Stacks in Poisson geometry

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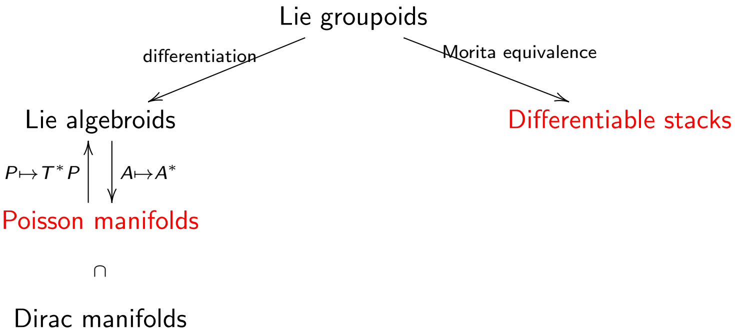
1. Introduction

Poisson and Stacks



Two problems

Linearization of Poisson structures

A Poisson structure induces a Lie algebroid
Integrate it to a Lie groupoid
Linearize the Lie groupoid
And linearization is Morita invariant!

$\Rightarrow$ simpler proof, generalization, rigidity

Desingularization of Dirac manifolds

Poisson manifolds ARE symplectic Lie algebroids
$\omega \in \Omega^2(A)$ IM, $\omega^\# : TA \to T^*A$ non-degenerate, ω closed
VB-algebroids are homotopy category
Dirac manifolds ARE symplectic up to homotopy Lie algebroids

$\Rightarrow$ revisiting pre-symplectic groupoids BCWZ

What is a stack?

Stacks are categorified spaces introduced by Grothendieck in 1959.

A manifold M yields a representable functor $\text{Manifolds}^\circ \xrightarrow{Hom(-,M)} \text{Sets}$

This functor is locally determined, for a function $X \to M$ can be reconstructed out of its restrictions $U_i \to M$ to an open over $\{U_i\}_i$.

A stack F is a locally determined functor $\text{Manifolds}^\circ \xrightarrow{F} \text{Groupoids}$

$$\boxed{\text{Yoneda: } F(X) = \{Hom(-,X) \to F\}}$$

Paradigmatic Example

M manifold $\rightsquigarrow$ $VB(M)$ vector bundles over M + isomorphisms

What is a differentiable stack?

A stack F sets a classification problem.

M is the universal / moduli / classifying space for F if $Hom(-, M) \cong F$
It does not exists in general!

A presentation is a surjective submersion $Hom(-, M) \to F$.

F is a differentiable stack if it admits a presentation.

Examples

Manifolds, orbifolds, orbit spaces of actions, leaf spaces of foliations, finite dimensional models for classifying spaces, ...

Differential geometry over singular quotients!

Groupoids from the stack viewpoint

- Given $G \rightrightarrows M$, the presheaf of groupoids F_G is a prestack.

$$F_G(X) = (\hom(X, G) \rightrightarrows \hom(X, M))$$

Its associated stack $\tilde{F}_G$ can be presented as $M \to \tilde{F}_G$.

- Given F a smooth stack and $M \to F$ a presentation of it, we construct a Lie groupoid by

$$M \times_F M \rightrightarrows M.$$

Theorem (Folklore)

A Lie groupoid is the same as a differentiable stack with a presentation.

Stacks from the groupoid viewpoint

Every Lie groupoid has an underlying differentiable stack

$$G \rightrightarrows M \qquad \mapsto \qquad [M/G]$$

When two groupoids have the same differentiable stack?
How to define stacks within the framework of Lie groupoids?

Morita equivalences

Theorem (Folklore)

A differentiable stack is the same as a Lie groupoid up to Morita equivalence.

Morita equivalences within Lie gpd theory $\begin{cases} \text{principal bibundles} \\ \text{Morita morphisms} \end{cases}$

2. Morita equivalences

Recalling Lie groupoids

A Lie groupoid $G \rightrightarrows M$ consists of:

- manifolds G, M and surjective submersions $s, t : G \to M$,
- a partial associative multiplication with units and inverses

$$m : G \times_M G \to G \quad (z \xleftarrow{h} y, y \xleftarrow{g} x) \mapsto (z \xleftarrow{hg} x)$$

Given $G \rightrightarrows M$ and $x \in M$ we have the isotropy group $G_x = \{x \leftarrow x\}$ and the orbit $O_x = \{y | \exists y \leftarrow x\} \subset M$. There is a normal representation $G_x \curvearrowright N_x O = T_x M / T_x O$. The orbit space is M/G.

