SECTIONS OF VECTOR BUNDLES OVER GROUPOIDS

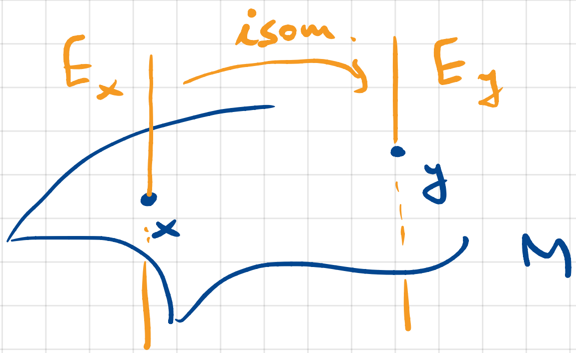
WORKSHOP ON POISSON GEOMETRY, GROUPOIDS & QUANTIZATION.

C. ORTIZ — SÃO PAULO

Why Lie Groupoids?

- Unifying objects:

  1. Lie groups $\dashrightarrow$ $G \rightrightarrows *$
  2. Manifolds $\dashrightarrow$ $M \rightrightarrows M$
  3. Global symmetries $\dashrightarrow$ $G \ltimes M \rightrightarrows M$
  4. Foliations $\dashrightarrow$ $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$
  5. Smooth quotients $\dashrightarrow$ $M \times_N M \rightrightarrows M$
  6. other symmetries $\dashrightarrow$ $GL(E) \rightrightarrows M$



- **Lie Theory:**

$$\text{Lie groupoids} \underset{\int}{\overset{\text{LIE}}{\rightleftarrows}} \text{Lie algebroids}$$

NOT ALWAYS POSSIBLE !! (Crainic–Fernandes)

- **Poisson Geometry:**

(M, π) Poisson manifold $\rightsquigarrow$ $T^*M \longrightarrow M$ Lie algebroid.

$\int$

$(G, \omega) \rightrightarrows M$

SYMPLECTIC GROUPOID

- Quantization
- Poisson maps as MOMENT MAPS !!

Why vector bundles over Lie groupoids?

(1) $(G, \omega) \rightrightarrows M$ symplectic groupoid

- $\omega \in \Omega^2(G)$ symplectic form
- MULTIPLICATIVE : $\underbrace{m^*\omega = pr_1^*\omega + pr_2^*\omega}$

$$
\begin{tikzcd}
G \,{}_s\!\times_t G \arrow[r, shift left=2, "pr_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "pr_2"'] & G
\end{tikzcd}
$$

$$
\begin{tikzcd}
TG \arrow[rr, "\omega^\#"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr] & & T^*G \arrow[d, shift left] \arrow[d, shift right] \arrow[dl] \\
TM & G & A^*
\end{tikzcd}
\quad ; \quad X \longmapsto \omega(X, \cdot)
$$

ω MULTIPLICATIVE iff $\omega^\#$ is a gpd. morphism.

... Geometric structures on Lie groupoids!!

(2) • $G \rightrightarrows M \rightsquigarrow \left[\frac{M}{G}\right]$ differentiable stack

(proper & étale) (orbifold)

• Vector bundles over $G \rightrightarrows M$ $\rightsquigarrow$ Vector bundles over $[M/G]$

(del Hoyo - O)

VECTOR BUNDLES OVER LIE GROUPOIDS

$$\begin{tikzcd} \mathcal{V} \arrow[r, shift left, "\tilde{s}"] \arrow[r, shift right, "\tilde{t}"'] \arrow[d] & E \arrow[d] \\ \mathcal{G} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M \end{tikzcd}$$

(vector bundle)

(+) COMPATIBILITY :

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\tilde{s}"] \arrow[d] & E \arrow[d] \\ \mathcal{G} \arrow[r, "s"] & M \end{tikzcd}$$

vector bundle map

... $\tilde{t}, \tilde{m}, \tilde{i}, \tilde{1}$.

RMK : VB-groupoid $\rightleftarrows$ Categorified vector bundle !!

EXAMPLES

① $E \longrightarrow M$ ⇢

vector bundle

$$\begin{tikzcd} E \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & E \arrow[d] \\ M \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

unit gpd.

unit gpd.

