Classical limits in field theory

Jethro van Ekeren

Universidade Federal Fluminense (UFF)
Niterói, Rio de Janeiro
Brazil

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Joint work with George Andrews (Penn State) and Reimundo Heluani (IMPA).

Classical (Hamiltonian) mechanics.

A collection A of "observables" like position, momentum, energy, etc.

These form a commutative algebra, i.e., we can add and multiply by scalars, and also multiply elements together.

Think of smooth functions over some manifold M.

There is an additional structure: A *Poisson bracket*

$$\begin{aligned} A \otimes A &\to A, \\ a, b &\mapsto \{a, b\}. \end{aligned}$$

Dynamics is given by Hamilton's equations: distinguished element $H \in A$, and the system evolves in such a way that

$$\frac{da}{dt} = \{H, a\}$$

for all $a \in A$.

Definition

A Poisson algebra is a commutative algebra A together with a Poisson bracket:

$$\{\cdot,\cdot\} : A \otimes A \to A,$$

i.e.,

(1) $\{b,a\} = -\{a,b\}$,
(2) $\{a,\{b,c\}\} - \{b,\{a,c\}\} = \{\{a,b\},c\}$,
(3) $\{a,bc\} = b\{a,c\} + a\{b,c\}$.

The first and second axioms here are the axioms of a Lie algebra, i.e.,

Definition

A Lie algebra is a vector space $\mathfrak{g}$ together with a Lie bracket:

$$[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g},$$

i.e.,

- $[b,a] = -[a,b]$,
- $[a,[b,c]] - [b,[a,c]] = [[a,b],c]$.

So a Poisson algebra is a vector space which is at once a commutative algebra and a Lie algebra, and the two structures interact according to the third axiom in the definition.

Classical field theory.

Now the fundamental objects which evolve in time are *fields*. A field F has a value $F(x)$ at each point x in space.

For any field F in the theory, and any point x in space, we could observe the value of F at x.

So our algebra contains an element $L_{F,x}$ for each field F and each point x.

This is a lot of observables!

Poisson bracket

$$\{L_{F,x}, L_{G,y}\} = ?$$

A new ingredient: locality.

The principle of *locality* says that fields at separate points $x \neq y$ should evolve independently.

So

$$\{L_{F,x}, L_{G,y}\} = 0 \qquad \text{whenever } x \neq y.$$

So

$$\{L_{F,x}, L_{G,y}\} = L_{K,y}\delta(x-y)$$

where K is some other field of the theory.

Actually the truth is a little more complicated

$$\{L_{F,x}, L_{G,y}\} = \sum_{j=0}^{N} L_{K^{(j)},y}\partial_y^j \delta(x-y)$$

So we have an structure of the type

$$F, G \mapsto (K^{(0)}, K^{(1)}, \ldots, K^{(N)}).$$

Collection A of fields.

Let's write

$$\{F_\lambda G\} = \sum_{j\in\mathbb{Z}_+} K^{(j)} \frac{\lambda^j}{j!}.$$

If $G(y) \in A$ is a field, then its space derivative $G'(y)$ is another field. So A comes equipped with an operation

$$T : A \to A, \qquad F \mapsto F'.$$

The operation T should satisfy the Leibniz rule, and (A, T) is called a *differential algebra*.

Now consider

$$\begin{aligned}\{L_{F,x}, L_{G',y}\} &= \frac{\partial}{\partial y}\left(L_{K,y}\delta(x-y)\right)\\ &= L_{K',y}\delta(x-y) + L_{K,y}\partial_y\delta(x-y).\end{aligned}$$

In terms of the λ-bracket this says

$$\{F_\lambda TG\} = (T+\lambda)\{F_\lambda G\}.$$

Definition

A Poisson vertex algebra is a commutative algebra A together with a derivation T, and a λ-bracket

$$\{a_\lambda b\} = \sum_{j\geq 0} \frac{\lambda^j}{j!} a_{(j)} b,$$

such that

- $\{a_\lambda Tb\} = (T+\lambda)\{a_\lambda b\}$.
- $\{b_\lambda a\} = -\{a_{-\lambda-T} b\}$.
- $\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} = \{\{a_\lambda b\}_{\lambda+\mu} c\}$.
- $\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}$.

Examples of Poisson vertex algebras...?

Easiest examples: arc spaces.

This is an algebra: $\mathbb{C}[x]$

This is a differential algebra: $\mathbb{C}[x, x', x'', \ldots]$ (the derivation T sends $x^{(n)}$ to $x^{(n+1)}$).

