

HYPERPOLYGONS SPACES:

A VIEW ON MODULI SPACES OF PARABOLIC HIGGS BUNDLES

ALESSIA MANDINI

joint with LEONOR GODINHO

POLSSON GEOMETRY, GRUPOIDS AND QUANTIZATION

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Plan of the talk:

- Polygon spaces (Hausmann-Knutson, Kapovich-Millsom, Godinho-Agapito, μ , etc)
- Hyperpolygon spaces (Konno, Horade - Proudfoot, Godinho-M, Fisher-Rayan, Rayan - Schaposnik...)
- Quasi-parabolic & parabolic Higgs bundles (Hitchin, Simpson, Komuro, García Prada, Gothen, Oliveira, Biswas, Florentino, Godinho, M, ...)
- Involutions & BRANES (Hitchin, Baraglia - Schaposnik, Jardim, Marchesi, Franco, BFGM, ...)

MODULI SPACE OF POLYGONS in \mathbb{R}^3

$n \geq 3$

Fix $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_{>0}^n$, let $(S_{\alpha_i}^2, \omega_i)$ be the sphere ω_i volume form

$$\begin{array}{ccc} \text{SO}(3) \curvearrowright \left(\prod_{i=1}^n S_{\alpha_i}^2, \omega \right) & \xrightarrow{\mu} & \text{SO}(3)^* \simeq \mathbb{R}^3 \\ \text{Ham} & & \sum_{i=1}^n v_i \\ & & (v_1, \dots, v_n) \end{array}$$

MODULI SPACE OF POLYGONS in \mathbb{R}^3

Fix $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_{>0}^n$, let $(S_{\alpha_i}^2, \omega_i)$ be the sphere ^{volume form}

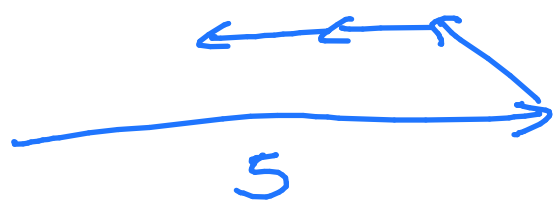
$$SO(3) \curvearrowright \left(\prod_{i=1}^n S_{\alpha_i}^2, \omega \right) \xrightarrow{\mu} SO(3)^* \simeq \mathbb{R}^3$$

$$(v_1, \dots, v_n) \longmapsto \sum_{i=1}^n v_i$$

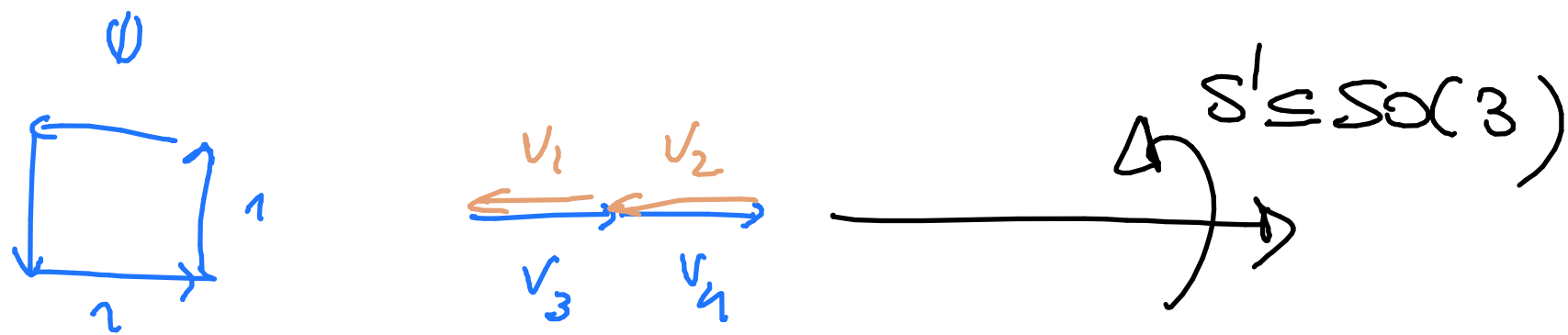
$$M(\alpha) = \prod_{i=1}^n S_{\alpha_i}^2 //_{SO(3)} = \mu_{SO(3)}^{-1}(0) //_{SO(3)} \quad \text{POLYGON SPACE}$$

$$(v_1, \dots, v_n) \in \mu_{SO(3)}^{-1}(0) \iff \begin{array}{c} v_n \\ \swarrow \\ v_1 \quad \searrow \quad v_2 \end{array} \quad \begin{array}{l} v_i \in \mathbb{R}^3 \\ \underline{\|v_i\| = \alpha_i} \end{array}$$

• $M(5, 1, 1, 1) = \emptyset$ $v_i > \sum_{i \neq j} v_i$



• $M(1, 1, 1, 1) \neq \emptyset$ singular



$$M(\alpha) \text{ is smooth} \iff \sum_{i \in I} \alpha_i \neq \sum_{i \in I^c} \alpha_i$$

