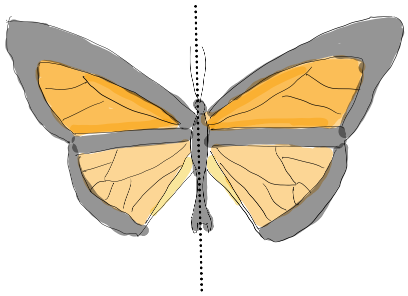
Groupoids: symmetries and Beyond



Workshop on Poisson Geometry, Groupoids and Quantization

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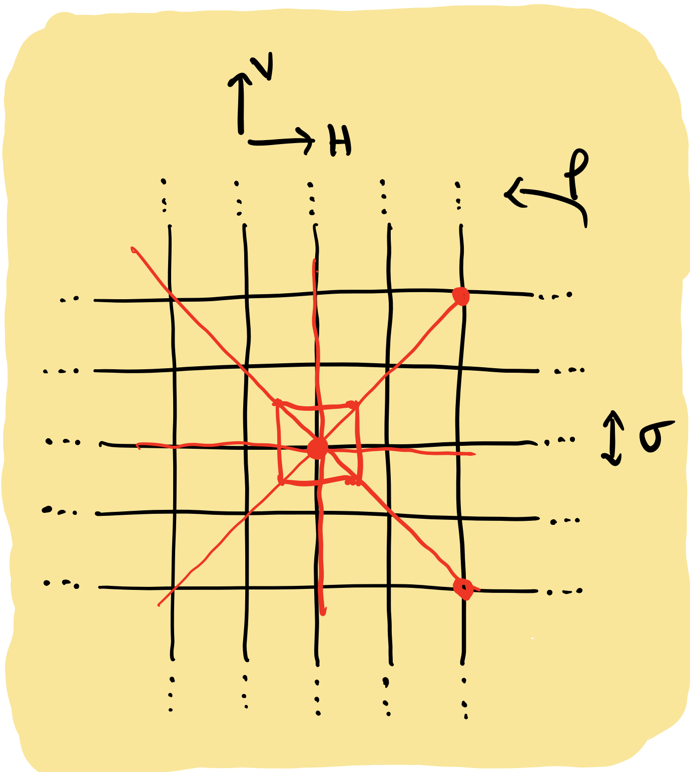
Object : Group of symmetries

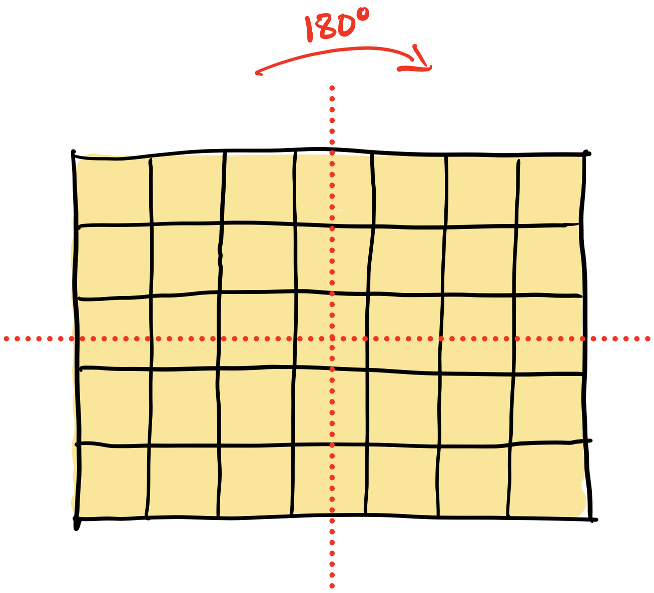


- $D_4 = \langle \rho, \sigma \mid \rho^4 = \sigma^2 = 1, \ \sigma\rho\sigma = \rho^3 \rangle$
- $\langle H \rangle = \mathbb{Z}$, $\langle V \rangle = \mathbb{Z}$

$\uparrow$

$\Gamma = D_4 \ltimes (\mathbb{Z} \times \mathbb{Z})$

The group of symmetries isn't enough...