Examples

Manifolds, Lie groups, Group actions, Submersions, Foliations, Principal bundles, pseudo-groups, flows, etc etc

Morita morphisms

A Lie groupoid morphism $\phi : (G \rightrightarrows M) \xrightarrow{\sim} (G' \rightrightarrows M')$ consists of maps $G \to G'$, $M \to M'$ preserving source, target, multiplication and units.

- ϕ is fully faithful if it induces a good pullback of manifolds,

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ \downarrow & & \downarrow \\ M \times M & \xrightarrow{\phi \times \phi} & M' \times M' \end{array}$$

- ϕ is essentially surjective if the following is a surjective submersion.

$$t\pi_1 : G' \times_{M'} M \to M' \qquad (x' \xleftarrow{g'} \phi(x), x) \mapsto x'$$

- ϕ is a Morita if it is both FF and ES.

There may not exists a quasi-inverse!

An example

A submersion $f : M \to B$ yields a Lie groupoid $M \times_B M \rightrightarrows M$ with an arrow $y \xleftarrow{(y,x)} x$ iff $f(x) = f(y)$. The projection yields a morphism

$$\phi : (M \times_B M \rightrightarrows M) \to (B \rightrightarrows B)$$

Then:

- ϕ isomorphism iff f diffeomorphism;
- ϕ equivalence iff f admits a global section;
- ϕ Morita iff f surjective.

Particular case: M manifold, $\{U_i\}_i$ open cover, $\coprod U_i \to M$ surjective submersion. The Cech groupoid has objects (x, i) with $x \in U_i$ and arrows (x, j, i) with $x \in U_{ji}$.

$$\coprod U_{ji} \rightrightarrows \coprod U_i$$

Characterization

Intuition: Two groupoids are linked by Morita maps if they have the same transverse geometry.

In fact, we have:

Theorem (dH, 2013)

$\phi : (G \rightrightarrows M) \xrightarrow{\sim} (G' \rightrightarrows M')$ *Morita morphism if and only if:*

- $\phi_* : M/G \xrightarrow{\cong} M'/G'$ *homeomorphism,*
- $\phi_x : G_x \xrightarrow{\cong} G'_{x'}$, *isomorphism for all* x, *and*
- $d_x\phi : N_x O \xrightarrow{\cong} N_{x'} O'$ *isomorphism for all* x.

The family of Morita maps is a saturated multiplicative system.
$\rightsquigarrow$ good properties for localization (calculus of fractions).

Morita equivalence

Two Lie groupoids $G \rightrightarrows M$, $G' \rightrightarrows M'$ are Morita equivalent $\sim$ if there is a third one $\tilde{G} \rightrightarrows \tilde{M}$ and Morita morphisms:

$$(G \rightrightarrows M) \underset{\sim}{\overset{\alpha}{\leftarrow}} (\tilde{G} \rightrightarrows \tilde{M}) \underset{\sim}{\overset{\beta}{\rightarrow}} (G' \rightrightarrows M')$$

Example

- Morita equivalent to a manifold iff submersion gpd
- Morita equivalent to a Lie group iff transitive groupoid
- Mon and Hol are not Morita equivalent in general!

Orbifold $->$ many orbifold charts $->$ Morita equivalent Lie groupoids

Generalized maps

A generalized map $\beta/\alpha : G \to G'$ of Lie groupoids is the class of a fraction β/α with α Morita:

$$(G \rightrightarrows M) \underset{\sim}{\xleftarrow{\alpha}} (\tilde{G} \rightrightarrows \tilde{M}) \xrightarrow{\beta} (G' \rightrightarrows M')$$

Two fractions β/α, β'/α' are equivalent if they fit in a diagram commutative up to isom.

Fractions can be composed (HFP) and yield a well-defined category:

Differentiable stacks

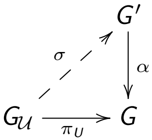
β/α invertible $\iff$ β Morita morphism $\iff$ β, α Morita equivalence

Relation with cocycles

$G \rightrightarrows M$ Lie groupoid, $\mathcal{U} = \{U_i\}$ open cover of M. The **Cech fibration** is a Morita morphism.

$$\left(\coprod G(U_j, U_i) \rightrightarrows \coprod U_i \right) \xrightarrow{\pi_{\mathcal{U}}} (G \rightrightarrows M)$$

They generate all the others: $\forall \alpha$ Morita $\exists\, \mathcal{U}, \sigma$ st



Proposition

Every generalized map can be defined as by a cocycle $\beta/\pi_{\mathcal{U}}$.