$\hat{=} \iff$

α is generic

$\forall I \subseteq \{1, \dots, n\}$

MODULI SPACE OF POLYGONS in \mathbb{R}^3

Fix $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_{>0}^n$, let $(S_{\alpha_i}^2, \omega_i)$ be the sphere ↖ volume form

$$SO(3) \curvearrowright \left(\prod_{i=1}^n S_{\alpha_i}^2, \omega \right) \xrightarrow{\mu} SO(3)^* \cong \mathbb{R}^3$$

$$(v_1, \dots, v_n) \longmapsto \sum_{i=1}^n v_i$$

$$M(\alpha) = \prod_{i=1}^n S_{\alpha_i}^2 //_{SO(3)} = \mu_{SO(3)}^{-1}(0) / SO(3) \quad \text{POLYGON SPACE}$$

$$(v_1, \dots, v_n) \in \mu_{SO(3)}^{-1}(0) \iff \begin{array}{c} v_n \\ \swarrow \\ v_1 \quad \searrow \quad v_2 \\ \vdots \end{array} \quad \begin{array}{l} v_i \in \mathbb{R}^3 \\ \|v_i\| = \alpha_i \end{array}$$

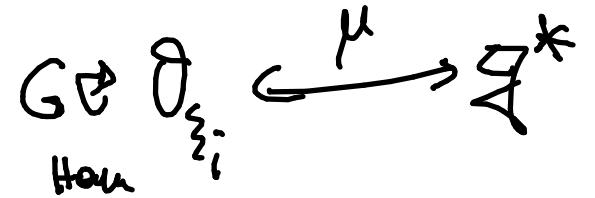
$$M(\alpha) \text{ smooth Kähler} \iff \varepsilon_I(\alpha) := \sum_{i \in I} \alpha_i - \sum_{j \in I^c} \alpha_j \neq 0$$

α GENERIC $\forall I \subseteq \{1, \dots, n\}$

MODULI SPACE OF POLYGONS IN \mathfrak{g}^*

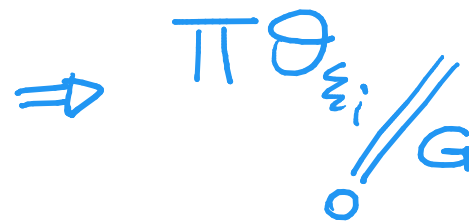
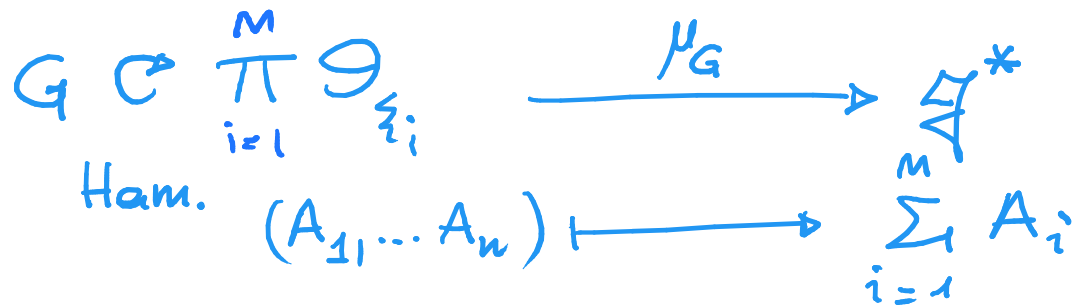
G Lie group, $\mathfrak{g} = \text{Lie}(G)$, \mathfrak{g}^* its dual

$G \curvearrowright \mathfrak{g}^*$ coadjoint action



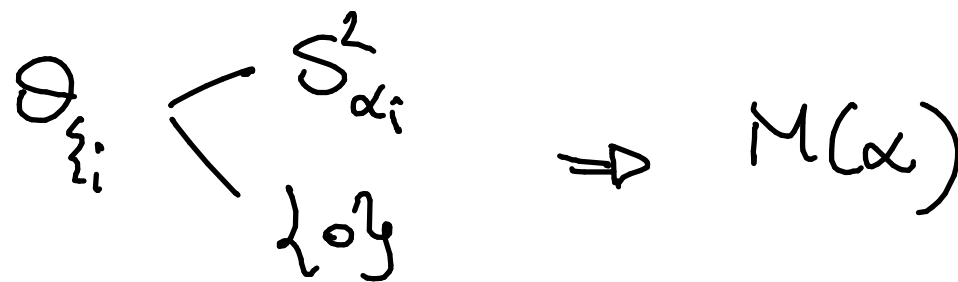
Choose $\xi_i \in \mathfrak{g}^*$, $\xi_i \neq 0 \forall i=1, \dots, n \rightsquigarrow \mathcal{O}_{\xi_i} = \text{coadjoint orbit}$

ω_{KKS}



G -polygon space

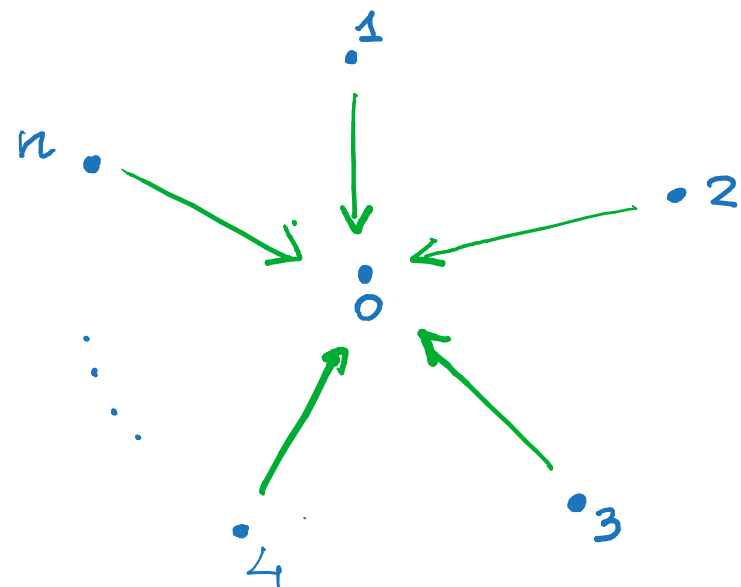
$G = \text{SU}(2)$



POLYGON SPACES AS QUIVER VARIETIES

Star-shaped quiver

$$V_0 = \mathbb{C}^2, \quad V_i = \mathbb{C} \quad \forall i=1, \dots, n$$



$$\bullet \text{ Rep } Q = \bigoplus_{i=1}^n \text{Hom}(\mathbb{C}, \mathbb{C}^2) \cong \mathbb{C}^{2n}$$