More generally, let

$$E = \mathbb{C}[x_1, x_2, \ldots, x_r]/(f_1, f_2, \ldots, f_s).$$

The *arc algebra* of E is obtained by adjoining a derivation to E in the "freest possible way".

Explicitly

$$JE = \frac{\mathbb{C}[x_i, x_i', x_i'', \ldots]}{(\partial^n(f_j))}.$$

Example: If $E = \mathbb{C}[x]/(x^3)$ then

$$JE = \frac{\mathbb{C}[x, x', x'', \ldots]}{(x^3, 3x^2x', 6x(x')^2 + 3x^2x'', \ldots)}.$$

Even for very simple algebras E, the arc algebra JE can be very complicated.

Structure of $J(\mathbb{C}[x]/(x^3))$

		0	1	2	3	4	5	6	p
	0	1							
	1								
	2		x 1						
	3		x' 1						
	4		x'' 1	1 x^2					
	5		x''' 1	1 xx'					
	6		1	2					
	7		1	2					
n	8		1	3	1 x^2x''				
	9		1	3	2				
	10		1	4	3				
	11		1	4	4				
	12		1	5	6	1			
	13		1	5	7	1			
	14		1	6	9	3			
	15		1	6	11	4			
	16		1	7	13	7			
	17		1	7	15	9			
	18		1	8	18	13	1		

Now let E be a Poisson algebra, and let $A = JE$.

Theorem

A carries a unique structure of Poisson vertex algebra such that

$$Tx = x' \quad \text{and} \qquad \{a_\lambda b\} = \{a, b\}$$

for $x \in A$ and $a, b \in E \subset A$.

It's clear the structure is unique: for example we must have

$$\{a_\lambda b^{(n)}\} = (\lambda + T)^n \{a, b\},$$

and $\{\cdot_\lambda \cdot\}$ extends to products by the Leibniz axiom.

Two dimensional conformal (quantum) field theory.

In quantum theory the states of a system are vectors in a vector space (more precisely a Hilbert space) V.

Now fields should be defined over the complex plane (or more generally over a Riemann surface).

The physics should be independent of the choice of coordinates used to describe it.

In this case, invariant under conformal transformations. (Preserve angles.) Infinitesimally (and locally) such a transformation is given by

$$z \mapsto w = \exp(\varepsilon f(z)\frac{d}{dz})z$$

The Lie algebra of local vector fields has a basis

$$z^m \frac{d}{dz}, \qquad \text{for } m \in \mathbb{Z}.$$

More conventionally

$$L_n = -z^{n+1} \frac{d}{dz}, \qquad \text{for } n \in \mathbb{Z},$$

with commutation relation

$$[L_m, L_n] = (m - n)L_{m+n}.$$

This Lie algebra:

$$W = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n, \qquad [L_m, L_n] = (m - n)L_{m+n},$$

is called the Witt algebra.

So our space of states V should carry a *representation* of W.

Representations of a group G:

$$\Sigma(g) : V \to V$$

for each $g \in G$, such that

$$\Sigma(g_1 g_2) = \Sigma(g_1)\Sigma(g_2).$$

Actually in quantum theory, the state of the system is described by a vector $v \in V$ (the Hilbert space), only up to rescaling. That is

$$v \qquad \text{and} \quad \alpha v$$

describe the same state.

So in principle "projective representations"

$$\Sigma(g_1 g_2) = \alpha(g_1, g_2)\Sigma(g_1)\Sigma(g_2),$$

where

$$\alpha : G \times G \to \mathbb{C}^\times$$

is a 2-cocycle, i.e.,

$$\alpha(g_1, g_2)\alpha(g_1 g_2, g_3) = \alpha(g_1, g_2 g_3)\alpha(g_2, g_3),$$

can appear.

A representation Σ of a Lie group G induces a representation σ of the Lie algebra $\mathfrak{g} = T_e G$.

That is, $\sigma(X) : V \to V$ for each $X \in \mathfrak{g}$, such that

$$\sigma([X_1, X_2]) = \sigma(X_1)\sigma(X_2) - \sigma(X_2)\sigma(X_1).$$

If Σ is only a projective representation of G then σ is only a projective representation of $\mathfrak{g}$, i.e.,

$$\sigma([X_1, X_2]) = \sigma(X_1)\sigma(X_2) - \sigma(X_2)\sigma(X_1) + \varepsilon(X_1, X_2) I_V,$$

where

$$\varepsilon : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

is a 2-cocycle, i.e., satisfying

$$\varepsilon(X_1, [X_2, X_3]) + \varepsilon(X_2, [X_3, X_1]) + \varepsilon(X_3, [X_1, X_2]) = 0.$$

Suppose V finite dimensional, and σ is a projective representation.