Example

Fractions $(M \rightrightarrows M) \leftarrow (\coprod U_{ji} \rightrightarrows \coprod U_i) \to (G \rightrightarrows *)$ are G-cocycles
Generalized maps $M \dashrightarrow G$ are principal bundles.

Cohomology

Given $G \rightrightarrows M$ Lie gpd, its nerve (NG_k, d_i, s_k) is the simplicial manifold

$$NG_k = \{x_k \xleftarrow{g_k} x_{k-1} \xleftarrow{g_{k-1}} \cdots \xleftarrow{g_2} x_1 \xleftarrow{g_1} x_0\}$$

- Faces $d_i : NG_k \to NG_{k-1}$ compose two arrows (or forget one)
- Degeneracies $s_j : NG_k \to NG_{k+1}$ insert an identity

The cohomology is $H^k(G) = H^k(\oplus_{p+q=k}\Omega^p(NG_q), d_{DR} \pm \sum_i (-1)^i d_i^*)$

$$\begin{array}{ccccccc}
\dots & & \dots & & \dots & & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^2(M) & \to & \Omega^2(G) & \to & \Omega^2(G \times_M G) & \to & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^1(M) & \to & \Omega^1(G) & \to & \Omega^1(G \times_M G) & \to & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^0(M) & \to & \Omega^0(G) & \to & \Omega^0(G \times_M G) & \to & \dots
\end{array}$$

Proposition

$H^k(G)$ is Morita invariant ($\Rightarrow$ cohomology of stacks!)

Representations

Given $G \rightrightarrows M$ a Lie groupoid and $E \to M$ a vector bundle, a **representation** $(G \rightrightarrows M) \curvearrowright (E \to M)$ is a map $\rho : G \times_M E \to E$, $\rho(g, v) = \rho_g(v)$, such that $\rho_h \rho_g = \rho_{hg}$, $\rho_{1_x} = \mathrm{id}_{E_x}$ and ρ_g linear.

$$y \xleftarrow{g} x \qquad \rightsquigarrow \qquad \rho_g : E_x \to E_y$$

Examples

- Vector bundles $(M \rightrightarrows M)$ and group representations $(K \rightrightarrows *)$
- Equivariant vetor bundles $(K \ltimes M \rightrightarrows M)$
- Vector bundle + descent datum $(M \times_B M \rightrightarrows M)$

$$(G \rightrightarrows M) \curvearrowright (E \to M) \qquad \rightleftharpoons \qquad (G \rightrightarrows M) \to (GL(E) \rightrightarrows M)$$

Proposition

Representations are a Morita invariant ($\Rightarrow$ vector bundles over stacks?)

3. Linearization and metrics

Linearization of Poisson manifolds

(M, π) Poisson, $x \in M$ a zero.
Then $T_x^* M$ is Lie algebra with $[v, w] = \pi(\tilde{v}, \tilde{w})$, and $T_x M$ linear Poisson.
(M, π) linearizable around x if it is locally equivalent to $(T_x M, \{,\}_{lin})$.

Conn's Theorem

(M, π) Poisson, $x \in M$ a zero, $T_x M^*$ semisimple of compact type, then π is linearizable around x.

- Geometric proof (Crainic-Fernandes, Annals 2011)
- Key: linearization of Lie groupoids Weinstein 2002, Zung 2004
- New approach to lin. of groupoids: dH-Fernandes (2014-2018)
- Linearization around a symplectic leaf: Vorobiev (2006), Crainic-Marcut (2012)

Riemannian groupoids

$f : M \to N$ submersion. A metric η^M on M is f-transverse if for all x, x' in the same fiber F there is an isometry $N_x F \cong T_y N \cong N_{x'} F$.

$$\boxed{\eta^M \text{ is } f\text{-transverse} \Leftrightarrow \exists\ \eta^N \text{ making } f \text{ a Riemannian submersion}}$$

Definition [dH-Fernandes]

A groupoid metric on $G \rightrightarrows M$ is a metric $\eta^{(2)}$ on

$$G^{(2)} = G \times_M G = \{z \xleftarrow{h} y \xleftarrow{g} x\} = \{\text{commutative triangles}\}$$

such that:

- $\eta^{(2)}$ is invariant under $S_3 \curvearrowright G^{(2)}$, and
- $\eta^{(2)}$ is transverse to $m : G^{(2)} \to G$.