$$q_1, \dots, q_n$$

$$q_i: \mathbb{C} \longrightarrow \mathbb{C}^2$$

$$\bullet K = U(2) \times U(1)^n / \Delta$$

diagonal S^1
in $U(2) \times U(1)^n$

$K \curvearrowright \mathbb{C}^{2n}$ by $[q](A, \lambda_1, \dots, \lambda_n) = [\dots, A^{-1} q_i \lambda_i, \dots]$ with moment map

Ham

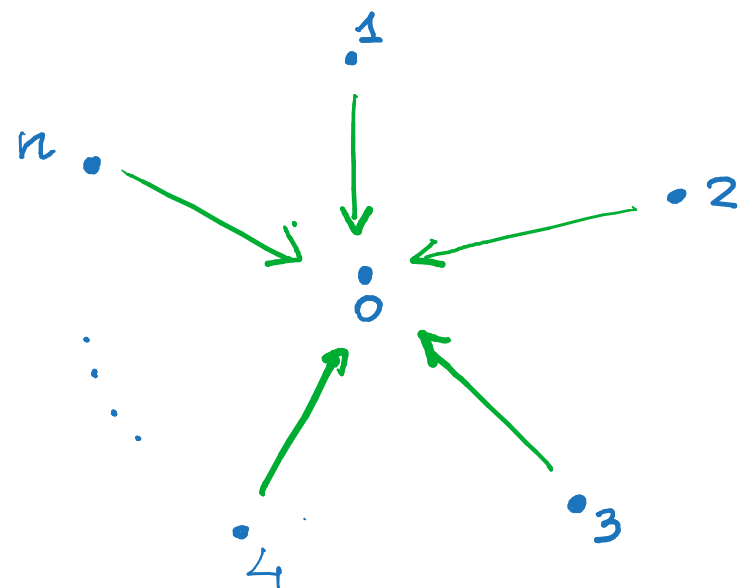
$$\mu(q_1, \dots, q_n) = \left(\frac{\sqrt{-1}}{2} \sum_{i=1}^n (q_i q_i^*)_0, \dots, \frac{1}{2} |q_i|^2, \dots \right)$$

$$\mathbb{C}^{2n} /_{(0, \alpha)} K := \mu^{-1}(0, \alpha) / K$$

POLYGON SPACES AS QUIVER VARIETIES

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$$\mathbb{C}^{2n} /_{(0, \alpha)} K := \mu^{-1}(0, \alpha) / K = M(\alpha)$$

$$\mathbb{R}^3 \cong \mathfrak{su}(2)^*$$

LENGTH OF V_i
 α_i

CLOSING
CONDITION

$$\sum_i v_i = 0$$

$$[q_1, \dots, q_n] \rightarrow [v_1, \dots, v_n] \text{ where}$$

$$v_i = (q_i q_i^*)_0$$

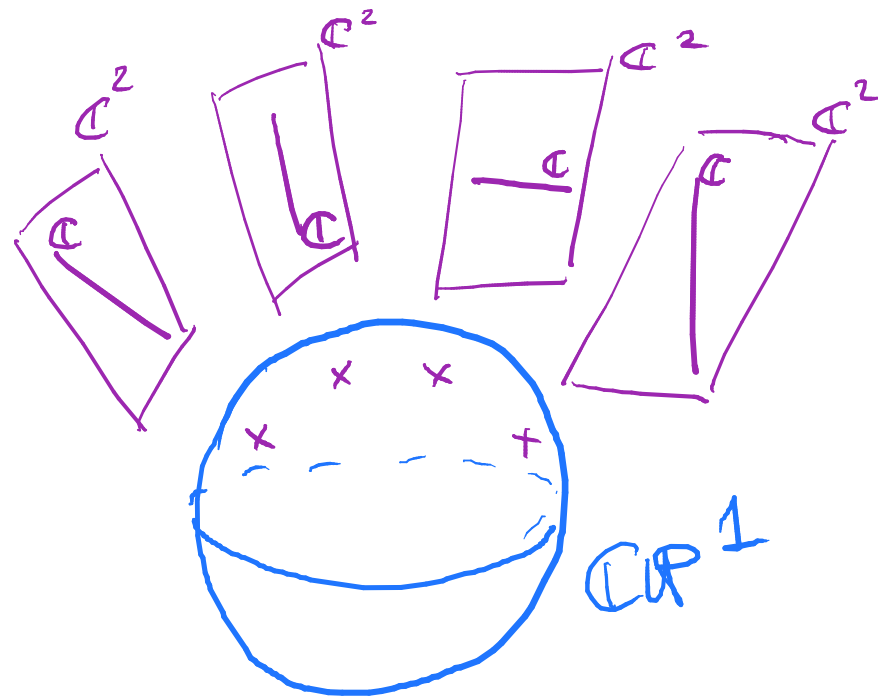
PARABOLIC BUNDLES

$E \longrightarrow \mathbb{C}P^1$ hol. $\text{rank}(E) = r = 2$

$D = \{x_1, \dots, x_n\}$ marked points on $\mathbb{C}P^1$, $x_i \neq x_j \ \forall i \neq j$