② 2- VECTOR SPACE ⇢

$$\begin{tikzcd} V_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & V_0 \arrow[d] \\ * \arrow[r, shift left] \arrow[r, shift right] & * \end{tikzcd}$$

$$V_1 \rightrightarrows V_0$$

(i.e $K \xrightarrow{\partial} V_0$; $K = \ker s$, $\partial = t|_K$)

③ Representations:

$$\begin{tikzcd} \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & E \arrow[dll] \\ M & & \end{tikzcd} \quad \rightleftharpoons \quad y \xleftarrow{\;g\;} x \ : \ E_x \xrightarrow{\cong} E_y$$

(+) ACTION PROPERTY.

i.e.

$$\mathcal{G} \xrightarrow{\Delta} GL(E) \qquad \text{groupoid morphism.}$$

$$g \longmapsto \Delta_g .$$

VB-gpd.

$$\begin{tikzcd} s^*E \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & E \arrow[d, "\pi"] \\ \mathcal{G} \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & M \end{tikzcd}$$

- $s^*E = \mathcal{G} {}_s\!\times_\pi E \ni (g, e) \longmapsto e \in E_{s(g)}$

$$\phantom{s^*E = \mathcal{G} {}_s\!\times_\pi E \ni (g, e)} \longmapsto \Delta_g e \in E_{t(g)}$$

... $\underbrace{\mathrm{Rep}(\mathcal{G}) \text{ category of representations of } \mathcal{G} \rightrightarrows M}$

$$\begin{aligned} \mathrm{Rep}(\mathcal{G}) &\longrightarrow VB(\mathcal{G}) \\ E &\longmapsto \mathcal{G} \ltimes E \end{aligned}$$

FUNCTOR

$\mathrm{Rep}(\mathcal{G})$ CONCRETELY:

1. $\mathrm{Rep}(M \rightrightarrows M) = \mathrm{Vect}(M)$

2. $\mathrm{Rep}(G \rightrightarrows *) = \mathrm{Rep}(G)$

3. $\mathrm{Rep}(G \ltimes M \rightrightarrows M) = \mathrm{Vect}(M)^G$ equiv. vector bundles

4. $\mathrm{Rep}(M \times_N M \rightrightarrows M) \cong \mathrm{Vect}(N)$

ABOUT $\quad \mathrm{Rep}(\mathcal{G}) \longrightarrow \mathrm{VB}(\mathcal{G})$

Does any
$$\begin{tikzcd} \mathcal{V} \arrow[r, shift left, "\tilde{s}"] \arrow[r, shift right] \arrow[d, "\pi_{\mathcal{V}}"'] & E \arrow[d, "\pi_E"] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$
come from $\mathrm{Rep}(\mathcal{G})$?

... NO.

LOOK AT:
$$\mathcal{V} \xrightarrow{\;\tilde{\pi}_{\mathcal{V}}\;} \mathcal{G}\,{}_s\!\times_{\pi_E} E$$
$$v \longmapsto \left(\pi_{\mathcal{V}}(v), \tilde{s}(v)\right)$$

$$\underbrace{\tilde{\pi}_{\mathcal{V}} \quad \text{isomorphism}}_{\displaystyle \text{iff} \quad C := (\mathrm{Ker}\,\tilde{s})|_M \longrightarrow M \text{ is zero.}} \iff \mathcal{V} = \mathcal{G} \ltimes E$$

CORE BUNDLE

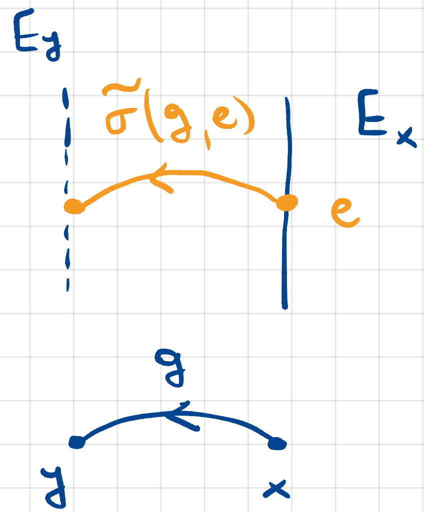
EXAMPLE:

$$\begin{tikzcd} T\mathcal{G} \arrow[r, shift left, "Ts"] \arrow[r, shift right, "Tt"'] \arrow[d] & TM \arrow[d] \\ \mathcal{G} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M \end{tikzcd} \dashrightarrow \underbrace{C = (\operatorname{Ker} Ts)\big|_M}_{\text{Lie algebroid of } \mathcal{G} \rightrightarrows M \;!!}$$

RMK:

$$\begin{tikzcd} \sigma \arrow[r, "\pi_\sigma"] & \mathcal{G}\, {}_s\!\times_{\pi_E} E \arrow[l, dashed, bend left, "\tilde{\sigma}"] \end{tikzcd}$$

(CLEAVAGE)



$$\dots \quad \Delta^E_g(e) = \tilde{t}\big(\tilde{\sigma}(g,e)\big)$$ is NOT A REPRESENTATION.