This extends/corrects the previous definitions [Gallego et al, 1989], [Glickenstein, 2007] and [Pflaum et al, 2011].

A system of metrics

Proposition

Given $G \rightrightarrows M$ and $\eta^{(2)}$, there exist metrics $\eta^{(1)}$ on G and $\eta^{(0)}$ on M such that:

- $G^{(2)} \overset{\pi_1}{\underset{\pi_2}{\overset{m}{\rightrightarrows\!\!\!\to}}} G \overset{s}{\underset{t}{\rightrightarrows}} M$ are Riemannian submersions,
- $i : G \to G$ is an isometry,
- $u : M \to G$ is totally geodesic, and
- the normal representations $G_x \curvearrowright N_x O$ are by isometries.

Examples

Manifolds , submersions , Lie groups , transitive groupoids , action groupoids of isometric actions , etal groupoids with orthogonal effect , orbifolds , etc etc

Proper groupoids

Recall $G \rightrightarrows M$ is proper if $(s,t) : G \to M \times M$ is a proper map.
This extends the notion of proper action.
Being proper is Morita invariant!!

Proper groupoids $\leftrightarrows$ Separated differentiable stacks

If $G \rightrightarrows M$ is proper then:

- the orbit space M/G is Hausdorff,
- the isotropy groups G_x are compact for all x.

Proper groupoids admit Haar systems
(generalization of Haar measures on groups).

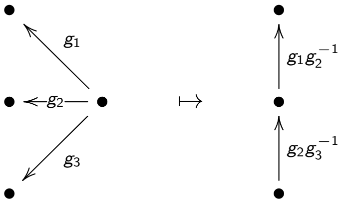
Construction of metrics

Theorem (dH-Fernandes, 2018)

Every proper *Lie groupoid admits a metric.*

Sketch of proof.

Averaging argument.
We see $G^{(2)}$ as a quotient of $\{(g_1, g_2, g_3) | s(g_1) = s(g_2) = s(g_3)\}$:



There are three commuting free proper actions $G \curvearrowright G^{(2)}$. They induce quasi-actions $G \curvearrowright TG^{(2)}$. Pick any metric on $G^{(2)}$ and average it with respect to the three actions. □

Linearization by exponential flows

In a Riemannian submersion $p : (M, \eta^M) \to (N, \eta^N)$ there is a correspondence between horizontal geodesics on M and geodesics on N.

Theorem (Ehresmann)

A proper submersion $p : M \to N$ is locally trivial.

Proof.

Endow M with a transverse metric, making p a Riemannian submersion. Then use the exponential flow.

$$\begin{array}{ccccc}
F \times \mathbb{R}^n & \cong & NF = TF^{\perp} & \overset{\exp}{\dashrightarrow} & M \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle dp} & & \downarrow{\scriptstyle p} \\
\mathbb{R}^n & \cong & T_y N & \overset{\exp}{\dashrightarrow} & N
\end{array}$$

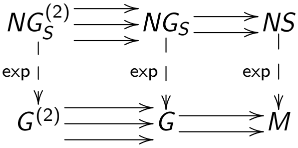
□

Linearization of groupoids

Theorem (dH-Fernandes, 2018)

A Riemannian groupoid $(G \rightrightarrows M, \eta^{(2)})$ *is* *linearizable* *around a saturated embedded submanifold* $S \subset M$.

Sketch of proof.



□

- Infinitesimal linearization.
- If G proper $\rightsquigarrow$ linearization,
- If G is s-proper then $\rightsquigarrow$ strict linearization.

Simpler proof and stronger version of Linearization Theorem [Weinstein, 2001], [Zung, 2004], [Crainic-Struchiner, 2011].

Morita invariance: metrics on stacks

Two metrics on $G \rightrightarrows M$ are equivalent if they induce the same metric on $N_x O$ for all $x \in M$.

Theorem (dH-Fernandes, 2019)

Equivalence classes of metrics is a Morita invariant of $G \rightrightarrows M$.
($\Rightarrow$ metrics on stacks!)

A Morita morphism $\phi : (G \rightrightarrows M, \eta^{(2)}) \to (G' \rightrightarrows M', \eta^{(2)'})$ is isometric if it induces isometries $N_x O \to N_{x'} O'$.

Pullback/pushforward of metrics

Given $\phi : (G \rightrightarrows M) \to (G' \rightrightarrows M')$ Morita map and a metric on G'/G, there exists a metric on G/G' for which ϕ is isometric.