A quasi-parabolic structure on E is a full flag

$$E_{x_i} = E_{x_i,1} \supset E_{x_i,2} \supset \{0\} \quad \forall x_i \in D$$

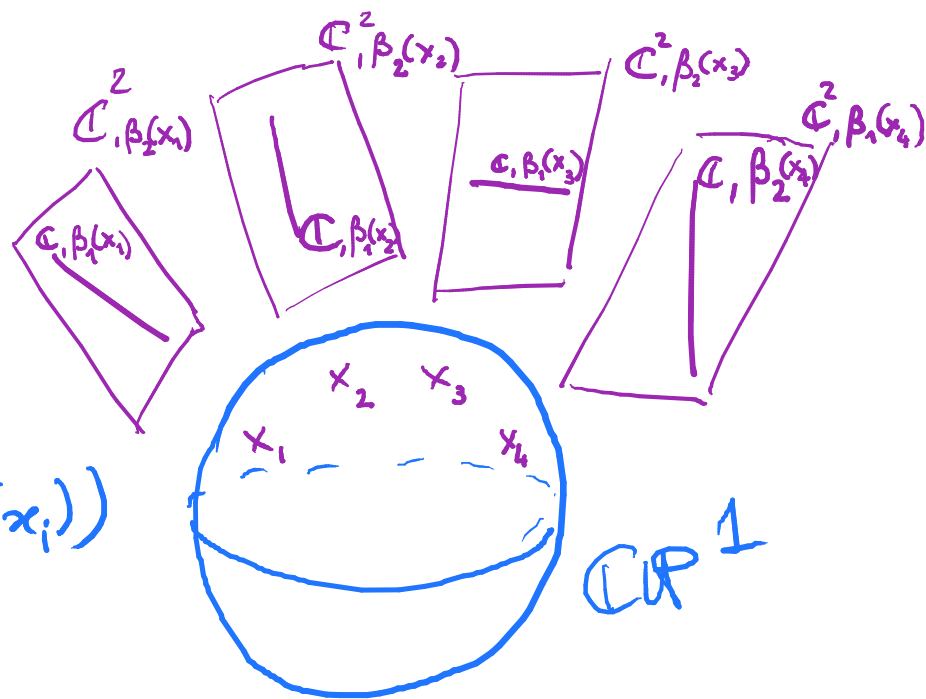


PARABOLIC BUNDLES

Boden - Yokogawa

A parabolic structure on a quasi-parabolic bundle E is the choice of parabolic weights

$$0 \leq \beta_1(x_i) < \beta_2(x_i) < 1 \quad \forall x_i \in D$$



$$\text{par deg}(E) = \text{deg}(E) + \sum_{i=1}^M (\beta_2(x_i) + \beta_1(x_i))$$

$$\text{par } \mu(E) = \frac{\text{par deg}(E)}{2}$$

E is semi-stable $\Leftrightarrow \mu(E) \geq \mu(L) \quad \forall L \subset E$ parab. subbundle

$\leadsto \mathcal{M}(\beta) =$ moduli space of \checkmark stable PBs over \mathbb{P}^1 , rk 2, deg 0

$\mathcal{M}(\beta)$ is smooth \Leftrightarrow (stability \Leftrightarrow semistability)

polygons

M.A.P.B.s

$$M(\alpha) \cong M(\beta) \quad \text{whenever } \alpha_i = \beta_2(x_i) - \beta_1(x_i)$$

$M(\alpha)$

$M(\beta)$



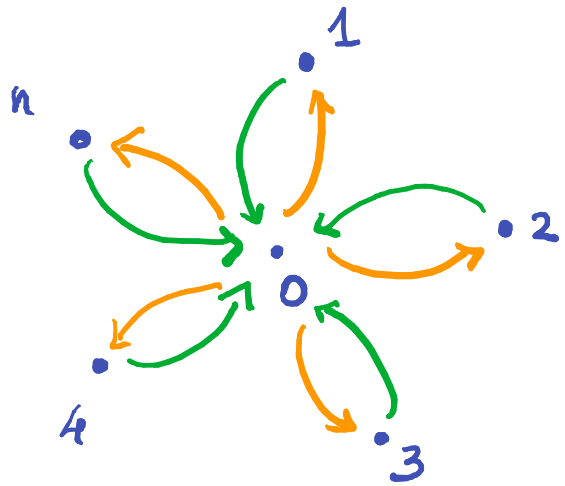
$$[v_1, \dots, v_n] = [q_1, \dots, q_n] \longrightarrow E = \mathbb{C}P^1 \times \mathbb{C}^2$$

$$E_{x_i} = E_{x_{i,1}} \supset E_{x_{i,2}} \supset 0$$

$$\mathbb{C}^2 \xleftarrow{q_i} \mathbb{C}$$

Godinho 13

α -HYPERPOLYGON SPACES



$$V_0 = \mathbb{C}^2$$

$$V_i = \mathbb{C}$$

$$\forall i = 1, \dots, m$$

$$\text{Rep } \mathcal{A} = \bigoplus_{i=1}^m \text{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \bigoplus_{i=1}^m \text{Hom}(\mathbb{C}^2, \mathbb{C}) = T^* \mathbb{C}^{2m}$$