Then we could redefine

$$\overline{\sigma}(X) = \sigma(X) - \frac{\mathrm{tr}_V\, \sigma(X)}{\dim(V)} I_V.$$

This is again a projective representation of $\mathfrak{g}$ on V.

But by construction

$$\mathrm{tr}_V\, \overline{\sigma}(X) = 0,$$

and so

$$\begin{aligned} 0 &= \mathrm{tr}_V\, \overline{\sigma}([X_1, X_2]) \\ &= \mathrm{tr}_V\, (\overline{\sigma}(X_1)\overline{\sigma}(X_2) - \overline{\sigma}(X_2)\overline{\sigma}(X_1)) + \mathrm{tr}_V\, \overline{\varepsilon}(X_1, X_2) I_V, \\ 0 &= 0 + \mathrm{tr}_V\, \overline{\varepsilon}(X_1, X_2) I_V, \end{aligned}$$

So $\overline{\varepsilon} = 0$ and $\overline{\sigma}$ is a true representation of $\mathfrak{g}$.

Moral: on finite dimensional vector spaces, there are no nontrivial projective representations of Lie algebras.

For infinite dimensional representations, things are different.

A projective representation of the Witt algebra

$$W = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n, \qquad [L_m, L_n] = (m-n)L_{m+n}$$

yields a true representation of the *Virasoro algebra*

$$\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C, \qquad [L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}C, \qquad [C, \mathcal{L}] = 0.$$

In the state space V (assuming it is an irreducible representation of $\mathcal{L}$), the element C acts by a constant, called the *central charge*.

Vacuum representations of $\mathcal{L}$.

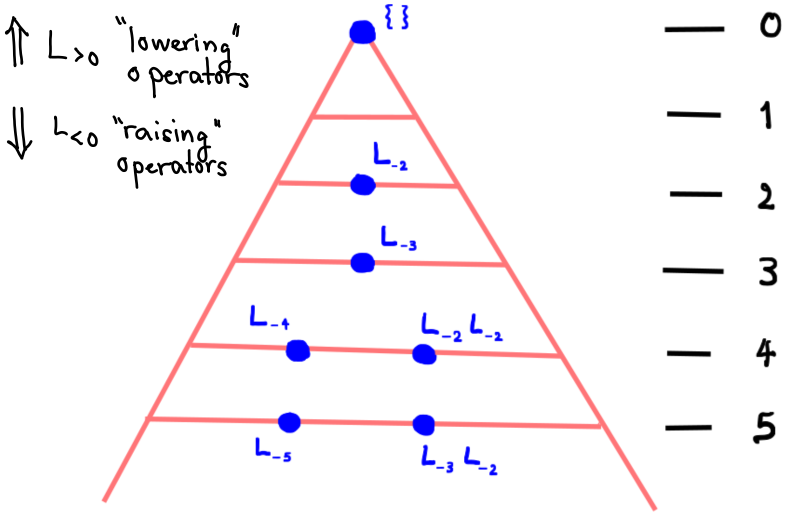
Positive half $\mathcal{L}_+ = \mathbb{C}C \oplus \bigoplus_{n \geq -1} \mathbb{C}L_n$.

Vacuum modules

$$V(c) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_+)} \mathbb{C}v$$

generated by v on which $\mathcal{L}_+$ acts via:

$$Cv = cv, \quad \text{and} \quad L_{\geq -1}v = 0.$$


⇑ $L_{>0}$ "lowering" operators
⇓ $L_{<0}$ "raising" operators
{ }
L_{-2}
L_{-3}
L_{-4}
$L_{-2}L_{-2}$
L_{-5}
$L_{-3}L_{-2}$
0
1
2
3
4
5

Natural $\mathbb{Z}_+$-grading on $V(c)$ with $V(c)_n$ spanned by the monomials

$$L_\lambda = L_{-\lambda_1} L_{-\lambda_2} \dots L_{-\lambda_k} v, \quad \deg = \sum \lambda_i = n,$$

where

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 1.$$

runs over the partitions λ of n into parts greater than 1.