BUT :

$$y \xleftarrow{g} x \;:\; \overbrace{\left(C_x \xrightarrow{\partial} E_x\right)}^{\mathcal{V}_x \text{ 2-vect. space}} \xrightarrow{\Delta_g} \overbrace{\left(C_y \xrightarrow{\partial} E_y\right)}^{\mathcal{V}_y \text{ 2-vect. space.}}$$

quasi - isomorphism of 2-term complexes:

$$\Delta_{gh} \xRightarrow{\Omega_{g,h}} \Delta_g \circ \Delta_h .$$

THEOREM: (Gracia Saz - Mehta , del Hoyo - O)

$$\mathrm{Rep}_2(\mathcal{G}) \longrightarrow \mathrm{VB}(\mathcal{G})$$

equivalence of categories.

SECTIONS OF VB-GROUPOIDS

- $E \longrightarrow M$ vector bundle $\dashrightarrow$ (i) fibers are vector spaces
  (ii) $\Gamma(E)$ vector space

- $$\begin{tikzcd} \mathcal{V} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & E \arrow[d] \\ \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$ $\dashrightarrow$ (i) fibers are 2-vector spaces
  (2-term complexes)
  (ii) $\underbrace{\text{What about sections?}}_{\text{2-vector space}}$

Def: A ***MULTIPLICATIVE*** section of $\mathcal{V}$ is a pair of sections $v : \mathcal{G} \longrightarrow \mathcal{V}$; $e : M \longrightarrow E$ such that:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "v"] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{V} \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "e"'] & E \end{tikzcd}$$

groupoid morphism.

NOTATION : $\Gamma_{mult}(\mathcal{V})$ multiplicative sections.

EXAMPLES

(1) $$\Gamma_{mult}\left(\begin{array}{ccc} V_1 & \rightrightarrows & V_0 \\ \downarrow & & \downarrow \\ * & \rightrightarrows & * \end{array}\right) \cong V_0$$

vector fields
on stacks
(Hepworth)

(2) $$\Gamma_{mult}\left(\begin{array}{ccc} T\mathcal{G} & \rightrightarrows & TM \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array}\right) = \mathfrak{X}_{mult}(\mathcal{G})$$

MULTIPLICATIVE
VECTOR FIELDS.

(3) $$\Gamma_{mult}\left(\begin{array}{ccc} T^*\mathcal{G} & \rightrightarrows & A^* \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array}\right) = \Omega^1_{mult}(\mathcal{G})$$

MULTIPLICATIVE
1-FORMS.

(4) $$\Gamma_{mult}\left(\begin{array}{ccc} \mathcal{G}\ltimes E & \rightrightarrows & E \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array}\right) = \Gamma(E)^{\mathcal{G}} \ni e$$

$$\Delta_g e_{s(g)} = e_{t(g)}$$

$$\forall g \in \mathcal{G}.$$

Given
$$\begin{array}{ccc} \mathcal{V} & \rightrightarrows & E \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array}$$
with core $C \longrightarrow M$

$$\left(\begin{array}{ccc} T\mathcal{G} & \rightrightarrows & TM \\ \downarrow & & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{array} \qquad C = A \right)$$

Proposition : (O - Waldron)

If $c \in \Gamma(C)$ then $c^r - c^l : \mathcal{G} \longrightarrow \mathcal{V}$ is multiplicative.

$(a \in \Gamma(A))$ $\quad (a^r - a^l : \mathcal{G} \longrightarrow T\mathcal{G})$ (vector field on $\mathcal{G}$.)

covers $\partial c \in \Gamma(E)$

$\partial : C \longrightarrow E$

$$\Downarrow$$

$$\Gamma(C) \xrightarrow{\ \delta\ } \Gamma_{mult}(\mathcal{V})$$
$$c \longmapsto c^r - c^l.$$

Category of multiplicative sections

$$
\begin{tikzcd}
\mathcal{G} \arrow[r, bend left=50, "v'"{name=U}] \arrow[r, bend right=50, "v"'{name=D}] \arrow[from=D, to=U, Rightarrow, "\tau"] & \mathcal{V} \arrow[r, bend left=50, "\pi_{\mathcal{V}}"{name=U2}] \arrow[r, bend right=50, "\pi_{\mathcal{V}}"'{name=D2}] \arrow[from=D2, to=U2, Rightarrow, "1_{\pi_{\mathcal{V}}}"] & \mathcal{G}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{G} \arrow[r, bend left=50, "id"{name=U}] \arrow[r, bend right=50, "id"'{name=D}] \arrow[from=D, to=U, Rightarrow, "1_{id}"] & \mathcal{G}
\end{tikzcd}
$$