$$q_i : \mathbb{C} \hookrightarrow \mathbb{C}^2$$

$$p_i : \mathbb{C}^2 \rightarrow \mathbb{C}$$

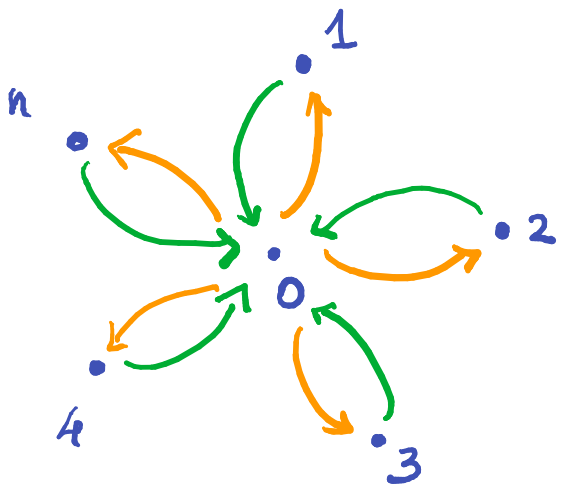
$T^* \mathbb{C}^{2n}$ has a natural hyperkähler structure $g, \omega_I, \omega_J, \omega_K$



$$K \subset T^* \mathbb{C}^{2n}, [p, q](A, \lambda_1, \dots, \lambda_n) = [\dots, \lambda_i^{-1} p_i A, \dots, A^{-1} q_i \lambda_i, \dots]$$

Hyperham.

α -HYPERPOLYGON SPACES



$$V_0 = \mathbb{C}^2$$

$$V_i = \mathbb{C}$$

$$\forall i = 1, \dots, m$$

$$\text{Rep } \alpha = \bigoplus_{i=1}^m \text{Hom}(\mathbb{C}, \mathbb{C}^2) \oplus \bigoplus_{i=1}^m \text{Hom}(\mathbb{C}^2, \mathbb{C}) = T^* \mathbb{C}^{2n}$$

$$q_i: \mathbb{C} \longleftrightarrow \mathbb{C}^2$$

$$p_i: \mathbb{C}^2 \longrightarrow \mathbb{C}$$

$T^* \mathbb{C}^{2n}$ has a natural hyperkähler structure

$$K \curvearrowright T^* \mathbb{C}^{2n}, [p, q](A, \lambda_1, \dots, \lambda_n) = [\dots, \lambda_i^{-1} p_i A, \dots, A^{-1} q_i \lambda_i, \dots]$$

hyperhamiltonian

$$\mu_{\mathbb{R}}(p, q) = \left(\frac{\sqrt{-1}}{2} \sum_{i=1}^n (q_i q_i^* - p_i^* p_i)_0, \dots, \frac{1}{2} (|q_i|^2 - |p_i|^2), \dots \right)$$

$$\mu_{\mathbb{C}}(p, q) = \left(\sum_{i=1}^m (q_i p_i)_0, \dots, p_i q_i, \dots \right)$$

$$\rightsquigarrow X(\alpha) := \frac{T^* \mathbb{C}^{2n}}{(0, \alpha)(0, 0)} \Big/ K = \mu_{\mathbb{R}}^{-1}(0, \alpha) \cap \mu_{\mathbb{C}}^{-1}(0, 0) \Big/ K$$

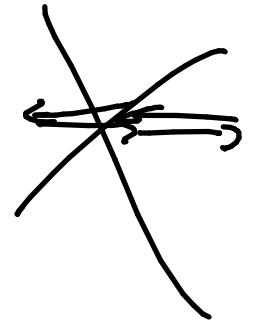
HYPERPOLYGON SPACE

• $X(\alpha)$ is a smooth hyperkähler manifold of $\dim = 4(n-3)$

$$\varepsilon_I(\alpha) \neq 0 \quad \forall I \subseteq \{1, \dots, n\}$$

$$\varepsilon_I(\alpha) = \sum_{i \in I} \alpha_i - \sum_{j \in I^c} \alpha_j$$

• $M(\alpha) = \{[p, q] \in X(\alpha) \mid p = 0\}$



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... ok great ... so ...

what's an hyperpolygon ??

• $X(\alpha)$ is a smooth hyperkähler manifold of $\dim = 4(n-3)$
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... ok great ... so ...

what's an hyperpolygon ??

A hyperpolygon $[p, q]$ is a polygon in $\text{Lie}(SL(2, \mathbb{C}))^*$

[Baalch]

$$G = SL(2, \mathbb{C})$$

PARABOLIC HIGGS BUNDLES

$E \rightarrow \mathbb{CP}^1$ as before, with parabolic points at $D = \{x_1 \dots x_n\}$

HIGGS FIELD $\phi \in H^0(\mathbb{P}^1, \text{SPEnd}(E) \otimes K(D))$

$f: E \rightarrow E$ is a strongly parabolic endomorphism
iff $f(E_{x_{i,j}}) \subseteq E_{x_{i,j+1}}$

(E, ϕ) PARABOLIC HIGGS BUNDLE

$$E_{x_{i,j}} \supset E_{x_{i,j+1}}$$

(E, ϕ) is semi-stable $\Leftrightarrow \mu(E) \geq \mu(L) \quad \forall L \subset E$ parabolic Higgs subbundles

$\rightsquigarrow \mathcal{H}(\beta) =$ moduli space of β -stable, rk 2, hdom. trivial PHBs over \mathbb{CP}^1 w/ fixed determinant and trace-free Higgs field.