Moving forward: Stacky Hopf-Rinow Thm [dH, de Melo]

4. Vector bundles over stacks

VB-groupoids

A VB-groupoid is a diagram of Lie groupoids and vector bundles

$$\begin{array}{ccc} \Gamma & \rightrightarrows & E \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array}$$

Hidden piece of data: the core $C = \ker(s : \Gamma|_M \to E)$.

$$\Gamma|_M \rightrightarrows E \quad \underset{Dold-Kan}{\rightleftharpoons} \quad C \xrightarrow{\partial} E$$

Core sequence: $0 \to t^*C \to \Gamma \to s^*E \to 0$ plays a key role in duality.

Theorem (Bursztyn-Cabrera-dH 2014)

A VB-groupoid is the same as a regular action $(\mathbb{R}, \cdot) \curvearrowright (\Gamma \rightrightarrows E)$.

Useful in differentiation and integration: VB-groupoids $\leftrightharpoons$ VB-algebroids

Examples of VB-groupoids

Examples

- The tangent groupoid $TG \rightrightarrows TM$ has core A_G;
- The cotangent groupoid $T^*G \rightrightarrows A_G^*$ has core T^*M.

Proposition

A representation is the same as a VB-groupoid with trivial core:

$$(G \rightrightarrows M) \curvearrowright (E \to M) \quad \leftrightharpoons \quad \begin{array}{ccc} G \times_M E & \rightrightarrows & E \\ \downarrow & & \downarrow \\ G & \rightrightarrows & M \end{array} \qquad \text{core } 0_M$$

If $\phi : G' \to G$ Morita then $\phi^* : Rep(G) \to Rep(G')$ equivalence.

Problem

Are VB-groupoids a Morita invariant?
What is the geometry induced by VB-groupoids at the level of stacks?

RUTH

A representation up to homotopy $(G \rightrightarrows M) \curvearrowright (C \oplus E \to M)$ is

- $\partial : C \to E$ a linear map;
- $\rho : G \curvearrowright (C \oplus E)$ a pseudo-representation;
- $\gamma^2_{h,g} : E_x \to C_z$ a chain homotopy between ρ_{hg} and $\rho_h \rho_g$;

satisfying certain coherence axioms.

Theorem (dH, Stefani 2017)

A ruth $(G \rightrightarrows M) \curvearrowright (C \oplus E \to M)$ is the same as a pseudo-functor into the general linear 2-groupoid

$$(G \rightrightarrows M) \dashrightarrow GL(C \oplus E \to M)$$

Grothendieck correspondence

Theorem (GraciaSaz-Mehta 2010, dH-Ortiz 2016)

Correspondence between RUTHs $(G \rightrightarrows M) \curvearrowright (C \oplus E \to M)$ *and VB-groupoids* $(\Gamma \rightrightarrows E) \to (G \rightrightarrows M)$.

A (linear) cleavage Σ on a VB-groupoid $\Gamma \rightrightarrows E$ is a section for $\Gamma \to s^*E$.

$$w \overset{\Sigma(g,v)}{\leftarrow - - -} v$$

$$y \overset{g}{\longleftarrow} x$$

Σ unital if $\mathrm{Id} \subset \Sigma$ and flat if $\Sigma\Sigma \subset \Sigma$.

From VB-groupoids to RUTH: Pick unital cleavage and define:

- ∂ is the target map;
- ρ pushes forward by using the cleavage;
- γ curvature of Σ.

Linear Morita morphisms

The fiber of a VB-groupoid $\Gamma \rightrightarrows E$ over $G \rightrightarrows M$ at $x \in M$ is a 2-vect $\Gamma^x \rightrightarrows E^x$. We have exact sequences

- $1 \to \Gamma_e^x \to \Gamma_e \underset{\leftarrow - -}{\to} G_x \dashrightarrow 1$
- $0 \to T_{u(e)}\Gamma_e^x \to T_{u(e)}\Gamma_e \underset{\leftarrow - -}{\to} T_x G_x \xrightarrow{\partial\ (0)} N_e E_x \to N_e E \underset{\leftarrow - -}{\to} N_x M \to 0$

Theorem (dH-Ortiz 2016)

$\phi : (\Gamma \rightrightarrows E) \to (\Gamma' \rightrightarrows E')$ linear Morita iff φ Morita and ϕ^x Morita for all x:

$$\begin{array}{ccc} \Gamma \rightrightarrows E & \overset{\phi}{\searrow} & \\ \bar{\phi}\downarrow & & \\ \varphi^*\Gamma' \rightrightarrows \varphi^* E' & \longrightarrow & \Gamma' \rightrightarrows E' \\ & & \\ G \rightrightarrows M & \underset{\varphi}{\longrightarrow} & G' \rightrightarrows M' \end{array}$$

Proof.

