in \{1,\ldots,n\} \mid w_L^i(\epsilon_L^\perp) < 0\}.$$

We can write

$$\left(\frac{1}{1-t^{-w_L^1}}\right)^{s_L^1} \cdots \left(\frac{1}{1-t^{-w_L^n}}\right)^{s_L^n} \delta(1-t^{\mu_L}) = (-1)^{|W_L^+(\epsilon_L^\perp)|} \sum_{\mu \in \mathbb{Z}_G^* \cap K_L^\#(\epsilon^\perp)} t^\mu,$$

since $\{\mu_L, w_L^1, \ldots, w_L^n\}$ is an integral basis of $\mathbb{Z}_G^*$, and $K_L^\#(\epsilon^\perp)$ is the cone defined by

$$K_L^\#(\epsilon^\perp) = \mathbb{R}\mu_L + \sum_{i \in W_L^+(\epsilon_L^\perp)} \mathbb{R}_{>0} w_L^i + \sum_{i \in W_L^-(\epsilon_L^\perp)} \mathbb{R}_{\leqslant 0} w_L^i.$$

Since $\{\mu_L, w_L^1, \ldots, w_L^n\}$ is the dual basis of $\{v_0^L, \ldots, v_n^L\}$, the cone $K_L^\#(\epsilon^\perp)$ can also be written as

$$K_L^\#(\epsilon^\perp) = \bigcap_{i \in W_L^+(\epsilon_L^\perp)} \{w \in \mathfrak{g}^* \mid w(v_i^L) > 0\} \bigcap_{i \in W_L^-(\epsilon_L^\perp)} \{w \in \mathfrak{g}^* \mid w(v_i^L) \leqslant 0\}.$$

The following lemma gives yet another description of the cone $K_L^\#(\epsilon^\perp)$.

Lemma 5.18. *Let $\{\eta^1, \ldots, \eta^n\} \subset \mathfrak{g}^*$ be a set of vectors satisfying $\eta^i(v_j^L) = \delta_{ij}$, $i, j = 1, \ldots, n$ and let K be the cone*

$$K = \mathbb{R}\mu_L + \sum_{i \in W_L^+(\epsilon_L^\perp)} \mathbb{R}_{>0}\eta^i + \sum_{i \in W_L^-(\epsilon_L^\perp)} \mathbb{R}_{\leqslant 0}\eta^i.$$

Then $K = K_L^\#(\epsilon^\perp)$.

Proof. Let $w \in K$ and write

$$w = r\mu_L + \sum_{i \in W_L^+(\epsilon_L^\perp)} a_i \eta_L^i + \sum_{i \in W_L^-(\epsilon_L^\perp)} b_i \eta_L^i,$$

where $r, a_i, b_i \in \mathbb{R}$, $a_i > 0$ and $b_i \leqslant 0$, $i = 1, \ldots, n$. Computing $w(v_i^L)$ for $i = 1, \ldots, n$ we get

$$w(v_i^L) = a_i > 0, \text{ for } i \in W_L^+(\epsilon_L^\perp) \text{ and } w(v_i^L) = b_i \leqslant 0 \text{ for } i \in W_L^-(\epsilon_L^\perp).$$

Therefore

$$K \subset \bigcap_{i \in W_L^+(\epsilon_L^\perp)} \{w \in \mathfrak{g}^* \mid w(v_i^L) > 0\} \bigcap_{i \in W_L^-(\epsilon_L^\perp)} \{w \in \mathfrak{g}^* \mid w(v_i^L) \leqslant 0\} = K_L^\#(\epsilon^\perp).$$

Let $w \in K_L^\#(\epsilon^\perp)$, since $\{\mu_L, \eta^i, \ldots, \eta^n\}$ forms a basis for $\mathfrak{g}^*$ we can write

$$w = r\mu_L + \sum_{i \in W_L^+(\epsilon_L^\perp)} a_i \eta_L^i + \sum_{i \in W_L^-(\epsilon_L^\perp)} b_i \eta_L^i.$$

Since $w \in K_L^\#(\epsilon^\perp)$, computing $w(v_i^L)$, $i = 1, \ldots, n$, we find that $a_i > 0$ and $b_i \leqslant 0$ which implies that $w \in K$. Therefore $K = K_L^\#(\epsilon^\perp)$. $\qquad\square$

Let $\{\eta_L, \eta_L^1, \ldots, \eta_L^n\}$ be the dual basis of $\{R, v_1^L, \ldots, v_n^L\}$, then $\eta_L^i(v_j^L) = \delta_{ij}$ and Lemma 5.18 implies that

$$K_L^\#(\epsilon^\perp) = \mathbb{R}\mu_L + \sum_{i \in W_L^+(\epsilon_L^\perp)} \mathbb{R}_{>0}\eta_L^i + \sum_{i \in W_L^-(\epsilon_L^\perp)} \mathbb{R}_{\leqslant 0}\eta_L^i.$$

The index sets $W_L^+(\epsilon_L^\perp)$ and $W_L^-(\epsilon_L^\perp)$ can also be expressed in terms of the basis $\{\eta_L, \eta_L^1, \ldots, \eta_L^n\}$.