Theorem [Godinho-M. 2013 + 2021]

(i) $\mathcal{M}(\alpha) \simeq \mathcal{M}(\beta)$ whenever $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$ generic

(ii) $\mathcal{X}(\alpha) \simeq \mathcal{H}(\beta)$ 4



[p, q] α -hyperpolygon \rightarrow

$E = \mathbb{C}P^1 \times \mathbb{C}^2$ ✓

$E_{x_i} = E_{x_{i,1}} \supset E_{x_{i,2}} \supset 0$ ✓

$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{q_i} & \mathbb{C} \\ & & \uparrow \\ & & \mathbb{C} \end{array}$

remember

$\sum_i (q_i p_i)_0 = 0$

"RESIDUE THEOREM"

$\exists! \phi$ s.t. $\text{Res}_{x_i} \phi = (q_i p_i)_0$

INVOLUTIONS - 1

[Biswas, Florentino, Godinho, M.]

$$i_1: X(\alpha) \longrightarrow X(\alpha)$$

$$[p, q] \mapsto [-p, q]$$



$$i_1: \mathcal{H}(\beta) \longrightarrow \mathcal{H}(\beta)$$

$$[E, \phi] \mapsto [E, -\phi]$$

The components of the fixed locus $\mathcal{H}(\beta)^{i_1}$ are:

- $\mathcal{H}(\beta) = \{(E, \phi \equiv 0)\}$ m.s. of parabolic bundles
- Z_S , for all $S \subseteq \{1, \dots, n\}$ s.t. $|S| \geq 2$ and $\varepsilon_S(\alpha) < 0$: $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$

$$Z_S = \left\{ (E, \phi) : E = L_0 \oplus L_1 ; \text{Res}_{x_i} \phi = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \text{Res}_{x_i} \phi = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\}$$

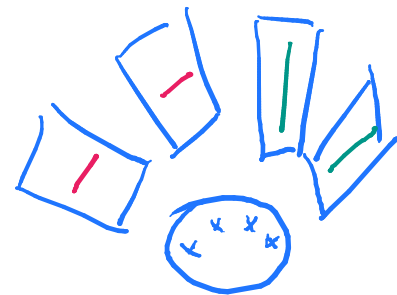
\downarrow $\forall i \in S$ $\forall i \in S^c$

$$(L_0)_{x_i} = E_{x_i, 2}$$

$i \in S$

$$(L_1)_{x_i} = E_{x_i, 2}$$

$i \in S^c$



G-POLYGON SPACE

G Lie group, $\mathfrak{g} = \text{Lie}(G)$, \mathfrak{g}^* its dual

Choose $\xi_i \in \mathfrak{g}^*$, $\xi_i \neq 0 \forall i=1, \dots, n \rightsquigarrow \mathcal{O}_{\xi_i} = \text{coadjoint orbit}$

$$G \hookrightarrow \prod_{i=1}^n \mathcal{O}_{\xi_i} \xrightarrow{\mu_G} \mathfrak{g}^* \Rightarrow \prod_{i=1}^n \mathcal{O}_{\xi_i} // G$$

$\text{Ham. } (A_1, \dots, A_n) \longmapsto \sum_{i=1}^n A_i$

G-polygon space

$G = \text{SU}(2) \rightsquigarrow \text{moduli space of polygons in } \mathbb{R}^3$

$G = \text{SL}(2, \mathbb{C}) \rightsquigarrow \text{HYPERPOLYGONS}$

$\text{SL}(2, \mathbb{R})$

SL(2, ℝ) - POLYGON SPACES

Lie (SL(2, ℝ)) $\simeq \mathbb{R}^{2,1} = (\mathbb{R}^3, (-, -, +))$ MINKOWSKI 3-SPACE

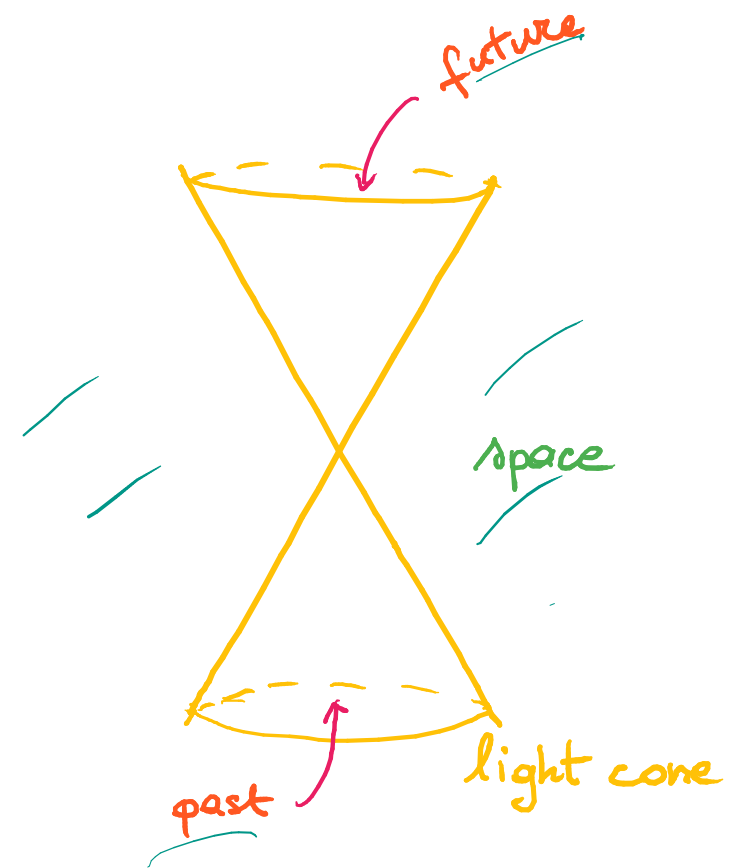
i.e. $u = (x, y, t) \in \mathbb{R}^{2,1}$, $u \cdot u = -x^2 - y^2 + t^2$