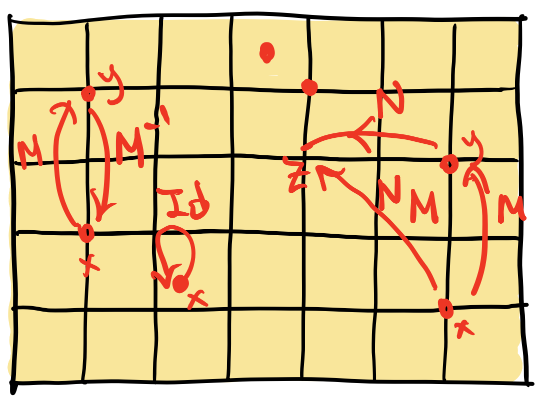
Klein's group $\mathbb{Z}_2 \times \mathbb{Z}_2$

Only 4 symmetries...

But we have many visible patterns !!

Solution: use groupoids!

R



(Γ = group of symmetries of … …)

$\mathcal{G} = \{(x, M, y)\}$

- $x, y \in R$
- $M \in \Gamma$
- $M(x) = y$

$y \xleftarrow{M} x$

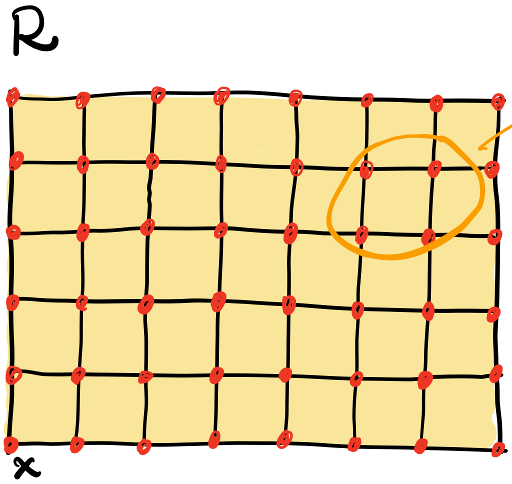
- $z \xleftarrow{N} y \xleftarrow{M} x \;=\; z \xleftarrow{NM} x$
- $x \xleftarrow{Id} x \;=\; 1_x$
- $y \xleftarrow{M} x \xleftarrow{1_x} x \;=\; y \xleftarrow{M} x$
- $y \overset{M}{\underset{M^{-1}}{\rightleftarrows}} x$

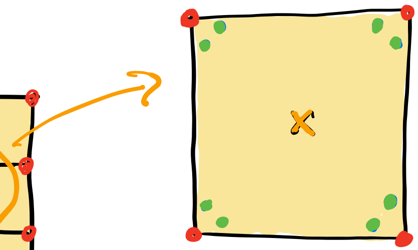
Groupoid Structure

"G - arrows between points of R"

- $G \overset{s}{\underset{t}{\rightrightarrows}} R$ $\qquad t(g) \xleftarrow{g} s(g)$
- Partial multiplication (associative): $G\,{}_{t}\!\times_{s} G \to G$:
$$\begin{tikzcd} t(g) & s(g)=t(h) \arrow[l, "g"'] & s(h) \arrow[l, "h"'] \arrow[ll, bend left, "g\cdot h"] \end{tikzcd}$$
- Units: $u: R \to G$:
$$\begin{tikzcd} x \arrow[loop above, "1_x"] & y \arrow[l, "g"'] \end{tikzcd} = \begin{tikzcd} x & y \arrow[l, "g"'] \end{tikzcd} = \begin{tikzcd} x & y \arrow[l, "g"'] \arrow[loop above, "1_y"] \end{tikzcd}$$
- Inverses: $i: G \to G$:
$$\begin{tikzcd} y \arrow[r, bend right, "g^{-1}"'] & x \arrow[l, bend right, "g"'] \end{tikzcd} \qquad g\cdot g^{-1} = 1_y, \qquad g^{-1} g = 1_x$$

Symmetries: Orbits & Isotropies





← Orbits

...

→ Isotropies at x: $\{$  $\}$

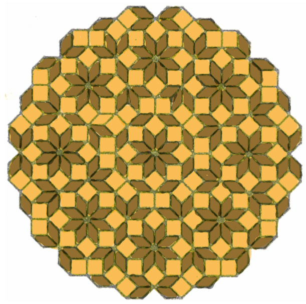
At ● & × : D_4

At ● : $\{1\}$

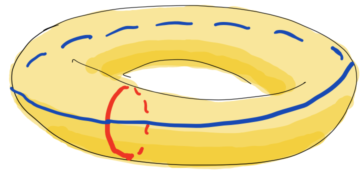
...