Char. of Morita maps + sequences above + 5 lemma. □

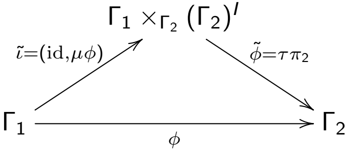
The derived category of VB-groupoids

Question: If $\phi : G \to G'$ Morita then $\phi^* : VB(G') \xrightarrow{\sim} VB(G)$? A: No!

$$E \times E \rightrightarrows E \qquad\qquad \nexists$$

$$S^1 \times S^1 \rightrightarrows S^1 \quad \to \quad * \rightrightarrows *$$

Given G, $VB[G] = VB(G)[\text{Morita}^{-1}]$ derived category of VB-groupoids.
Fact: Every linear epimorphic Morita map over G admits a section.



Using *canonical factorization* we conclude $\Gamma_1 \sim \Gamma_2$ iff there are acyclic Ω_1, Ω_2 such that $\Gamma_1 \oplus \Omega_1 \cong \Gamma_2 \oplus \Omega_2$.

Morita invariance of VB-groupoids

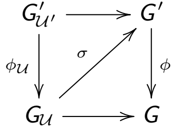
Theorem (dH-Ortiz 2018)

$\phi : G' \to G$ Morita then $\phi^* : VB[G] \to VB[G']$ equivalence.

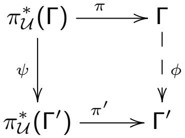
- Notion of 2-vector bundle over differentiable stacks.

Proof.

Step 1:



Reduction to the case of Cech fibration



Step 2: Show it is full by averaging with partition of 1

Step 3: Show faithful by averaging homotopies.

Step 4: Essentially surjective comes from strictification of cleavage.

□

VB-cohomology

A VB-groupoid $\Gamma \to G$ has VB-cohomology $H^\bullet_{VB}(\Gamma)$ obtained by considering linear cochains ϕ that are projectable:

i) $\phi(v_1, \ldots, v_{p-1}, 0_g) = 0$, and

ii) $\phi(v_1, \ldots, v_p 0_g) = \phi(v_1, \ldots, v_p)$.

Combining Morita invariance of $H^\bullet(\Gamma)$ (Crainic 2001) and a section for inclusion $C^\bullet_{lin}(\Gamma) \to C^\bullet(\Gamma)$ (Cabrera-Drummond 2016) we get:

Proposition

Morita invariance of VB-cohomology.

$\Rightarrow$ Deformation cohomology of Lie groupoids is a Morita invariant.

In progress: Morita invariance of cohomology with coefficients in RUTH [dH, Studzinski, Ortiz]

Dirac structures

A Dirac structure on M is $L \subseteq TM \oplus T^*M$ such that:

- Lagrangian for $\langle (X,\alpha),(Y,\beta)\rangle = \alpha(Y) + \beta(X)$, and
- involutive for $[\![(X,\alpha)(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - i_Y \mathrm{d}\alpha)$.

Examples

Poisson structures, pre-symplectic structures, foliations, etc

Proposition

A Dirac structure L over M is the same as:

- A Lie algebroid A;
- a closed IM-2-form $\Lambda \in \Omega^2(A)$;

such that $\Lambda^\# : TA \to T^*A$ quasi-isomorphism of VB-algebroids.

Integrating Dirac structures

Globlal counter-part of Dirac structures:
Pre-symplectic groupoids (Bursztyn-Crainic-Weinstein-Zhu 2004).

- $G \rightrightarrows M$ with $\dim(G) = 2\dim(M)$;
- a closed multiplicative 2-form $\omega \in \Omega^2(G)$

such that $ds(x) \cap dt(x) \cap \ker \omega_x = \{0\}$.

Proposition

$G \rightrightarrows M$ Lie groupoid, ω closed 2-form on G. Then (G, ω) pre-symplectic groupoid iff $\omega^\# : (TG \rightrightarrows TM) \to (T^*G \rightrightarrows A^*)$ linear Morita map.

Corollary

Simple proof of integration of Dirac structures: (i) multiplicative forms correspond to IM forms + (ii) core sequence is preserved.

Thanks!