$u \in \mathbb{R}^{2,1}$ is

- SPACE-LIKE, if $u \cdot u < 0$
- LIGHT-LIKE, if $u \cdot u = 0$
- TIME-LIKE, if $u \cdot u > 0$

$u = (x, y, t) \in \mathbb{R}^{2,1}$ is

- future if $t > 0$
- past if $t < 0$



Coadjoint orbits of $SL(2, \mathbb{R}) \simeq \mathbb{R}^{2,1}$

$$S_R := \{(x, y, t) \in \mathbb{R}^{2,1} \mid -x^2 - y^2 + t^2 = R^2\} \begin{cases} S_R^+ & \text{if } t > 0 \\ S_R^- & \text{if } t < 0 \end{cases} \begin{array}{l} \text{FUTURE} \\ \text{PSEUDOSPHERE} \\ \\ \text{PAST} \\ \text{PSEUDOSPHERE} \end{array}$$

$$H_R := \{(x, y, t) \in \mathbb{R}^{2,1} \mid +x^2 + y^2 - t^2 = R^2\} \quad \underline{\text{SPACE PSEUDOSPHERE}}$$

$$C^- = \{(x, y, t) \in \mathbb{R}^{2,1} : x^2 + y^2 - t^2 = 0, t < 0\} \leftarrow \text{the past light cone}$$

$$C^+ = \{(x, y, t) \in \mathbb{R}^{2,1} : x^2 + y^2 - t^2 = 0, t > 0\} \leftarrow \text{the future light cone}$$

$$\{0\}$$

LIGHT-LIKE CASE

[Godinho-M 2021]

$C^- = \{(x, y, t) \in \mathbb{R}^{2,1} : x^2 + y^2 - t^2 = 0, t < 0\}$ ← the past light cone

$C^+ = \{(x, y, t) \in \mathbb{R}^{2,1} : x^2 + y^2 - t^2 = 0, t > 0\}$ ← the future light cone

$$SL(2, \mathbb{R}) \curvearrowright \prod_{i \in S} C^- \times \prod_{i \in S^c} C^+ \xrightarrow{\mu_{SL(2, \mathbb{R})}} \mathbb{R}^{2,1}$$

$$(u_1, \dots, u_n) \longmapsto \sum_{i=1}^n u_i$$

$$\mu^{-1}(0)_{\text{reg}} := \mu_{SL(2, \mathbb{R})}^{-1}(0) \cap \left(\prod_{i \in S} C^- \times \prod_{i \in S^c} C^+ \right)_{\text{reg}}$$

$$\rightarrow M_0^{|S|} = \mu^{-1}(0)_{\text{reg}} / SL(2, \mathbb{R}) = \mathbb{Z}_S \quad \text{MODULI SPACE OF CLOSED NULL POLYGONS in } \mathbb{R}^{2,1}$$

is a symplectic non-compact manifold, non-empty $\Leftrightarrow n \geq 4$.

Theorem [BFGM] $i_1([p, q]) = [-p, q]$ $i(\varepsilon, \phi) = (\varepsilon, -\phi)$

The components $Z_S \subseteq X(\alpha)$ and $Z_S \subseteq \mathcal{H}(\beta)$ are isomorphic to $M^S(\alpha)$



$$(X(\alpha) \cong) Z_S \longrightarrow M^S(\alpha)$$

$$[p, q] \longmapsto [u_1, \dots, u_n]$$

where

$$u_i = \frac{\sqrt{-1}}{2} (p_i^* p_i - q_i q_i^*) + \frac{1}{2} (p_i^* q_i^* + q_i p_i)$$

NULL HYPERPOLYGON SPACES

As before, consider

$$K = (U(2) \times U(1)^n) / \Delta \hookrightarrow T^*\mathbb{C}^{2n} \xrightarrow{\mu_{HK} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}} \mathcal{K}^* \oplus \mathcal{K}_{\mathbb{C}}^*$$

$$\mathcal{P}_0^n := \left\{ (p, q) \in \mu_{\mathbb{R}}^{-1}(0, 0) \cap \mu_{\mathbb{C}}^{-1}(0, 0) : |p_i|^2 + |q_i|^2 \neq 0 \quad \forall i=1, \dots, n \right\}$$

PROP. (Godinho-M., 2021): K acts freely on \mathcal{P}_0^n and

$$X_0^n := \mathcal{P}_0^n / K$$

is a smooth hyperkähler manifold of $\dim = 4(n-3)$

INVOLUTIONS - 0

[Godimho, Π 2021]

$$i_0: X_0^n \longrightarrow X_0^n$$

$$[p, q] \longmapsto [q^t, p^t]$$

The fixed point set of i_0 has $2^{n-1} - (n+1)$ components Z_S

$$Z_S = \{ [p, q] : S \text{ \& } S^c \text{ are } \underline{\text{straight}} \text{ at } [p, q] \}$$

$$\forall S \subseteq \{1, \dots, n\} \text{ s.t. } 2 \leq |S| \leq n-2 \text{ and } 1 \in S.$$