(*) $\boxed{1_{\pi_{\mathcal{V}}} \cdot \tau = 1_{id}}$ ← - - Horizontal composition

Def: $Sec(\mathcal{G}, \mathcal{V})$ category:

- objects : $\Gamma_{mult}(\mathcal{V})$
- $v \overset{\tau}{\Longrightarrow} v'$ natural transf. with (*)

THEOREM (O - Waldron)

$\mathrm{Sec}(\mathcal{G}, \mathcal{V})$ is a 2-vector space

isomorphic to :

$$\Gamma(C) \ltimes \Gamma_{mult}(\mathcal{V}) \rightrightarrows \Gamma_{mult}(\mathcal{V})$$

$$(c, v) \xmapsto{s} v$$
$$\xmapsto{t} (c^r - c^l) + v$$

- $\mathcal{V} \underset{ME}{\simeq} \mathcal{V}' \Longrightarrow \mathrm{Sec}(\mathcal{G}, \mathcal{V}) \simeq \mathrm{Sec}(\mathcal{G}, \mathcal{V}')$

- Cohomology of $\Gamma(C) \xrightarrow{d} \Gamma_{mult}(\mathcal{V})$ $= \begin{cases} \Gamma(C)^{\mathcal{G}} & \text{deg. zero} \\ H^1_{diff}(\mathcal{G}, \mathcal{E}_{\mathcal{V}}) & \text{deg. one} \end{cases}$

EXAMPLES

$W = \operatorname{Ker} s$

$\partial = t|_W$

① $\mathrm{Sec}(V_1 \rightrightarrows V_0) \rightleftharpoons W \xrightarrow{\partial} V_0$

② $\mathrm{Sec}(T\mathcal{G} \rightrightarrows TM) \rightleftharpoons \Gamma(A) \longrightarrow \mathfrak{X}_{\mathrm{mult}}(\mathcal{G})$

$$a \longmapsto a^r - a^l$$

③ $\mathrm{Sec}(T^*\mathcal{G} \rightrightarrows A^*) \rightleftharpoons \Omega^1(M) \longrightarrow \Omega^1_{\mathrm{mult}}(\mathcal{G})$

$$\alpha \longmapsto s^*\alpha - t^*\alpha$$

④ $\mathrm{Sec}(\mathcal{G} \ltimes E \rightrightarrows E) \rightleftharpoons 0 \longrightarrow \Gamma(E)^{\mathcal{G}}$

$$\cong$$

$$\Gamma(E)^{\mathcal{G}} \rightrightarrows \Gamma(E)^{\mathcal{G}}$$

MULTIPLICATIVE VECTOR FIELDS

- $\Gamma(A) \longrightarrow \mathfrak{X}_{mult}(\mathcal{G})$ in deformation theory of Lie groupoids (Crainic – Struchiner – Mestre)

  $a \longmapsto a^r - a^l$

- $\mathrm{Sec}(\mathcal{G}, T\mathcal{G}) \cong \Gamma(A) \ltimes \mathfrak{X}_{mult}(\mathcal{G}) \rightrightarrows \mathfrak{X}_{mult}(\mathcal{G})$

  is a Lie 2-algebra !!

  (MORITA INVARIANT $\rightsquigarrow$ Vector fields on $[M/\mathcal{G}]$)

  (Hepworth)

  (i) Berwick Evens – Lerman

  (ii) O – Waldron (Any LA – groupoid !! e.g. $T\mathcal{G} \to \mathcal{G}$)

- $\mathfrak{X}(M) \curvearrowright C^\infty(M)$ Lie algebra module

DERIVATIONS

$$\left(\Gamma(A) \longrightarrow \mathfrak{X}_{mult}(\mathcal{G}) \right) \curvearrowright \left(C^\infty(M) \longrightarrow C^\infty_{mult}(\mathcal{G}) \right)$$

DERIVATIONS

Vector fields on $\left[\frac{M}{\mathcal{G}}\right]$ $\curvearrowright$ Functions on $\left[\frac{M}{\mathcal{G}}\right]$

(ongoing joint with Sebastián Herrera – USP & James Waldron – Newcastle)

OBRIGADO !!