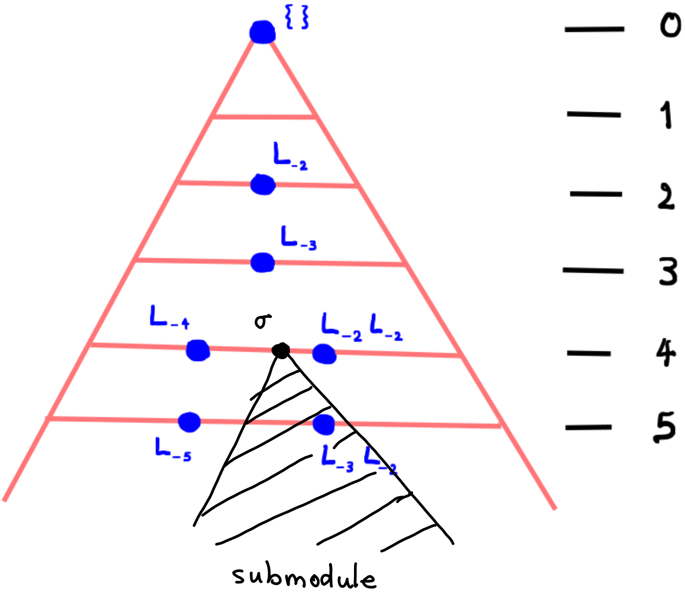
Generating function (character) of $V(c)$:

$$\chi_{V(c)}(q) = \sum_{n=0}^{\infty} \dim V(c)_n q^n = \prod_{m=2}^{\infty} \frac{1}{1-q^m}$$
$$= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + 8q^9 + 12q^{10} \cdots$$

Minimal Models: certain special values

$$c = c_{p,p'} = 1 - 6\frac{(p-p')^2}{pp'}, \quad p, p' \geq 2, \quad \gcd(p, p') = 1,$$

at which a singular vector $\sigma_{p,p'}$ appears in degree $s = (p-1)(p'-1)$.



{ }
L_{-2}
L_{-3}
L_{-4}
σ
$L_{-2} L_{-2}$
L_{-5}
$L_{-3} L_{-2}$
submodule
0
1
2
3
4
5

Singular vector generates the maximal nontrivial submodule.

Consider quotient $\mathcal{L}$-module

$$\mathrm{Vir}_{p,p'} = V(c)/U(\mathcal{L})\sigma_{p,p'}.$$

The character of $\mathrm{Vir}_{p,p'}$ (determined by Feigin and Fuchs) is very interesting:

$$\chi_{\mathrm{Vir}_{p,p'}}(q) = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{m\in\mathbb{Z}} \left(q^{\frac{(2pp'm+p-p')^2-(p-p')^2}{4pp'}} - q^{\frac{(2pp'm+p+p')^2-(p-p')^2}{4pp'}} \right).$$

Example:

$$\chi_{\mathrm{Vir}_{2,7}}(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 5q^8 + 6q^9 + 8q^{10} + \cdots .$$

Classical limits.

In classical physics observables commute with each other.

In quantum theory, famously, this is not so.

Already in our case we see, for example

$$\begin{aligned} L_{-2}L_{-3}v &= (L_{-3}L_{-2} + (L_{-2}L_{-3} - L_{-3}L_{-2}))\, v \\ &= (L_{-3}L_{-2} + [L_{-2}, L_{-3}])\, v \\ &= L_{-3}L_{-2}v + L_{-5}v. \end{aligned}$$

So the operators L_{-2} and L_{-3} do not commute!

In classical physics we would expect

$$L_{-2}L_{-3}v \equiv L_{-3}L_{-2}v$$

Note that the difference between the two sides is a shorter monomial. This holds in general.

A paradigm for classical limits.

The vector space $V = V(c)$ carries a filtration:

$$G_0 V \subset G_1 V \subset G_2 V \subset \dots$$

where

$$G_p V = \langle L_\lambda v \mid \lambda = (\lambda_1, \dots, \lambda_k), \;\; k \leq p \rangle \, .$$

Now

$$L_{-2} L_{-3} v \text{ and } L_{-3} L_{-2} v \text{ lie in } G_2 V$$

and

$$L_{-2} L_{-3} v \equiv L_{-3} L_{-2} v \pmod{G_1 V}$$

Associated graded of a filtered vector space

$$\mathrm{gr}_G\, V = \frac{V_1}{V_0} \oplus \frac{V_2}{V_1} \oplus \frac{V_3}{V_2} \oplus \cdots$$

Then $\mathrm{gr}_G\, V$ is naturally a commutative algebra. (Product is union of partitions)

And A carries a derivation (Induced by $L_{-1} : V \to V$).

Let $x = [L_{-2}v] \in A$. Denote by $A^0 \subset A$ the subalgebra generated by x.

Then $x' = [L_{-3}]$, $x'' = 2[L_{-4}]$, etc.

So A is generated as a differential algebra by A^0.