$\mathcal{G} = \{(x, M, y)\}$ $\quad y \xleftarrow{M \in \Gamma} x$

$\Gamma = D_4 \ltimes (\mathbb{Z} \times \mathbb{Z})$



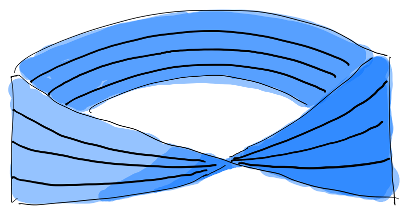
Quasicrystals



Fundamental Groupoid

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0$$

Germs of pseudogroups

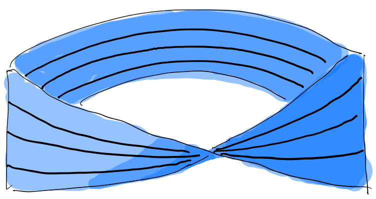


Holonomy groupoid

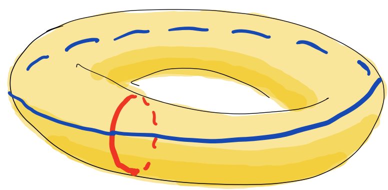
Lie Groupoids

"Smooth groupoids"

- G, R – manifolds
- smooth structural maps
- $G \overset{s}{\underset{t}{\rightrightarrows}} R$ surjective submersions

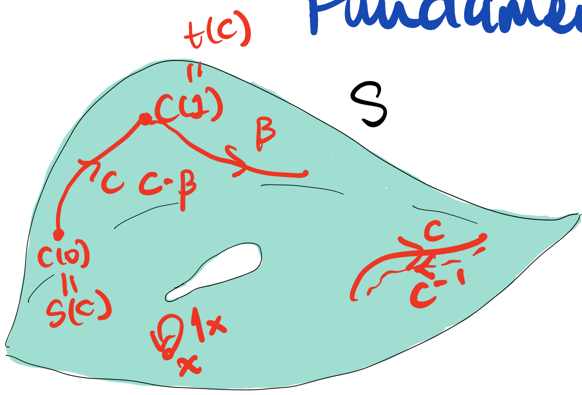


Monodromy & Holonomy Groupoid



Fundamental Groupoid

Fundamental Groupoid



$$\Pi(S) = \frac{\{[0,1] \xrightarrow{c} S \text{ continuous}\}}{\text{homotopy}}$$

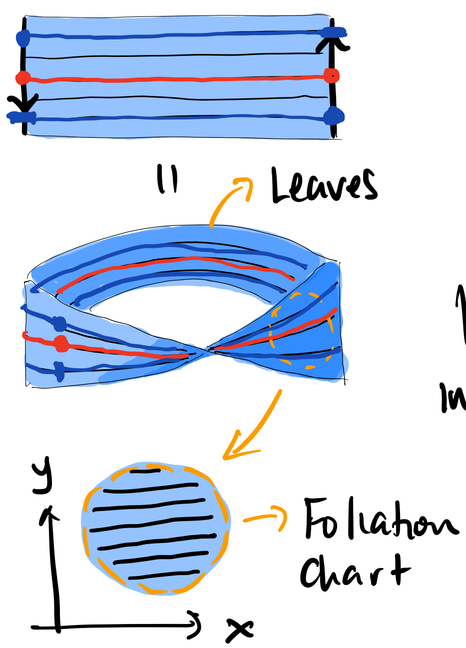
$$\Pi(S) \overset{t}{\underset{s}{\rightrightarrows}} S$$

- Multiplication: concatenation of curves
- Units : 1_x = constant path x
- Inverse of c: c traversed in opposite direction

→ Isotropy at $x = \pi_1(S, x)$

→ $s^{-1}(x) \xrightarrow{t} S$ Universal cover

Foliations $(M,\mathcal{F})$



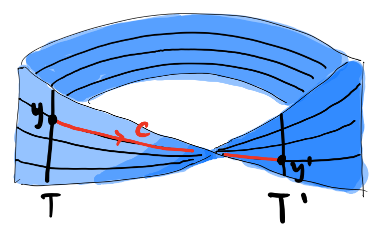
$$\left\{ \begin{array}{c} c:[0,1]\to M \\ \text{inside leaves} \end{array} \right\} := \mathcal{P}(M,\mathcal{F})$$