Lemma 5.19.

$$W_L^+(\epsilon_L^\perp) = \{i \in \{1, \ldots, n\} \mid w_L^i(\epsilon_L^\perp) > 0\} = \{i \in \{1, \ldots, n\} \mid \eta_L^i(\epsilon) > 0\}$$

and

$$W_L^-(\epsilon_L^\perp) = \{i \in \{1, \ldots, n\} \mid w_L^i(\epsilon_L^\perp) < 0\} = \{i \in \{1, \ldots, n\} \mid \eta_L^i(\epsilon) < 0\}.$$

Proof. Since the weights $\{\alpha_1, \ldots, \alpha_n\}$ of the G_L-action on $\mathbb{C}^n$ form a dual basis to $\{v_1^L, \ldots, v_n^L\}$ in $\mathfrak{g}_L^*$ and

$$\epsilon_L^\perp = \eta_L^1(\epsilon)v_1^L + \cdots + \eta_L^n(\epsilon)v_n^L,$$

we have $\alpha_i(\epsilon_L^\perp) = w_L^i(\epsilon_L^\perp) = \eta_L^i(\epsilon)$, for $i = 1, \ldots, n$. $\qquad\qquad \square$

Theorem 5.20. *The index of the horizontal Dolbeault operator $\overline{\partial}_H$ is given by*

$$\mathrm{ind}_G^M(\sigma(\overline{\partial}_H))(t) = (-1)^n \sum_{\mu \in C^\circ \cap \mathbb{Z}_G^*} t^\mu + \sum_{\mu \in (-C) \cap \mathbb{Z}_G^*} t^\mu.$$

Proof. We note that $\{\eta_L^1, \ldots, \eta_L^n\}$ are primitive vectors determining the edge directions of P at p, since $\eta_L^i(R) = 0$ for $i = 1, \ldots, n$ and $\eta_L^i(v_j^L) = \delta_{ij}$. Let $\epsilon \in \mathfrak{g}$ be a polarizing vector, as in Proposition 4.1, that is, ϵ satisfies $\eta_L^i(\epsilon) \neq 0$ for $i = 1, \ldots, n$ and for all $L \subset M_n$. A polarizing vector for the polytope P is a vector in the dual vector space $\mathcal{H}^*$. The vector $\epsilon \in \mathfrak{g}$ determines a polarizing vector ϵ_H for the polytope P by

$$\epsilon_H(\eta) = \eta(\epsilon), \text{ for all } \eta \in \mathcal{H}.$$

Since the edge vectors $\{\eta_L^1, \ldots, \eta_L^n\}$ satisfy $\epsilon_H(\eta_L^i) = \eta_L^i(\epsilon) \neq 0$, $i = 1, \ldots, n$, ϵ_H is a polarizing vector for the polytope P. Since $\eta_L^i(\epsilon) = \epsilon_H(\eta_L^i)$, Lemma 5.19 implies that

$$W_L^-(\epsilon_L^\perp) = \{i \in \{1, \ldots, n\} \mid \eta_L^i(\epsilon) < 0\} = E_p^-(\epsilon_H)$$

and

$$W_L^+(\epsilon_L^\perp) = \{i \in \{1, \ldots, n\} \mid \eta_L^i(\epsilon) > 0\} = E_p^+(\epsilon_H),$$

where $E_p^-(\epsilon_H)$ and $E_p^+(\epsilon_H)$ correspond to the edges η_L^i of P such that $\eta_L^i(\epsilon_H) < 0$ and $\eta_L^i(\epsilon_H) > 0$, respectively. The vector ϵ_H determines a cone $\check{K}_p^\#$, as in Definition 5.15. Since the vectors η_L^i are the edge vectors of P meeting at p, the identity

$\check{K}_p^{\#} = K_L^{\#}(\epsilon^{\perp})$ holds. It follows that

$$\operatorname{ind}_G^M \sigma(\overline{\partial}_H)(t) = \sum_{L \subset M_n} \left(\frac{1}{1-t^{-w_L^1}}\right)^{s_L^1} \cdots \left(\frac{1}{1-t^{-w_L^n}}\right)^{s_L^n} \delta(1-t^{\mu_L})$$

$$= \sum_{L \subset M_n} (-1)^{\left|W_L^+(\epsilon_L^{\perp})\right|} \sum_{\mu \in \mathbb{Z}_G^* \cap K_L^{\#}(\epsilon^{\perp})} t^{\mu}$$

$$= \sum_{p \in \operatorname{Vert}(P)} (-1)^{\left|E_p^+(\epsilon_H)\right|} \sum_{\mu \in \mathbb{Z}_G^* \cap \check{K}_p^{\#}} t^{\mu}$$

and the result follows by applying Proposition 5.14. $\qquad\square$

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