Associated graded of $V = \mathrm{Vir}_{2,7}$:

	$\mathrm{gr}^p_G V_n$	0	1	2	3	4	5	6	p
	0	1 {}							
	1								
	2	(2)	1						
	3	(3)	1						
	4	(4)	1	1 (2,2)					
	5		1	1 (3,2)					
	6		1	2					
	7		1	2					
n	8		1	3	1 (4,2,2)				
	9		1	3	2				
	10		1	4	3				
	11		1	4	4				
	12		1	5	6	1			
	13		1	5	7	1			
	14		1	6	9	3			
	15		1	6	11	4			
	16		1	7	13	7			
	17		1	7	15	9			
	18		1	8	18	13	1		

For $V = \mathrm{Vir}_{2,7}$ we have $A^0 \cong \mathbb{C}[x]/(x^3)$.

This is because

$$\sigma_{2,7} = L_{-2}L_{-2}L_{-2}v + \alpha_1 L_{-3}L_{-3} + \alpha_2 L_{-4}L_{-2} + \alpha_3 L_{-6}$$

and so in the associated graded

$$[\sigma_{2,7}] \equiv [L_{-2}L_{-2}L_{-2}] = x^3.$$

Comparing A with $J(A^0)$

In general there is a surjection

$$\pi : J(A^0) \to A.$$

In particular

$$\pi : J(\mathbb{C}[x]/(x^3)) \to \mathrm{gr}_G \, \mathrm{Vir}_{2,7} \, .$$

Definition

We say V (and generally any vertex algebra V) is *classically free* if the surjection

$$\pi : J(A^0) \to A$$

is an isomorphism.

Proposition

The $(2, 2s+1)$ *minimal models* $\mathrm{Vir}_{2,2s+1}$ *are classically free.*

All other (p, q) minimal models are not classically free. However...

Proposition

The $(3, 4)$ *minimal model is as "close" to classically free as possible.*

Let $s = (p-1)(q-1)/2$.

Idea: compare the graded dimensions of

$$J(\mathbb{C}[x]/(x^s)) \qquad \text{and} \qquad \mathrm{gr}_G \, \mathrm{Vir}_{p,q} \, .$$

For example for $(p,q) = (2,5)$ this is the Rogers-Ramanujan identity:

$$\sum_{k \in \mathbb{Z}_+} \frac{q^{k^2+k}}{(q)_k} = \frac{1}{(q)_\infty} \sum_{m \in \mathbb{Z}} \left(q^{\frac{(20m+3)^2-9}{40}} - q^{\frac{(20m+7)^2-9}{40}} \right).$$

Notation:

$$(q)_\infty = \prod_{m=1}^{\infty} (1-q^m), \quad \text{and} \quad (q)_n = \prod_{m=1}^{n} (1-q^m).$$

The Ising model: $(p, p') = (3, 4)$. Central charge $c = 1/2$.

Singular vector in $V(1/2)$:

$$\sigma_{3,4} = L_{-2}^3 + \frac{93}{64} L_{-3}^2 - \frac{27}{16} L_{-6} - \frac{33}{8} L_{-4} L_{-2}.$$

So

$$\pi : J(\mathbb{C}[x]/(x^3)) \twoheadrightarrow \operatorname{gr} \mathrm{Vir}_{3,4} .$$

Compare

$$\chi_{J(\mathbb{C}[x]/(x^3))}(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 5q^8 + 6q^9 + 8q^{10} + 9q^{11} + \cdots$$
$$\chi_{\mathrm{Vir}_{3,4}}(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 5q^8 + 5q^9 + 7q^{10} + 8q^{11} + \cdots .$$

Hence $\ker \pi$ contains a 1-dimensional component in degree 9.

By computer calculation we find that this vector is

$$b = L_{-5}L_{-2}L_{-2} + 6L_{-4}L_{-3}L_{-2}.$$

Write $I = J(a, b)$ the ideal in

$$J\mathbb{C}[x] = \mathbb{C}[L_{-2}, L_{-3}, L_{-4}, \ldots]$$

generated by $a = L_{-2}^3$, b and all their derivatives. Now have

$$\pi' : J\mathbb{C}[x]/I \twoheadrightarrow \mathrm{gr}\,\mathrm{Vir}_{3,4}\,.$$

Theorem (Andrews, vE, Heluani, 2020)

The morphism π' is an isomorphism.

Proof follows from

Theorem (Andrews, vE, Heluani, 2020)

$$\sum_{k_1,k_2=0}^{\infty} \frac{q^{4k_1^2+3k_1k_2+k_2^2}}{(q)_{k_1}(q)_{k_2}} \left(1 - q^{k_1} + q^{k_1+k_2}\right)$$

$$= \frac{1}{(q)_\infty} \sum_{m\in\mathbb{Z}} \left(q^{\frac{(24m+1)^2-1}{48}} - q^{\frac{(24m+7)^2-1}{48}} \right).$$