$\sim$

homotopy inside leaves

holonomy

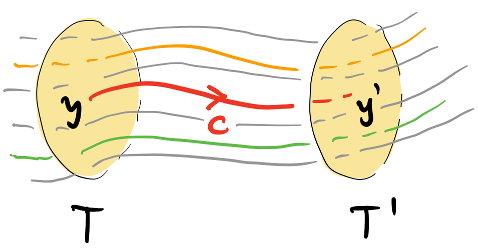
$\mathrm{hol}(c):(T,y)\to(T',y')$



Holonomy

Locally:

$(A \subset T, y) \xrightarrow{f} (T', y')$



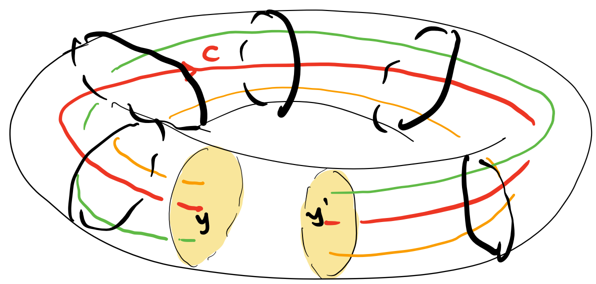
$\mathrm{germ}_y(f) = \mathrm{hol}(c)$

$c \overset{\mathrm{hol}}{\sim} c' \ : \ \mathrm{hol}(c) = \mathrm{hol}(c')$

$c \overset{\mathrm{hom}}{\sim} c' \Rightarrow c \overset{\mathrm{hol}}{\sim} c'$

Globally:

$\mathrm{hol}(c) = \mathrm{germ}_y(f)$

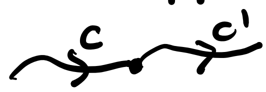


$\mathrm{hol}(c) = \mathrm{germ}_y(f)$

Monodromy & Holonomy groupoids

$$\mathrm{Mon}(M,\mathcal{F}) = \frac{\mathcal{P}(M,\mathcal{F})}{\text{homotopy}} \dashrightarrow \mathrm{Hol}(M,\mathcal{F}) = \frac{\mathcal{P}(M,\mathcal{F})}{\text{holonomy}}$$

$$\mathrm{Mon}(M,\mathcal{F}) \rightrightarrows M \qquad \mathrm{Hol}(M,\mathcal{F}) \rightrightarrows M$$

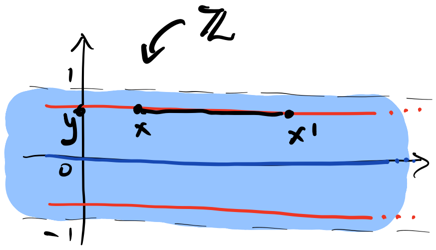


Proposition:
- Orbits = Leaves
- Isotropies – discrete

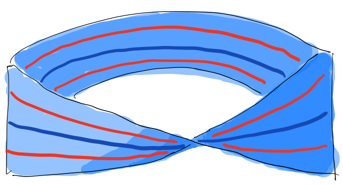
- $\mathrm{Mon}(M,\mathcal{F})(x,\cdot) \longrightarrow \mathrm{Hol}(M,\mathcal{F})(x,\cdot)$ coverings

$$\begin{tikzcd} \mathrm{Mon}(M,\mathcal{F})(x,\cdot) \arrow[rr] \arrow[dr] & & \mathrm{Hol}(M,\mathcal{F})(x,\cdot) \arrow[dl] \\ & L_x & \end{tikzcd}$$

Möbius Band



$$\Big\downarrow \quad (x,y)\sim(x+k,(-1)^k y)$$



$$\parallel$$

$$\mathbb{R}\times(-1,1)/\mathbb{Z} = M$$

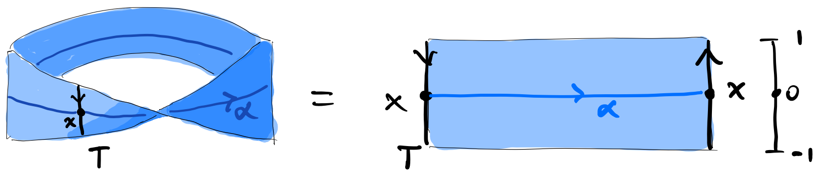
$$\begin{array}{c} \mathrm{Mon}(M,\mathcal{F}) = \mathbb{R}\times\mathbb{R}\times(-1,1)/\mathbb{Z} \\ s\big\downarrow\big\downarrow t \\ M \end{array}$$