TIME-LIKE CASE

$$S_R := \{(x, y, t) \in \mathbb{R}^{2,1} \mid -x^2 - y^2 + t^2 = R^2\}$$

S_R^+ if $t > 0$

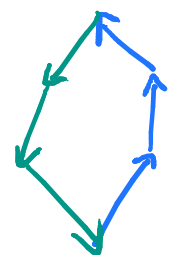
S_R^- if $t < 0$

FUTURE PSEUDOSPHERE

PAST PSEUDOSPHERE

$$SL(2, \mathbb{R}) \overset{\text{Ham.}}{\curvearrowright} \prod_{i \in S} S_{\alpha_i}^+ \times \prod_{i \in S^c} S_{\alpha_i}^- \xrightarrow{\mu_{SL(2, \mathbb{R})}} \text{Lie } SL(2, \mathbb{R})^* \simeq \mathbb{R}^{2,1}$$

$(\alpha_1, \dots, \alpha_n)$ $\longmapsto \sum_{i=1}^n \mu_i$



$$M^S(\alpha) := \prod_{i \in S} S_{\alpha_i}^+ \times \prod_{i \in S^c} S_{\alpha_i}^- \Big/_{\text{SL}(2, \mathbb{R})} = \mu_{SL(2, \mathbb{R})}^{-1}(0) \Big/_{\text{SL}(2, \mathbb{R})}$$

[Foth] $M^S(\alpha)$ is smooth $\Leftrightarrow \varepsilon_I(\alpha) \neq 0 \quad \forall I \subseteq \{1, \dots, n\}$

$M^S(\alpha)$ is non-compact unless $|S|=1$ or $|S^c|=1$

Theorem [Godimko - II. 2019] $i_p([p, q]) = [q^t, q^t]$

For any $S \subseteq \{1, \dots, n\}$ s.t. $1 \in S$ and $2 \leq |S| \leq n-2$,
 the components $Z_S \subseteq (X_0^n)^{i_0} \simeq (\mathcal{H}_0^n)^{i_0}$ are
 isomorphic to $M_0^{|S|}$.



$$\begin{array}{ccc} Z_S & \longrightarrow & M_0^{|S|} \\ [p, q] & \longmapsto & (u_1, \dots, u_n) \end{array}$$

where, if $p_i = (0, b_i)$ & $q_i = \begin{pmatrix} c_i \\ 0 \end{pmatrix} \quad \forall i \in S$

$p_j = (a_j, 0)$ & $q_j = \begin{pmatrix} 0 \\ d_j \end{pmatrix} \quad \forall j \in S^c$

$$\Rightarrow u_i = (\operatorname{Re}(b_i c_i), \operatorname{Im}(b_i c_i), -|c_i|^2) \quad \forall i \in S$$

$$u_j = (\operatorname{Re}(a_j d_j), \operatorname{Im}(a_j d_j), |a_j|^2) \quad \forall j \in S^c$$

QUASI-PARABOLIC HIGGS BUNDLES

$$E \longrightarrow \mathbb{CP}^1 \quad \text{rank}(E) = r = 2$$

$D = \{x_1, \dots, x_n\}$ marked points on \mathbb{CP}^1 , $x_i \neq x_j \quad \forall i \neq j$

A quasi-parabolic structure on E is a full flag

$$E_x = E_{x_i,1} \supset E_{x_i,2} \supset \{0\} \quad \forall x_i \in D$$

$$q\text{-par deg}(E) = \text{deg}(E) + 2n; \quad q\text{-par } \mu(E) = r + \frac{\text{deg}(E)}{2}$$

A quasi-parabolic Higgs bundle over \mathbb{CP}^1 , rk 2,

is a pair (E, ϕ) with E as above and

$$\phi \in H^0(\mathbb{CP}^1, \text{SQPEnd}(E) \otimes K_{\mathbb{CP}^1}(D))$$

where $f: E \rightarrow E$ is a strongly quasi-parabolic end.
iff $f(E_{x_i,1}) \subset E_{x_i,2}$ and $f(E_{x_i,2}) \subset 0 \quad \forall x_i \in D$

DEF, A quasi-parabolic $SL(2, \mathbb{C})$ -Higgs bundle over $\mathbb{C}P^1$ at D is a rank-2 q -parabolic Higgs bundle (E, ϕ) , where E has trivial determinant and ϕ is traceless

DEF (stability)

A quasi-parabolic Higgs bundle (E, ϕ) is stable if

① $\text{Res}_{x_i} \phi \neq 0 \quad \forall x_i \in D$

② $q\text{-par } \mu(E) > q\text{-par } \mu(L) \quad \forall L \subset E \text{ quasi-parab. subb. that is preserved by } \phi$

\mathcal{M}_0^n = moduli space of stable $q.p.$ $SL(2, \mathbb{C})$ -Higgs bundles

Theorem [Godinho-M. 2013 + 2021]

- (i) $X(\alpha) \simeq \mathcal{H}(\beta)$ whenever $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$ generic
- (ii) $X_0^n \simeq \mathcal{H}_0^n$



[P, q] α -hyperpolygon
 0 -hyperpolygon

$\rightarrow E = \mathbb{C}P^1 \times \mathbb{C}^2$

$$E_{x_i} = E_{x_{i,1}} \supset E_{x_{i,2}} \supset 0$$

$$\begin{matrix} | & & | \\ \mathbb{C}^2 & \xrightarrow{q_i} & \mathbb{C} \end{matrix}$$

remember

$\sum_i (q_i p_i)_0 = 0$

"RESIDUE THEOREM"

$\exists! \phi$ s.t. $\text{Res}_{x_i} \phi = (q_i p_i)_0$

BASED ON :

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THANKS!