- $[x',x,y] \overset{s}{\longmapsto} [x,y]$, $\quad [x',x,y] \longmapsto [x',y]$
- $[x',x,y]\cdot[x,x'',y] = [x',x'',y]$
- $s^{-1}[x_0,y_0] = [t,x_0,y_0] \cong \mathbb{R}_t$

$$\begin{array}{c} \big\downarrow \\ S^1 = L_{[x_0,y_0]} \end{array}$$

Möbius Band

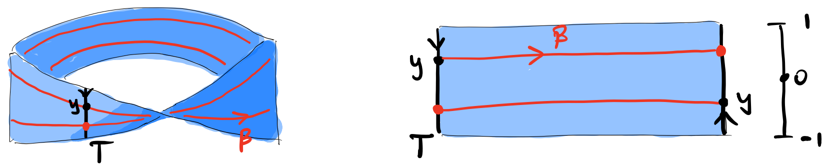
- $hol(\alpha) : (T, x) \to (T, x)$, $hol(\alpha) = -Id$, $hol(\alpha^2) = Id$



Isotropy x

$\langle [\alpha] \rangle = \mathbb{Z}_2$

- $hol(\beta) : (T, y) \to (T, y)$, $hol(\beta) = Id$



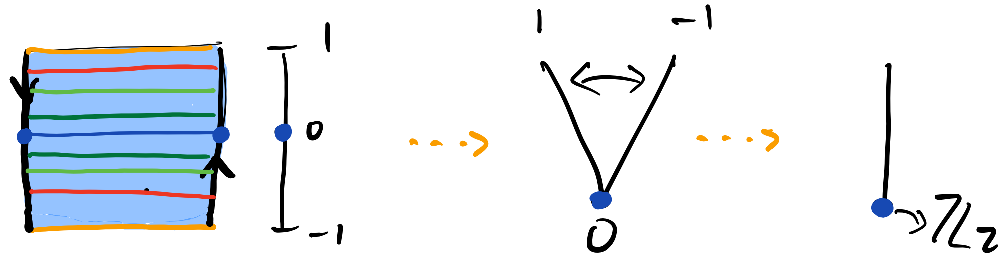
Isotropy y

$\langle \underset{\underset{[\beta]}{\shortparallel}}{[1_y]} \rangle = \langle e \rangle$

- $Hol(M, \mathcal{F})(x, \cdot) = S^1$

Foliations & Holonomy

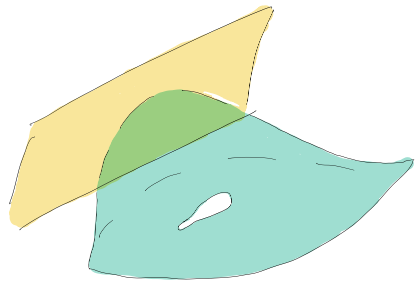
Theorem $(M, \mathcal{F})$ has compact leaves & $\mathrm{Hol}(M,\mathcal{F})(x,x)$ – finite, then $M/\mathcal{F}$ – orbifold.



What are Lie groupoids for?

- Foliations $\longleftrightarrow$ Holonomy, Monodromy
- Pseudogrups & PDE $\longleftrightarrow$ Effective groupoids
- Orbifolds $\longleftrightarrow$ Effective proper groupoids
- Poisson, Jacobi, Dirac... $\longleftrightarrow$ Symplectic groupoids, contact groupoids...

Lie Theory for groupoids



Lie groups

"Continuous symmetries can be reconstructed using their linear approximation" Sophus Lie

Lie algebras

Lie Theory

Differential geometry for groups $\longleftrightarrow$ Linear algebra, representation theory

Lie Theory for groupoids

Multiplicative		Linear
Lie groupoids	$\rightleftarrows$ C&F	Lie algebroids
Local Lie groupoids	$\rightleftarrows$	Lie algebroids
Maps of groupoids	$\rightleftarrows$	Maps of algebroids

There is still a lot to understand...

Semisimplicity, reductive groupoids...